Educational Studies in Mathematics (2024) 115:271–287
https://doi.org/10.1007/s10649-023-10287-0

Seeing number relations when solving a three-digit subtraction task

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Accepted: 6 December 2023 / Published online: 17 January 2024

Abstract
The decomposition of numbers when solving subtraction tasks is regarded as more powerful than counting-based strategies. Still, many students fail to solve subtraction tasks despite using decomposition. To shed light upon this issue, we take a variation theoretical perspective (Marton, 2015) seeing learning as a function of discerning critical aspects and their relations of the object of learning. In this paper, we focus on what number relations students see in a three-digit subtraction task, and how they see them. We analyzed interview data from 55 second-grade students who used decomposition strategies to solve $204 - 193 =$. The variation theory of learning was used to analyze what number relations the students experienced and how they experienced them, aiming to explain why they made errors even though they used presumably powerful strategies in their problem-solving. The findings show that students who simultaneously experienced within-number relations and between-number relations when solving the task succeeded in solving it, whereas those who did not do this failed. These findings have importance for understanding what students need to discern in order to be able to solve subtraction tasks in a proficient way.

Keywords Subtraction · Early arithmetic skills · Variation theory · Three-digit subtraction · Decompose · Number relations

1 Introduction

Research has identified a plethora of strategies used by students in solving subtraction tasks, and efforts have been made to identify powerful strategies as key in teaching arithmetic skills (e.g., Baroody, 2016; Fuson, 1992). In particular, it has been shown that the ability to decompose numbers facilitates the solving of subtraction tasks, for example, solving $13 - 5 =$ by decomposing 5 into 3 and 2 and then making use of 10 as a benchmark

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in $13-3-2=$, thus solving the task in an effective way. Knowledge of number relations, such as how 13 relates to 10 and the fact that 5 is composed of 3 and 2, is seen as an important cornerstone in the development of arithmetic skills (Baroody, 2016; Fuson, 1992; Resnick, 1983). However, research has not yet sufficiently explored what makes such strategies successful, or rather why the use of decomposition sometimes fails. In the present paper, we therefore direct attention to what number relations students experience when solving subtraction tasks, and how they experience these number relations within the task. This may offer insight into why some students do not succeed in solving subtraction tasks in a proficient way even when using strategies (e.g., decomposition) that are known to be powerful.

The study population comprised eight-year-old students who had participated in an intervention focusing on the decomposition of numbers when solving subtraction tasks in the number range 1 to 20; in the present study, we examined how these students decomposed numbers when solving a task with three-digit numbers ($204-193=$) 1 year later. Since these students had used decomposition in the first grade, we expected that they would also use this strategy for subtraction tasks in a higher number range. We were interested in how they used decomposition in tasks where there are multiple ways of decomposing the minuend and the subtrahend in the solving process. Decomposition of three-digit numbers involves strategies such as splitting numbers into hundreds, tens, and ones in order to solve a subtraction task; this includes splitting both the minuend and the subtrahend, and splitting the subtrahend alone. Selter et al. (2012) separate strategies that only split digits into hundreds, tens, and ones, which they call decomposition, from strategies by which the student sequentially subtracts tens and ones from the minuend. In this study, we chose not to separate these strategies because according to Selter et al. (2012), they are often combined. We therefore use the term “decomposition” for all strategies that include splitting into tens, ones, and hundreds.

In our sample, 55 students used decomposition of units in one way or another. Not all of them managed to solve the task correctly, even though they made use of strategies that are suggested to be powerful and that should lead to success in arithmetic problem-solving. Earlier findings (see Björklund & Runesson Kempe, 2022) suggest that one way to uncover the reasons for different outcomes in solving subtraction tasks might be to study what number relations the students discern, that is, to focus on how the task appears to a student rather than what strategies are used. The present study used the variation theory of learning (Marton, 2015) as a theoretical framework to analyze students’ ways of experiencing the number relations. Our analysis focused on how the task appeared to each student, which logically related to what the student did in solving the task. We investigated why some students ended up having difficulty when using decomposition, and how their ways of solving might be explained. The aim is thus to identify and discuss what is behind the decomposition strategy in terms of the number relations that the students discerned. Our research question is: How do students experience number relations when solving a specific three-digit subtraction task using decomposition?

2 Literature review

Numerous studies in different cultural contexts have examined how students use and develop various strategies to mentally solve multi-digit addition and subtraction problems, resulting in various classifications of strategies (see, e.g., Blöte et al., 2000; Carpenter et al., 1997; Fuson et al., 1997; Selter, 2001; Thompson, 1999; Threlfall, 2002). Although these strategies are named differently, there are commonalities among some of them. For instance, they imply "splitting," "partitioning," or "decomposing" the numbers in the operation into hundreds, tens, and ones and then "recomposing" the decomposed numbers (e.g., Blöte et al., 2000; Selter, 2001; Selter et al., 2012; Torbeyns et al., 2009a, 2009b). While there is an extensive literature on the various strategies students can use, there is less to be found on the difficulties and challenges these strategies may imply, particularly for younger students, and hence there is little knowledge about why students may fail to solve the tasks correctly.

