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Using the onto-semiotic approach to analyze novice algebra learners' meaning-making processes with different representations

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Abstract
Representations are key to mathematical activities and meaning-making processes as they are part of modeling, connecting, communicating, and understanding mathematical ideas and concepts. The current study sought to examine a group of novice algebra learners' interactions with different representations from an onto-semiotic approach. A case study method was employed to understand how different algebraic practices (abstracting, generalizing, justifying, and operating on symbols) and functional thinking types (recursive, covariational, and correspondence) were facilitated through working with multiple representations. Three 6th graders participated in the study by completing 12 algebra tasks and taking part in two interviews. The onto-semiotic approach guided the data analysis process that involved the identification of mathematical objects that emerged in the participating students' mathematical practices. Then, the configuration of objects and semiotic functions established by the students in the functional situations was examined to understand the role of representations in the students' development of algebraic thinking and practices. Findings showed that abstraction is an essential process for generalization. Thinking about far figures facilitated abstraction and generalization through helping students construct non-ostensive concrete/pictorial representations. Verbal representations interacted with all representations and preceded symbolic representations. Working with near figures promoted recursive and covariational thinking while examining the far figures usually resulted in correspondence thinking. Implications for the school curriculum are discussed in the paper.

Keywords Algebraic practices · Functional thinking · Novice algebra learners · Representations · Onto-semiotic approach

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1 Introduction

Students' interactions with multiple representations give us important clues regarding what and how mathematical meanings are constructed in a particular mathematical activity (Font et al., 2007). Using representations is related to modeling, understanding, connecting, and meaning-making processes (Radford, 2014a). A relatively recent framework that can be used to examine students' meaning-making processes with different representations is the onto-semiotic approach to mathematical cognition (Font et al., 2007; Godino et al., 2007). Algebra, a domain of mathematics viewed as a gatekeeper to higher levels of mathematics learning (Kaput, 2008; Wu, 2001), offers students opportunities to work with different representations. In this study, using the onto-semiotic approach as the theoretical framework, we examined how a group of first-time formal algebra learners interacted with multiple representations.

2 Theoretical background

2.1 Algebraic thinking and content strands

According to Kaput (2008), algebraic thinking involves two core aspects: (A) generalizing regularities and constraints and representing these generalizations; (B) reasoning with and acting on generalizations expressed in a conventional system. Related to core aspect A, Wu (2001) discussed abstraction in addition to generalization, noting that they are closely linked processes. Abstraction reifies similarities underlying mathematical structures as it is the process of conceptualizing the essence of particulars/examples by focusing on their similarities rather than differences, while generalization involves building properties or relationships that work for all cases under investigation (Mitchelmore, 2002; White & Mitchelmore, 2010). Radford (2003) examined students' work with figural patterns and found that they abstracted features of ostensive figural patterns and the actions performed on them to reach a generalization. Radford (2014b), however, cautions that not all acts of generalizing involve algebraic thinking (e.g., finding a formula by using the trial and guess strategy). Core aspect B involves applying rule-based actions on symbols and knowing how different forms of symbols relate to each other. Radford's (2014b) definition of analyticity is closely linked to core aspect B. Analyticity refers to treating unknown quantities as if they were known and starting with the unknown quantity when performing four operations on algebraic equations. Justifying is a common skill for both core aspects A and B, as students need to explain the reasons behind their mathematical actions when engaged in algebraic thinking (Blanton et al., 2018). The current study derived four mathematical practices from the related literature on algebraic thinking (Blanton et al., 2018; Kaput, 2008; Mitchelmore, 2002; Radford, 2014b; Wu, 2001): abstracting, generalizing, justifying, and operating on symbols.

Kaput (2008) defined three content strands of algebra: (1) study of structures and systems; (2) study of functional relationships; and (3) application of modeling. The first strand is based on generating expressions, equalities, and inequalities; working flexibly with the equal sign; and generalizing arithmetic relationships (Blanton et al., 2018; Eriksson & Sumpter, 2021). The second strand refers to generalizing functional relations between co-varying quantities. The third strand includes modeling a given situation with equality or inequality; generalizing regularities that arise within or outside mathematics; and comparing different models to each other.

