Computable Constants for Korn's Inequalities on Riemannian Manifolds

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Abstract

A method is presented for the explicit construction of the non-dimensional constant occurring in Korn's inequalities for a bounded two-dimensional Riemannian differentiable simply connected manifold subject to Dirichlet boundary conditions. The method is illustrated by application to the spherical cap and minimal surface.

Keywords Korn's inequalities · Riemannian manifolds · Computable Korn's constants

Mathematics Subject Classification 26D10 · 49J40 · 49Q05 · 53A99 · 74A99

1 Introduction

Korn's inequalities, first formulated in 1906 [26, 27] and further treated by Friedrichs in 1947 [15], have attracted an extensive literature comprehensively reviewed in surveys that include those by Horgan [20], Gurtin [18], Kondratiev and Oleinik [23, 24], and Ciarlet [9, Chap. 6]. Nonlinear versions, partly in the form of rigidity theorems, have been developed by Frieseke, James, and Müller [16], and by Ciarlet, Gratie and Mardare [10], amongst others. The main application of both linear and nonlinear inequalities is to proofs of existence, uniqueness, and continuous data dependence of solutions to boundary value problems in continuum mechanics and related theories.

The inequalities hold for bounded and unbounded regions of Euclidean space $\mathbb{R}^n$, $n = 2, 3$, and interrelate the gradient ∇u of a suitably differentiable vector field u and its symmetric and antisymmetric parts S , A given by

$$\nabla u = S + A, \quad (1.1)$$

where

$$S = \frac{1}{2} (\nabla u + (\nabla u)^T), \quad A = \frac{1}{2} (\nabla u - (\nabla u)^T),$$

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and a superscript T denotes transposition.

For bounded regions $\Omega \subset \mathbb{R}^n$, the *first* Korn's inequality is of the form

$$\int_{\Omega} |\nabla u|^2 dx \leq C \int_{\Omega} |S|^2 dx, \quad (1.2)$$

in which C is a non-dimensional positive constant dependent upon the geometry of Ω . Its optimum value is usually known as Korn's constant, but this term is conveniently used here to signify all constants C that generically feature in Korn's inequalities. Insertion of the decomposition (1.1) into (1.2) leads to the sequence

$$\int_{\Omega} |S|^2 dx \leq \int_{\Omega} (|S|^2 + |A|^2) dx = \int_{\Omega} |\nabla u|^2 dx \leq C \int_{\Omega} |S|^2 dx, \quad (1.3)$$

which shows that $C \geq 1$. Inequality (1.3) also implies both the *second* Korn's inequality (see Friedrichs [15])

$$\int_{\Omega} |A|^2 dx \leq (C - 1) \int_{\Omega} |S|^2 dx, \quad (1.4)$$

and the *third* Korn's inequality

$$\int_{\Omega} |A|^2 dx = \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} |S|^2 dx \leq \left(\frac{C-1}{C} \right) \int_{\Omega} |\nabla u|^2 dx. \quad (1.5)$$

The vector field u in these inequalities must satisfy conditions apart from differentiability. For example, the first Korn's inequality is invalidated when u is a rigid body rotation as then S , but not ∇u , identically vanishes. Various normalisations must be introduced, or the inequalities modified, to account for rigid body displacements and in particular pure rotations. This is achieved here and throughout by confining attention to bounded regions and to admissible functions that vanish on the boundary. Subject to these restrictions, the divergence theorem yields

$$\int_{\Omega} |tr \nabla u|^2 dx + \int_{\Omega} |\nabla u|^2 dx = 2 \int_{\Omega} |S|^2 dx, \quad (1.6)$$

where tr denotes the trace operator. We immediately conclude that

$$\int_{\Omega} |\nabla u|^2 dx \leq 2 \int_{\Omega} |S|^2 dx, \quad (1.7)$$

and therefore (1.2) is established with $C = 2$, which, as remarked by Horgan [20], is the best possible constant.

Other inequalities are obtained by the alternate elimination of $|S|^2$ and $|\nabla u|^2$ between (1.3) and (1.6) but do not improve (1.4) and (1.5) when $C = 2$. For example, elimination of $|S|^2$ (cp. [18, p. 38]) gives

$$2 \int_{\Omega} |A|^2 dx = \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} |tr \nabla u|^2 dx \quad (1.8)$$

$$\leq \int_{\Omega} |\nabla u|^2 dx, \quad (1.9)$$

and (1.5) is recovered when $C = 2$.

Korn's inequalities can be augmented by Poincaré's inequality. For the admissible vector functions just stated, the appropriate Poincaré inequality is

$$\Lambda \int_{\Omega} |u|^2 dx \leq \int_{\Omega} |\nabla u|^2 dx, \quad (1.10)$$

where Λ denotes to the first eigenvalue for the fixed membrane problem on Ω . In conjunction with inequality (1.2), we then have

$$\int_{\Omega} |u|^2 dx \leq \frac{C}{\Lambda} \int_{\Omega} |S|^2 dx. \quad (1.11)$$

Inequalities (1.2), (1.4), and (1.5) have counterparts for other sets of admissible vector functions that, for example, satisfy homogeneous Neumann boundary data. They have also been established for unbounded regions $\Omega \subseteq \mathbb{R}^n$, notably by Kondratiev and Oleinik [25]. A generalised version has been derived by Neff, Pauly and Witsch [34] that separately specialises to the Poincaré's and Korn's inequalities and presents exact constants subject to Dirichlet boundary conditions. Nevertheless, respective Korn's constants in other problems are explicitly known only for a few special geometries discussed, for example, in [20]. In particular, for plane regions in $\mathbb{R}^2$, Horgan and Payne [22] relate Korn's constants to estimated eigenvalues of various membrane problems. Furthermore, Lewicka and Müller [30] discuss the constant for thin regions, while Paroni and Tomassetti [37] examine its asymptotic behaviour for thin cylinders. Explicit constants are also obtained by Ryzhak [41] for parallelepipeds. These aspects are not considered further.

The extension of Korn's inequalities to Riemannian manifolds with or without boundary in an Euclidean space has been achieved mainly by Ciarlet and co-workers [7–9] and the cited references. The proof of these results, which applies also to two-dimensional manifolds or surfaces, employs a lemma of Duvaut and Lions [13] and is intrinsic. While the corresponding constants are shown to exist and to depend upon the geometry of the manifold, they are not explicitly determined.

In other investigations of manifolds, for example, by Chen and Jost [6], Dain [12] (see also Holst, Kommemi and Nagy [19]), Neff [33] and Ovtchinnikov and Xanthis, [35, 36], the constant is likewise not completely computed. Moreover, the analysis of a thick spherical shell, by Andreou, Dassios and Polyzos [2] using variational arguments developed by Payne and Weinberger [38] and by Dafermos [11], demonstrates that when pure rotations are excluded, Korn's constant tends to infinity as the thickness of the shell tends to zero, a result confirmed by [36].

Computable Korn's constants are essential for quantitative analyses, and in consequence the last conclusion alone motivates the need for a separate study of general surfaces.

The present task therefore is to derive explicit Korn's constants and Korn's inequalities for vectors in the tangent space to a general two-dimensional bounded simply-connected differentiable Riemannian manifold in an ambient three-dimensional Euclidean space subject to homogeneous Dirichlet boundary conditions. Conceptually simple arguments, that take into account curvature, are developed from classical differential geometry and differential equations to generalise not only inequalities (1.2)–(1.9), and (1.11), but also identity (1.6) and Poincaré's inequality (1.10). Conclusions are illustrated in varying detail by the spherical cap and minimal surface. Although Korn's constants may be explicitly computed for several other special surfaces, the calculations are often protracted and preferably performed only as necessary.