Decomposing numbers into hundreds, tens, and ones requires several complicated steps, particularly in three-digit subtraction tasks. For example, Selter (2001) describes several variations in how the task $701-698=$ could be solved mentally, including the "stepwise" strategy ($701-600; -90; -8$, keeping the minuend unaltered), the "hundreds, tens, and units" strategy ($700-600$; $0-90$; $1-8$; $100-90; -7$), and a mix of the two. Such decompositions require the student to keep each of the partitions of the decomposed numbers in mind and to then recompose them correctly. Moreover, the ability to decompose numbers requires an understanding of the part-whole relation of numbers (Cheng, 2012; Peters et al., 2013; Resnick, 1983). Being aware of part-whole relations may allow students to decompose numbers ($c = a + b$), to make use of commutativity ($(a + b) = (b + a)$), and to apply the complement principle ($a + b = c$ implies $c-a=b$) when solving addition and subtraction tasks (Zhou & Peverly, 2005). For instance, learning that addition and subtraction tasks represent the same number relation is suggested to be a prerequisite for understanding how to use decomposition strategies, such as when using addition and decomposition of numbers to solve a subtraction task (Björklund et al., 2021). Moreover, studies have shown that students' understanding of place value impacts their ability to solve subtraction tasks with multidigit numbers (e.g., Fuson, 1990). Knowing about place value is necessary when decomposing numbers into hundreds, tens, and ones, and when operating on one position at a time.

However, for solving a subtraction task where the difference between minuend and subtrahend is small, such as $701-698=$, neither the "stepwise" strategy nor the "hundreds, tens, and units" strategy (Selter, 2001) is the most effective. Instead, adding 3 to 698 (decomposing the difference into 2 and 1, $698+2+1$) is more appropriate (e.g., Linsen et al., 2014; Torbeyns et al., 2009a). Although this strategy also requires a decomposition of numbers ($3=2+1$), there are fewer partitions to keep in mind. Several names are used for this strategy in the literature: "indirect addition" (Linsen et al., 2014), "subtraction as addition" (e.g., Selter, 2001; Van Der Auwera et al., 2023), and "counting up" (e.g., Torbeyns et al., 2009a).

Although it is effective and leads to fewer errors, subtraction as addition is used less frequently by younger students (Heinze et al., 2009; Selter, 2001). This may be due to the strong influence of the curriculum. For instance, if students are taught one single strategy, they tend to use this even when another (e.g., subtraction as addition) is more effective (Van Der Auwera et al., 2023). Selter (2001) found that after being introduced to the standard algorithm, the number of students using the standard algorithm methods increased and this became their most commonly used method.

The point of departure in the present study was that using the subtraction as addition strategy requires an understanding of certain number relations. A similar point has been suggested by Linsen et al. (2014), who argue that subtraction as addition requires a solid understanding of number magnitude (hence being aware of the magnitudes of the numbers in the task and their relations) along with an understanding of the roles of the subtrahend and minuend.

This indicates that there is even more to learning to solve subtraction tasks than being able to decompose and recompose numbers, or seeing a task as an addition or subtraction problem. It seems that the number relations and the role that numbers play in a specific subtraction task (as minuend or subtrahend) should also be emphasized in powerful arithmetic problem-solving.

A recent study (Björklund & Runesson Kempe, 2022) examined the connection between the way students mentally encounter and solve an orally presented subtraction task, the strategies they use, and how they see and experience the number relations in the problem. For instance, the authors argued that a student who treats the task $32-25=$ as an addition problem (subtraction as addition) and states that the answer is "Seven, cause five and two is seven" ($25+5+2=32$) may experience the minuend, the subtrahend, and the difference as a relation (Björklund & Runesson Kempe, 2022). This perspective on students' ways of solving the task and the strategies they demonstrate—not simply categorizing the strategies they use but exploring how they experience numbers and relations between numbers—may reveal why some students fail to solve the task despite using strategies regarded as effective.

The various strategies students use when solving problems and how they adapt these to different problems have sometimes been studied through a choice/no choice method, where students solve some tasks by using their preferred strategy and other tasks by using predetermined strategies (e.g., Van Der Auwera et al., 2023). In this study, however, we closely analyzed students' ways of reasoning when solving a single subtraction task with only a small difference between minuend and subtrahend. Our aim was not to study their strategies *per se,* but to find explanations for why some students succeed and others fail despite using the same strategy and demonstrating skill in decomposing numbers. We explored this by focusing on how the numbers and number relations appeared to each student in a phenomenological sense. This resonates with the work of Threlfall (2002), who argues that how students notice characteristics of numbers and their relation determines "how a solution path emerges" (p. 45) and "what is noticed about the specific problem" (p. 42). Mason (2021) argues in a similar way that solving a problem is not simply the application of strategies at hand, but rather the result of what is paid attention to or noticed in the problem. Taking his own experience of approaching a problem as an object of analysis, he describes problem-solving as a dynamic process of changes of attention to various features of the problem. There is a shift in what is foregrounded and what remains in the background, in the attention between the whole and the details, in perceiving properties of the numbers and recognizing relations between them. Thus, solving arithmetic tasks in powerful ways also seems to include a way of experiencing and noticing the characteristics and numbers in the task, namely as relational.

The theoretical position taken in this paper is in line with Mason's proposal. We studied how eight-year-old students solved a three-digit subtraction task with a small difference between minuend and subtrahend. Our analysis went beyond the observed strategies in order to investigate and interpret what was attended to and noticed in the numbers and number relations in the task.