The current study included teaching tasks/situations based on the three strands defined by Kaput (2008) with a focus on the second strand, the study of functional relationships. The reason for this choice was the flexibility of the functional relationships to offer opportunities to study the other strands as well. Pattern tasks/situations were used to introduce a formal study of algebra to the students. Studying patterns offers students opportunities to make abstractions and generalizations (English & Warren, 1998; Radford, 2003; Rivera, 2010). In a typical pattern situation, students analyze special cases, compare how they are similar or different, organize data collected from the cases, generate a rule that could be applied to all cases (relates to the third strand), and then use the rule to solve the initial and/or further questions (Stacey, 1989). In working with patterns, students also practice using symbols in meaningful ways such as examining the equality of different expressions derived from the same pattern, thus engaging in the first strand as well.

A functional relation expresses the dependent variable in terms of the independent variable using a representational tool (Pittalis & Zacharias, 2019). Prior research studies have shown that students demonstrate three types of thinking when working with functional relations: (1) recursive thinking; (2) covariational thinking; and (3) correspondence thinking (Blanton & Kaput, 2011; Stephens et al., 2017). Recursive thinking focuses on change in one variable (dependent or independent) without relating the two variables to each other (e.g., y values increase by 1). Covariational thinking involves examining how the two variables vary simultaneously (e.g., as y increases by 1, x increases by 4). Correspondence thinking refers to generating a closed-form rule that directly relates one variable to the other variable (e.g., y is 2 times x plus 1). In the study of functional relationships, typically, students use a variable as a varying quantity (Blanton et al., 2011; Usiskin, 1988).

Students might use a variety of representations to explore functional relations (Iori, 2017; Schwartz & Yerushalmy, 1992; Wilkie, 2016). Palatnik and Koichu (2017) examined ninth-grade students' sense making while working with a numerical sequence. The students first related pictorial and numerical representations to each other and then constructed algebraic representations. Using a survey method, Wilkie (2016) examined 102 upper primary students' functional thinking across different representations and found that students were able to generalize linear relationships using symbolic representations when the question was based on a figural growing pattern (23%), a pictorial and a tabular representation (25%), and a real-life related verbal representation (46%). The prior studies reported students' successes and difficulties in using different representations. To promote conceptual learning of algebra at all grade levels, further investigation of how students learn initial formal algebra concepts using multiple representations is warranted (Wilkie, 2016, 2021). In particular, there is a need to clarify the types of functional thinking that different representations can support and how these relate to algebraic practices.

The current study aims to contribute to the existing knowledge on multiple representations by conducting an in-depth analysis of a group of novice formal algebra learners' experiences in functional relation situations using figural patterns and their engagement in algebraic practices of abstracting, generalizing, justifying, and operating on symbols. The onto-semiotic approach (OSA) provides rich analytic tools to examine these processes, as will be explained in the next section. Research in line with the theory of objectification revealed how students engage in these algebraic practices, in particular in abstracting and generalizing (Radford, 2003). In research studies that examined algebraic thinking from an onto-semiotic perspective, generalization has been at the center of the analysis (e.g., Aké et al., 2013; Godino et al., 2015). The OSA lacks explicit theorization of abstraction. However, it provides tools (e.g., ostensive/non-ostensive and extensive/intensive dualities) that can be used to make elements of the abstraction process visible (Font et al., 2013). Thus, the current study uses the OSA to understand the role of algebraic practices as students' individual and group knowledge emerge in social practice. We hope that this new application of the OSA will allow a wider application for this theory in mathematics education and thus a deeper understanding of students' algebraic practices for educators.

2.2 The onto-semiotic approach (OSA)

The OSA draws on sociocultural theories and views mathematics as a logically organized system that has its own symbolic language and is based on collaborative problem solving (Font et al., 2013; Godino et al., 2007). Mathematical practice is defined as any type of action or representation used by someone in the process of solving mathematical problems, communicating the solution to others, justifying the answer, and generalizing it to other situations or problems. The OSA broadly defines mathematical objects as any entity that emerges from mathematical practice and introduces the following primary objects (Font et al., 2013; Godino et al., 2007; Montiel et al., 2009):

- *Linguistic elements*: words, expressions, notations, or graphs expressed through speaking, writing, or gestures. In this study, material elements (e.g., algebra tiles) are also included in this category (Godino et al., 2011).
- *Situations*: problems, exercises, tasks, or applications.
- *Concepts*: mathematical notions (e.g., slope) given with a definition or description.
- *Propositions*: statements about the properties of concepts.
- *Procedures*: algorithms, calculations, or techniques used to solve the problem.
- *Arguments*: statements used to justify or refute propositions or procedures.