Section 2 introduces notation and basic notions from differential geometry. Section 3 is devoted to three versions of Poincaré's inequality, while Sect. 4 describes how the Korn's inequalities are constructed for the general manifold and the corresponding Korn's constants explicitly estimated. The spherical cap and minimal surface are considered in Sect. 5. Section 6 contains brief concluding observations.

2 Notation and Basic Differentiable Geometry

Elements of differential geometry used later are summarised in this section and serve to introduce notation. Standard textbooks, for example [4, 29, 32, 44] and more recently [7–9], may be consulted for complete treatments. The convention is adopted of summation over repeated indices with Greek indices ranging over 1, 2 while Latin indices have the range 1, 2, 3. The comma notation indicates covariant differentiation and not partial differentiation. Scalar, vector and tensor quantities are not typographically distinguished. Sufficient differentiability is assumed to justify all operations.

Consider a two-dimensional simply-connected bounded differentiable Riemannian manifold M embedded in an ambient three-dimensional Euclidean space $\mathbb{R}^3$ and let ∂M denote the Lipschitz smooth boundary curve of M . When specialised to be part of a spherical surface of radius r , M is denoted by $\Sigma(r)$ with boundary $\partial \Sigma(r)$.

Let x^α , $\alpha = 1, 2$, be contravariant components of the vector x that describes a curvilinear coordinate system on M , and let $y \in M$ be the three dimensional position vector of a point on M . We have

$$y(x) = y^i(x^1, x^2)e_i, \quad (2.1)$$

where $x \in D \subset \mathbb{R}^2$, D is a bounded plane region, and e_i , $i = 1, 2, 3$, are the orthogonal unit basis vectors of a rectangular three-dimensional Cartesian coordinate system. The map (2.1) is assumed to be an immersion.

Partial differentiation with respect to the variables x^α is denoted by $\partial/\partial x^\alpha$. The covariant basis vectors in the tangent plane to M at the point y , given by

$$a_\alpha = \frac{\partial y}{\partial x^\alpha}, \quad \alpha = 1, 2,$$

are linearly independent since the map (2.1) is an immersion. The contravariant tangent basis vectors a^α , orthogonal to a_α , satisfy

$$a^\alpha \cdot a_\beta = \delta^\alpha_\beta,$$

where δ^α_β is the Kronecker delta of mixed order, and an interposed dot represents the scalar product.

The metric tensor, or first fundamental form, has covariant components

$$a_{\alpha\beta} = a_\alpha \cdot a_\beta = a_{\beta\alpha},$$

whose determinant, a , given by

$$a = \det(a_{\alpha\eta}) = a_{11}a_{22} - a_{12}^2 > 0,$$

is positive by linear independence of a_α . Linear independence additionally implies not only the existence of a non-trivial unit normal vector $a_3 = a^3$ at the point $y \in M$ specified by

$$a_3 = a^3 = \frac{a_1 \times a_2}{\sqrt{a}},$$

where the vector product is indicated by $\times$, but also that the matrix $(a_{\alpha\beta})$ with components $a_{\alpha\beta}$ is positive-definite in the sense that there exist non-negative constants $0 \leq \lambda_0 \leq \lambda_1$ such that

$$\lambda_0 \xi^\alpha \xi^\alpha \leq a_{\alpha\beta} \xi^\alpha \xi^\beta \leq \lambda_1 \xi^\alpha \xi^\alpha,$$

for all contravariant vectors with components ξ^α .

The contravariant components $a^{\alpha\beta}$ of the metric tensor, defined by

$$a^{\alpha\beta} = a^\alpha \cdot a^\beta = a^{\beta\alpha},$$

satisfy the relations

$$\begin{aligned} a^{\alpha\gamma} a_{\gamma\beta} &= \delta_\beta^\alpha, \\ \det(a^{\alpha\beta}) &= a^{-1}, \end{aligned}$$

and

$$a_{11} = a a^{22}, \quad a_{12} = -a a^{12}, \quad a_{22} = a a^{11}. \quad (2.2)$$

The matrix $(a^{\alpha\beta})$ is also positive-definite in the sense that

$$\lambda_1^{-1} \xi_\alpha \xi_\alpha \leq a^{\alpha\beta} \xi_\alpha \xi_\beta \leq \lambda_0^{-1} \xi_\alpha \xi_\alpha, \quad (2.3)$$

for all covariant vectors with components ξ_α .

The symmetry of $a^{\alpha\beta}$ together with (2.3) ensures the existence of a unique positive-definite symmetric square root $c^{\alpha\beta}$ such that

$$c^{\alpha\gamma} c^{\gamma\beta} = a^{\alpha\beta}.$$

Quantities ψ_β^α , $\xi_{\alpha\beta}$, related by

$$\psi_\beta^\alpha = c^{\lambda\alpha} \xi_{\lambda\beta},$$

satisfy the relations

$$\begin{aligned} \psi_\beta^\alpha \psi_\beta^\alpha &= c^{\lambda\alpha} \xi_{\lambda\beta} c^{\mu\alpha} \xi_{\mu\beta} \\ &= a^{\lambda\mu} \xi_{\lambda\beta} \xi_{\mu\beta}, \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} a^{\alpha\beta} \psi_\alpha^\lambda \psi_\beta^\lambda &= a^{\alpha\beta} c^{\mu\lambda} \xi_{\mu\alpha} c^{\sigma\lambda} \xi_{\sigma\beta} \\ &= a^{\alpha\beta} a^{\mu\sigma} \xi_{\mu\alpha} \xi_{\sigma\beta}. \end{aligned} \quad (2.5)$$

We conclude from (2.3), (2.4), and (2.5) that

$$\begin{aligned}
 \lambda_1^{-2} \xi_{\alpha\beta} \xi_{\alpha\beta} &\leq \lambda_1^{-1} a^{\lambda\mu} \xi_{\lambda\beta} \xi_{\mu\beta} \\
 &= \lambda_1^{-1} \psi_\beta^\alpha \psi_\beta^\alpha \\
 &\leq a^{\lambda\mu} \psi_\lambda^\alpha \psi_\mu^\alpha \\
 &= a^{\lambda\mu} a^{\alpha\beta} \xi_{\alpha\lambda} \xi_{\beta\mu}.
 \end{aligned} \tag{2.6}$$

A vector u in the tangent plane to M at the point $y(x)$ has the representation

$$u = u^\alpha a_\alpha = u_\alpha a^\alpha, \tag{2.7}$$

in terms of the contravariant u^α and covariant u_α components. It follows that indices may be lowered or raised on multiplication by $a_{\alpha\beta}$ and $a^{\alpha\beta}$. Covariant differentiation of the covariant and contravariant components of u , indicated by subscript comma, is defined respectively to be

$$u_{\alpha,\beta} = \frac{\partial u_\alpha}{\partial x^\beta} - u_\sigma \Gamma_{\alpha\beta}^\sigma, \tag{2.8}$$

$$u_{,\beta}^\alpha = \frac{\partial u^\alpha}{\partial x^\beta} + u^\sigma \Gamma_{\beta\sigma}^\alpha, \tag{2.9}$$

where $\Gamma_{\alpha\beta}^\sigma$, the non-tensorial Christoffel symbols of the second kind, are obtained from the Gauss relations

$$\frac{\partial a^\alpha}{\partial x^\beta} = -\Gamma_{\sigma\beta}^\alpha a^\sigma + b_\beta^\alpha a_3, \tag{2.10}$$

in which $b_\beta^\alpha = a^{\alpha\lambda} b_{\lambda\beta}$, and $b_{\alpha\beta}$ are the symmetric components of a covariant tensor whose related quadratic form is the second fundamental form. Unlike the first fundamental form, it is not necessarily positive-definite.

