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Generalizing actions with the subtraction-compensation property: primary students' algebraic thinking with tasks involving vertical towers of blocks

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Abstract

An important approach for developing children's algebraic thinking involves introducing them to generalized arithmetic at the time they are learning arithmetic. Our aim in this study was to investigate children's attention to and expression of generality with the subtraction-compensation property, as evidence of a type of algebraic thinking known as relational thinking. The tasks involved subtraction modelled as difference and comparing the heights of towers of blocks. In an exploratory qualitative study, 22 middle primary (9–11-year-old) students from two schools participated in individual videoed interviews. The tasks were designed using theoretical perspectives on embodied visualization and concreteness fading to provide multiple opportunities for the students to make sense of subtraction as difference and to advance their relational thinking. Twelve out of 22 students evidenced conceptual understanding of the comparison model of subtraction (subtraction as difference) and expression of the compensation property of equality. Four of these students repeatedly evidenced relational thinking for true/false tasks and open equivalence tasks. A proposed framework for levels of attention to/expression of generality with the subtraction-compensation property is shared and suggestions for further research are presented.

Keywords Generalized arithmetic · Early algebra · Mathematical structure · Relational thinking · Subtraction as difference · Primary mathematics

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1 Introduction

Historically, school mathematics has been conceptualized as involving arithmetic in primary school *then* algebra in secondary school—students first learn to calculate and later learn to generalize (Cai & Moyer, 2008; Carpenter et al., 2003; Carraher et al., 2006). Since the 1990s, however, the abilities and inclinations of primary school students to generalize about arithmetic at the same time as learning arithmetic have come to the fore (Lins & Kaput, 2004). Kaput (2008) argued that children benefit from being introduced to algebra as generalized arithmetic early on their mathematics learning, such that they learn to abstract structures and systems from computations and relations at the same time they are learning arithmetic. They can be given opportunities to notice the structures of equations and expressions, to look for and describe patterns, and to use mathematical properties in their development of and justification of strategies. Emerging research suggests that learning arithmetic through an algebraic lens is an effective way to build computation fluency during primary school (Chesney et al., 2018), as well as foster an understanding of equivalence and numerical relationships that form a solid foundation for learning more formalized algebra in secondary school (Britt & Irwin, 2008; Kindrat & Osana, 2018).

This project is designed to build on a growing research base investigating how children come to attend to and express generality about arithmetic properties when given appropriate opportunities to do so (Blanton et al., 2015; Cooper & Warren, 2008, 2011; Mason, 2008). Of special interest is children's attention to properties involving the operation of subtraction, and in particular, the compensation property of equality relating to the subtraction of two operands, which can be explained as follows: what you add to (or subtract from) one operand must be added to (or subtracted from) the second operand for the difference to remain the same (Russell et al., 2011). Prior research has highlighted the difficulties children experience with generalizing activities involving this property of subtraction (Cooper & Warren, 2008, 2011).

The overall purpose of our research program is to gain insights into how children make sense of the mathematical properties of addition and subtraction, and develop a learning task sequence that fosters their arithmetic and algebraic thinking. The data were drawn from 22 students' individual task-based interviews and included video recordings and written student work samples. The central research question was as follows: *How do middle primary (9–11-year-old) students evidence relational thinking in the context of comparing heights of towers of blocks?*

2 Background

In the following three sub-sections, we overview theoretical perspectives on developing early algebraic thinking, our characterization of relational thinking for this study, and prior research on relational thinking with the subtraction-compensation property of equality.

2.1 Theoretical perspectives on developing early algebraic thinking

The process of generalizing—noticing how the features of a specific mathematics situation apply in general to many situations—is at the heart of algebra (Mason, 2017). In efforts to define and conceptualize algebra and algebraic thinking, it has been found that researchers use different combinations of terms for similar phenomena and vice versa: similar terms for substantively different phenomena (Kieran, 2022; Venkat et al., 2019). Such terms include structure, relationships, generality or generalization, and properties (Venkat et al., 2019). Additionally, there are differing perspectives on approaches for developing students' algebraic thinking at school including a functional perspective and a generalized arithmetic perspective (Kieran, 2022).

A functional perspective of algebraic thinking focuses on children learning to generalize relationships between co-varying quantities. For example, with figural growing patterns, one variable is the item/sequence number, and another variable is the number of shapes (quantifiable aspect) that comprise each figure. Children evidence algebraic thinking by attending to and expressing generality about how the features of particular figures relate to the item number and how the relationship between variables (equation or rule) holds true for all figures in that pattern. They use gesturing, such as between item number and figural aspects or from one figure to another, to support their verbal explanations of the functional relationship. Researchers have theorized different levels of thinking related to the process of children's attention to the particular instances of a functional relationship shifting to its general relationship. For example, Radford's (2010) framework comprises four types of generalizing actions, each of which evidences a type of thinking about functional relationships:

- Arithmetic generalization (*arithmetic thinking*)
- Factual generalization (*emerging algebraic thinking*)
- Contextual generalization (*algebraic thinking* evidenced by generality expressed linguistically)
- Symbolic generalization (*algebraic thinking* evidenced by generality expressed with alphanumeric semiotic system of algebra

A generalized arithmetic perspective (overviewed in the Introduction) focuses on children learning to notice the structure of equations and expression, to find patterns, and use mathematical properties (Kaput, 2008). As with Radford's framing of algebraic thinking, Venkat et al.'s (2019) theorization of the process of children attending to and expressing generality about *mathematical structure* highlights progress from perceiving relationships within particular (local) cases (apprehending/conjecturing an emergent structure) to generalizing actions, across a class of examples or working with a particular case while viewing it as generic of the class.

In our study, we investigated children's attention to the subtraction-compensation property as well as expressions of it in generalizing actions across different cases, which we considered evidence of relational thinking. In the next sub-section, we seek to explain our understanding and use of the term "relational thinking" in this study.

2.2 Characterizing relational thinking for this study

Kieran (2022) categorized and synthesized research activity around children's early algebraic thinking related to generalized arithmetic using two dimensions: structural thinking and analytic thinking. The structural thinking dimension included studies investigating how children see and express structure, and properties of number and operations. The analytic thinking dimension included studies investigating how children deal with unknowns as if they were known, to transform and solve equations. Somewhat common to both

Table 1 Basic properties of addition, subtraction, and equality

	Name	Representation
Addition	Associative	$(a+b)+c=a+(b+c)$
	Commutative	$a+b=b+a$
Subtraction	Order-relevant	$(a-b)-c\neq a-(b-c)$
	Order-relevant	$a-b\neq b-a$
Equality	Reflexive	$a=a$
	Transitive	If $a=b$ and $b=c$, then $a=c$
	Symmetric	If $a=b$ then $b=a$

dimensions is thinking related to the properties of equivalence, which is captured in the notion of having a *relational* view of equivalence: that is, viewing the equal sign as an indicator of a relation and applying a knowledge of equivalence to simplify calculations (Jacobs et al., 2007) and solve and evaluate equations (Rittle-Johnson et al., 2011). A relational view of equivalence is referred to by some as relational thinking (e.g., Carpenter et al., 2003; Venenciano et al., 2021), whereas others use the phrase structural thinking in a similar way to relational thinking (e.g., Mason et al., 2009). Kieran (2022) referred to a possible general dimension encompassing both structural and analytic thinking, which she referred to as relational thinking. In this paper, we refer to structural thinking as thinking pertaining to properties of operations and structures (e.g., additive and multiplicative structures), and relational thinking as thinking pertaining to properties of equality.