3 Theoretical framework

The theoretical point of departure for this study is phenomenography and its theoretical extension, variation theory (Marton, 1981, 2015; Marton & Booth, 1997). Phenomenographic studies describe variation in how the same thing can be experienced. This implies making statements about the world as experienced by others, which in phenomenography is known as adopting a second-order perspective. One might ask how this is possible, given that we do not have access to the thinking or experiences of others. However, taking a second-order perspective means adopting an approach that implies a special way of looking at the relationship between logic, understanding, and acting. Smedslund (1970) argues that there is a circular relationship between understanding and logic. If someone gives an answer to a question that may seem incomprehensible or strange, according to Smedslund (1970), we should assume that the answer is logical from that person's way of understanding, not that they have a different logic. For example, a child who solves the problem "You have 10 candies and eat 6, how many are left?" might do so in the following way:

The child raises six fingers, comprising all the fingers on his left hand along with the thumb on his right. He says: "Ate six (looks at his six fingers). This is six. And then I took the thumb [on his right hand] away. So it's five" (Björklund et al., 2018).

This seemingly odd answer, and the way the numbers are modeled with fingers, could be understood in a second-order perspective as an expression of how "six" appears to the child. The child shows six as five and one, but most likely without comprehending that "six" denotes all the fingers, not just the last (the thumb). His answer of "five" is therefore a logical answer in relation to how he experiences the numbers.

How something is experienced is seen as relational. Our awareness is directed to the learning object, and how this is experienced depends on what aspects are attended to and discerned. Learning something means discerning aspects that have not previously been discerned. Hence, learning originates from making distinctions (Gibson & Gibson, 1955), and discernment and differentiation are the basis for learning and experiencing in a new way. For instance, to solve the task $95-7=$ by subtracting $95-5-2$, one must be able to see and make distinctions between parts and wholes as well as parts within wholes, and to understand how 95 and 7 are related to 90 ($95=90+5$; $7=5+2$). If we are to be able to make such distinctions, some things must come into the focus, or foreground, of our attention. This is in line with Gurwitsch (1964), who states that our awareness is structured. We are aware of everything all the time, albeit in different ways and to different degrees. Whereas some things are in the foreground of our attention, others are in the background and hence are not attended to in the same way. This is why students may approach a task like $95-7=$ in different ways. How a student approaches a task can tell us about what is foregrounded in their awareness, and thus discerned. Relating experience to how a task is solved (i.e., to how one acts) is based on the conjecture that acting and experiencing are logically related:

> Actually, such a stance follows from considering acts and ways of seeing as being intertwined, being two aspects of the same whole. Such a stance implies that what people do is consistent with what they see (the two are logically related). (Marton, 2015, p. 88)

In this paper, when we relate students' strategies to their ways of experiencing, we are taking the stance that powerful ways of acting stem from powerful ways of experiencing and discerning critical aspects of a situation.

4 Method

This study draws on data from a more extensive study in which a total of four classes from three schools in the same municipality (middle-class socioeconomic status with some immigrant students in each class) near a large Swedish city participated in an eight-month-long intervention in the first grade (Kullberg et al., 2022). The intervention focused on decomposition of numbers, for example, $13 = 8 + 5$ and $13 = 8 + 2 + 3$. To understand more about the number knowledge gained by the intervention group, we directed attention to how the students encountered and were able to solve one particular task in a higher number range in the second grade. The students were interviewed three times: in the first grade before the intervention started, in the first grade after the intervention, and in the second grade a year after the intervention ended. The present study focused on the analysis of the task $204 - 193 =$ in the second grade (third interview). Each student's legal guardian provided written consent for their participation in the study.

4.1 Context of the study

One goal of the intervention had been for the students to be able to solve tasks like $15 - 7 =$ using decomposition of numbers with 10 as a benchmark ($15 - 5 - 2 = 8$), and subtraction as addition ($7 + 3 + 5 = 15$). Hence, the students had been taught both strategies (Van Der Auwera et al., 2023). During the intervention, they had learned to compose and decompose numbers, and to use part–whole relations to solve addition and subtraction tasks in the number range 1 to 20. We knew from meetings with the teachers in the second grade that they continued to emphasize number relations and decomposition in addition and subtraction in higher number ranges. The textbook used in the second grade introduced the written algorithm with two- and three-digit numbers including tasks carrying over ten. Strategies such as "constant difference" (e.g., solving $204 - 193 =$ by adding 7 to both terms to make an easier calculation) had not yet been taught to the students, since this is usually done in higher grades in Sweden.

4.2 The data

Semi-structured interviews were conducted individually by three researchers with significant experience in interviewing young students. Each interview lasted 20–30 min and was video-recorded. The tasks were asked numerically, written on a card. Since our interest was in the way the task and the number relations were experienced by the students, follow-up questions were posed, such as "How do you know it is… [e.g., 11]?" and "Please tell me how you solved the task." No manipulatives or paper and pencil were used during the interviews.