Most partial differentiation operations apply to covariant differentiation and are facilitated by Ricci's lemma which states that

$$a_{\alpha\beta,\gamma} = a_{,\gamma}^{\alpha\beta} = a_{,\gamma} = 0. \tag{2.11}$$

The second covariant derivatives, however, do not commute but satisfy the Ricci identity

$$u_{\alpha,\beta\sigma} - u_{\alpha,\sigma\beta} = u_\gamma R_{\alpha\beta\sigma}^\gamma, \tag{2.12}$$

where $R_{\alpha\beta\sigma}^\gamma$, the mixed components of the Ricci-Christoffel curvature tensor, are related to the covariant components by

$$R_{\alpha\beta\delta\gamma} = a_{\alpha\lambda} R_{\beta\delta\gamma}^\lambda. \tag{2.13}$$

Various symmetries result in only four non-vanishing covariant components, namely

$$R_{1212} = R_{2121} = b = b_{11}b_{22} - (b_{12})^2, \tag{2.14}$$

$$R_{1221} = R_{2112} = -b, \tag{2.15}$$

in terms of which the Gaussian curvature, K , of M is expressed as

$$K = \frac{R_{1212}}{a} = \frac{b}{a}. \quad (2.16)$$

The mean curvature, H , specifically of the two-dimensional manifold $M \subset \mathbb{R}^3$, is defined by

$$2H = a^{\alpha\beta} b_{\alpha\beta} = b_{\alpha}^{\alpha}, \quad (2.17)$$

where, as previously stated, $b_{\beta}^{\alpha} = a^{\alpha\gamma} b_{\gamma\beta}$.

Remark 2.1 When the manifold is a plane region in $\mathbb{R}^2$, we may select $x^{\alpha} = y^{\alpha}$ and $x^3 = y^3 = 0$ to show that partial and covariant differentiation coincide and that

$$\begin{aligned} a_{\alpha} &= e_{\alpha}, & a_3 &= e_3, & a_{\alpha\beta} &= \delta_{\alpha\beta}, & a &= 1, & b_{\alpha\beta} &= 0, \\ K &= H = 0. \end{aligned} \quad (2.18)$$

Certain integrals that frequently recur in the following sections are listed here for reference. They involve the covariant and contravariant components of the symmetric linear strain tensor and antisymmetric rotation tensor defined by

$$e_{\alpha\beta} = \frac{1}{2} (u_{\alpha,\beta} + u_{\beta,\alpha}), \quad \omega_{\alpha\beta} = \frac{1}{2} (u_{\alpha,\beta} - u_{\beta,\alpha}), \quad (2.19)$$

$$e^{\alpha\beta} = a^{\alpha\lambda} a^{\beta\mu} e_{\lambda\mu}, \quad \omega^{\alpha\beta} = a^{\alpha\lambda} a^{\beta\mu} \omega_{\lambda\mu}, \quad (2.20)$$

and the element of surface area of M denoted by

$$dS = \sqrt{a} dx^1 dx^2. \quad (2.21)$$

We set

$$D_1 = \int_M u^{\alpha} u_{\alpha} dS, \quad D_2 = \int_M K u^{\alpha} u_{\alpha} dS, \quad (2.22)$$

$$E = \int_M e^{\alpha\beta} e_{\alpha\beta} dS, \quad W = \int_M \omega^{\alpha\beta} \omega_{\alpha\beta} dS, \quad (2.23)$$

$$I_1 = \int_M a^{\alpha\lambda} a^{\beta\sigma} u_{\beta,\alpha} u_{\sigma,\lambda} dS, \quad I_2 = \int_M a^{\alpha\lambda} a^{\beta\sigma} u_{\alpha,\beta} u_{\sigma,\lambda} dS, \quad (2.24)$$

$$I_3 = \int_M u_{\lambda} u_{\sigma} a^{\alpha\beta} b_{\alpha}^{\lambda} b_{\beta}^{\sigma} dS, \quad (2.25)$$

where $D_1 \geq 0$, $E \geq 0$, $W \geq 0$, while by (2.6) on taking $\xi_{\alpha\beta} = u_{\alpha,\beta}$, we have

$$I_1 \geq \lambda_1^{-2} \int_M u_{\alpha,\beta} u_{\alpha,\beta} dS \geq 0. \quad (2.26)$$

The integrals D_2 , I_2 , I_3 , are indefinite for unspecified geometries of M .

For plane regions $\Omega \subset \mathbb{R}^2$, the integrals reduce to

$$D_1 = \int_{\Omega} |u|^2 dx, \quad E = \int_{\Omega} |S|^2 dx, \quad I_1 = \int_{\Omega} |\nabla u|^2 dx, \quad I_2 = \int_{\Omega} (\operatorname{tr} \nabla u)^2 dx, \\ D_2 = I_3 = 0.$$

We conclude this section by recording the form of the divergence theorem appropriate to a bounded manifold M . Let w^α be the contravariant components of a differentiable surface vector w defined on M . Then

$$\int_M w_{,\alpha}^\alpha dS = \int_{\partial M} w_\alpha n^\alpha ds, \quad (2.27)$$

where n^α are the contravariant components of the unit outward normal to the boundary curve ∂M , ds is the element of its length, and dS is given by (2.21).

3 Poincaré Inequalities

We sketch proofs based on variational arguments of three inequalities that are required subsequently but which are of general importance in the geometry and analysis of manifolds.

3.1 First Poincaré Inequality

The following lemma corresponds to inequality (1.10).

Lemma 3.1 (Poincaré's inequality for scalar functions) *Let $\phi(x)$ be a differentiable scalar invariant function defined at each point $y(x) \in M$ where $(x^1, x^2) \in D \subset \mathbb{R}^2$ are curvilinear coordinates defining $y(x)$. Let*

$$\phi(x) = 0, \quad y(x) \in \partial M. \quad (3.1)$$

Then

$$\Lambda \int_M \phi^2 dS \leq \int_M \nabla_s \phi \cdot \nabla_s \phi dS, \quad (3.2)$$

where the tangential surface gradient $\nabla_s \phi$ on M is given by

$$\nabla_s \phi = a^\alpha \frac{\partial \phi}{\partial x^\alpha}, \quad (3.3)$$

and Λ , a positive eigenvalue of the corresponding self-adjoint elliptic linear Laplace-Beltrami differential operator L specified by

$$L\phi = \frac{1}{\sqrt{a}} \frac{\partial}{\partial x^\beta} \left(\sqrt{a} a^{\alpha\beta} \frac{\partial \phi}{\partial x^\alpha} \right), \quad (3.4)$$

satisfies

$$\Lambda \phi + L\phi = 0, \quad x \in D, \quad (3.5)$$

and is dependent upon the geometry of M and the boundary condition (3.1).

Proof Write inequality (3.2) as the Rayleigh quotient

$$\Lambda = \inf_{x \in D} \frac{\int_M \nabla_s \phi \cdot \nabla_s \phi \, dS}{\int_M \phi^2 \, dS}. \quad (3.6)$$

Standard variational arguments then show that the minimum for (3.6) subject to boundary conditions (3.1) satisfies the Euler-Lagrange equations (3.5).

Consequently, Λ and $\phi(x)$ are the eigenvalues and eigenfunctions for the Laplace-Beltrami differential operator L on the plane bounded region $D \subset \mathbb{R}^2$.

Exact expressions for the eigenvalue Λ are known for only certain geometries, and under various conditions. See, for example Grigor'yan [17], Alencar and Neto [1], Li [31], and Saloff-Coste [42]. By contrast, lower bounds for Λ are comparatively abundant and become significant in Sect. 4 and Sect. 5 for explicit estimates of Korn's constants. A useful result is that Λ decreases as $|M|$ increases for homogeneous Dirichlet boundary conditions. Moreover, Sperb [45], on improving a lower bound by Cheeger [5], shows that

$$\Lambda \geq h^2(M), \quad h(M) = \inf_{\tilde{M} \subset M} \frac{|\partial \tilde{M}|}{|\tilde{M}|}, \quad (3.7)$$

where $|\partial M|$ and $|M|$ are the length and surface area of the bounding curve ∂M and surface M respectively.