Structural thinking and relational thinking make use of *mathematical structure*, which encompasses general properties that are instantiated in particular situations as relationships between elements (Mason et al., 2009). Basic (axiomatic) properties associated with numbers, operations, and relations, often cited in studies on early algebraic thinking and generalized arithmetic, include properties of addition: associativity, commutativity, and identity; and properties of equality: reflexivity, transitivity, and symmetry. Note that when it comes to studies involving subtraction, however, there is often no explicit mention of the operation's properties. This may be due to difficulties in phrasing—since the commutative and associative properties are not properties of subtraction. One away around this is to refer to subtraction as having order-relevant properties. These properties are outlined in Table 1.

The addition-compensation property of equality[1] can be given as follows: If $a+b=d$ then $(a+c)+(b-c)=d$; the subtraction-compensation property of equality can be given as follows: If $a-b=d$, then $(a+c)-(b+c)=d$ (Molina & Castro, 2021).

Despite notable differing conceptualizations in the field of early algebra, there is general consensus that early algebraic thinking involves (i) acts of deliberate generalization and expression of generality, and (ii) reasoning based on generalizations (often as a separate endeavour) that are communicated semiotically, such as through speech, gestures, and written symbols (Kieran, 2022; Lins & Kaput, 2004). It is important to note that *early* algebraic thinking does not necessarily involve communicating generalizations using alphanumeric symbols (Kieran, 2022; Mason, 2017; Radford, 2011). Children can evidence their attention to generality in their actions (Radford, 2011) and verbalizations (Venkat et al., 2019). In this study, we looked for evidence of middle primary students' relational thinking in

[1] Also termed the addition-compensation principle by Cooper and Warren (2008)

verbalizing and applying the subtraction-compensation property with tasks designed to elicit such generalizing.

2.3 Prior research on relational thinking with the subtraction-compensation property

We found numerous generalized arithmetic studies on developing children's understanding of the equal sign and equivalence, but very few studies on the compensation properties of equality. In research with Grade 3 students, Cooper and Warren (2011) found that the subtraction-compensation property was particularly difficult for students to grasp. They began with activities drawing students' attention to the addition-compensation property, which they explained as "do the opposite"—"If the first number is increased / decreased by an amount, then the second number is oppositely decreased/increased by the same amount respectively to keep the sum of the two numbers the same" (p. 204). They found that exploring the property with unnumbered paper strips and number lines but not measuring cylinders helped their students attend to generality. After addition, they attempted to teach the subtraction-compensation property, which they explained as "do the same": "If the first number is increased/decreased by an amount, then the second number is increased/decreased by the same amount respectively to keep the difference of the two numbers the same" (p. 204). They found that students experienced confusion. The researchers speculated that the opposite effect of the compensation property for subtraction problems compared to the compensation property for addition problems was problematic for developing children's relational thinking.

Schifter (2018) also investigated Grade 3 students exploring the addition-compensation property and then the subtraction-compensation property. The students expressed their attention to generality with addition by inventing and acting out stories, for example people at the beach are in the water or on the sand. They may go in or out of the water but the total number of people remains the same. They were prompted to invent a subtraction story and test their rule for addition and found it did not work. Although the rule for subtraction-compensation emerged from the activities, it is unclear if it was the result of trial and error with calculations, or relational thinking.

In our study, we chose the representation of vertical towers of (joinable) blocks and subtraction modelled as difference to investigate how tasks might elicit their attention to and expression of generality.

3 Research design

In a qualitative collective case study, we explored middle primary (Year 3 or 4, 9–11-year-old) students' attention to and expression of generality with the subtraction-compensation property, as evidence of their relational thinking. The students were interviewed using a sequence of tasks involving subtraction modelled as difference with towers of blocks. The unit of case study analysis was each student (Creswell, 2013). The theoretical perspective of embodied visualization informed the design of the interview tasks. Such a perspective views children's thinking to be "a tangible social practice materialized in the body (e.g., through kinaesthetic actions, gestures, perception, visualization)". Their thinking involves "the use of signs (e.g., mathematical symbols, graphs, written and spoken words), and artifacts of different sorts (rulers, calculators and so on)" (Radford, 2011, pp. 17–18). In this study, an artifact of physical (joinable plastic) blocks was provided to the students and semiotic data were collected from video recordings and written work samples.

The choice of subtraction model and tool, task design considerations, participant demographic information, and data analysis are outlined in the four following sub-sections.

3.1 Choice of subtraction model and tool

In this study, we investigated how middle primary students made sense of the compensation property of equality in the context of subtraction by providing them with a model of two towers of blocks to compare. We analysed their responses to a range of tasks involving different representations, including symbolic subtraction expressions and equations. We reasoned that the comparison model of subtraction would be more helpful than the take-away model for making sense of the compensation property of equality. These two models are described by Usiskin (2007) as follows:

- Take-away model: If a quantity b is taken away from an original quantity a, the quantity left is $a-b$ (the *remainder*).
- Comparison model: The quantity $a-b$ tells how much b is less than the quantity a (the *difference*).

The comparison model has also been termed "determining the difference" (Selter et al., 2012) or subtraction as difference. The take-away model of subtraction has traditionally received more attention in school mathematics than the comparison model (Selter et al., 2012; Usiskin, 2007). This is of concern since both models are needed to successfully interpret additive problem-solving situations and both underpin the flexible use of strategies for subtraction computation (Selter et al., 2012).

Carraher (1993) argued that length expresses magnitude more directly and unambiguously than other attributes. Yet prior research found that children experienced confusion with the compensation property of equality when utilizing horizontal linear bars (Cooper & Warren, 2008, 2011). We chose in this study to investigate children working with a *vertical* linear representation. We provided Unifix blocks (joinable plastic blocks) for the students to build their own vertical towers of blocks because they provide potential for attention to a proportional qualitative (taller or shorter) and/or quantitative length dimension (e.g., shorter or taller by 3 blocks) for comparing the difference in heights. The blocks have high physicality and perceptual richness (Fyfe & Nathan, 2019) and comparing tower heights is grounded for children in a visual familiar life context of tall buildings (Bofferding, 2018). As with the use of vertical number lines (representing temperature, financial net worth, altitude, etc.) for learning directed number concepts (Stephan & Akyuz, 2012), both measurement and specified discrete quantities are present to support student thinking. We speculated that the idea of buildings having underground parking levels might also be a potential real-life analogy for exploring subtraction as difference with a negative subtrahend if the opportunity afforded.

