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Competencies and fighting syllabusism

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Abstract
For decades, mastery ambitions related to processes like problem-solving, modelling, and reasoning have been incorporated in mathematics curricula around the world. Meanwhile, such ambitions are hindered by syllabusism, a term I use to denote a conviction that results in mastery of a subject being equated with proficiency in a specific subject matter and making that equation the fulcrum of educational processes from teaching to curriculum development. In this article, I argue that using an open two-dimensional structure for curricular content that comprises a set of subject-specific competencies and a modest range of subject matter can help fight syllabusism. I explore and motivate the concept of syllabusism, using the development of a width-depth model of possible curricular ambitions within a given period of time to visualise the detrimental consequences for the attained depth of student learning. In the final part of the article, I illustrate the use of the width-depth model by analysing a specific mathematics curriculum. This analysis leads to two conclusions. Firstly, by highlighting mastery ambitions at the structural level, an open two-dimensional content structure is a powerful means to fight syllabusism. Secondly, using such an approach requires the explicit expression of these mastery ambitions and their conceptualisation independent of the subject matter. In the case of mathematics education, this has taken the form of a set of mathematical competencies.

Keywords Mastery ambitions · Syllabusism · Width-depth model · Annotated syllabus content structure · Two-dimensional content structure · Mathematical competencies

1 Introduction

Over the course of the last 40 years, mathematics curricula around the world have been subject to reforms characterised by a shift towards a stronger focus on various descriptions of student mastery; however, not in the sense of mastering techniques and rote learning. On the contrary, mastery in this context means being very good at mathematics, whatever that means. There are a wide array of reports and books describing and attempting to frame this ambition from countries including (Niss et al., 2016) Denmark (Niss & Jensen, 2002), England

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(Cockcroft et al., 1982; Blausten et al., 2020), Japan (IMPULS, 2017), Portugal (Abrantes, 2001), and the USA (National Research Council, 2001). The mastery ambitions described in these curricular frameworks are conceptualised in different ways, but the frameworks share the same fundamental principle: The important thing is not just what you know, but how you know it—and what you can do with what you know (Niss & Højgaard, 2019).

One of the main findings emerging from various studies of such reforms is that they often do not have the desired impact when it comes to actual teaching practices in schools (Cuban, 2013; Hopmann, 2003), resulting in a gap between the intended and the implemented curriculum (Bauersfeld, 1980). Why is that? Clearly, there is no single answer to this question. An abundance of national case studies have examined variations of this phenomenon. For example (Højgaard & Sølberg, 2023), Innabi and Sheikh (2007) compare perceptions of critical thinking among schoolteachers in Jordan before and after an educational reform, Mwakapenda (2002) provides a critical analysis of the context and status of reforms of secondary mathematics education in Malawi, and Hirsch and Reys (2009) discuss the introduction of the Common Core State Standards for Mathematics in the USA as a vehicle for the development of school practices. I mention these examples to illustrate that the existence of such a gap is evidently a highly complex educational problem with many interwoven roots whose role and importance vary between countries and educational cultures.

Nevertheless, it is relevant from both a development and a research perspective to investigate what can be done to reduce such gaps. When doing so, instead of trying to devise gap-diminishing "recipes", one approach is to start by asking a different question: Why not? Or to elaborate slightly: What is hindering the practice of mathematics teaching in schools from following the strongly advocated focus on various descriptions of student mastery? As examples of this approach, Bergqvist and Bergqvist (2017) use Sweden as a case to discuss why teachers might or might not adapt their teaching to the "message" of a given curriculum reform, while Lerman (2014) argues that, in England, a reform process has resulted in strong regulation of the practices of both teachers and teacher educators.

In this article, I argue that a likely—and from my experience common—hindrance to the pursuit of mastery ambitions in mathematics education of the kind mentioned above is a conviction among those who provide the framework for mathematics education (the developers of curricula, textbooks, test regimes, etc.) that results in mastery of a subject being equated with proficiency related to the specific subject matter, and in using that equation as the hub of educational processes from teaching to curriculum development—a conviction I refer to as *syllabusism*. This conviction has been briefly addressed in previous research publications, both by others (e.g., Lewis, 1972; Jensen & Jankvist, 2018) and by myself (e.g., Højgaard & Sølberg, 2019; Højgaard, 2022). However, besides a preliminary version of parts of the analysis presented in this article (Højgaard, 2012), the background for the conviction has only been properly discussed in a Danish-language publication by Jensen (1995) and has never been thoroughly analysed and modelled from a curricular perspective.

Hence, this article's contribution is to present a discussion of the background for syllabusism to an international audience and conduct a thorough analysis of the detrimental consequences for mathematics education from a curricular perspective. Based on Danish curricular reforms and developmental and research projects examining their implementation, I argue that using an open two-dimensional structure for curricular content that comprises a set of subject-specific competencies and a modest range of subject matter can help fight syllabusism. I explore and motivate the concept, using the development of a width-depth model of possible curricular ambitions within a given period of time to visualise the detrimental consequences for the attained depth of student learning. In the final part of the article, I illustrate the use of the width-depth model by analysing a specific mathematics curriculum. A key finding is that, to use the open two-dimensional approach, the mastery ambitions must be expressed explicitly and conceptualised independent of subject matter. In the case of mathematics education, this has taken the form of a set of mathematical competencies.

2 Mathematical mastery and syllabusism

What does it mean to master mathematics? As the most simple and straightforward answer, it is tempting to equate mastery with proficiency in mathematical subject matter—that is, in using certain mathematical concepts (fractions, area, functions, etc.) and procedures (adding fractions, calculating area from a formula, drawing graphs of certain functions, etc.). While sometimes convenient, this is a highly reductionist approach that can have serious consequences, including shifting the focus of teaching and learning away from the essential complexity of mastering mathematics as a subject. I believe everyone with a sense of mastering a subject will agree that there is much more to it than becoming proficient in relation to its subject matter, but this "much more" is forgotten (or neglected) in an educational system that revolves around proficiency in the subject matter specified in various curricula (Blomhøj & Jensen, 2007).