Other lower bounds employed subsequently are described by Protter and Weinberger [39], based upon the maximum principle, and by Bandle [3] that include those exploiting the equivalence of (3.5) to the inhomogeneous fixed membrane problem. Of course, when M is a plane region Ω , a lower bound is provided by the Faber-Krahn estimate:

$$\Lambda \geq \frac{\pi j_0^2}{|\Omega|}, \quad (3.8)$$

where $j_0 \approx 2.4048$ is the smallest positive zero of the Bessel function J_0 . Equality holds for the circular disc.

For our general discussion, however, we assume that either Λ or a lower bound are explicitly known.

The next result extends the first Poincaré's inequality to surface tangent vectors.

Proposition 3.1 (A Poincaré inequality for surface vectors) *Let $u(x)$ be a smooth tangential surface vector at $y(x) \in M$ that satisfies the boundary condition*

$$u(x) = 0, \quad y(x) \in \partial M. \quad (3.9)$$

Then

$$\Lambda \int_M u^\alpha u_\alpha \, dS \leq \int_M \left[a^{\alpha\beta} a^{\lambda\sigma} u_{\lambda,\alpha} u_{\sigma,\beta} + u_\lambda u_\sigma a^{\alpha\beta} b_\alpha^\lambda b_\beta^\sigma \right] dS, \quad (3.10)$$

where Λ is the constant appearing in inequality (3.2).

Recall that covariant differentiation is denoted by subscript comma.

Remark 3.1 In terms of the integrals (2.22)-(2.25), inequality (3.10) is expressed as

$$\Lambda D_1 \leq I_1 + I_3. \quad (3.11)$$

Proof Let $v_i(x)$ be the components of the surface vector $u(x)$ with respect to an orthogonal Cartesian set of axes with unit basis vectors e_i , $i = 1, 2, 3$. We have the identity

$$u^\alpha a_\alpha = u_\alpha a^\alpha = v_i e_i. \quad (3.12)$$

The Poincaré inequality (3.2) applied separately to each component v_i leads to

$$\begin{aligned} \Lambda \int_M u^\alpha u_\alpha dS &= \Lambda \int_M v_i v_i dS \\ &\leq \int_M \nabla_s v_i \cdot \nabla_s v_i dS \\ &= \int_M a^{\alpha\beta} \frac{\partial v_i}{\partial x^\alpha} \frac{\partial v_i}{\partial x^\beta} dS \\ &= \int_M a^{\alpha\beta} \frac{\partial}{\partial x^\alpha} (u_\lambda a^\lambda \cdot e_i) \frac{\partial}{\partial x^\beta} (u_\sigma a^\sigma \cdot e_i) dS \\ &= \int_M a^{\alpha\beta} \frac{\partial}{\partial x^\alpha} (u_\lambda (a^\lambda \cdot e_i) e_i) \cdot \frac{\partial}{\partial x^\beta} (u_\sigma (a^\sigma \cdot e_i) e_i) dS \\ &= \int_M a^{\alpha\beta} \frac{\partial}{\partial x^\alpha} (u_\lambda a^\lambda) \cdot \frac{\partial}{\partial x^\beta} (u_\sigma a^\sigma) dS \\ &= \int_M [a^{\alpha\beta} a^{\lambda\sigma} u_{\lambda,\alpha} u_{\sigma,\beta} + u_\lambda u_\sigma a^{\alpha\beta} b_\alpha^\lambda b_\beta^\sigma] dS, \end{aligned}$$

which is the desired inequality (3.10). The derivation has used expressions (2.10) for the partial derivative of a basis vector a^α and (2.8) for covariant differentiation. $\square$

Remark 3.2 Under certain conditions the last term on the right of (3.10) may be removed. From (2.3) and Schwarz's inequality we successively have

$$\begin{aligned} a^{\alpha\beta} u_\lambda u_\sigma b_\alpha^\lambda b_\beta^\sigma &\leq \lambda_0^{-1} u_\lambda u_\lambda b_\alpha^\mu b_\alpha^\mu \\ &\leq \frac{\lambda_1}{\lambda_0} u^\alpha u_\alpha N^2, \end{aligned}$$

where

$$N^2 = \max_M b_\alpha^\lambda b_\alpha^\lambda, \quad (3.13)$$

and therefore

$$I_3 \leq \frac{\lambda_1}{\lambda_0} N^2 D_1. \quad (3.14)$$

Consequently, for manifolds that satisfy both $N < \infty$ and the condition

$$\Lambda_1 = \left(\Lambda - \frac{\lambda_1}{\lambda_0} N^2 \right) > 0, \quad (3.15)$$

inequality (3.10) reduces to

$$\Lambda_1 \int_M u^\alpha u_\alpha dS \leq \int_M a^{\alpha\beta} a^{\lambda\sigma} u_{\lambda,\alpha} u_{\sigma,\beta} dS, \quad (3.16)$$

which may be equivalently written as

$$\Lambda_1 D_1 \leq I_1. \quad (3.17)$$

3.2 Second Poincaré Inequality

For certain manifolds, for example the hemispherical surface, inequality (3.10) becomes inappropriate when computing the corresponding Korn's constant and must be replaced. The alternative Poincaré inequality, which also holds generally, involves only the first term on the right of (3.10). We have

Proposition 3.2 *Let $u(x)$ be a smooth tangential surface vector at $y(x) \in M$ that satisfies the homogeneous boundary condition (3.9). Then*

$$\Lambda \int_M u^\alpha u_\alpha dS \leq \int_M a^{\alpha\lambda} a^{\beta\sigma} u_{\beta,\alpha} u_{\sigma,\lambda} dS, \quad (3.18)$$

or

$$\Lambda D_1 \leq I_1, \quad (3.19)$$

where the constant Λ is the eigenvalue occurring in inequality (3.2).

Proof As before, regard inequality (3.18) as a Rayleigh quotient and apply variational arguments to conclude that the Euler-Lagrange equations consist of the uncoupled elliptic system

$$\Lambda u_\alpha + a^{\beta\sigma} u_{\alpha,\sigma\beta} = 0, \quad (3.20)$$

or

$$\Lambda u_\alpha + \frac{1}{\sqrt{a}} \frac{\partial}{\partial x^\beta} \left(\sqrt{a} a^{\beta\sigma} \frac{\partial u_\alpha}{\partial x^\sigma} \right) = 0, \quad x \in D. \quad (3.21)$$

Properties of the corresponding eigenvalues and eigenfunctions consequently are those for the Laplace-Beltrami operator (3.4). $\square$

Note that inequality (3.11) is preferable to (3.19) when $I_3 < 0$.

4 Korn's Inequalities

We extend the Korn's inequalities described in Sect. 1 to the general two-dimensional manifold defined in Sect. 2. Korn's constants are shown to depend upon the eigenvalue Λ for the Laplace-Beltrami operator (3.4) and also upon the Gaussian curvature and components of the second fundamental form for M . We first establish a basic inequality.

4.1 Basic Korn's Inequality on M

Proposition 4.1 *Let M be as defined in Sect. 2 and let the smooth tangent surface vector u vanish on ∂M . Then u satisfies the Korn's inequality*

$$\int_M a^{\alpha\lambda} a^{\beta\sigma} u_{\beta,\alpha} u_{\sigma,\lambda} dS \leq 2 \int_M e^{\alpha\beta} e_{\alpha\beta} dS + \int_M K u^\alpha u_\alpha dS, \quad (4.1)$$

$$u(x) = 0, \quad y(x) \in \partial M, \quad (4.2)$$

where K is the Gaussian curvature, and the covariant and contravariant strain components are given by (2.19)₁ and (2.20)₁.