3.2 Design of the subtraction interview tasks

We designed fifteen sequenced tasks (see Appendix and Table 2) increasing in difficulty according to the concepts of concreteness fading (Fyfe & Nathan, 2019; Goldstone & Son, 2005) and to the size of the minuend, subtrahend, and/or difference. We drew on

Table 2 Interview questions: representation by task type

	Depict	Odd one out*	True/false# (small minuend)	Open equivalent## (small minuend)	True/false (large minuend)	Open equivalent (large minuend)
Concrete	Q1 Q2					
Contextualized		Q3 Q4 Q5	Q6a	Q7a	Q8a Q10a Q12a Q14a	Q9a, Q11a Q13a Q15a
Symbolic equation			Q6b, Q6c, Q6d	Q7b, Q7c	Q8b, Q8c Q10b, Q10c Q12b, Q12c Q14b, Q14c	Q9b, Q9c Q11b, Q11c Q13b, Q13c Q15b, Q15c

*E.g., Circle the pair of towers that doesn't belong: 10 blocks & 8 blocks; 11 blocks & 9 blocks; 15 blocks & 13 blocks; 7 blocks & 4 blocks

#E.g., True or false: 20 – 16 = 22 – 14

##E.g., Fill in the numbers: 34 – 28 = …… – ……

Goldstone and Son's (2005) definition of concreteness fading as "the process of successively decreasing the concreteness of a simulation with the intent of eventually attaining a relatively idealised and decontextualised representation that is still clearly connected to the physical situation that it models" (p. 70). This definition resonates with a process that can occur within a single time period, as with our task-based interviews, and does not require mastery at each stage.

Both making and drawing towers were included to help students visualize subtraction as difference—a comparison of heights and determining the difference. We assumed that children of this age may not have developed this meaning for subtraction but would have had some prior experience of comparing length and discrete numbers of objects. The second task also provided an opportunity for initial evidence of generalizing through attention to the compensation property of equality, i.e., changing the original towers' heights in a way that keeps the difference constant. Our iconic representations also included the drawings made by the students themselves of pairs of towers and written numeric pairs of tower heights (Q3 and Q4). Q5 included symbolic subtraction expressions and later questions included full equations (e.g., True or false: 20 – 16 = 10 – 6 and Fill in the numbers: 20 – 16 = __–__) all in the problem context of comparison or determining the difference.

The true/false questions with a matching open task (Q6–Q15) increased in the size of minuend and subtrahend, with the intent of providing repeated opportunities for students to evidence generalizing by using the compensation property of equality. These true/false and open question formats were recommended by Carpenter et al. (2003) for early algebraic thinking with generalized arithmetic.

All students were asked to respond to the first seven tasks, and some students, based on their given responses, were asked to continue with further tasks. Although the task representations changed, Unifix blocks were available throughout the interview and some students used them beyond the first few tasks.

3.3 Participants and data collection

Twenty-two Year 3 or 4 (9–11-year-old) students from two schools in metropolitan Melbourne (low-medium and medium–high socioeconomic status (SES); one Catholic sector, one independent) were interviewed. (Nearly 40% of Australian students in Victoria attend non-government schools.) These middle primary students were in their fourth or fifth year of primary school. They were selected randomly from a range of backgrounds related to SES, English as an additional language, and prior mathematics achievement levels in subtraction. The intent was to maximize the range of levels of understanding about subtraction. Table 3 presents demographic information about the student participants and interview questions attempted. For ease of reading the study's findings, gender-preserving pseudonyms have been assigned alphabetically according to level of generalizing actions evidenced.

The students were each interviewed for approximately half an hour in their own school setting during a school day and were encouraged to explain their thinking verbally during the interview (Booth et al., 2017; Radford, 2010). A video camera was positioned overhead and pointed downwards to capture a student's hand gestures, writing, use of blocks, and

Table 3 Student pseudonyms, demographic information, and interview questions attempted

Pseudonym	Year level	Sex	English as additional language?	Interview questions attempted
Abby	3	F	Y	Q1–7
Bronte	3	F		Q1–7
Cathy	4	F		Q1–7
Darren	4	M	Y	Q1–7
Elijah	3	M		Q1–7
Freddie	3	M		Q1–11
Georgia	3	F		Q1–7
Helen	3	F		Q1–9
Ian	3	M		Q1–7
Jia	4	F	Y	Q1–7
Karen	4	F	Y	Q1–7
Lucy	3	F		Q1–7
Melanie	3	F	Y	Q1–7
Nerinda	3	F		Q1–9
Ollie	4	M	Y	Q1–9
Pam	4	F	Y	Q1–11
Quentin	4	M		Q1–9
Rosalie	4	F		Q1–9
Soren	4	M	Y	Q1–13
Timothy	4	M		Q1–15
Ulrich	4	M		Q1–15
Vincent	3	M		Q1–15

voice, but not their face. A4-size handouts of each question were given to students to draw and write on, and these were used alongside the video data in the analysis.

3.4 Data analysis

For our analysis of each student's responses in the interviews, we drew on embodied visualization for researching algebraic thinking (Radford, 2011) and the Student Noticing framework (Lobato et al., 2013). An embodied perspective of visualization considers thinking as not purely mental but intertwined with the body and world (Radford, 2011): visual perception, kinaesthetic actions, gestures, signs, and artifacts do not merely mediate thinking but are actually part of it. The Student Noticing framework supports fine-grained analysis of how mathematics learners select, interpret, and work with features of a task that are salient to them from multiple sources of information. The framework was used in prior research on secondary students on algebraic generalizing tasks (e.g., Wilkie, 2022). Students' Centres of Focus (CoF)—noticing of properties, features, regularities, or conceptual objects—are identified moment-by-moment throughout the interview to provide insights into their thinking processes. Specific types of focusing interactions, which are semiotic data (verbal utterances, physical gestures, written markings) that give rise to or contribute to a particular CoF, are documented

Table 4 Three Centres of Focus and associated codes that emerged from the interview analysis (source: Wilkie & Hopkins, 2024)