I label the conviction behind such an approach *syllabusism*. While discussing curricula in centralised educational systems, Basdemir (2013) mentions this term, but without properly defining syllabusism as a concept. I therefore propose the following definition: Syllabusism denotes a conviction that results in mastery of a subject being equated with proficiency in a specific subject matter, and in making that equation the fulcrum of educational processes from teaching to curriculum development. Lewis (1972) and Jensen (1995) introduced a similar understanding under the label syllabusitis, but, as I consider it more productive to explore and address the both explicit and implicit convictions and ideologies than to diagnose a disease or condition and prescribe treatment, I prefer the term syllabusism.

3 Syllabusism and curriculum structure

In line with Kilpatrick (1996), I use the term *curriculum* to denote a vector composed of the following six components (Niss, 2016; Niss & Højgaard, 2024): goals, content, materials, forms of teaching, student activities, and assessment. This is a traditional approach in the sense that it maintains a focus on goals, content, and guidelines concerning different aspects of teaching, learning, and assessment, with a conceptualisation focusing on curriculum as cultural practice (Kanu, 2003, 2006) being just one alternative.

However, the chosen conceptualisation intentionally breaks with tradition by adopting a broad approach to what aspects of content, teaching, and learning can and should be included in a curriculum. Historically, mathematics teaching and learning have given primacy to the mathematical subject matter (e.g., numbers, geometric shapes, equations, functions) and related procedural skills; that is (Niss & Højgaard, 2019), the ability to perform methodologically well-defined procedures, routines, and techniques using that subject matter (e.g., adding numbers, calculating areas, solving equations, and drawing graphs).

This tradition has curricular implications. Contrary to the broader approach outlined above, mathematics curricula have often been defined in terms of the subject matter students are expected to know and the procedural skills they are expected to demonstrate in dealing with this subject matter. In many countries, Denmark being one of them, this emphasis has been facilitated by structuring mathematics curricula around the following components (Niss & Jensen, 2002):

- The purpose of the teaching.
- A syllabus, that is, an outline of the subject matter to be covered.
- Assessment and testing instruments.

Sometimes the purpose is determined first and used as a basis for the development of the syllabus and modes of assessment and testing. Often, however, the syllabus comes first, with the purpose added later as a sort of politically oriented foreword, and the modes of assessment and testing are often only presented with reference to syllabus-specific objectives and formal settings ("A 4-h written test", etc.). The syllabus hereby becomes the fulcrum of curriculum development and, consequently, the central arena for discussion between teachers and the developers of curricula, textbooks, test regimes, etc. In such a system, syllabusism is systematically nurtured.

My intention here is not to discuss the trustworthiness and generalisability of this description of the traditions in mathematics education. Instead, it leads me to a more constructive and forward-looking analysis addressing the following question: *How can the content of mathematics education be structured in a way that facilitates mastery ambitions by fighting syllabusism?*

4 Modelling the depth and width of curricular ambitions

In a famous statement that supports my impression that syllabusism is widespread in mathematics education, Schmidt et al. (1997, p. 2) referred to the situation in the USA in the 1990s as characterised by a "splintered vision":

> These splintered visions produce unfocused curricula and textbooks that fail to define clearly what is intended to be taught. They influence teachers to implement diffuse learning goals in their classrooms. They emphasise familiarity with many topics rather than concentrated attention to a few. And they likely lower the academic performance of students who spend years in such a learning environment. Our curricula, textbooks, and teaching are all "a mile wide and an inch deep." (Schmidt et al., 1997, pp. 1-2)

The emphasis on "familiarity with many topics rather than concentrated attention to a few" is an instance of syllabusism. If an educational system is inclined to equate the mastering of a subject with proficiency in a certain subject matter, then it is very tempting to measure the level of ambition by the amount of subject matter to be covered. In the passage above, Schmidt et al. address this as the "width" of the curriculum, which I believe essentially boils down to the "width" of the syllabus.

They also talk about the "depth" of the curriculum, which I interpret as a metaphor for the ambitions concerning the students' understanding of and ability to apply the content of the syllabus to which they are introduced. Sticking with this metaphor, I consider "depth" in terms of intended learning outcomes outlined in the curriculum (Bauersfeld, 1980).

The dual concerns of an overly broad syllabus and a lack of depth of student learning can be addressed by a model focusing on both these dimensions of curricular ambitions, cf. the proposal in Fig. 1.

This model represents a fundamental shift in focus—from a one-dimensional width-of-the-syllabus approach to a two-dimensional width-times-depth approach—when discussing

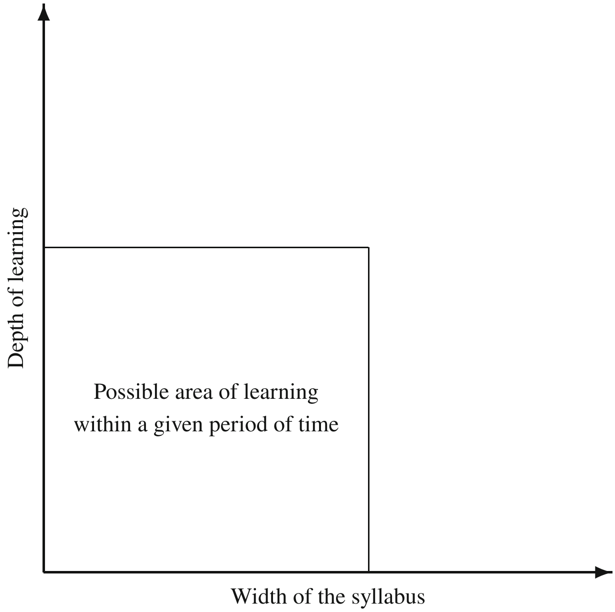

Fig. 1 A two-dimensional width-depth model of curricular ambitions within a given period of time

and describing a curriculum's ambitions regarding student learning outcomes. In the width-depth model in Fig. 1, such ambitions are represented as an "area of learning". Paraphrasing the SOLO taxonomy (Biggs & Collis, 1982), we might refer to this as a shift in *the structure of the intended learning outcome*—SILO.