Proof By means of (2.19)₁ and interchange of indices, we successively obtain

$$\begin{aligned} e^{\alpha\beta} e_{\alpha\beta} &= a^{\alpha\lambda} a^{\beta\sigma} e_{\alpha\beta} e_{\lambda\sigma} \\ &= \frac{1}{2} a^{\alpha\lambda} a^{\beta\sigma} u_{\beta,\alpha} u_{\sigma,\lambda} + \frac{1}{2} a^{\alpha\lambda} a^{\beta\sigma} u_{\alpha,\beta} u_{\sigma,\lambda}. \end{aligned} \quad (4.3)$$

The Ricci relations (2.11) and commutativity identity (2.12) are now used to write the second term on the right of the last expression as

$$\begin{aligned} a^{\alpha\lambda} a^{\beta\sigma} u_{\alpha,\beta} u_{\sigma,\lambda} &= (a^{\alpha\lambda} a^{\beta\sigma} u_{\alpha} u_{\sigma,\lambda})_{,\beta} - a^{\alpha\lambda} a^{\beta\sigma} u_{\alpha} u_{\sigma,\lambda\beta} \\ &= (a^{\alpha\lambda} a^{\beta\sigma} u_{\alpha} u_{\sigma,\lambda})_{,\beta} - (a^{\alpha\lambda} a^{\beta\sigma} u_{\alpha} u_{\sigma,\beta})_{,\lambda} \\ &\quad + a^{\alpha\lambda} a^{\beta\sigma} u_{\alpha,\lambda} u_{\sigma,\beta} - a^{\alpha\lambda} a^{\beta\sigma} u_{\alpha} u_{\sigma} R^{\nu}_{\lambda\beta} \\ &\geq (a^{\alpha\lambda} a^{\beta\sigma} u_{\alpha} u_{\sigma,\lambda})_{,\beta} - (a^{\alpha\lambda} a^{\beta\sigma} u_{\alpha} u_{\sigma,\beta})_{,\lambda} - K u^\alpha u_\alpha, \end{aligned} \quad (4.4)$$

where (2.14), (2.15) and (2.16) are employed together with the inequality

$$a^{\alpha\lambda} a^{\beta\sigma} u_{\alpha,\lambda} u_{\sigma,\beta} = (a^{\alpha\lambda} u_{\alpha,\lambda}) (a^{\sigma\beta} u_{\sigma,\beta}) \geq 0.$$

The divergence theorem (2.27) and boundary condition (4.2) then lead to

$$\int_M a^{\alpha\lambda} a^{\beta\sigma} u_{\alpha,\beta} u_{\sigma,\lambda} dS \geq - \int_M K u^\alpha u_\alpha dS, \quad (4.5)$$

which may be written

$$I_2 \geq -D_2. \quad (4.6)$$

Insertion of (4.5) into the integrated version of (4.3), re-arranged as

$$\int_M a^{\alpha\lambda} a^{\beta\sigma} u_{\beta,\alpha} u_{\sigma,\lambda} dS = 2 \int_M e^{\alpha\beta} e_{\alpha\beta} dS - \int_M a^{\alpha\lambda} a^{\beta\sigma} u_{\alpha,\beta} u_{\sigma,\lambda} dS, \quad (4.7)$$

recovers the required Korn's inequality (4.1) analogous to (1.2). $\square$

Remark 4.1 (Non-negative I_2) Note that $I_2 \geq 0$ on manifolds M for which the Gaussian curvature K is everywhere non-positive.

4.2 Other Korn's Inequalities

Abbreviated forms of the basic Korn's inequality (4.1), the subsidiary inequality (4.5), and the identity (4.7) are

$$I_1 \leq 2E + D_2, \quad (4.8)$$

$$I_2 \geq -D_2 \quad (4.9)$$

$$2E = I_1 + I_2, \quad (4.10)$$

from the last of which we have

$$I_2 \leq 2E. \quad (4.11)$$

Other forms of Korn's inequalities for general M are now easily derived. Elimination of I_1 between (4.10) and either (3.11) or (3.19) respectively yields

$$\Lambda D_1 \leq 2E + D_2 + I_3, \quad (4.12)$$

and

$$\Lambda D_1 \leq 2E + D_2, \quad (4.13)$$

which are analogous to (1.11).

Expanded versions of inequalities (4.12) and (4.13) are

$$\Lambda \int_M u^\alpha u_\alpha dS \leq 2 \int_M e^{\alpha\beta} e_{\alpha\beta} dS + \int_M u_\lambda u_\sigma a^{\alpha\beta} b_\alpha^\lambda b_\beta^\sigma dS + \int_M K u^\alpha u_\alpha dS, \quad (4.14)$$

$$\Lambda \int_M u^\alpha u_\alpha dS \leq 2 \int_M e^{\alpha\beta} e_{\alpha\beta} dS + \int_M K u^\alpha u_\alpha dS. \quad (4.15)$$

Subject to boundedness assumptions (3.13) and (3.15), we conclude from (3.14) and (3.17) that

$$I_3 \leq \left(\frac{\lambda_1}{\lambda_0} \right) N^2 D_1 \leq \left(\frac{\lambda_1 N^2}{\lambda_0 \Lambda_1} \right) I_1. \quad (4.16)$$

On the other hand, we always have

$$D_2 \leq \bar{K} D_1, \quad (4.17)$$

where

$$\bar{K} = \max_M |K|. \quad (4.18)$$

In consequence, geometries for which

$$\Lambda_2 := \Lambda_1 - \bar{K} > 0, \quad \Lambda_3 := \Lambda - \bar{K} > 0, \quad (4.19)$$

allow Korn's inequalities (4.12) and (4.13) to be written

$$D_1 \leq 2\Lambda_\gamma^{-1} E, \quad \gamma = 2, 3, \quad (4.20)$$

which corresponds to (1.11). Moreover, (4.8) implies

$$I_1 \leq 2E + \tilde{K} D_1 \leq 2C_\gamma E, \quad C_\gamma = (1 + \tilde{K} \Lambda_\gamma^{-1}), \quad (4.21)$$

analogous to the first Korn's inequality (1.2).

4.3 Second and Third Korn's Inequalities. Friedrich's Inequality

We complete the extension of Korn's inequalities by considering types corresponding to (1.4) and (1.5).

The rotation of the surface tangential vector $u(x)$ has antisymmetric covariant components (see (2.19)₂) given by

$$\omega_{\alpha\beta} = \frac{1}{2} (u_{\alpha,\beta} - u_{\beta,\alpha}). \quad (4.22)$$

Interchange of indices leads to

$$\begin{aligned} W &:= \int_M \omega^{\alpha\beta} \omega_{\alpha\beta} dS \\ &= \frac{1}{4} \int_M a^{\alpha\lambda} a^{\beta\sigma} (u_{\lambda,\sigma} - u_{\sigma,\lambda}) (u_{\alpha,\beta} - u_{\beta,\alpha}) dS \\ &= \frac{1}{2} \int_M a^{\alpha\lambda} a^{\beta\sigma} (u_{\sigma,\lambda} u_{\beta,\alpha} - u_{\sigma,\lambda} u_{\alpha,\beta}) dS, \end{aligned}$$

or to the equivalent expression

$$W = \frac{1}{2} (I_1 - I_2). \quad (4.23)$$

The last identity in combination with (4.9) and (4.10) yields

$$E - W = I_2 \geq -D_2, \quad (4.24)$$

$$E + W = I_1. \quad (4.25)$$

Accordingly, from (4.8) and (4.25), we deduce the relations

$$2W = 2I_1 - 2E \leq I_1 + D_2, \quad (4.26)$$

$$W \leq E + D_2. \quad (4.27)$$

Furthermore, we have $E \geq 0$ and $W \geq 0$. Consequently, we conclude from (4.23)-(4.25) that

$$E \leq I_1, \quad W \leq I_1, \quad I_2 \leq I_1, \quad I_2 \leq E, \quad -I_2 \leq W. \quad (4.28)$$

These inequalities reduce to those stated in Sect. 1 when $M = \Omega \subset \mathbb{R}^2$ and $C = 2$.