Three of the items in the interview consisted of subtraction tasks in which the minuend was greater than 100 ($204 - 193 =$, $204 - 12 =$, and $134 - 78 =$); from these, we chose the first one. When a single task is used for analysis, the choice of task and the numbers in the task become critical. One reason for choosing this particular task was that there was only a small difference between the two three-digit numbers, making it tedious to solve the task by counting backward in single units, which could cause trouble for students (Selter, 2001). In addition, we wanted to present a task where there were several ways to decompose the numbers, where knowledge about place value and "zero" was necessary, and where the minuend had a zero in the tens position, which has been shown to be difficult for some students (Selter, 2001). Furthermore, this task required bridging through tens and hundreds, which was likely to be a challenge since the task was to be solved without paper and pencil.

From the 86 students in the original intervention group, we selected a subgroup ($N = 55$) who used a decomposition strategy when solving the selected task. Data from the remaining 31 students were not analyzed in this study, either because they did not use decomposition of numbers in units larger than ones or they used counting strategies such as counting single units ($N = 21$), or they did not attempt to solve the task, saying, for example, “this is too hard, we haven’t worked with big numbers like this yet” ($N = 5$), or their explanations did not provide sufficient data to make a stable interpretation ($N = 5$). The sample ($N = 55$) included in this paper was therefore representative of students who had used some sort of decomposition strategy.

4.3 Analysis

Video observation data from the interviews was first coded with regard to the strategies enacted by the students when solving the task $204 - 193 =$. Attention was given both to the students’ utterances and to their gestures (e.g., pointing at specific numbers on the task sheet), in order to get a comprehensive picture of how they were encountering the task. Our focus was initially on how the students handled the numbers and number relations when solving the task (Björklund & Runesson Kempe, 2022).

We identified four main strategies that involved some sort of decomposition: *decompose into digits*, *decompose the minuend and the subtrahend*, *decompose the subtrahend*, and *decompose the difference* (Table 1).

We initially analyzed whether a “taking away” or “determining the difference” model was used (Selter et al., 2012). After having noted the solutions to a task, we analyzed how the students had solved it stepwise. For the purpose of our study and in line with our research focus, we were particularly interested in *how* the numbers in the operation were decomposed and thus in what ways the students structured the numbers in parts and as wholes in the operation.

The *decompose into digits* strategy involved starting with the smallest value. Students who used this strategy often ended up solving the task incorrectly, for example, as $204 - 193 = 191$, starting with the ones ($4 - 3 = 1$), then subtracting the tens incorrectly ($0 - 9 = 9$), and finishing with the hundreds ($2 - 1 = 1$). This strategy was based on handling the numbers as “ones” only.

Students who “determined the difference” by partitioning the difference (11) into two parts (7 and 4) and noting that $193 + 7 + 4 = 204$ were coded as using *decompose the difference.* These students decomposed the missing part in the part-whole relation.

Decomposing also involved a “taking away” in which the numbers were decomposed but were also handled as hundreds, tens, and ones (e.g., the student saying “Two hundred four minus one hundred is one hundred four”). For this coding, we conducted a deeper analysis in order to distinguish whether both or only one of the integers were decomposed, which resulted in two categories: *decompose the minuend and the subtrahend* and *decompose the subtrahend* (Table 2).

Table 1 Main strategies and decompositions (in bold) observed among students when solving 204–193

Strategy	Decompositions of 204 – 93
Decompose into digits	20**4** – 19**3**, 2**0**4 – 1**9**3, **2**04 – **1**93
Decompose the minuend and the subtrahend	204 = **200 + 4,** 193 = **100 + 90 + 3**
Decompose the subtrahend	193 = **100 + 90 + 3**
Decompose the difference	193 + **7 + 4** = 204

Table 2 Students' decomposition strategies and answers to the problem 204 – 193 =. The correct answer (11) is highlighted in bold

	Decompose into *digits*					Decompose the *minuend & subtrahend*					Decompose the *subtrahend*	Decompose the *difference*		
Answer	**11**	89	101	191	197	3	9	**11**	19[1]	189	**11**	**11**	13	14
N = *55*	1	1	3	7	1	4	1	3	1	8	16	7	1	1
Total	13					17					16	9		

[1]The answer 19 included repeated miscalculations and did not end up in a distinct category related to the number relations discerned

For our main research interest, to shed light on how the students experienced number relations and why they failed or succeeded in solving the task correctly, it was necessary to pay attention to the variation in how the students described their solutions to the task. We (the authors) did this by adopting a second-order perspective, trying to interpret and determine which number relations within the task were attended to, and how the students experienced these relations.

We applied variation theory and the conjecture of the logical relation between act and experience as an analytical tool to examine what emerged in the students' ways of reasoning and solving. For instance, one of the students said: "11, because I thought 193 has **7** up to 200 and then I thought, well … 11. It was 7 between to 200 from 193. So I thought **7 + 4** is 11." We interpreted this as meaning that the student was paying attention to the difference between the minuend and subtrahend and that this difference was decomposed into seven and four. This implies that the student experienced both the relation between the minuend and the subtrahend (between-number relation) and the relation within the difference (within-number relation). Conversely, a student who said: "204… I take away 100, 104, and next 90, that's 14, and then 3. 11." was interpreted as having discerned other number relations, starting with the within-number relation (in 193) followed by between-number relations (e.g., 204 – 100, 104 – 90, 14 – 3).