The size of the "area of learning" in Fig. 1 represents what is possible *within a given period of time*, for example, the number of lessons available for the teaching of a subject during a school year. The idea is to invoke realism when discussing and setting curricular ambitions by acknowledging the inevitable balancing of width and depth: Any increase in depth involves a corresponding reduction in width, and vice versa. To provide a concrete example: Imagine a teacher compiling a plan for the next year's maths lessons in a Year 5 class. To follow up on previous years' work with a particular mathematical concept, such as area, and associated techniques, the teacher can plan to spend the allotted lessons on widening what we, inspired by Stein et al. (2007), might call the *enacted syllabus* related to that concept. Sticking with the example of area, this might be done by letting students work with more complex geometric shapes. Alternatively, the teacher can plan to focus on deepening the students' understanding of the concept by means of concept maps, reasoning about the concept ("what is the relation between the area and circumference of different geometric shapes?"), or modelling with the concept ("how big is the school playground?")—to mention just a few examples of ways of letting students work "in depth" with a topic with which they are already familiar. However, as there are only a limited number of lessons available, the teacher cannot do both—they must prioritise.

4.1 Syllabusism in the width-depth model

As previously mentioned, syllabusism denotes a conviction that results in mastery of a subject being equated with proficiency in specific subject matter, and in making that equation the fulcrum of educational processes from teaching to curriculum development. Consequently, a main and easily identifiable component when specifying ambitions regarding the level of mastery is the amount of subject matter—or, in curricular terms, the width of the syllabus—to be covered in a specific educational setting. Hence, when specifying curricular ambitions, syllabusism naturally implies a tendency to "boost" the width of the syllabus, because "insisting" on a wide syllabus is considered ambitious. Arguments such as "we *need* to include the important concepts of …in the syllabus, anything else would be unambitious" made by various parties lead to a conviction that a certain minimal width of the syllabus is imperative.

The problem with syllabusism is not its favouring of a wide syllabus per se, but that it inevitably reduces the *attained* depth of learning (Stein et al., 2007), if we follow the logic of the width-depth model. If a relatively wide syllabus is seen as a premise in the (based on my involvement in Danish curricular processes, often implicit) discussions of how to balance width and depth when it comes to curricular ambitions, the logical consequence is a reduction in the attained depth of learning, regardless of the curriculum's *intended* depth of learning, cf. the visualisation in Fig. 2.

This logical consequence of a wide syllabus, I suggest, is not something that anyone considers desirable; nor does it seem to be acknowledged by most curriculum designers—if there is any truth to my impression that syllabusism is widespread within mathematics education around the world, and if we assume that this is not because of a conscious preference for a wide syllabus as opposed to greater depth of learning. A more plausible explanation for the common focus on width seems to be the ease of "measuring", comparing, and communicating curricular ambitions by means of something as concrete as the width of a syllabus, compared to the difficulty of doing so by means of something as fuzzy as the depth of learning. Therefore, the alarm bells ring whenever even minor reductions in the number of topics included in the syllabus are suggested, whereas the gulf between the intended and attained

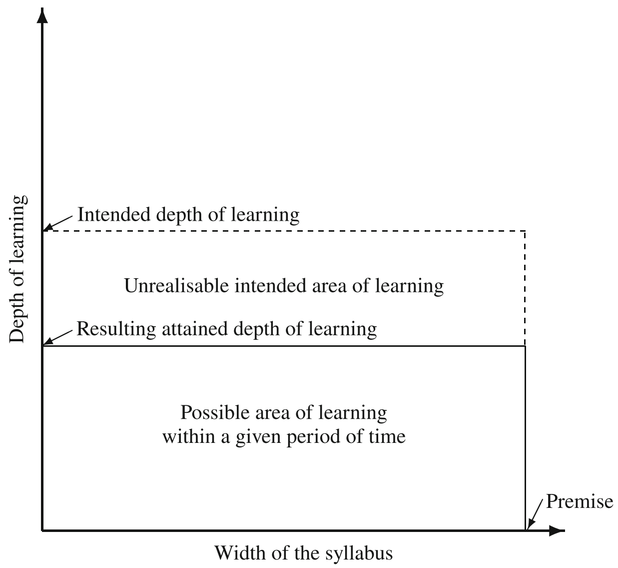

Fig. 2 Syllabusism visualised by means of the width-depth curriculum model: a syllabus with a large width is used as a premise for curriculum design, resulting in a reduction of the attained depth of learning compared to intentions in the curriculum

depth of learning often remains hidden when designing a new curriculum. Elaborating on their characterisation of curricula as “a mile wide and an inch deep”, Schmidt et al. (1997, p. 2) state:

> Reforms have already been proposed by political, business, educational and other leaders. Extensive efforts are underway to implement these standards, but the implementation process itself is shaped by the prevailing culture of inclusion. Like the developers of curricula and the publishers of textbooks, teachers add reform ideas to their pedagogical quivers without asking what should be taken away.

4.2 Prioritising mastery in the width-depth model

An alternative approach to curriculum design is to use a certain depth of learning as a premise, based on arguments such as “we *need* to favour relational understanding as, unlike instrumental understanding, it is robust and self-perpetuating”; or “mastery-oriented ambitions should be given priority to align the curriculum with societal expectations”.

Seen in isolation, there are probably few who oppose such arguments. If we once again follow the logic of the width-depth model, cf. the visualisation in Fig. 3, the challenge is to gain acceptance for a fairly narrow but essentially flexible syllabus as a necessary consequence of a greater focus on depth (as exemplified by, e.g., Lew et al., 2012).

In contrast to the simple “ambition is measured by the width of the syllabus”-logic of syllabusism, the basic principle here is difficult to grasp and challenging to convey: Less [width of a syllabus] is more [depth of learning].