On supposing the bound (4.19), we may employ (4.21) and (4.28)₂ to establish the second general Korn's inequality given by

$$W \leq I_1 \leq 2C_\gamma E, \quad (4.29)$$

corresponding to (1.4).

Two particular geometries are considered in the next section for which explicit computations are possible.

5 Examples

Inequalities for general manifolds derived in Sect. 4 are now applied to two special surfaces for which the corresponding Korn's constants or their lower bounds may be explicitly computed. A detailed discussion is presented for the spherical cap and hemispherical surface as examples of certain confocal conical sections. Less extensively discussed are minimal surfaces.

5.1 Spherical Cap and Hemispherical Surface

Let $\Sigma(r)$ denote the surface of a spherical cap of radius r whose bounding curve $\partial\Sigma(r)$ subtends an angle 2ω , $0 < \omega \leq \pi/2$, at the centre of the corresponding sphere taken as origin of a rectangular Cartesian coordinate system y . Let (r, x^α) be spherical polar coordinates for a point $y \in \Sigma(r)$ so that

$$\begin{aligned} y^1 &= r \sin x^1 \cos x^2, \\ y^2 &= r \sin x^1 \sin x^2, \\ y^3 &= r \cos x^1. \end{aligned}$$

The covariant components of the metric tensor and mixed components of the second fundamental form become

$$\begin{aligned} a_{11} &= r^2, & a_{12} &= a_{21} = 0, & a_{22} &= r^2 \sin^2 x^1, \\ a &= r^4 \sin^2 x^1, \\ b_1^1 &= b_2^2 = -r^{-1}, & b_2^1 &= b_1^2 = 0, \end{aligned} \quad (5.1)$$

from which follows

$$\begin{aligned} b &= b_{11}b_{22} - b_{12}^2 = r^2 \sin^2 x^1, \\ K &= b/a = r^{-2}. \end{aligned} \quad (5.2)$$

The eigenvalue Λ is determined from the associated Laplace-Beltrami equation (3.5), which in spherical surface polar coordinates becomes

$$\frac{1}{r^2 \sin x^1} \left[\frac{\partial}{\partial x^1} \left(\sin x^1 \frac{\partial \phi}{\partial x^1} \right) + \frac{1}{\sin x^1} \frac{\partial^2 \phi}{(\partial x^2)^2} \right] + \Lambda \phi = 0, \quad (5.3)$$

where ϕ is subject to the homogeneous boundary condition (3.1). Consider the separable solution

$$\phi(x^1, x^2) = \Theta(x^1)\Phi(x^2), \quad (5.4)$$

and put

$$\Phi''(x^2) = -m^2 \Phi(x^2), \quad \Phi(0) = \Phi(2\pi), \quad (5.5)$$

where m is some non-negative integer and a superposed prime indicates differentiation with respect to the argument of the function. Insertion into (5.3) leads to the associated Legendre equation for $\Theta(x^1)$

$$(1 - \mu^2)\Theta'' - 2\mu\Theta' + \left(\Lambda r^2 - \frac{m^2}{(1 - \mu^2)}\right)\Theta = 0, \quad (5.6)$$

where we have put $\mu = \cos x^1$. When Λ is chosen to be

$$\Lambda = n(n + 1)r^{-2}, \quad (5.7)$$

for positive integer n , the corresponding eigenfunctions, related to surface spherical, or tesseral, harmonics, are Ferrer's functions (see, for example, [14, p. 75] or [43, p. 78])

$$T_n^m(\mu) = (1 - \mu^2)^{m/2} \frac{d^m P_n(\mu)}{d\mu^m}, \quad (5.8)$$

where $P_n(\mu)$ is the Legendre polynomial of order n , and for temporary convenience, differentiation is denoted by the operator d .

The integers m, n are selected to ensure that the boundary condition (3.1) is satisfied. In particular, for given m , we choose n to be the smallest value for which

$$T_n^m(\cos \omega) = 0. \quad (5.9)$$

First set $m = 0$ to conclude from (5.8) and (5.9) that n is the least value for which

$$P_n(\cos \omega) = 0. \quad (5.10)$$

The discussion is continued by separate consideration of the spherical cap and hemispherical surface.

For the spherical cap, we have $0 < \omega < \pi/2$, and in consequence (5.10) is satisfied for $n > 1$.

In spherical polar coordinates, the integrals D_2 and I_3 defined in (2.22)₂ and (2.25) reduce to

$$D_2 = I_3 = r^{-2} D_1, \quad (5.11)$$

and consequently the augmented Korn's inequality (4.12) may be written

$$D_1 \leq 2C_1 r^2 E, \quad (5.12)$$

or

$$\int_{\Sigma(r)} u^\alpha u_\alpha dS \leq 2C_1 r^2 \int_{\Sigma(r)} e^{\alpha\beta} e_{\alpha\beta} dS, \quad (5.13)$$

where

$$C_1 = \frac{1}{[n(n + 1) - 2]}. \quad (5.14)$$

For the hemispherical surface, we have $\omega = \pi/2$ and the boundary condition (5.10) becomes

$$P_n(0) = 0,$$

which holds for all positive odd integers n . Therefore, we take $n = 1$. But then the computable constant C_1 in (5.14) is unbounded and (5.13) remains meaningful only for the spherical cap with $0 < \omega < \pi/2$. The difficulty is resolved by using (4.13) to obtain

$$D_1 \leq 2C_2 r^2 E, \quad (5.15)$$

or

$$\int_{\Sigma(r)} u^\alpha u_\alpha dS \leq 2C_2 r^2 \int_{\Sigma(r)} e^{\alpha\beta} e_{\alpha\beta} dS, \quad (5.16)$$

where

$$C_2 = \frac{1}{[n(n+1) - 1]}. \quad (5.17)$$

On taking $n = 1$, we have $C_2 = 1$ for the hemispherical surface. Furthermore, since $C_2 < C_1$ for all positive integers n , Korn's augmented inequality (5.16) is preferable to (5.13) even for the spherical cap ($0 < \omega < \pi/2$) but with n now determined from the boundary condition (5.10).

Remark 5.1 Let the manifold M contain the spherical cap $\Sigma(r)$. Monotonicity of the eigenvalues subject to homogeneous Dirichlet boundary conditions gives in an obvious notation

$$\Lambda_M \geq \Lambda_{\Sigma(r)},$$

and Korn's inequality corresponding to (5.13) is

$$\int_M u^\alpha u_\alpha dS \leq 2C_1 r^2 \int_M e^{\alpha\beta} e_{\alpha\beta} dS.$$

Remark 5.2 While Sperb's [45] improvement (3.7) of Cheeger's lower bound for Λ [5] may be used in (4.12), or (4.14), and yields a valid result, nevertheless when applied to the spherical cap, the Korn's constant is necessarily greater than the constants computed in either (5.13) or (5.16). The bound (4.12) remains important, however, for non-spherical surfaces.