These primary objects are related to each other and may come together to form configurations. A configuration is a network of mathematical objects that connect to each other through relationships formed among them. For example, a student can justify (*argument*) a formula (*proposition*) of a functional relation (*concept*) using geometric figures and words (*linguistic element*), forming a configuration about the functional relation.

To better understand how mathematical objects exist, the OSA examines these objects from five dual facets (Font et al., 2013; Godino et al., 2005; Vergel et al., 2021). Below, we explain each facet and make connections to the current study where appropriate:

- *Personal/institutional:* Personal objects are the results of individuals' mathematical practices while institutional objects emerge from a practice community (institution) that involves dialogue, argumentation, and agreements. The algebra content that the teacher in the current study aimed to teach constituted the institutional objects while the students' individual conceptions of this content generated personal objects.
- *Ostensive/non-ostensive:* Ostensive objects are shown with material representations (e.g., a table showing a linear relationship), while non-ostensive objects are mental processes or rules employed in institutional objects (e.g., non-existence of the multiplication sign in 2y).

- *Extensive/intensive:* An extensive object is a specific case (example) of a more general form which is an intensive (type) object. The algebraic practices of abstracting and generalizing are closely related to this dual facet. Generalization is the process of generating an intensive object (Montiel et al., 2009). In figural patterns, the process of abstracting might support generalization by identifying similarities and features in the given extensive objects.
- *Unitary/systemic*: Unitary objects are treated as a whole system that was learned previously while systemic objects can be decomposed into parts to be studied.
- *Expression/content:* Mathematical objects form semiotic functions in which one object (expression) represents another object (content). This dual facet has particular importance for the current study as the study focuses on representations. The OSA disagrees with the opinion that mathematical objects exist apart from their different representations (Font et al., 2013). Rather, the argument is that mathematical objects are related to each other through semiotic functions (Font et al., 2007). A semiotic function is a correspondence between a signifier (expression) and a signified (meaning, content). Each object/representation pair allows people to use different systems of practices, leading to a new meaning of the object. The richness of semiotic functions established between mathematical objects indicates the richness of meanings created by the subjects (Font et al., 2007; Radford, 2014a).

Learning to think algebraically requires students' appropriation of institutional algebraic objects. The mathematical objects and dual facets that OSA proposes are useful to understand and analyze students' algebraic practices. Based on these theoretical tools, Aké et al. (2013) defined the algebraization levels given in Table 1 for mathematical practices in primary education. This categorization of algebraic thinking is used in the current study to understand the algebraic thinking levels of the participants.

The current study embraces the OSA to examine the mathematical practices of novice algebra learners as the OSA perspective brings together the individual and social aspects of mathematical activity and provides rich theoretical tools such as mathematical objects and dual facets to examine students' algebraic thinking (Aké et al., 2013; Godino et al., 2015) and meaning-making processes when using representations. The research question that guided this study is as follows:

How are algebraic practices and functional thinking facilitated through working with different representations in functional relation situations for a group of novice formal algebra learners from an onto-semiotic approach?

Table 1 Algebraization levels

Level	Description
0	A mathematical activity at this level does not include algebraic features. No intensive object is involved
1	This level includes intensive objects (e.g., Fig. 100 includes $1+2+3+\ldots+100$ beads); however, the generalization is based on the objects at the concrete level. It resembles factual generalization defined by Radford (2003)
2	This level involves intensive objects expressed in algebraic symbols; however, operations with variables are not carried out
3	This level is considered properly algebraic. It involves intensive objects expressed in algebraic symbols, and operations with variables are carried out. This level requires analyticity defined by Radford (2014b)

3 Methodology

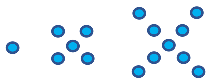
This study employed a case study approach using qualitative data. The case study approach is suitable when researchers are interested in understanding and interpreting the processes involved in a bounded system/case (Merriam, 1998). In the current research, we aimed to understand how a group of first-time formal algebra learners used different representations as they engaged in algebra tasks/situations. This group was the case of the study. With "formal algebra," we mean a systematic study of symbolic expressions and equations that typically take place in secondary schools in many countries (Stephens et al., 2017; Wilkie, 2016).