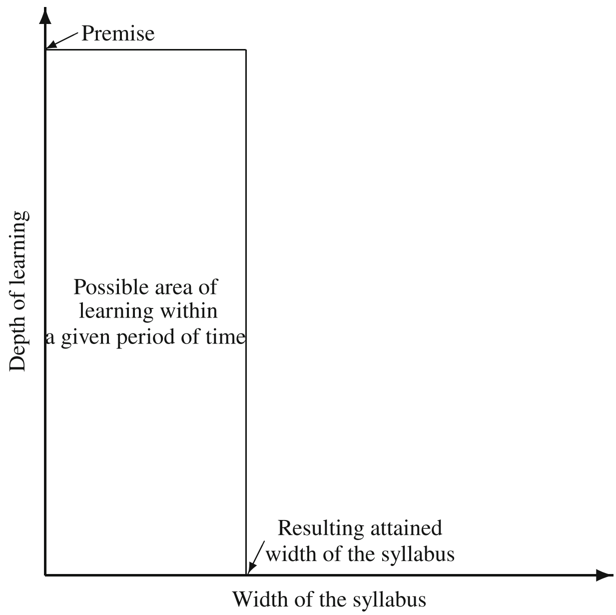

Fig. 3 Prioritising mastery visualised by means of the width-depth curriculum model: a great intended depth of learning is used as a premise for curriculum design, resulting in a narrowing of the syllabus compared to an approach that does not prioritise mastery ambitions

5 Explicit mastery objectives as "the missing link"

One of the reasons for the traditional syllabus-focused curricular structure might be that there has been no viable alternative when searching for a suitable vocabulary for communicating curricular ambitions among the various bodies involved in mathematics education. Without such a vocabulary, it is difficult to discuss and more concretely describe the depth being missed in a syllabusism-driven regime, and consequently also difficult to argue in favour of a more balanced approach as illustrated by the width-depth model in Fig. 1. One alternative is a curriculum that gives primacy to the purpose of teaching; however, this is conceptually a much more general kind of statement (Niss, 1996) with no direct relation to the planning and organisation of teaching. The result is a "missing link" between the purpose of teaching and the planning of classroom activities, with research and development projects involving hundreds of Danish teachers (Højgaard & Sølberg, 2023) showing that only a small minority have the time and professional background to create such a link themselves.

5.1 Competency descriptions in the Danish KOM Project

This hypothesis regarding the reasons for the traditional syllabus-focused curricular structure, supported by our personal experience, was the main motivation when the chairman of the committee Mogens Niss and I accepted the Danish Ministry of Education's invitation to be the driving forces in the so-called KOM Project around the turn of the millennium. The project, conducted in the years 2000–2002, and its findings are thoroughly reported in Niss and Jensen (2002), with Niss and Højgaard (2019) presenting an updated extract in English, while Niss and Højgaard (2024) includes an English translation of much of the original report.

The fundamental idea at the core of the KOM Project was that a set of mathematical competencies can replace the syllabus as the main focus in mathematics education, thereby offering an alternative description of what it means to master mathematics (Blomhøj & Jensen, 2007; Højgaard & Sølberg, 2023). To achieve this, the project sought to identify, explicitly formulate, and exemplify a set of mathematical competencies as independent dimensions of mathematical competence, cf. the visualisation in Fig. 4, with an emphasis on the role of such competencies in the teaching and learning of mathematics.

The KOM Project's conceptual framework has (Niss & Højgaard, 2019) been used and sometimes criticised in many subsequent studies, generating extensive discussion and additional conceptual developments, as reflected in many publications both in this journal and elsewhere (e.g., Abrantes, 2001; Aguilar et al., 2021; Alpers et al., 2013; Boesen et al., 2014; Højgaard, 2021; Jankvist & Niss, 2015; Niss, 2015). A central issue has been discussions of the similarities and differences between mathematical competence and competencies and related notions such as mathematical proficiency, mathematical practices, fundamental mathematical capabilities, mathematical literacy, quantitative literacy, and numeracy. In Sect. 7 I will return to one characteristic of competence that I believe constitutes a critical distinction from similar concepts.

Another issue has concerned the nature of competence, conceptualised as someone's insightful readiness to act appropriately in response to the challenges of given situations (Niss & Højgaard, 2019). One question that has been raised is whether this makes competence something individual persons possess, or whether it is inherently of a more social nature? My reply can be summed up as "both": Among other characteristics, the conceptualisation of competence outlined above contains an inherent duality between subjective and social/cultural properties (Wedege, 2000; Højgaard, 2009). Subjective, because a competency

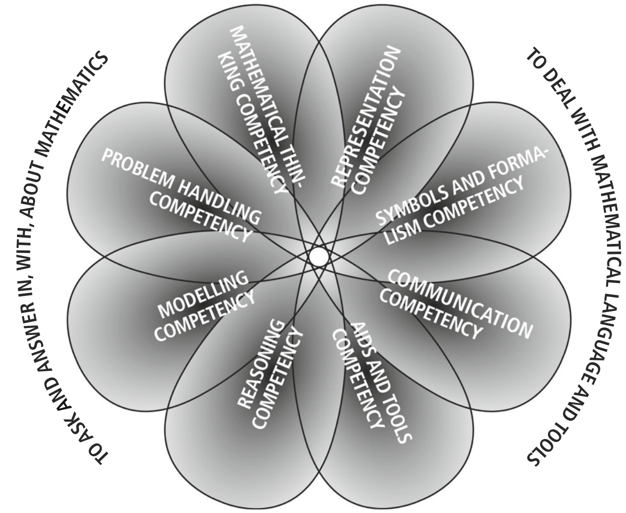

Fig. 4 A visual representation—known as the "KOM flower"—of the eight mathematical competencies identified in the KOM Project (Niss & Højgaard, 2019, p. 19)

is always someone's—competencies do not exist by themselves, what exists are competent people. Social/cultural, because the degree to which specific actions "meet the challenges" is always relative to the meaning and legitimacy ascribed to these actions by others (Jørgensen, 1999).