Our analysis focused on identifying how students who used decomposition of numbers handled the numbers and number relations when solving the task. Specifically, we directed attention to what the students foregrounded in their problem-solving, which came through in their ways of reasoning about the task and how they solved it. Interpreting students' sometimes very short and scant descriptions opened the door to different interpretations. In the analysis, therefore, various interpretations have been carefully considered in order to ultimately suggest those that we consider most reasonable and in accordance with the theoretical perspective taken. For instance, arriving at the answer 7 instead of 11 by subtracting the ones in the minuend (204 – 100, 104 – 90, 14 – 3 – 4 = 7, or 200 – 100, 100 – 90, 10 – 3 – 4 = 7) instead of adding (e.g., 200 – 100, 100 – 90, 10 – 3 = 7, 7 + 4 = 11) could be interpreted as not being aware that addition is possible in a subtraction task. However, our interpretations were guided by our theoretical assumptions about the directedness of our awareness. Hence, we interpreted how the student's awareness was directed in the moment. Consequently, errors like the one above were not explained in terms of lack of knowledge or cognitive ability, but rather by how the student's awareness was directed in the moment and what came to the fore in their awareness and was discerned.

5 Results

Four ways of approaching the subtraction task were observed (Table 2). There was a great variety among the students' answers; in total, 11 different answers were given by the 55 students. The incorrect answers found in one category were unique for that category. Students who used the *decompose into digits* strategy ($N=13$) solved the task by subtracting the hundreds, tens, and ones separately, usually starting with the ones, as in the standard algorithm. These students subtracted the digits in each unit starting with the ones ($4-3$, $0-9$, and $2-1$) and ended up with various answers (11, 89, 101, 191, 197). The most common incorrect answers were 191 ($N=7$), which was produced by students who calculated the tens as $9-0=9$, and 101 ($N=3$), which was produced by students who calculated the tens as $0-9=0$. Only one student in this category gave a correct answer. This student started by subtracting the ones, but then changed to decomposing only the subtrahend ($4-3=1$, $200-100=100$, $100-90=10$, $10+1=11$).

Taken together, the two categories *decompose the minuend and the subtrahend* and *decompose the subtrahend* were coded for more than half ($N=33$) of the students in our sample. In this case, 204 and 193, or only 193, were decomposed into smaller units (e.g., 204 into 200 and 4, and 193 into 100, 90, and 3) to calculate the difference between the numbers in steps. We found a variety of incorrect answers among students who decomposed both the minuend and the subtrahend (3, 9, 19, 189). However, all students who decomposed only the subtrahend (e.g., $204-100=104$, $104-3=101$, $101-90=11$) ended up with a correct answer.

Decompose the difference is considered a powerful way of solving tasks in which the difference is small, as in $204-193=$. The students ($N=9$) who used this strategy identified the numbers in the task as being close together, and so addition could be used to determine the difference between them (e.g., $193+7=200$, $200+4=204$, $7+4=11$). Only two of the nine students who used this strategy did not end up with a correct answer; they were able to find 7 and 4 to solve the task, but made calculation errors such as $7+4=13$.

Our specific interest in this study was to find reasons why decomposition of numbers results in incorrect answers even though the literature in the field assumes it to be an effective strategy. Thus, in the following section, we present the analysis of students who used decomposition of the minuend and the subtrahend in terms of how these students discerned number relations.

5.1 Beyond decomposing the minuend and the subtrahend

As noted above, 33 of the 55 students used either *decompose the minuend and the subtrahend* or *decompose the subtrahend* in solving the problem (Table 2). Among these students, we made two distinct observations: that decomposition was not a simple, straightforward way of solving the subtraction task; and that even though the students were observed to make use of decomposition in solving the task, many failed to arrive at a correct answer. As shown in Table 2, there was a clear difference in how the students operated with numbers; those who kept the minuend unaltered and decomposed the subtrahend were able to handle the necessary number relations both within and between numbers, but those who decomposed not only the subtrahend but also the minuend had a substantially smaller success rate in solving the task (14 of 17 were incorrect).

Consequently, we see that some ways of operating with part-whole relations in the task induced difficulty, particularly when both minuend and subtrahend were altered during the decomposing operation. In the following, we describe the critical difference in the students' ways of experiencing the subtraction task as two distinct ways of experiencing number relations in the task: *simultaneously experiencing the number relations* and *dissolving number relations.*

5.1.1 Simultaneously experiencing the number relations

Only three of the 17 students in our study produced the correct answer by decomposing both minuend and subtrahend. Our interpretation is that these students succeeded in solving the subtraction task because they experienced the minuend as being related to the subtrahend, and kept this between-number relation in the foreground when finding the difference through decomposing strategies in which the within-number relations were also foregrounded. To complete the task correctly, these students *simultaneously* foregrounded the within-number and between-number relations. A typical way of reasoning can be noted in the following example.

One student, Michael, decomposed the subtrahend while keeping the minuend in the foreground as a composed set of 200 and 4 (Table 3). He succeeded in solving the task because he experienced the minuend as being related to the subtrahend (i.e., the between-number relation) while at the same time keeping track of the within-number relations (decomposed parts of the minuend and subtrahend). Michael experienced 4 as continuously being a part of the minuend 204 throughout the subtraction operation with the 4 and 3. He solved the task because the number relation between minuend and subtrahend and, *at the same time,* the within-number relations as connected to minuend and subtrahend, respectively, were foregrounded. When number relations were experienced in this simultaneous way, the students solved the tasks correctly.