3.1 Participants and context

Three sixth graders voluntarily participated in the study after obtaining child assent and parental consent forms. The study took place during the students' winter break. The students (three boys) were in the same class and were attending a public middle school in southwestern Turkiye. Participants' pseudonym names used throughout this paper are Arda, Baha, and Emir. Arda's average mathematics test score in the first term of the sixth grade was 88 out of 100. This average was 94 for Baha and 85 for Emir. These scores were based on their school report cards managed by their mathematics teacher.

As part of this study, the participants attended five algebra lessons in 2 weeks. Each lesson was video tape-recorded and lasted between 1 and 1.5 hr. They engaged in 12 algebra situations in these five lessons. None of the participants had learned formal algebra prior to these lessons. Figure 1 presents situation 3 as an example of the functional relation tasks used in the study, and Table 2 shows brief information about the situations used. Based on the related literature (Blanton & Kaput, 2011; English & Warren, 1998; Radford, 2014b; Stacey, 1989), the instructional sequence started with figural patterns and introduced variables as varying quantities. The worksheet questions were staged to go from simpler to more difficult similar to algebraization levels (Aké et al., 2013). In lessons 3 and 4, the students worked with situations that involved

Examine the growing letter X built by your teacher.



- Construct the fourth and fifth figures using the color chips.
- How many chips will there be in the 10th figure?
- How many chips will there be in the 100th figure?Compare your answer to those of your friends.
- Explain how you would find the number of chips in any step to another friend. The answer to this question will be written collaboratively.
- Write a mathematical sentence that shows how to calculate the number of chips in any figure.

Fig. 1 Situation 3: The Growing Letter X

Table 2 The situations used in the study

Lesson/situation number	Situation title	Algebra content strand	The representation used to introduce the situation
Lesson 1			
Situation 1	The Growing Letter T	Study of functional relationships (SFR)	Manipulatives: two-color chips
Situation 2	Growing Rectangles	SFR	Figural drawings
Situation 3	The Growing Letter X	SFR	Manipulatives: two-color chips
Situation 4	Create Your Own Pattern	SFR	Students created their own patterns using figural drawings and/or chips
Lesson 2			
Situation 5	A Growing Pattern	SFR	Figural drawings
Situation 6	Toothpick Squares	SFR	Manipulatives: toothpicks
Situation 7	Tiles around the Pool	SFR	Figural drawings
Lesson 3			
Situation 8	Model a Given Situation	Application of modeling	Manipulatives: algebra tiles and envelopes
Situation 9	Make a Rectangle	Study of structures and systems (SSS)	Manipulatives: algebra tiles
Situation 10	Exchange Tiles	SSS	Manipulatives: algebra tiles
Lesson 4			
Situation 11	Writing Stories and Algebraic Equations	SSS	Verbal representations and symbolic representations
Lesson 5			
Situation 12	Block Patterns	SFR	Virtual manipulative (created by Utah University, involved several patterns)

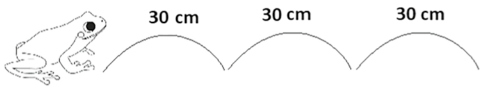
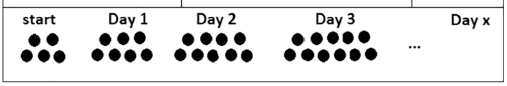
variables as fixed unknowns. At this phase of the instruction, the students were asked to perform addition and subtraction operations on algebraic expressions expressed in symbols. The last lesson engaged the students in functional relations tasks.

The first author taught the lessons. Her role in this study was a researcher-teacher (Tabach, 2011). On the one hand, she was the teacher of the research lessons; she tried to build a learning environment where the students shared ideas with each other and challenged and contributed to each other's thinking. She facilitated the emergence of personal and group knowledge and introduced institutional knowledge. On the other hand, she observed the participants' learning process closely and collected data during the lessons from a researcher's point of view.