Centre of Focus (CoF) and codes		Description	Illustrative examples from the interviews
CoF1 attending to difference			
COM	Comparison of heights of towers or numbers	Student attends to concept of differing tower heights or numbers	Verbal, gesture: holds towers and says "One is higher, one is lower" (*Bronte*); "This tower has 3 and this one has 4" (Melanie); "One is shorter and one is taller" (*Karen*)
AMO	Amount of difference between tower heights or numbers	Student specifies amount of difference (more than, less than, both directions)	Verbal, gesture, written: points to towers in turn and says "This one is 2 more than this one"; writes "2" and draws towers with difference of 2 (*Bronte*)
NEG	Differences involving zero/negative numbers	Student attends to concept of difference involving 0 and/or negative number	Written: fills in 92 – 38 = 54 – 0; fills in 230 – 46 = 184 – 0 and also 230 – 46 = 0 – -184 (*Vincent*)
CoF2 using a subtraction strategy			
IA	Indirect addition	Student adds on to smaller tower/number to find difference (e.g., 19 + ? = 30)	Verbal, written: for 15 – 5 says "I know 5 plus 10 equals 15" (*Bronte*); writes "10" as the difference; "I did a tower… I went 5, 6, 7" (*Lucy*)
TA	Take away (direct subtraction)	Student takes away smaller tower/number from larger tower/number to find difference (e.g., 30 – 19 = ?)	Written: writes and uses vertical take away algorithm for 35 – 8, 34 – 9, 30 – 3, and 45 – 18 (*Jia*); for "difference between 92 and 38" writes "subtract 2 from 92", then "subtract 6 from 90 = 84", then "84 – 30 = 54" (*Pam*)
IS	Indirect subtraction	Student takes away from larger tower/number to reach smaller tower/number (e.g., 30 – ? = 19)	Verbal, gesture: points to numbers in turn in "30 blocks & 19 blocks" and says "This one is minusing 11" (*Timothy*)
OTH	Other	Student uses invalid or irrelevant strategy	Written: writes difference between unit digits in minuend and subtrahend (*Vincent*)
CoF3 evidencing relational thinking (expressing generality)			

Table 4 (continued)

Centre of Focus (CoF) and codes		Description	Illustrative examples from the interviews
G-ADD	Adding the same amount only	Student explains that adding the same amount to each tower/number keeps the difference the same	Gesture: adds 3 blocks to each tower to keep same difference (*Ollie*) Verbal, gesture, written: explains adding 5 to minuend and subtrahend in 30 – 3 to assess if 35 – 8 is equivalent; says "yes" for 20 – 16 = 30 – 26 and says "you just add 10 to both of them"; writes "T" for true (*Soren*)
G-SUB	Subtracting the same amount only	Student explains that subtracting the same amount to each tower/number keeps the difference the same	Verbal, gestures written: takes 2 blocks off each tower and says "I took the 2 off the 6 and took 2 more off the 4"; draws towers with removed blocks above (*Lucy*)
G-DIR	Adding and subtracting: Changes to the minuend and subtrahend need to be the same magnitude and direction	Student explains that the difference changes if amounts are added to/subtracted from 1 tower/number	Verbal, gesture, written: writes "F" for false; points to each minuend pair and then subtrahend pair in turn (20 – 16 = 22 – 14) and says "because that one got added and that one got subtracted" (*Rosalie*)
G-MAG	Adding and subtracting: Changes to the minuend and subtrahend need to be the same magnitude	Student attends (invalidly) to only the magnitude of changes to minuend and subtrahend and not direction	Written, verbal: writes in the gaps "The difference between 20 and 16 is the same as the difference between 19 and 17" and says "because 20 take away 1 equals 19 and add the 1 to 16 equals 17" (*Jia*)

to analyse student actions and interactions (Lobato et al., 2013). Examples of focusing interactions found in the study are presented in “Section 4”: in Table 4, in the figures, and in direct quotes of students’ utterances.

The data analysis process was iterative, interpretive, and collaborative (Corbin & Strauss, 2008), and similar steps to Lobato et al. (2013) were followed. A list of provisional CoF and codes was developed from the literature (Miles & Huberman, 1994) and the first author’s prior research on students’ generalizing (e.g., Wilkie, 2022). This list was then refined and added to through constant comparative analysis of the video data and students’ written work. Focusing interactions were documented and coded moment-by-moment (Lobato et al., 2013). Multiple passes were made, initially by the first author, then with a subset by the second author, and then jointly with the same subset. Once all the students’ focusing interactions were documented and coded according to CoF (see Table 4), we analysed those interactions which had been interpreted as CoF3—evidencing relational thinking (expressing generality). It is important to note that for CoF3, a student needed to express verbally (not just implicitly attend to) the subtraction-compensation property, albeit in a partial or context-specific way. These students’ CoF3 responses were then further categorized according to the nature of their generalizing actions (see Description column in Table 4) and this led to our emergent framework shared in “Section 4” (Table 5).

4 Findings

The “Section 4” is structured with three sub-sections. In the first sub-section, we present three CoF and 11 related codes which emerged from our analysis. In the next two sub-sections, we share about students’ attention to subtraction as difference and different levels of expressing generality about the subtraction-compensation property as evidence of their relational thinking.

4.1 Centres of focus evidenced

Table 4 presents the three Centres of Focus along with 11 associated codes and illustrative examples.

As seen in Table 4, the CoF relate to three types of attention in the students’ task responses: conceptual attention to differing heights of towers of blocks, computational attention to calculating the difference, and relational attention to the subtraction-compensation property. Additionally, CoF3 involved students expressing their attention to generality verbally.

4.2 Students’ attention to the concept of difference

At the beginning of the interview, each student was asked to make two towers of different heights with Unifix blocks. When prompted, “What is the difference between them?”, 7 out of 22 students attended to the heights as qualitatively different (CoF1 COM) whereas the others evidenced conceptual understanding of the difference as a certain number of blocks (CoF2 AMO). For example, Abby made a tower of 3 and 5 blocks and stood them on the table vertically. She said, “This is taller and this one is shorter”, pointing to each in turn (CoF1 COM). Her drawing of the towers, presented in Fig. 1, evidences conceptual

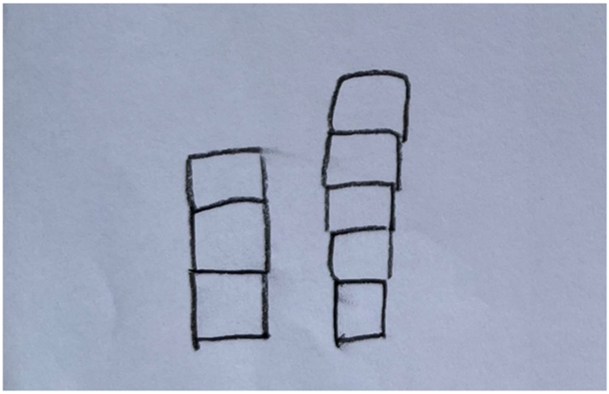

Fig. 1 Abby's drawing of her towers

attention to towers standing on the ground side by side, but not to the correspondence of block heights.

Abby did not evidence that she understood the concept of quantitative difference (CoF1 AMO) and her direct counting of small numbers of blocks was frequently inaccurate, suggestive of the need for more secure one-to-one correspondence concepts.

For the same task (Q1), another student Melanie initially made two towers of three blocks but was prompted again to make them different heights. She added a block to one tower. When asked about the difference, she said, "This one has 3 and this one has 4" (CoF1 COM). She was prompted, "How much taller is this one?" She was then able to quantify the difference as 1 block (CoF2 AMO). Her picture, presented in Fig. 2, evidences conceptual attention to the towers standing on the ground, correspondence of block heights, and the difference in heights.