The analogue of the first Korn's inequality is obtained by substitution of (5.11) and (5.15) in (4.8), and assumes the form

$$I_1 \leq 2(1 + C_2) E, \quad (5.18)$$

where

$$\begin{aligned} I_1 &= r^{-4} \int_{\Sigma(r)} [u_{1,1}^2 + \sin^{-2} x^1 \{u_{1,2}^2 + u_{2,1}^2\} + \sin^{-4} x^1 u_{2,2}^2] dS \\ &\geq r^{-4} \int_{\Sigma(r)} u_{\alpha,\beta} u_{\alpha,\beta} dS, \end{aligned} \quad (5.19)$$

since $0 \leq \sin x^1 \leq 1$. Inequality (5.18) accordingly may be rewritten as

$$\int_{\Sigma(r)} u_{\alpha,\beta} u_{\alpha,\beta} dS \leq 2(1 + C_2) r^4 \int_{\Sigma(r)} e^{\alpha\beta} e_{\alpha\beta} dS. \quad (5.20)$$

Another version is derived on noting the lower bound

$$I_1 \geq r^{-2} \int_{\Sigma(r)} u_{,\beta}^\alpha u_{\alpha,\beta} dS,$$

so that

$$\int_{\Sigma(r)} u_{,\beta}^\alpha u_{\alpha,\beta} dS \leq 2(1 + C_2) r^2 \int_{\Sigma(r)} e^{\alpha\beta} e_{\alpha\beta} dS. \quad (5.21)$$

The corresponding second Korn's inequality follows immediately from (4.26), (5.11) and (5.18) and is given by

$$\int_{\Sigma(r)} \omega^{\alpha\beta} \omega_{\alpha\beta} dS \leq (1 + 2C_2) \int_{\Sigma(r)} e^{\alpha\beta} e_{\alpha\beta} dS, \quad (5.22)$$

or

$$W \leq (1 + 2C_2) E. \quad (5.23)$$

The third Korn's inequality analogous to (1.5) results from combining (4.25) with (5.18) to obtain

$$\begin{aligned} \int_{\Sigma(r)} \omega^{\alpha\beta} \omega_{\alpha\beta} dS &\leq \left(\frac{1 + 2C_2}{2(1 + C_2)} \right) I_1 \\ &\leq \left(\frac{1 + 2C_2}{2(1 + C_2)} \right) r^{-4} \max \{ \sin^2 \bar{X}^1, \sin^4 \hat{X}^1 \} \int_{\Sigma(r)} u_{\alpha,\beta} u_{\alpha,\beta} dS, \end{aligned} \quad (5.24)$$

by virtue of the mean value theorem applied to (5.19) where $0 < \bar{X}^1, \hat{X}^1 < \pi/2$.

Remark 5.3 Families of confocal conicoids may be treated similarly provided there is a curvilinear coordinate system for which the corresponding associated Laplace-Beltrami equation admits a separable solution.

Remark 5.4 (Additional normalisation) Korn's constant for the hemispherical surface may be improved by further normalisation of the surface tangent vector u . For example, for each covariant component $u_\alpha(x^1, x^2)$ the normalisation

$$\int_{-1}^1 u_\alpha(\cos^{-1} \mu, x^2) P_1(\mu) d\mu = 0, \quad 0 \leq x^2 \leq 2\pi, \quad (5.25)$$

excludes both the eigenfunction $P_1(\mu)$ and the choice $n = 1$ when considering the boundary condition (5.10) with $\mu = 0$. On letting $m \neq 0$ in the general solution (5.8) to the associated Legendre equation (5.6), we determine $n > 1$ to satisfy the boundary condition (5.9) which replaces (5.10). When $m = 1$, the boundary condition (5.9) becomes

$$\frac{dP_n(\mu)}{d\mu} = 0,$$

and for $\mu = 0$ the least value of n is $n = 2$. Korn's constant in (5.15) has the decreased value $2C_2r^2 = 2r^2/5$.

Remark 5.5 (Families of spherical caps) Certain problems of spatial stability on unbounded regions in $\mathbb{R}^3$ require a continuous sequence of Korn's inequalities belonging to a one parameter family of spherical caps contained in a half-space. The previous analysis may adapted to the problem. The half-space remains specified by $y^3 \geq 0$, but the centres of the respective spherical caps are on the negative y^3 -axis at successively increasing distances d from the origin $y = 0$. Let r , the radius of the corresponding spherical cap $\Sigma(r)$, be related to d by $d = \lambda r$ where λ is a fixed positive constant and $0 < \lambda < 1$. Then $\Sigma(r)$ intersects the plane $y^3 = 0$ in a circle that subtends at the centre of $\Sigma(r)$ an angle $\omega = \cos^{-1} \lambda$ independent of any particular $\Sigma(r)$. Korn's constant, likewise independent of $\Sigma(r)$, is determined by appropriate choice of m , ω and therefore n . For example, when $\omega = \pi/6$ and $m = 0$, boundary condition (5.10) becomes

$$P_n(\sqrt{3}/2) = 0,$$

and we take $n = 4$ to obtain for each r the Korn's constant $2C_2r^2 = 2r^2/19$.

5.2 Minimal Surface

Suppose that the bounded manifold has no umbilic points and that through each point the orthogonal lines of curvature are taken to be the curvilinear coordinate directions. With respect to this coordinate system, we have

$$b_{12} = a_{12} = 0, \quad (5.26)$$

and

$$K = b_1^1 b_2^2 = \frac{b_{11} b_{22}}{a_{11} a_{22}}, \quad 2H = b_1^1 + b_2^2. \quad (5.27)$$

A different curvilinear coordinate system is adopted for the minimal surface of revolution discussed below.

Further suppose that M is a minimal surface so that $H = 0$ and therefore

$$b_1^1 = -b_2^2, \quad K = -(b_1^1)^2 \leq 0. \quad (5.28)$$

Substitution in (2.22)₂ and (2.25) yields

$$D_2 = -I_3 = - \int_M (b_1^1)^2 u^\alpha u_\alpha dS \leq 0. \quad (5.29)$$

Consequently, inequalities (4.12) or (4.13) lead to the augmented first Korn's inequality for a minimal surface in the form:

$$\Lambda D_1 \leq 2E. \quad (5.30)$$

The analogous first Korn's inequality, immediate from (4.8) and (5.29), is

$$I_1 \leq 2E, \quad (5.31)$$

where

$$I_1 = \int_M [a^{11} u_{,1}^\sigma u_{,1}^\sigma + a^{22} u_{,2}^\sigma u_{,2}^\sigma] dS \geq 0.$$

Elimination of I_2 between (4.10) and (4.23) combined with (5.31), or directly from (4.27) and (5.29), leads to the second Korn's inequality for the minimal surface in the form

$$W \leq E. \quad (5.32)$$

On appeal to (4.25), we conclude from (5.32) that the corresponding third Korn's inequality is

$$2W \leq I_1. \quad (5.33)$$

Explicit Korn's constants are therefore determined for all three Korn's inequalities (5.31), (5.32), and (5.33) on a minimal surface. They are exactly equivalent to the constants specified in inequalities (1.2), (1.4) and (1.5) for which $C = 2$. Indeed, these inequalities may be regarded as holding on the special minimal surfaces that are portions of flat hypersurfaces.