A third issue stems from a critique of curricular reforms as remaining enclosed within the subject of mathematics. The danger of such a "compartmentalised" approach is that mathematics education fails to address and engage with social consciousness, politics, ethics, etc. Does the idea of emphasising mathematical competencies make any difference in this regard? Not itself, no. As mentioned, it is an attempt to characterise mathematical mastery, so as a conceptual framework, it follows the compartmentalised approach. However, stressing the ability to "act appropriately in response to the challenges of given situations" as a pivotal element in mathematics education has shown potential as a way of "opening up" the subject towards broader societal issues by choosing such situations carefully (e.g., Jensen, 2007; Gibbs et al., 2022). Hence, focusing on competencies might be a way to incorporate cultural practices as a key element in mathematics curricula, and to respect the importance of practice in cultural inquiry (Kanu, 2003). This is an issue that calls for more research.

It is beyond the scope of this article to give a thorough account of these discussions. Readers are referred to Niss (2015), Niss et al. (2016), and Niss and Højgaard (2024) for discussion of competencies and related notions, and to Blomhøj and Jensen (2003), Højgaard (2009), and Niss and Højgaard (2019) for elaborations on the conceptualisation of competence in the KOM framework.

5.2 Approaches to curricular descriptions of mastery

As mentioned in the introduction, the approach laid out in the KOM report is one of many attempts to generate a broader and more ambitious framing of the design of mathematics curricula. In a similar vein, many countries have developed similarly concrete mathematics curricula, where expectations of a certain depth of learning are made explicit—often as a direct follow-up to preceding framework documents. The most internationally well-known example is probably the American "Principles and Standards for School Mathematics" (NCTM, 2000, 2003), now further developed into "Common Core State Standards for Mathematics" (NGA Center & CCSSO, 2010), but China (Ministry of Education of the People's Republic of China, 2018), Denmark (Undervisningsministeriet, 2009), and Germany (Blum et al., 2012/2006) provide examples of other, similar developmental processes at the national level.

6 An "annotated syllabus" structuring of curricular content

As an example illustrating my point, Australia has been in the process of developing a new national curriculum for general education, including a new framework for mathematics education (National Curriculum Board, 2009). The main feature of this framework is to distinguish between content strands, which I interpret as parallel to a syllabus, and proficiency strands, which I (National Research Council, 2001) interpret as a way of addressing the "depth of learning" part of the curricular content. The approach chosen in the Australian framework is described as follows:

> The content strands describe the "what" that is to be taught and learnt while the proficiency strands describe the "how" of the way content is explored or developed, i.e., the thinking and doing of mathematics. *Each of the "content descriptions" in the mathematics curriculum will include terms related to understanding, fluency, problem-solving, or reasoning.* (National Curriculum Board, 2009, p. 7, my emphasis)

The italicised part of the quote is an unusually clear example of what I believe to be a very common way of linking syllabus and "depth of learning" with regard to curricular content (Højgaard, 2012). The so-called proficiency strands use specific depth-related objectives ("acquire computational fluency with…", "develop a general understanding of…", etc.) to annotate the syllabus, cf. the model in Fig. 5.

The problem with such an "annotated syllabus" approach is that it does not fundamentally break with the tradition of linear descriptions of the content of mathematics education; it just "dresses it up" with some new "depth of learning" ambitions. It is even explicit in the name, "content strands", that the content of the teaching—the things on the agenda when planning lessons—is still considered synonymous with the subject matter chosen for the syllabus.

Hence, an "annotated syllabus" structure does not encourage a reappraisal of the balance between the width of the syllabus and the depth of learning—quite the contrary. From a concrete planning perspective, it is still possible to forget or neglect the curriculum's new ambitions in terms of depth of learning, instead translating the linear descriptions of content into what is, by nature, a likewise linear annual teaching plan. Here, an arithmetic

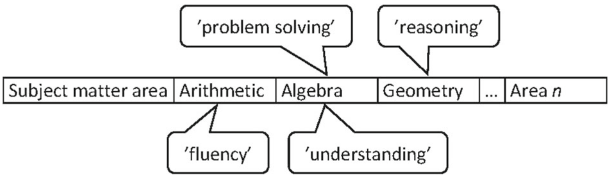

Fig. 5 A model of an "annotated syllabus" structuring of the content of mathematics education (adapted from Højgaard, 2012, p. 6416)

concept—“addition” or “multiplication”, for example—is the heading for part one of the plan, a geometric concept—“length” or “area”, for example, is the heading for part two. As such, this approach to curriculum development *still* encourages syllabusism at the system level.

7 A matrix structuring of curricular content

In the KOM Project, a different approach was suggested, based on the following aforementioned conceptualisation (Niss & Højgaard, 2019): *Competence* is someone’s insightful readiness to act appropriately in response to the challenges of given situations. In line with this conceptualisation, we can think of *a subject-specific competency* as someone’s insightful readiness to act appropriately in response to a certain kind of subject-specific challenge of given situations. As an example, *a mathematical competency* is defined as someone’s insightful readiness to act appropriately in response to a certain kind of mathematical challenge of given situations. Each of the petals of the “KOM flower” in Fig. 4 represents what we in the KOM Project considered to be one such kind of mathematical challenge.

This conceptualisation includes an inherent focus on the *exercise* of mathematics; i.e., the *enactment* of mathematical activities and processes. By choosing this approach, KOM intentionally distinguished between what it means to *know and understand* mathematics and what it means to *enact* mathematics. The former is related to mathematical subject matter, while the latter is spanned by mathematical competencies. Hence, by definition and by design, the mathematical competencies go across and beyond—and cannot be subsumed under—specific mathematical subject matter areas. Instead, competencies on the one hand, and knowledge and understanding related to subject matter areas on the other hand, constitute two independent but interacting dimensions of mastery of mathematics.

One can represent this relationship by a matrix structure. At a given educational level for which $a_1, \ldots, a_n$ have been selected as areas of subject matter to be covered, the relationship between competencies and subject matter areas can be depicted as in Fig. 6.