3.2 Data sources

The data sources included videotapes of the five lessons, the students' worksheets completed during the lessons, a mid-interview with each participant after the second lesson, and a final interview with each participant after all the lessons were completed. Each interview lasted between 10 and 20 min. In these interviews, our goal was to understand the students' emerging personal knowledge about functional relation situations and how they used different representations during this process. Accordingly, we posed questions that would reveal students' functional thinking when using different representations. The mid-interview included two questions and the final interview included four questions. Table 3 shows sample questions from each interview.

Table 3 Sample questions from the mid- and final interviews

<table>
<tr><td>Mid-interview question 1</td><td>A frog jumps 30 centimeters in each step. How many centimeters will it jump in n steps?

</td></tr>
<tr><td>Mid-interview question 2</td><td>Use an applet to analyze a pattern (Chairs around a Table, created by the National Council of Teachers of Mathematics).</td></tr>
<tr><td>Final interview question 3</td><td>How do the following expressions relate to each other?
Ali had 5 liras in his piggy bank. He adds 2 liras to the piggy bank every day. How much money will there be in the piggy bank after x days?
<table><tr><th>x</th><th>y</th></tr><tr><td>0</td><td>5</td></tr><tr><td>1</td><td>7</td></tr><tr><td>2</td><td>9</td></tr><tr><td>3</td><td>11</td></tr></table>
y=5+2.x

</td></tr>
</table>

3.3 Data analysis

The onto-semiotic approach (Font et al., 2007; Godino et al., 2007) guided the data analysis process. Firstly, all lesson videos were watched, and the mathematical objects included in each situation were determined. For each situation, a detailed report was created based on the first view of the videos, making 12 reports in total. The worksheets that the students completed were used to confirm the identification of the mathematical objects in the teaching episodes. Like the lesson videos, the videos of the student interviews were analyzed by identifying the mathematical objects generated by each student. The general format of the reports and sample entries for each section is given in Table 4 for The Growing Letter X situation shared in Fig. 1.

Secondly, all lesson videos were watched again. In this second round of analysis, the focus was on understanding the configuration of objects and semiotic functions established by the students in functional relation situations. Particular attention was paid to instances where four algebraic practices (generalizing, abstracting, justifying, and operating on symbols) and three thinking types involved in functional relationships (recursive thinking, covariational thinking, correspondence thinking) took place. Additionally, the levels of algebraization of the mathematical activity carried out by each student were identified using the framework proposed by Aké et al. (2013). The role of representations in algebraic thinking was shown in diagrams. This process involved the situation reports created in the first step. The diagram of the Growing Rectangles situation is given in Fig. 2 as an example.

Finally, all the diagrams were compared to each other to identify commonalities across the situations used. Common themes about how the representations were used by the students in the meaning-making process of algebraic situations were identified. Data from the individual interviews and students' worksheets were used to triangulate the conclusions.

Table 4 Part of a sample situation report

Report section	Sample entries
Situation	Growing Letter: The linear growth of the letter X is presented with one chip in step 1, 5 chips in step 2, 9 chips in step 3, and so on
Linguistic elements	Color chips, gestures, spoken language, written language, symbols (letters, equal sign), numbers
Concepts	Pattern-linear growth, near figure, far figure, variable as varying quantity, equality, functional relationship
Procedures	Procedure A: Constructed the fourth figure by placing 1 chip at the center and 3 chips to each of the four parts of X (Baha) Procedure B: Constructed the fourth figure by placing 1 chip at the center and then 1 chip to every four parts of X. Placed 3 groups of 4 chips (Emir)
Propositions	Proposition A: The number of chips in the 10th figure is 37 (all students)
Arguments	Argument A1: Proposition A is correct because of commonalities in figures 2 and 3. Figure 2 has $1\times4+1$ chips, figure 3. has $2\times4+1$ chips so figure 10 will have $9\times4+1$ chips. (Arda) Argument A2: Proposition A is correct because of rhythmic counting up to the 10th figure (1, 5, 9, …, 37). (Emir)