The CoF1 code on difference involving negative numbers (CoF1 NEG) emerged when the interviewer asked a few students if they could use negative numbers for creating equivalent expressions in the open equation tasks (later in the interview). They had each chosen to use zero previously in the same task and probing their understanding about numbers smaller than zero was considered appropriate. For Q11, Thomas had filled in the statement "The difference between 92 and 38 is the same as the difference between 84 and 0" (calculation error; should have been 54 and 0). He was asked if he could use negative numbers at all. He wrote 83 and -1, and then 82 and -2, as in Fig. 3. He said, "I think about it as like trees and roots".

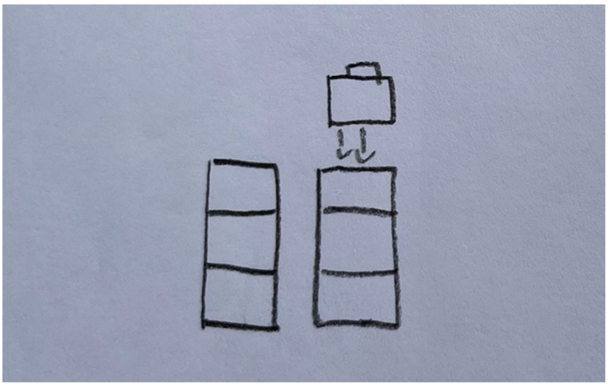

Fig. 2 Melanie's drawing of her towers

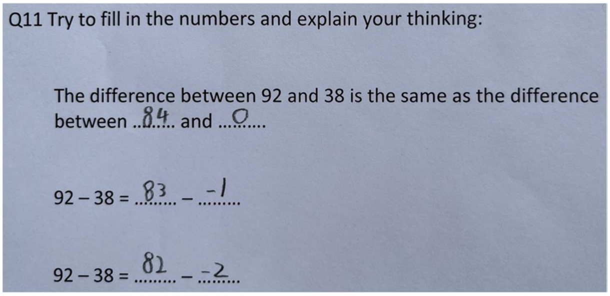

Fig. 3 Thomas's inclusion of negative numbers when writing equivalent expressions

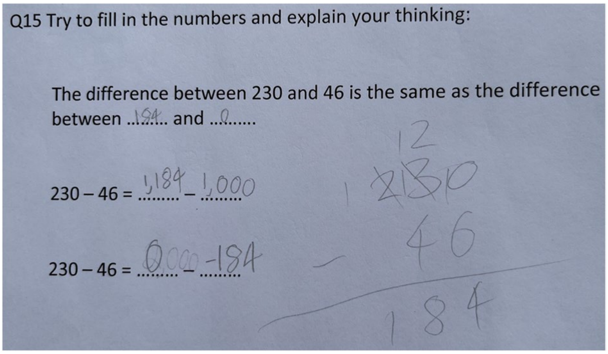

Fig. 4 Vincent's inclusion of negative numbers when writing equivalent expressions

For Q15, Vincent had filled in the statement "The difference between 230 and 46 is the same as the difference between 184 and 0", and then "1184 – 1000". He was asked if he could use negative numbers, and he wrote "0 – -184", as shown in Fig. 4.

The interviewer commented on the difficulty of representing negative numbers with blocks. Vincent placed one of his towers on the edge of the table and gestured with an open horizontal hand level with the table. He said, "This is the floor and this is how you do it". He then gestured with a horizontal hand to show an increasing depth of levels underneath the table. Although this code was only recorded for a few students (who had used zero in their open tasks), their use of analogies (trees and roots, or underground tower levels) is suggestive of a vertical representation supporting conceptual meaning for directed number, specifically the subtraction of negative numbers.

Overall, all but one student in this cohort evidenced conceptual understanding of the quantitative difference in heights of pairs of towers of blocks. Additionally, a few students incorporated negative numbers (as hoped).

4.3 Evidence of students' relational thinking

Table 5 presents an emergent framework of four proposed levels of attention to or expression of generality with the subtraction-compensation property of equality. Out of the 22 students interviewed, nearly half of them (10) evidenced reasoning predominantly focused on making sense of the concept of difference itself but not yet

Table 5 Proposed framework of levels of attention to/expression of generality of the subtraction-compensation property (source: Wilkie & Hopkins, 2024)

Level of attention to/expression of generality		Description	Num. students
Pre-1	Conceptual attention to difference	Is developing/has developed conception of difference between 2 tower heights or pairs of numbers	10
1	Concrete context only	Explains that adding or subtracting the same amount to each tower/number keeps the difference the same (expresses subtraction-compensation property of equality in their own words but only with physical towers)	2
2	Partial or context-specific	Expresses aspects of generality related to the subtraction-compensation property of equality but partial/context-specific: - "Nice numbers" only (e.g., add/subtract 10, 20 to both minuend and subtrahend) - Adding/subtracting only small amounts to both minuend and subtrahend - Lack of attention to direction of change to minuend and subtrahend (evidence of invalid relational thinking)	6
3	Intentional	Evidences relational thinking in numerous tasks to assess equivalence of symbolic expressions and to create own equivalent expressions	4

relational thinking (pre-1 level). The remaining students evidenced some attention to and expression of generality: a few with the concrete representation of the towers of blocks (level 1), and six partially or in a specific context (level 2). Four students provided repeated evidence of relational thinking to both assess symbolic equations (true/false) and to create their own equivalent expressions with open symbolic equations (level 3).

These three levels that emerged from the data analysis are described with illustrative evidence in the following three sub-sections.

4.3.1 Level 1: attention to and expression of generality in a concrete context (only)

Two students evidenced reasoning in a concrete context about what happens to towers of blocks when their heights are changed. In Q1, Karen had added one block to each of her original towers (rather than making new towers) to keep the difference the same. Yet she did not reason similarly with Q3 and 4 when moving to a different representation. It is possible that the change of representation from concrete to written numeric pairs (e.g., 10 blocks and 8 blocks) was a source of confusion for her. She seemed to see the four parts of each question like a table of values and compared the numbers in vertical columns rather than the (horizontal) numbers in each pair. She responded similarly in Q5, seeming to ignore or not interpret the subtraction symbol in each expression. The lack of an equal sign might have been an issue here. In Q6, Karen did evidence a take-away strategy to assess the equivalence of some expression (with $20-16$). Now with this subsequent change in representation (worded sentence and then equations), she seemed to interpret the subtraction symbol as "take away". For Q7 (open task), Karen asked if she could use the blocks and it was in this question, using concrete blocks, that she evidenced attention to generality again. She had proceeded to make towers of 20 and 16 blocks. The interviewer asked her if she could make them into other towers with the same difference. Karen added 2 blocks to each of her towers and said, "They still equal, have 4 [sic]… that'll be 18, that'll be 22". She wrote 20 – 16 = "22 – 18". She then added 3 blocks to each tower, saying, "it still equal 4" and writing 20 – 16 = "21 – 25". She then added 6 blocks to each tower, saying again, "it still equal 4" and writing 20 – 16 = "27 – 31". Karen's unconventional ordering of her chosen numbers (smaller number first) suggests that she was no longer attending to the subtraction symbol as "take away" but to recording the changing heights of her pairs of towers.