Competency \ **Subj. matter area**	Area 1	Area 2	$\cdots$	Area n
Mathematical thinking				
Problem handling				
Modelling				
Reasoning				
Representation				
Symbols and formalism				
Communication				
Aids and tools				

Fig. 6 A matrix structuring of the competencies × subject matter area of mathematics education (adapted from Niss & Jensen, 2002, p. 114)

Each of the cells in this matrix can be conceptualised in two ways. Firstly, as an answer to the following question: What is the specific role of competency i ($i = 1, 2, \ldots, 8$) in dealing with subject matter area j ($j = 1, 2, \ldots, n$) at the educational level under consideration? Secondly, as an answer to the following question: What is the specific role of subject matter area j ($j = 1, 2, \ldots, n$) in the activation of competency i ($i = 1, 2, \ldots, 8$) at the educational level under consideration?

Posing and answering such questions represents a two-dimensional approach to determining the learning ambitions that should be part of any given curriculum. That in itself is an important step in the fight against syllabusism, since the presence of competencies as a separate dimension is a way of making depth of learning a structuring and thereby defining element when expressing curricular ambitions. However, the two-dimensional approach only really becomes binding if it is also used as a model for determining curricular content—that is, if *both* competencies and subject matter areas are on the agenda on equal terms when the planning of teaching takes place. Only then has there been genuine resistance to syllabusism as the logic framing teaching practices.

8 An open two-dimensional structuring of curricular content

A two-dimensional content model, where competency goals and subject matter areas are separated as two independent dimensions of content, can a priori both facilitate and challenge adherence to the "less is more" mantra (Højgaard & Sølberg, 2023). In itself, the priority given to competencies by representing them as a separate content dimension is a way of structurally acknowledging the relevance of paying attention to the depth of the content descriptions in the curriculum. This was the rationale for suggesting a two-dimensional approach in the KOM Project.

On the other hand, adding competencies to content descriptions just worsens the problem of a surfeit of content in the allotted time if not accompanied by a reduction in the width of the syllabus. This is not least the case if one chooses to work with a matrix-structured content model as originally suggested in the KOM report. If teachers find it stressful having to work with n different subject matter areas within the lessons allotted for mathematics during a school year, they are unlikely to welcome a model that they may feel requires them to work with $8 \times n$ combinations of competencies and subject matter areas (cf. Fig. 6). Contrary to the ambition of fighting syllabusism in mathematics education, there is a risk of turning the "too-much-content-to-cover situation" from bad to worse by systematically combining syllabusism with "competenceism".

In the KOM report, it is argued that not every cell in the matrix-structured model of curricular ambitions needs to be taught, but when the model was introduced to Danish teachers in the years following the report's publication, they consistently reported feeling overwhelmed by the large number of cells. In response to this feedback, I developed the idea of removing the grid in the matrix-structured content model, thereby transforming it into a more open two-dimensional structure, as depicted in Fig. 7.

The open two-dimensional structure invites curriculum designers to make room for and facilitate alternative approaches to curricular content where teachers and other educators tailor lessons in ways that link the purpose of teaching to the specific context and circumstances. This is intended to encourage careful consideration of what constitutes an appropriate balance between width and depth with regard to curricular ambitions, cf. the model in Fig. 1 (Højgaard & Sølberg, 2023).

Competency \ Subj. matter area	Number domains	Arith-metic	Alge-bra	Geo-metry	...	Area *n*
Mathematical thinking						
Problem handling						
Modelling						
Reasoning						
Representation						
Symbols and formalism						
Communication						
Aids and tools						

Fig. 7 An open two-dimensional structuring of the content of mathematics education (Højgaard, 2012, p. 6416)

9 An example of an open two-dimensional structure of mathematical content

In Denmark, the national curriculum for compulsory mathematics education—grades k-9—incorporated the competency framework developed in the KOM Project since a reform in 2009 (Undervisningsministeriet, 2009). The content structure currently being used was developed as part of a reform in 2014, with adjustments in 2019 providing the current mathematics curriculum. It consists of a series of tables presenting common learning objectives at different levels and for different subject matter areas (Undervisningsministeriet, 2019) and an accompanying booklet providing descriptions of the various competencies and subject matter areas, as well as positioning them within the context of the broader purpose of compulsory mathematics education (Børne– og Undervisningsministeriet, 2019). Højgaard (2024) describes this structure in detail, and Højgaard and Sølberg (2023) offer an analysis of its genesis as part of a 20-year curriculum development process.

Here, I will merely note that the first thing you encounter in the accompanying curriculum booklet is a recommendation to base the planning of mathematics teaching on the content model in Fig. 8. Hence, the curriculum for compulsory mathematics education in Denmark uses the open two-dimensional content model depicted in a generalised form in Fig. 7.

	Problem handling	Modelling	Reasoning and math. thinking	Representation and symbol handling	Communication	Aids and tools
Numbers and algebra						
Geometry and measuring						
Statistics and probability						

Fig. 8 The official two-dimensional content model for compulsory mathematics education (grades k–9) in Denmark (Børne– og Undervisningsministeriet, 2019, p. 8, my translation)

10 Using the open two-dimensional content structure—an example

The open two-dimensional content structure can be considered an invitation to work with individual competencies in two different ways: either independently of any predetermined subject matter, thereby exploiting the open nature of the two-dimensional model, or by charting the role of a particular competency across the entire range of different subject matter areas to be covered at a given educational level. In both cases, content is approached via the horizontal dimension in Fig. 7, i.e., from the perspective of the various competencies. Meanwhile, it is also possible to approach curricular content via the vertical dimension, working with individual subject matter areas either in relation to one or more competencies or independent of any predetermined competency (Niss & Højgaard, 2019).

Such a four-stringed approach was empirically tested in the longitudinal research and development project *KOMPIS* (Højgaard & Sølberg, 2019), for which the open two-dimensional model in Fig. 7 was developed. Based on previous developmental research (Jensen, 2007), one of the participating teachers and I agreed to use the four-stringed approach to the open two-dimensional model in the development of her annual teaching plan for the experimental mathematics class observed in the KOMPIS Project, cf. the visualisation of the planning model in Fig. 9. More specifically (Sørensen, 2010), one component of the teaching with a "horizontal" approach was planned as project work aimed at developing either mathematical modelling competency or mathematical reasoning competency from a holistic perspective (Blomhøj & Jensen, 2003). Another component of the teaching with a "vertical" approach was planned as teacher-guided coursework aimed at developing the students' understanding of a selected mathematical concept and a particular mathematical competency—e.g., problem handling or symbols and formalism—considered "well suited" to conceptually focused teaching in general and to the selected mathematical concept in particular.