Table 3 Michael's ways of reasoning and our interpretation of the number relations foregrounded

Michael's utterances	Interpretation	
	Strategy	*Number relations foregrounded* (between M and S and within M and S, respectively)
Is it 11 then? Start by taking away the units, 200 minus 100 is 100,	*200 – 100 = 100, breaks up 204 into 200 + 4 and 193 into 100 + 90 + 3, operating on the hundreds*	*Within- and between-number relations*
thinking 200 [sic] minus 90 is 10	*100 – 90 = 10, continues operating with the partitions 100 and 90, finding the difference 10 between them*	*Between-number relation*
Then I took those (pointing at the 4 and the 3), that's 1	*4 – 3 = 1, operating on the decomposed units 4 from the minuend and 3 from the subtrahend*	*Between-number relation*
Makes it 11	*10 + 1 = 11, relating the difference 10 to the difference 1, adding to get the difference between the original minuend and subtrahend*	*Within- and between-number relations*

5.1.2 Dissolving number relations

The 17 students who decomposed not only the subtrahend but also the minuend had a much smaller success rate in solving the task (14 of 17 were incorrect). An explanation for this discrepancy in success rate seems to be found in the contrast between students who simultaneously experienced the between- and within-number relations (see example of Michael, above), and those for whom the number relations dissolved while they were operating on the task.

The significance of simultaneity in experiencing the decomposed parts of minuend and subtrahend, constituting within-number relations of the minuend and subtrahend respectively *and* between-number relations (as in the example with Michael above), is illustrated in the contrast with students who primarily foregrounded between-number relations. This focus on the between-number relations induced observable difficulties in working out what the decomposed parts should be related to.

One student (Melker) started the operation by decomposing the minuend (204 into 200 + 4) and subtrahend (193 into 100 + 90 + 3), and then operating stepwise to find the difference between the remaining parts of the minuend and subtrahend (10) (Table 4). This was followed by an operation on the units 4 and 3 (4 − 3 = 1). However, the units—or rather, the outcome of the operation on the units (1)—were now no longer seen as parts of the minuend and subtrahend, but appeared as an independent unit that Melker had difficulty relating to the numbers; that is, he had difficulty working out what the 1 was a part of.

Melker operated on the partitions of both minuend (200 + 4) and subtrahend (100 + 90 + 3). He failed to arrive at a correct answer because the within-number relations fell into the background and the partitions were experienced as a between-number relationship. This meant that 1, which should have been seen as a part of the difference to be added to the other part of the difference, 10, was instead regarded as part of a new set of subtrahend and minuend to be operated on. The original minuend 204 was then disregarded and the student acted as 10 being a new minuend. Each part was related to other parts in a between-number relation, rather than remaining part of a composite number. That is, the within-number relation that is necessary to relate to the between-number relation (minuend

Table 4 Melker's ways of reasoning and our interpretation of the number relations foregrounded

Melker's utterances	Interpretation	
	Strategy	*Number relations foregrounded* (between M and S and within M and S, respectively)
I take 200 minus 100, that's 100	*200 – 100 = 100, breaks up 204 into 200 + 4 and 193 into 100 + 90 + 3, operating on the hundreds*	*Between- and within-number relations*
Minus 90 then I have 10	*100 – 90 = 10, continues operating with the partitions 100 and 90, finding the difference 10 between them*	*Between-number relation*
And then I take 3 minus 4; I mean, 4 minus 3, and then I have 1 left	*4 – 3 = 1, operating on the decomposed units 4 from the minuend and 3 from the subtrahend*	*Between-number relation*
So the answer's 9	*10 – 1 = 9 relating the difference 10 to the difference 1, but keeps subtracting the two differences*	*Between-number relation*

and subtrahend) had *dissolved*, and the student lost track of what the task's original minuend and subtrahend were.

When the minuend and subtrahend were both decomposed, the parts constituting the within-number relations were separated from the minuend and subtrahend, respectively. This led to another student, Mehmet (Table 5), becoming unsure as to whether the units were parts of the minuend or subtrahend. An example of this could be seen when Mehmet foregrounded the difference between the remaining parts of the minuend and subtrahend, but the decomposed part, 4, was no longer experienced as part of the minuend.

The parts that resulted from the decomposition were not simultaneously related to the minuend and subtrahend. When Mehmet calculated the difference between 93 and 100, 100 became a new minuend in the operation, and the new difference, 7, was then experienced as a new minuend from which the 4 from 204 (200 + 4) was used as a subtrahend, causing Mehmet to find 3 as the answer. In this way, the within-number relations dissolved, and numbers were rather seen as individual numbers that could be in a between-number relation with other numbers; this resulted in Mehmet conducting a series of operations which had lost their connection to the original minuend.

Similarly, when decomposing the minuend and subtrahend, students did not always experience the new between-number relation (finding the difference between 4 and 3), and instead might subtract them (the ones) step by step (subtracting 4 and subtracting 3) or add them (4 + 3 = 7), and then subtract the sum from the experienced difference, 10 (from the operation 100 – 90). Thus, the 4 was no longer seen as part of the minuend (Table 6).