Lucy had also evidenced attention to generality in Q2 by removing 2 blocks each from her original towers: "I took the 2 of the 6 and took 2 more off the 4. They are different heights with the same amount of ummm" (interviewer: "difference"). As with Karen, Lucy struggled with Q4, which could be related to the change in representation to numeric pairs. She attended to changes in numbers vertically down the pairs but not to the difference within each pair. In Q5, with the change to a symbolic expression, she calculated each expression using an indirect addition strategy but struggled with the size of the numbers. In Q6, she constructed and used towers of 20 and 16 blocks to assess equivalence. For example, she added 1 block to each tower and then wrote "T" (true) next to the statement "The difference between 20 and 16 is the same as the difference between 21 and 17". In Q7, she seemed to misunderstand the language of the question and made towers of 14 and 16 rather than 20 and 16 blocks, and filled in, "The difference between 20 and 16 is the same as the difference between 14 and 16". Lucy made another pair of towers of 8 and 6 blocks, writing 20 – 16 = 8 – 6. She then removed three blocks from each, writing 20 – 16 = 5 – 3. Interestingly, in Q7, Lucy only subtracted blocks whereas Karen had only added blocks to keep the difference the same. This pattern of either adding or subtracting blocks (but not both) to keep the difference the same was evidenced by other students with other representations, as shared in the next sub-section.

4.3.2 Level 2: partial or context-specific expression of generality

Students categorized as level 2 were found to express generality but only in particular contexts or ways. For example, some students generalized by adding or subtracting "nice" numbers to both subtrahend and minuend to keep the difference constant. Pam, when assessing 20 – 16 = 30 – 26 (true/false), said, "It's like 20 – 16 but like this is 10 more than usual" (points to 30 – 26). Yet she did not continue to generalize in Q7 to 11 but drew on indirect addition and take-away strategies to calculate the differences. Rosalie attended to adding or subtracting small numbers to keep the difference constant. In Q6, she explained:

> If we get 20 and add 1, it's 21 and if you have 16 and add 1, it's 17. It kind of makes sense. If you add up a number, you should add up something else too. (Rosalie)

When assessing 20 – 16 = 10 – 6 (true/false), she wrote "T" and said, "It's like minusing 10 off each number". When assessing 20 – 16 = 22 – 14, she noticed the direction of the change and wrote "F" saying "because that one got added and that one got subtracted" (CoF3 G-DIR). Yet when the size of numbers increased (Q10) or the amount subtracted was no longer a 10, she appeared confused and no longer evidenced relational thinking. Similarly, Ollie attended to generality by adding and subtracting multiples of 10 to create equivalent expressions in Q7 as seen in Fig. 5.

(Earlier with the towers in Q2, Ollie had evidenced noticing the subtraction-compensation property when he had written "you add 1 block on the small one that won't be the

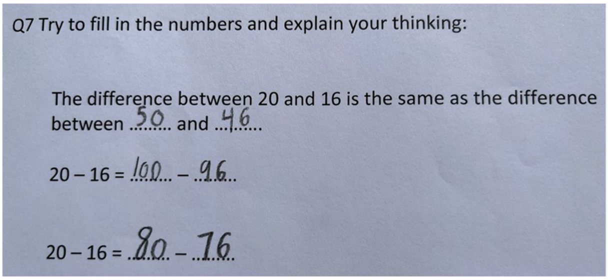

Fig. 5 Ollie's generation of equivalent expressions by adding/subtracting multiples of 10

same, so you have to add a block on the other one".) Some students evidenced valid and invalid relational thinking, related to only noticing the magnitude and not the direction of changes to matching minuends and subtrahends. For example, Nerinda, when assessing the differences between 34 and 28 and between 33 and 29 (written sentence in Q8), said, "True [drew tick], they've added one to both so it's going to be the same answer". It could have been the structure of the written sentence that obscured the change in direction, but Nerinda also incorrectly assessed the symbolic equation 34 – 28 = 35 – 27, this time misapplying the addition-compensation property: "True, because they minused 1 from here [pointed to 27] and then added it onto there [pointed to 35] so then it would be the same" (shown in Fig. 6).

In Q9, Nerinda wrote 34 – 28 = "33 – 27" and "35 – 29" but then invalidly wrote "30 – 32" after counting on her fingers, suggestive of subtracting 4 from 34 and adding 4 to 28. Her responses suggest that she was focusing on the magnitude of the changes being important, which was valid. Yet she began correctly with assessing the need to add the same number to both minuend and subtrahend, and then reverted to misapplying the addition-compensation property. We wonder if the symbolic subtraction expressions obscured her previous clarity since we think it likely she would not have made those errors with the physical towers of blocks. It is also of note that Nerinda did not double-check her attempts at relational thinking with actual calculations of the difference. For these students at level 2, more experience with generalizing actions in a concrete context or using more manageable numbers, along with encouragement to calculate the

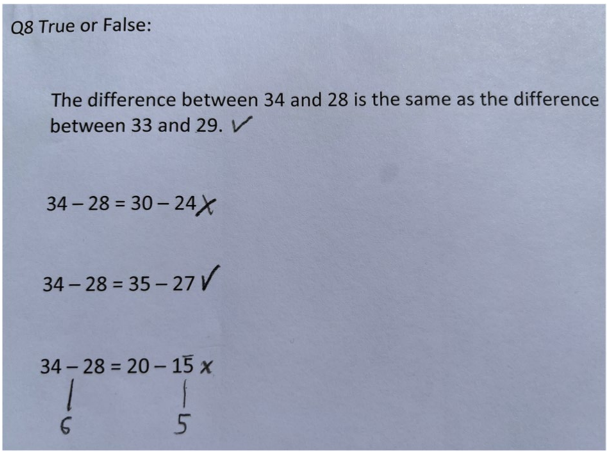

Fig. 6 Nerinda misapplying the addition-compensation property to subtraction

differences to check their conjectures, may be needed to consolidate their understanding of the subtraction-compensation property.

4.3.3 Level 3: intentional and repeated expression of generality (evidence of relational thinking)

Four out of 22 students' responses were categorized as level 3 because of repeated evidence of relational thinking, both with adding and subtracting the same amount, with larger numbers, and across different tasks. They also evidenced noticing (eventually for two students) and explaining the need for attention to the direction of changes, not just the magnitude. For example, in Q6, Soren correctly assessed 20 – 16 = 22 – 14 as false, saying, "They added 2 to this [pointed to 22] but they subtracted 2 to this [sic] [pointed to 14]". In Q12, he correctly assessed 510 – 485 = 500 – 495 as false, saying, "You just add 10 to them [pointed to 500 – 495] but this one [pointed to 495] is minus 10, so it's more close [sic]". Similarly, Timothy correctly assessed 34 – 28 = 35 – 27 as false, drawing an upward arrow above 35 and a downward arrow below 27. He also explained that the difference had increased by two instead of remaining constant.