The computation, however, of Korn's constant in the augmented first Korn's inequality (5.30) depends upon explicit expressions for the eigenvalue Λ or its lower bound subject to homogeneous Dirichlet boundary conditions. Computable lower bounds may be derived according to methods outlined in Sect. 3.1. In this respect, on recalling the bound $|\partial M|^2 \geq 4\pi|M|$ for a minimal surface (see, for example, [29, p. 245]), we obtain from (3.7)

$$\Lambda \geq \frac{4\pi}{|M|}. \quad (5.34)$$

Nevertheless, to be specific, we examine the minimal surface that is a surface of revolution, or right helicoid. Introduce polar coordinates (x^1, x^2) and represent each point $y \in M$ as

$$y^1 = x^1 \cos x^2, \quad (5.35)$$

$$y^2 = x^1 \sin x^2, \quad (5.36)$$

$$y^3 = (px^2 + q), \quad (5.37)$$

where p, q are prescribed positive constants. Suppose that $0 \leq r_1 \leq x^1 \leq r_2$, and that the minimum height of the helicoid occurs at $x^2 = 0$, and its maximum height h at $x^2 = \bar{x}^2$. In consequence, we have $0 \leq x^2 \leq \bar{x}^2$ and $h = p\bar{x}^2 + q$. Define the positive constant m by $m\bar{x}^2 = 2\pi$ so that

$$m = \frac{2\pi p}{(h - q)}. \quad (5.38)$$

Relations (5.35)–(5.37) imply

$$a_{11} = 1, \quad a_{12} = 0, \quad a_{22} = (x^1)^2 + p^2, \quad (5.39)$$

$$a = a_{11}a_{22} = (x^1)^2 + p^2,$$

$$b_{11} = b_{22} = 0, \quad b_{12} = p/\sqrt{a},$$

$$K = -\frac{p^2}{a^2}, \quad H = 0.$$

Furthermore, on setting

$$s_\alpha = r_\alpha^2 + p^2, \quad s_1 \leq a \leq s_2, \quad (5.40)$$

we may express D_2 , I_1 , I_2 and I_3 as

$$\begin{aligned} D_2 &= -I_3 \\ &= -p^2 \int_M a^{-2} u^\alpha u_\alpha dS \\ &\leq -p^2 s_2^{-2} \int_M u^\alpha u_\alpha dS \\ &\leq 0, \end{aligned} \quad (5.41)$$

$$\begin{aligned} I_1 &= \int_M [(a^{11})^2 (u_{1,1})^2 + a^{11} a^{22} (u_{1,2} u_{1,2} + u_{2,1} u_{2,1}) + (a^{22})^2 (u_{2,2})^2] dS \\ &= \int_M [(u_{1,1}^1 u_{1,1}^1 + u_{1,2}^1 u_{1,2}^1 + a^{-1} (u_{1,2}^1 u_{1,2}^1 + u_{2,2}^1 u_{2,2}^1))] dS, \end{aligned} \quad (5.42)$$

where in the expression for I_1 we have used (2.2) and (5.39). We also have

$$I_2 = \int_M [(a^{11})^2 (u_{1,1})^2 + 2a^{11} a^{22} u_{1,2} u_{2,1} + (a^{22})^2 (u_{2,2})^2] dS. \quad (5.43)$$

We deduce from (4.6), (5.41), and (5.43) that $I_2 \geq 0$, while from (5.42) and (5.40), we have

$$\min(1, s_2^{-1}) \int_M u_{,\beta}^\alpha u_{\alpha,\beta} dS \leq I_1 \leq \max(1, s_1^{-1}) \int_M u_{,\beta}^\alpha u_{\alpha,\beta} dS. \quad (5.44)$$

As remarked in Sect. 3.1, several lower bounds are available to estimate Λ . We first consider that established by Krahn [28] (see [3, p. 111]) which like (5.34) involves the area of the right helicoid M given by

$$\begin{aligned} |M| &= \int_0^{\bar{x}^2} \int_{r_1}^{r_2} [(x^1)^2 + p^2]^{1/2} dx^1 dx^2 \\ &= \left(\frac{h-q}{2p} \right) I_4, \end{aligned} \quad (5.45)$$

where

$$I_4 = \left(r_2 s_2^{1/2} - r_1 s_1^{1/2} \right) + p^2 \ln \left(\frac{r_2 + s_2^{1/2}}{r_1 + s_1^{1/2}} \right), \quad (5.46)$$

upon recalling the notation (5.40). Krahn's lower bound then may be stated as

$$\Lambda \geq \frac{\pi j_0^2}{|M|} = \frac{2\pi p}{(h-q)I_4} j_0^2 = m I_4^{-1} j_0^2, \quad (5.47)$$

where m is defined in (5.38) and we recall that $j_0 \approx 2.4048$ is the first positive zero of the Bessel function J_0 .

For comparison, observe that the lower bound (5.34) for the right helicoid becomes

$$\Lambda \geq 8\pi \left(\frac{p}{h-q} \right) I_4^{-1}. \quad (5.48)$$

It is instructive to compute a lower bound obtained from properties of separable solutions to the associated Laplace-Beltrami equation (3.5) which becomes

$$\Lambda a \phi + x^1 \frac{\partial \phi}{\partial x^1} + a \frac{\partial^2 \phi}{(\partial x^1)^2} + \frac{\partial^2 \phi}{(\partial x^2)^2} = 0, \quad (5.49)$$

subject to the Dirichlet boundary conditions

$$\phi(r_1, x^2) = \phi(r_2, x^2) = 0, \quad 0 \leq x^2 \leq \bar{x}^2, \quad (5.50)$$

$$\phi(x^1, 0) = \phi(x^1, \bar{x}^2) = 0, \quad r_1 \leq x^1 \leq r_2. \quad (5.51)$$

Put $\phi(x) = \Theta(x^1)\Phi(x^2)$, and choose $\Phi(x^2) = B \sin(mx^2)$, where B is constant and m is given by (5.38), to obtain $\Phi(0) = \Phi(\bar{x}^2) = 0$ and $\Phi''(x^2) = -m^2\Phi(x^2)$. Moreover, $\Theta(x^1)$ satisfies the ordinary differential equation

$$(\Lambda a - m^2)\Theta + x^1\Theta' + a\Theta'' = 0, \quad \Theta(r_1) = \Theta(r_2) = 0, \quad (5.52)$$

and consequently the lower bound for Λ established by Protter and Weinberger [39, p. 37] (independent of boundary conditions) may be employed to give

$$\Lambda > \inf_{r_1 \leq x^1 \leq r_2} \left\{ \frac{m^2}{a} \right\} \quad (5.53)$$

$$\geq \left(\frac{2\pi p}{h-q} \right)^2 \frac{1}{s_2}. \quad (5.54)$$

Inspection shows that the lower bound (5.47) is superior and therefore preferable to (5.48), whereas for fixed r_1, r_2, p , the lower bound (5.54) is superior to both for sufficiently small $(h-q)$.

Remark 5.6 (Reduction to the plane region) When $p = 0$, the right helicoid reduces to the circular annulus in the plane $y^3 = q$ and (5.52) becomes Bessel's equation. The eigenvalues for the circular annulus have been calculated, for example, by Ramm and Shivakumar [40], while for the circular disc (corresponding to $r_1 = 0$) the eigenvalue is the well-known Faber-Krahn expression j_0^2/r_2^2 . However, the lower bounds (5.47), (5.48) and (5.54) become vacuous when $p = 0$ unless p vanishes to order $(h-q)$ such that (5.38) remains satisfied. Then $I_4 = (r_2^2 - r_1^2)$, and we may take $m = 1$, $2\pi p = (h-q)$.

6 Concluding Remarks

Surfaces of constant mean curvature or constant Gaussian curvature are other special surfaces that appear amenable to the methods of this paper. A possible alternative general approach, motivated by the analyses of Payne and Weinberger [38] and of Dafermos [11]

(see also Horgan and Knowles [21]) for bounded regions in Euclidean space subject, for example, to homogeneous Neumann boundary data, similarly seeks to apply variational arguments to the bounded manifold. In Euclidean space, the Euler-Lagrange equations are the Navier equilibrium equations of linear isotropic homogeneous compressible elasticity and Korn's constant is related to values of Poisson's ratio for which uniqueness fails. Variational arguments applied to the bounded two-dimensional manifold should be expected to similarly lead to non-uniqueness in the equilibrium boundary value problem but for a linear elastic shell. The approach awaits investigation as does extension to bounded manifolds with Neumann boundary conditions, and to manifolds without boundary.

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