Mark decomposed the minuend into 200 (which he mentioned) and 4 (which he did not say out loud) and the subtrahend into 100, 90, and 3, and then operated with the decomposed hundreds and tens. Hence, at first, he foregrounded both the within-number relations *and* the between-number relations. However, when operating on the units in relation to the difference between the decomposed minuend and subtrahend, he saw 4 as separate from the minuend and instead as part of the subtrahend, as he continued to subtract all the decomposed parts and ended up with the remaining 3. In this operation, Mark expressed extensive skills in decomposing numbers (within-number relations), but *the within-number*

Table 5 Mehmet's ways of reasoning and our interpretation of the number relations foregrounded

Mehmet's utterances	Interpretation	
	Strategy	*Number relations foregrounded* (between M and S and within M and S, respectively)
I'm thinking 200 minus 100 and then I end up with 100	*200 – 100 = 100, breaks up 204 into 200 + 4 and 193 into 100 + 90 + 3, operating on the hundreds*	*Between- and within-number relations*
Then I take 100 minus 93 and then I end up with, hm, 100, I think 7	*100 – 93 = 7, calculating the difference between 93 and 100, 100 becoming a new minuend in the operation*	*Between-number relation*
And 7 minus 4 is 3	*7 – 4 = 3, the new difference now becoming another minuend from which the original part of 204 (200 + 4) is used as a subtrahend, ending up with 3 as the answer*	*Between-number relation*

Table 6 Mark's ways of reasoning and our interpretation of the number relations foregrounded

Mark's utterances	Interpretation	
	Strategy	*Number relations foregrounded* (between M and S and within M and S, respectively)
Those were big numbers… I split that one into 200	*Decomposing 204 to 200 + 4*	*Within-number relation*
200 minus 100 equals 100	*200 – 100 = 100, decomposing 193 into 100 + 90 + 3, operating only with the hundreds*	*Between- and within-number relations*
100 minus 90 is 10	*100 – 90 = 10, 100 is the new minuend, operating with the minuend and decomposed tens (100 + 90 + 3)*	*Between-number relation*
And 10 minus 4 is 6	*10 – 4 = 6, 10 is the new minuend, operating on the new minuend and the decomposed 4 (200 + 4) in the original minuend*	*Between-number relation*
And 6 minus 3 equals 3	*6 – 3 = 3, 6 is the new minuend, operating on the new minuend and remaining ones in the subtrahend*	*Between-number relation*

relations moved to the background in his awareness and the between-number relations were foregrounded in his acts. The operation stopped only when there were no more decomposed parts to subtract. He had lost track of what the original minuend and subtrahend were, and 100, then 10, and finally 6 became minuends in separate operations, and so the number relations necessary for solving the task had dissolved.

Another student, Martin, also decomposed both the minuend and subtrahend; he predominantly foregrounded the between-number relations, but the within-number relations that he first discerned were altered into new addends that he then operated on (Table 7). This was expressed first in the common way of reducing the hundreds: "I take 200 minus 100 and have 100." However, he then subtracted the decomposed 4 from 93; that is, 93 was seen as a new minuend of which 4 was a related part. After completing the operation 93 – 4, Martin added the 100 from the first operation with the decomposed parts of the minuend and subtrahend.

The within-number relation of the minuend 204 and the subtrahend 193 fell into the background. Martin experienced the decomposed parts as new between-number relations, and the original number relations dissolved. Thus, he ended up handling the parts found within the numbers, no longer considering the between-number relation in the original subtraction task. This way of solving 204 – 193 = was most frequently observed among the students who decomposed minuend and subtrahend. The reason for this, in accordance with our analysis, is that they lost track of what the respective parts of the minuend and the subtrahend were, leading them to compose new minuends and addends while operating with the decomposed parts.

The seemingly irrational student answers (e.g., 3, 9, or 189) did have a logic behind them, which could be understood in terms of how the students experienced the number relations. Our analysis makes it clear that experiencing the between- and within-number relations *simultaneously* is key to using decomposition strategies successfully. We have shown this by making comparisons with the unsuccessful attempts to solve the task in which the within-number relations dissolved in the act of decomposing.

Table 7 Martin's ways of reasoning and our interpretation of the number relations foregrounded

Martin's utterances	Interpretation	
	Strategy	*Number relations foregrounded* (between M and S and within M and S, respectively)
189, I have 200 minus 100; that's 100	*200 – 100 = 100, decomposing 204 into 200 + 4 and 193 into 100 + 93*	*Between- and within-number relations*
And then I take 4 minus 93; that's 89	*93 – 4 = 89, 93 is the new minuend (and not a part of the subtrahend), subtracts 4 (now subtrahend) from the original minuend (in 204)*	*Between-number relation*
Then I take 100 plus 89 is 189	*100 + 89 = 189, adding the 100 from the first operation to the new difference, 89*	*Between-number relation*

6 Discussion

We suggest that students' choice of strategy for solving tasks is related to how they notice characteristics of numbers and number relations (Mason, 2021; Threlfall, 2002). By taking the perspective of the students and studying what number relations came to the fore, our analysis reveals that how number relations are experienced when encountering the task seems to lead to different decomposing operations. Our analysis of the students' decomposition strategies revealed differences in success in solving the task, whereby *decompose the difference* and *decompose the subtrahend* led to correct answers, *decompose into digits* led to incorrect answers, and *decompose the minuend and the subtrahend* resulted in a broad variety of answers.