In Q8, when assessing the equivalence of differences between 34 and 28, and 33 and 29 (written sentence), Ulrich initially attended only to magnitude but then self-corrected:

> That and that [pointed to 35 – 8 and then 34 – 9 underneath] would equal the same thing—no! They *wouldn't* be the same because they *lowered* that number and *raised* that number. If you *add* more numbers to subtract, you need to *add* more numbers to the numbers you're subtracting from otherwise it won't equal the same thing. (Ulrich)

Because the numbers were small, Ulrich was able to check his initially incorrect answer. In Q12 when assessing the differences between 510 and 485 and between 515 and 480, Ulrich correctly applied his attention to direction saying, "That won't work because you're *increasing* that [pointed from 510 to 515] but *decreasing* that [pointed from 485 to 480]". He also double-checked by calculating the differences using IA, which seemed to help him confirm that his relational thinking was valid.

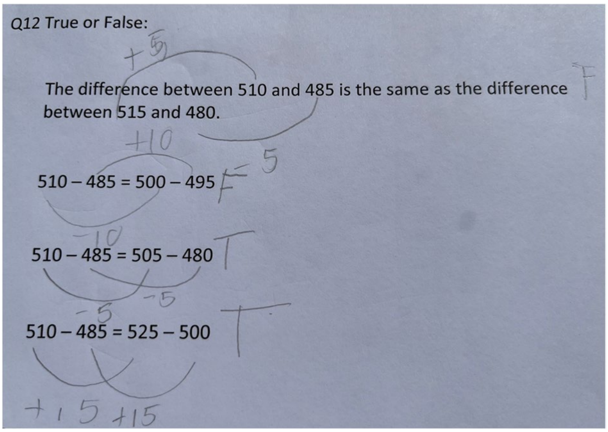

Fig. 7 Vincent self-correcting by adding directional signs to the changes between minuends and subtrahends

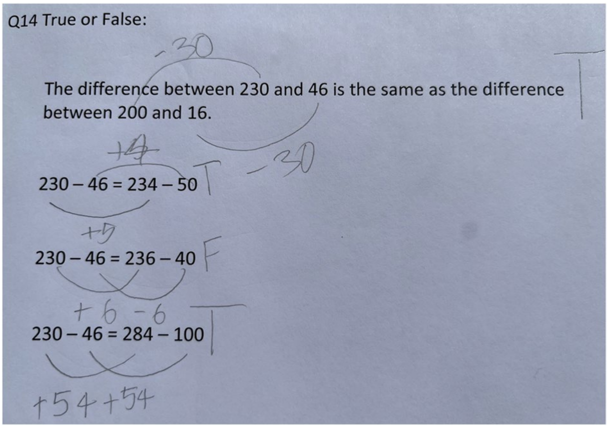

Fig. 8 Evidence of Vincent's deliberate generalizing with arcs and directional numbers

Vincent also evidenced eventual attention to both magnitude and direction of changes to minuend and subtrahend. In Q12, he had only paid attention to the magnitude of the changes and not the direction (CoF3 G-MAG) but then quickly self-corrected:

> True. I read the question. The difference between this and this is 5 [drew arc 510 to 515 and '5'] and this and this is also 5 [drew arc 485 to 480 and '5']. Wait! No, it doesn't actually! It actually doesn't because this is *plus* 5 and this is *minus* 5. (Vincent)

Vincent started to erase his drawn arcs, but the interviewer intervened and suggested adding the new information, so he wrote "+5" and "–5" instead, as shown in Fig. 7.

Vincent continued drawing arcs and directional +/– signs when assessing the rest of Q12 and Q14, as shown in Fig. 8.

These findings suggest that the students who could easily double-check their generalizing with calculations (indirect addition and take-away strategies) were able to make sense of both the direction and magnitude of change to counter any invalid thinking. Repeated questions that were designed to elicit this issue (by adding and subtracting the same amount) also appeared to support multiple opportunities for the students to try out their relational thinking and test their ideas. It is important to note that the interview questions increased in the sizes of numbers and that those students who struggled with calculating at those levels may have evidenced generalizing if given more questions where calculating was not a cognitive overload.

5 Discussion and conclusion

In an in-depth qualitative exploratory study, we investigated middle primary (9–11-year-old) students' relational thinking with subtraction-as-difference tasks involving towers of blocks. Twenty-two students responded in individual interviews to various tasks that shifted from concrete and numeric representations to symbolic expressions and equations. Video data and written task responses were analysed using the Student Noticing framework (Lobato et al., 2013) to look for evidence of student attention to and expression of generality with the compensation property of equality in the context of subtraction.

Three Centres of Focus (CoF) and 11 associated codes emerged from the analysis and these were categorized as attending to difference (subtraction as comparison model), using a subtraction strategy, and using relational thinking. All but one student evidenced making sense conceptually of the difference in heights of two towers; most students verbalized and wrote the difference quantitatively. A few students initially verbalized a qualitative difference, for example, “taller” or “shorter” but when prompted, could identify the difference as a certain number of blocks. We found that the pictures of towers drawn by the students also provided insights into their level of understanding, through variations in how accurately the blocks in each tower corresponded. When the interview tasks shifted to a numeric representation of tower heights (e.g., “10 blocks & 8 blocks”), most students continued to verbalize the concept of “difference”, suggestive of retaining their visualization through the transition (Fyfe & Nathan, 2019; Goldstone & Son, 2005).

In previous research on subtraction (e.g., Cooper & Warren, 2011; Ng & Lee, 2009; Yeo et al., 2019), pictorial representations of horizontal bars were used in tasks and involved length comparisons, but not vertical representations. In the literature on directed number, researchers found that vertical number lines supported development of students’ conceptual understanding of adding and subtracting negative numbers (Stephan & Akyuz, 2012). Bofferding (2018) explored the order-relevant property of subtraction (see Table 1) with lower primary (Year 1) students using the picture of a building with above- and below-ground levels. She highlighted the conceptual difficulties experienced by the students in differentiating between $4-1$ and $1-4$. In our study, we provided physical blocks for children to build their own concrete towers of quantifiable whole numbers. We found evidence that comparison of vertical heights of towers, and with a concrete representation, was supportive of students’ conceptual understanding of subtraction as difference. We suggest that the ability to quantify the number of blocks (rather than having unspecified heights) was also conceptually helpful for students to make sense of the subtraction as comparison model. This finding resonates with Usiskin’s (2007) argument that children need to encounter subtraction models (take away and comparison) in small whole-number situations.