It is complex and demanding for teachers to use such an approach when planning mathematics teaching. Danish developmental and research projects have shown this to be the case more generally when using the open two-dimensional content model in Fig. 7. There is a risk that the model's intentionally simplistic representation of content can downplay or

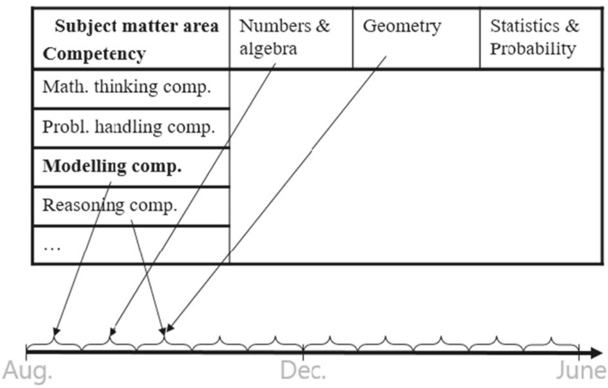

Fig. 9 The two-dimensional content structure in Fig. 7 applied in the specific circumstances of the KOMPIS Project as a tool for planning teaching by creating multi-week modules, each with explicit learning objectives consisting of subject-specific competencies and/or objectives related to specific subject matter (Højgaard & Sølberg, 2019, p. 55)

even obscure the complexity of working with mathematical competencies as a central part of mathematics teaching.

While demanding, implementing a two-dimensional content structure nevertheless remains possible. When enacting competency-oriented teaching, this model can provide a challenging but fruitful framework for determining and representing the content and ambitions of a specific educational context—and doing so in a way that structurally constitutes an explicit break with syllabusism, instead favouring a focus on mastery (Højgaard & Sølberg, 2023), cf the analysis in Sect. 7. The KOMPIS Project concluded that the open two-dimensional approach to planning described above allowed teachers to maintain an overview of the overall curricular goals while deciding what to teach and when (Højgaard & Sølberg, 2019). This conclusion has subsequently been validated by feedback from the many teachers who have participated in extensive in-service training in competency-oriented mathematics education as part of another longitudinal research and development project, where the open two-dimensional content structure was implemented at scale (Højgaard & Winther, 2021).

11 Analysing specific curricula

I will now exemplify how the distinction between an "annotated syllabus" and a two-dimensional structuring of content can be used to analyse specific existing curricula, which allows me to highlight a critical aspect of curriculum development. The example chosen concerns the aforementioned Common Core State Standards (CCSS) for Mathematics initiative in the USA, which I use to explore certain structural characteristics rather than the specifics of this curriculum document.

11.1 The US CCSS for Mathematics as an example of an "annotated syllabus" structure

The *Common Core State Standards Initiative* can—in the absence of a national US curriculum—be perceived as an initiative to form a "coalition of willing states" in order to create a set of common standards (Niss, 2016). The bulk of the CCSS-Mathematics document (NGA Center & CCSSO, 2010) is focused on how various mathematical subject matter areas should be taught at different k-12 grade levels. This priority is structurally emphasised by letting all but one (modelling at high school level) of the headings for each grade level refer to a particular subject matter area (e.g., geometry) or concept (e.g., functions).

The document also places an emphasis on eight *Standards for Mathematical Practice* (NGA Center & CCSSO, 2010, pp. 6-8):

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP4 Model with mathematics.
MP5 Use appropriate tools strategically.
MP6 Attend to precision.
MP7 Look for and make use of structure.
MP8 Look for and express regularity in repeated reasoning.

However, although it is not spelled out as explicitly as in the excerpt from the Australian framework quoted on page 10, the "annotated syllabus" structure in Fig. 5 is also used here.

This is evident from the integration of the eight standards for mathematical practice in more specific guidelines for the planning of teaching, as in the following example:

> In Grade 8, instructional time should focus on three critical areas: (1) formulating and reasoning about expressions and equations, including modeling an association in bivariate data with a linear equation and solving linear equations and systems of linear equations; (2) grasping the concept of a function and using functions to describe quantitative relationships; and (3) analyzing two- and three-dimensional space and figures using distance, angle, similarity, and congruence, and understanding and applying the Pythagorean Theorem. (NGA Center & CCSSO, 2010, p. 52)

This approach to content description seems in agreement with the approach envisioned by Schmidt et al. (2002), which the Common Core State Standards for Mathematics explicitly endeavour to follow (NGA Center & CCSSO, 2010, introduction). Referring to a study of mathematics curricula in high-achieving TIMSS countries, this approach suggests one "think of some math topics as part of a required core taught in particular grades [...]", more concretely based on "an evolution from an early emphasis on arithmetic in grades one through four to more advanced algebra and geometry beginning in grades seven and eight" (Schmidt et al., 2002, p. 5), as exemplified in the CCSS quote above. This is accompanied by other recommendations regarding the curricular organisation of subject matter, whereas standards for mathematical practice or similar mastery-oriented terms are not even mentioned.

11.2 The lack of an independent mastery dimension

The standards for mathematical practice in the CCSS describe different aspects of what is considered to be important when learning mathematics. Together, these "strings" are meant to entwine and form a strong "rope" of mathematical capability, to use a metaphor from the national report fostering this approach (National Research Council, 2001).

I consider every one of the aforementioned standards for mathematical practice both relevant and important as guidelines for the teaching and learning of mathematics, but they are not—and are not meant to be—a description of mastery ambitions independent of the chosen subject matter. Hence, the CCSS do not represent a two-dimensional approach, despite the apparent emphasis on both subject matter and standards for mathematical practice. Consequently, the CCSS content cannot be represented by an open two-dimensional structure in any meaningful way. It is—and is seemingly intended to be—a syllabus annotated with guidelines for practice. Very meaningful guidelines, but not a structural break with syllabusism.