In conclusion, we have identified differences in acting and discerning number relations when solving the task, but we have also shown what is necessary to discern in order to successfully make use of decomposition. Our knowledge contribution is both empirical and theoretical: it directs attention to the *errors* appearing in students' use of decomposition, based on empirical observations of their attempts to solve a specific task. The significance of this attention is obvious when we compare the outcome space of differences in which numbers the students decomposed (Table 2). The theoretical framework used here offers a way to interpret students' ways of solving a task (acts) as relating to how they experienced the task and the meaning they assigned to it, depending on the number relations. We explain the observed difference in terms of (un)discerned number relations, thus contributing a theoretical view on what it means to develop arithmetic skills. In this way, the findings add to previous studies on students' solving of multi-digit subtraction tasks (e.g., Heinze et al., 2009; Selter, 2001; Torbeyns et al., 2009a, b; Van Der Auwera et al., 2023), with detailed interpretations of the number relations these students experienced when using decomposition to solve a task. The findings thereby shed light on and could explain why the use of decomposition sometimes fails.

From our analysis, we conclude that all of the students who decomposed the subtrahend only ended up with the correct answer. Keeping the minuend unaltered and the between-number relation of minuend and subtrahend in the foreground of awareness while simultaneously operating with within-number relations of the subtrahend seems to be one successful way of attending to and experiencing the number relations when using decomposition.

Hence, these students experienced both between- and within-number relations. Several studies point to the importance of understanding the number relations involved in subtraction tasks in order to be able to solve such tasks (e.g., Linsen et al., 2014; Peters et al., 2013). However, these studies do not use number relations to give a detailed explanation of which relations are in the foreground from a student perspective when solving a specific task, as is done in this study.

Having taken part in the intervention in the first grade likely affected the students' solving of tasks in the interviews in the second grade. We draw this conclusion as many students in our sample successfully decomposed numbers when trying to solve addition and subtraction tasks. It is possible that the intervention influenced the students in our sample to overuse decomposition of numbers when solving subtraction tasks. Moreover, the written algorithm for addition and subtraction was introduced in the textbooks for the second grade. Being taught how to solve tasks using the algorithm (*decompose into digits*, Table 2) likely also influenced how the students solved the tasks in the interview (Selter, 2001). In this study, we did not analyze *decompose into digits* as discerned number relations, since only relations between single digits were in the fore of the students' attention, not number relations between or within the numbers, although it would have been possible using theoretical framework given that these students likely have not discerned within- or between-number relations.

There are some critical features of the study that must also be considered. We selected and focused specifically on the group of students who showed that they had mastered a decomposition strategy (even if it was not solved correctly) in one task with two three-digit numbers bridging over tens and hundreds that would be solved without paper and pencil. Using only one task may be seen as a limitation. However, the variety in incorrect answers and in ways of experiencing the number relations indicates that the chosen task contributed to fulfilling our purpose of closely studying what may lie behind the errors that students make when using this strategy. Furthermore, the experience of number relations among the same group of students has previously been analyzed using two-digit numbers ($32-25=$) in the first grade, showing similar differences in their foregrounding of number relations (Björklund & Runesson Kempe, 2022). The present study complements this by revealing the students' experiences of multiple number relations when solving a subtraction problem with two three-digit numbers.

The analysis was performed inductively; that is, the categories emerged from the empirical data, and are thereby descriptive of the current group of students. Therefore, we cannot claim generalizability, and we cannot rule out that there may be other ways of experiencing number relations among other groups of students solving other tasks. Further research is needed in order to examine how stable the findings are in relation to other students and to students in different grades. Moreover, different types of tasks solved by the same students may shed light on the consistency of the discerned number relations for individual students; however, this was not within the scope of the present paper. The variety of student answers identified suggests that subtraction involving three-digit numbers is challenging for students. Selter's (2001) finding that less than 50% of the fourth graders ($N=300$) in his study could solve the subtraction problem $701-698=$ suggests that this continues to be a challenge through the grades.

It could be possible to consider that young children lose track of their thinking in solving these kinds of subtraction by using a different theoretical frame, such as cognitive load theory. This might explain why students fail to solve the task correctly. However, this would possibly not give sufficient direction for teaching. Our results suggest that solving these subtractions in ways that requires fewer decompositions takes certain discernment of number relations, something that must be attended to when teaching. Students need to have the opportunity to experience subtraction as a part-whole relation, the complement principle ($a+b=c$ implies $c-a=b$), and the magnitude of the numbers, when learning to solve problems such as $204-193=$ in a powerful way (by using addition). If the numbers in the subtraction task are not considered to be close, students ought to become aware that the minuend (the whole) does not need to be decomposed in the solving process. These features, we suggest, are critical for student learning and need to be made explicit in teaching.

Funding Open access funding provided by University of Gothenburg. The project was funded by the Swedish Institute for Educational Research [Grant number 2018-00038].

Data availability The data that support the findings of this study are available on reasonable request from the corresponding author. The data are not publicly available as they contain information that could compromise the anonymity of the participants.

Declarations

Conflict of interest The authors declare no competing interests.

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