Additionally in our study, a few students extended the analogy to encompass negative numbers—the underground levels of a building and the roots of a tree. Unlike Bofferding’s (2018) study, the students extended beyond the provided towers of blocks to make sense of what a negative subtrahend might mean. Our findings suggest that concrete and vertical representations incorporating both the attribute of length and discrete quantities (of blocks) have the potential to support students’ relational thinking, even with negative numbers. Lower secondary students have been found to experience difficulties with integer addition and subtraction, particularly with the traditional abstract neutralization tool (chips or counters of different colours representing $+1$ and -1). In contrast, vertical number lines have been found to support students’ conceptual understanding of addition and subtraction with negative numbers (Stephan & Akyuz, 2012). Further research on the tool of vertical towers of blocks for helping students make sense of subtraction as difference, and with directed number, would be worthwhile.

This study contributes to the literature on early algebra and generalized arithmetic in providing evidence of students’ attention to the compensation property of equality and relational thinking when provided with a tool (towers of blocks) for making sense of subtraction as difference. An early emergent framework of proposed levels of attention to generality was shared in “Section 4”. Overall, with this cohort of 9–11-year-old students, just over half (12 students) evidenced attention to generality. A few students did so solely with a concrete representation. With towers of blocks in hand, they added or removed the same number of blocks from each tower and verbalized that doing so would keep the difference the same. Six students evidenced relational thinking suggestive of partial or context-specific understanding related to the magnitude of numbers and changes involved.

With Radford's (2010) theorized levels of algebraic thinking for generalizing figural patterns, students' emerging algebraic thinking can be evidenced at the levels of factual or contextual generalization through their gestures or natural language. Yet devising symbolic equations to express their generalizations is important for considering students' thinking as clearly algebraic in nature. In our study, we found that some students provided hints that they were implicitly attending to generality beyond a concrete context, but not clearly or repeatedly enough with symbolic subtraction equations to demonstrate unequivocal algebraic thinking. Four out of 22 students did give such evidence, verbally when describing how they were comparing numbers, gesturally when pointing to numbers in turn on either side of an equation's equal sign, and sometimes with invented written markings, such as vertical arrows and bridges between matching minuends and subtrahends (e.g., Figs. 7 and 8).

One conceptual difficulty that some students experienced (and initially by two of the four students on level 3) was recognizing the directional aspect of difference with equations (not with the physical towers). Some students recognized matching magnitudes of changes to minuend and subtrahend, but not direction, i.e., if amounts are being added or subtracted. We speculate that the representation of physical towers of blocks provides visual and tactile cues about magnitude and direction changes but abstract symbolic equations (e.g., $34 - 28 = 35 - 27$ in Q8) may highlight the magnitude but obscure the direction. Those students who were double-checking their relational ideas with actual calculations of the difference self-corrected from that point in the interview. Ongoing research is planned to investigate hindrances and affordances of the vertical tower model for distinguishing between the addition- and subtraction-compensations properties.

Overall, this study provides insights into the potential for developing children's structural and relational thinking with subtraction. The tool of physical joinable "Unifix" blocks for building and comparing towers of blocks was found to be supportive for students' sense-making about subtraction as difference, but there is more to understand about helping students attend to magnitude and direction of changes to matching minuends and subtrahends with symbolized equations. The interview tasks were designed to elicit students' attention to the subtraction-compensation property of equality, but these findings do not imply that students will choose such relational thinking of their own volition or that these students fully understand the subtraction as comparison model. Yet it is encouraging to note that there is the potential for developing such generalizing activity at middle primary levels of schooling. Hickendorff et al. (2019) overviewed empirical research on students' multi-digit addition and subtraction and highlighted that number-based (rather than digit-focused) teaching can increase students' efficient and adaptive use of indirect addition and compensation. Future research on learning tasks is needed to investigate if and when students might choose to apply relational thinking in different contexts and with different representations, among a choice of subtraction/addition strategies.

Appendix. Interview tasks used in the study

Q1. With the blocks try to make two towers with different heights.

What is the difference between them?
How can you write or draw this information? [line spaces removed]

Q2. Try to make another two towers that have different heights to your first two towers but that have the same difference in height between them (as your first two towers).

How can you write or draw this information?

Q3. Circle the pair of towers that doesn't belong:

10 blocks & 8 blocks
11 blocks & 9 blocks
15 blocks & 13 blocks
7 blocks & 4 blocks

Q4. Circle the pair of towers that doesn't belong:

25 blocks & 15 blocks
20 blocks & 10 blocks
30 blocks & 19 blocks
15 blocks & 5 blocks

Q5. Circle the pair of towers that doesn't belong:

35 – 8
34 – 9
30 – 3
45 – 18

Q6. True or False:

a) The difference between 20 and 16 is the same as the difference between 21 and 17.
b) 20 – 16 = 10 – 6
c) 20 – 16 = 22 – 14
d) 20 – 16 = 30 – 26

Q7. Try to fill in the numbers and explain your thinking:

a) The difference between 20 and 16 is the same as the difference between ……… and ……….
b) 20 – 16 = ……… – ………
c) 20 – 16 = ……… – ………

Q8. True or False:

a) The difference between 34 and 28 is the same as the difference between 33 and 29.
b) 34 – 28 = 30 – 24
c) 34 – 28 = 35 – 27
d) 34 – 28 = 20 – 15

Q9. Try to fill in the numbers and explain your thinking:

a) The difference between 34 and 28 is the same as the difference between ……… and ……….
b) 34 – 28 = ……… – ………
c) 34 – 28 = ……… – ………

Q10. True or False:

a) The difference between 92 and 38 is the same as the difference between 93 and 40.
b) 92 – 38 = 90 – 36
c) 92 – 38 = 93 – 37
d) 92 – 38 = 62 – 8

Q11. Try to fill in the numbers and explain your thinking:

a) The difference between 92 and 38 is the same as the difference between ……… and ……….
b) 92 – 38 = ……… – ………
c) 92 – 38 = ……… – ………

Q12. True or False:

a) The difference between 510 and 485 is the same as the difference between 515 and 480.
b) 510 – 485 = 500 – 495
c) 510 – 485 = 505 – 480
d) 510 – 485 = 525 – 500

Q13. Try to fill in the numbers and explain your thinking:

a) The difference between 510 and 485 is the same as the difference between ……… and ……….
b) 510 – 485 = ……… – ………
c) 510 – 485 = ……… – ………

Q14. True or False:

a) The difference between 230 and 46 is the same as the difference between 200 and 16.
b) 230 – 46 = 234 – 50
c) 230 – 46 = 236 – 40
d) 230 – 46 = 284 – 100

Q15. Try to fill in the numbers and explain your thinking:

a) The difference between 230 and 46 is the same as the difference between ……… and ……….
b) 230 – 46 = ……… – ………
c) 230 – 46 = ……… – ………

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Declarations

Conflict of interest The authors declare no competing interests.

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