12 Conclusion

At the beginning of this article, I posed the following research question, which has guided the inquiry: *How can the content of mathematics education be structured in a way that facilitates mastery ambitions by fighting syllabusism?* I believe the analysis presented above supports the following conclusions:

There are two reasons why an open two-dimensional content structure is a powerful way to facilitate mastery ambitions by fighting syllabusism: It makes mastery ambitions explicit and highlights them at the structural level; and it encourages teachers and other educational planners to reflect on what constitutes—and how to achieve—an appropriate balance between the width of the syllabus and the depth of learning given the inevitable time constraints. To make use of such an approach, mastery ambitions must be conceptualised independent of subject matter. In the case of mathematics education, this has been done by means of a set of mathematical competencies.

This highlighting of the importance of explicit mastery ambitions is nothing new; it has been on the agenda within curriculum research and development concerning mathematics education for decades. Meanwhile, the analysis above contributes to the field by showing that, while including explicit mastery ambitions in curricular content descriptions is a necessary and important step towards creating classroom cultures that share these ambitions, it is by no means sufficient to overcome the challenge of syllabusism. From a curriculum perspective, the pivotal developmental step does not concern the explicit description of objectives pertaining to the depth of learning in itself; rather, the necessity of balancing width and depth when describing curricular content must be accepted and highlighted at a structural level.

Conceptually, the point of this article is not to reintroduce the mathematical competencies developed by the KOM Project as a framework for discussing mathematical mastery. Such discussions have already been presented elsewhere, not least with the publication of Niss and Højgaard (2019), and will be thoroughly elaborated with the forthcoming publication of Niss and Højgaard (2024). Rather, the present analysis makes a second contribution to the field by highlighting the importance of conceptualising mathematical mastery as a pivotal educational ambition in itself, independent of subject matter. Without such an approach, there can be no break with syllabusism when describing curricular content and there is a risk that mastery ambitions come across as mere lip service—secondary addenda to a curriculum that remains structured around the syllabus.

13 Perspectives for future curriculum development

The structure of the interplay between competencies and subject matter seems to invite analysis that is both broader in scope and explores the issues raised here in greater depth. In the KOM report (Niss & Jensen, 2002), we proposed that such analysis should start with consideration of the cells in the matrix structure that was then on the table (depicted in Fig. 6). With the analysis above in mind, my proposal now is to skip the cells and use the open two-dimensional structure in Fig. 7 as a framework for analytical reflection:

- Are there specific competencies that should be given prominence across the various subject matter areas, either in general or at specific educational levels?
- Does the same apply to specific subject matter areas?
- How can/should a given competency be developed or practised in relation to different subject matter areas?
- How can/should a given subject matter area contribute to the development of the different competencies?

In relation to the last question, a first step might be to examine how a single concept is treated: What do you want students to be able to do with, e.g., the concept "fractions"? How can their work with that concept be organised, approached, challenged, etc., if you wish to contribute to the development of the students' reasoning competency/modelling competency/aids and tools competency, etc.?

All such considerations are related to a single two-dimensional structure and, thereby, a single educational level. To analyse more than one teaching level at a time, we would need to work with a three-dimensional structure, cf the model in Fig. 10. With the two-dimensional

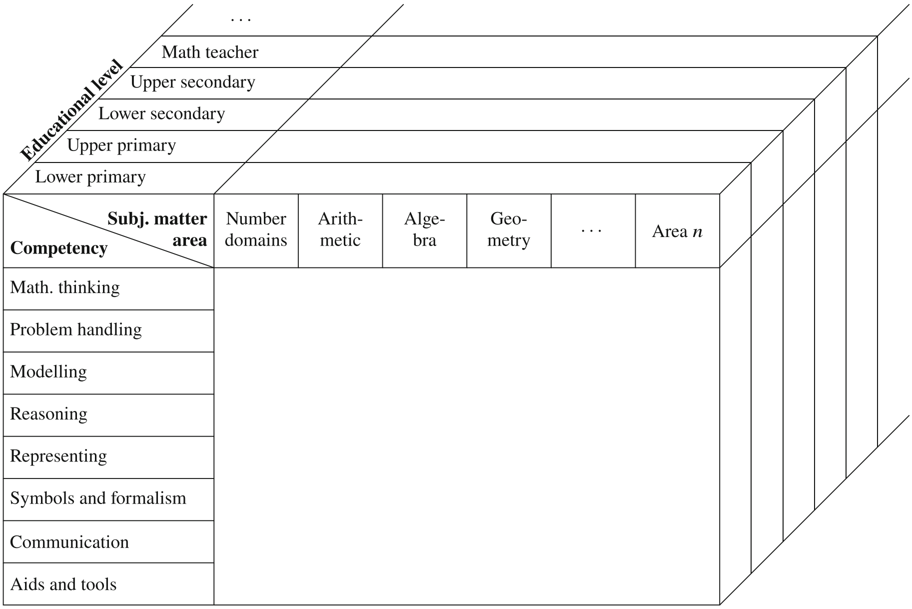

Fig. 10 A three-dimensional model for analysing progression and coherence as regards content in mathematics education

structure in Fig. 7 as the starting point, educational level forms the third, easily overlooked, dimension (cf. the textbook content model in Højgaard, 2019).

Analysis within this three-dimensional structure deals with competencies and their connection to longitudinal subject matter areas:

- How should the educational system ensure *longitudinal* coherence and progression with regard to working with competency X? …subject matter area Y?
- For example, how does the concept "fractions" percolate through the curriculum (e.g., underpinning concepts of ratio and of linear functions)? Or what trajectory can be envisioned for students' mathematical modelling competency (e.g., towards being able to take an informed stance concerning societal applications of mathematics)?

From the perspective of ensuring curricular coherence, such questions are perhaps the most important of all.

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