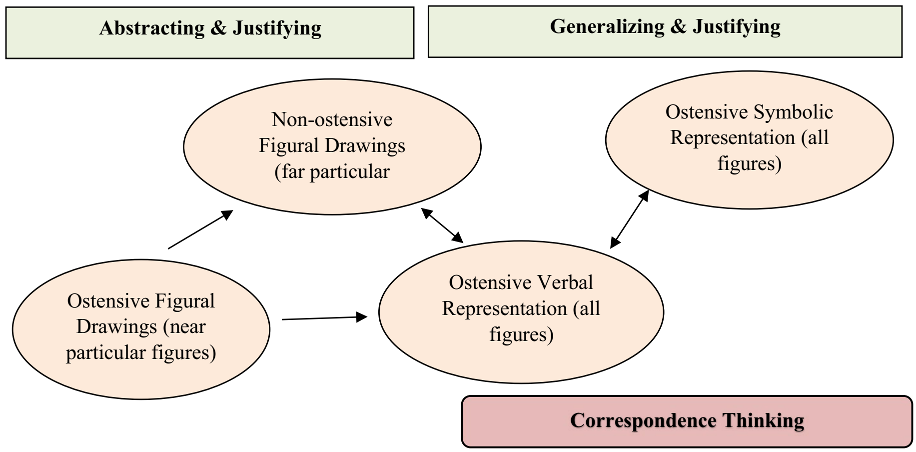


Fig. 2 A sample diagram for students' use of representations in functional relation situations

4 Findings

In this section, we first present the findings related to what role each representation played in the emergence of algebraic practices: abstracting, generalizing, justifying, and operating on symbols. Then, we share the findings on how each representation supported students' functional thinking. Figure 3 shows the overall interaction among the different forms of representations and algebraic practices as well as the functional thinking observed.

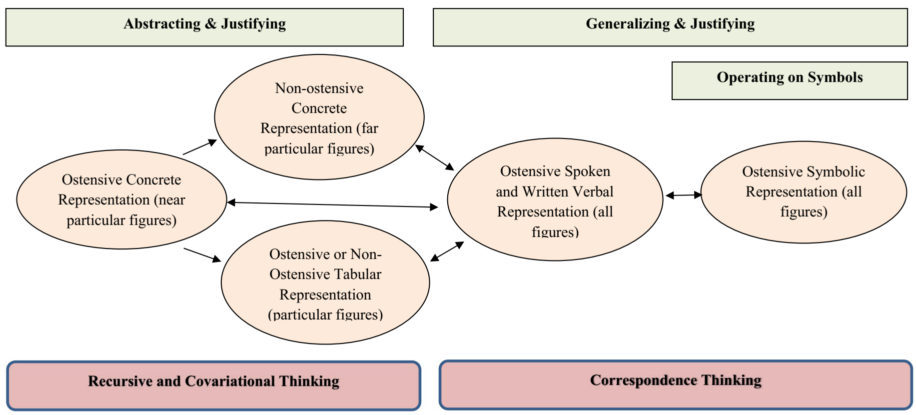


Fig. 3 The role of representations in algebraic practices and functional thinking

4.1 Representations and algebraic practices

In this study, functional relationship situations were presented to students in the form of figural patterns using pictorial, digital pictorial, or concrete representations. They were given the first three figures and were asked to create the fourth and fifth figures. These near figures helped students to notice the structure of the pattern. For example, for situation 3 shown in Fig. 1, Arda explained the geometric configuration of the pattern by saying “The fifth figure has 4 chips in each corner [arm/part of the letter X]. In each step, we add one chip to each corner.” Here, Arda formed a semiotic function between the geometric formation of the pattern and the verbal expression he used. He signified the geometric structure of the fifth figure with “…has 4 chips in each corner” and then referred to his physical action with chips (a procedure), “In each step, we add one chip to each corner,” noticing a regularity in the growth of the pattern.

The students started noticing similarities and differences in the near figures such as how the figure grows from one step to another as Emir said, “It goes by 3.” for situation 6—Toothpick Squares. In situation 2—Growing Rectangles (Fig. 4), while drawing the fourth and fifth figures, the students discussed that in each figure, the height is 2 more than the base of the rectangle. Emir said, “The fourth figure is 4 by 6, it [pointing to height] is 2 more.” In all situations, the ostensive concrete, pictorial, and digital pictorial representations initiated the abstraction process for the students. This initial abstracting included noticing the geometric configuration of the figural patterns (features of the extensive objects expressed in linguistic terms) and the regularity in the growth of the pattern (procedures performed on the extensive objects).

After creating and examining the near figures, the students answered questions about far figures. For instance, in situation 1, the students examined the pattern of The Growing Letter T. Figure 5 shows the first three steps of the pattern. Here is the conversation from

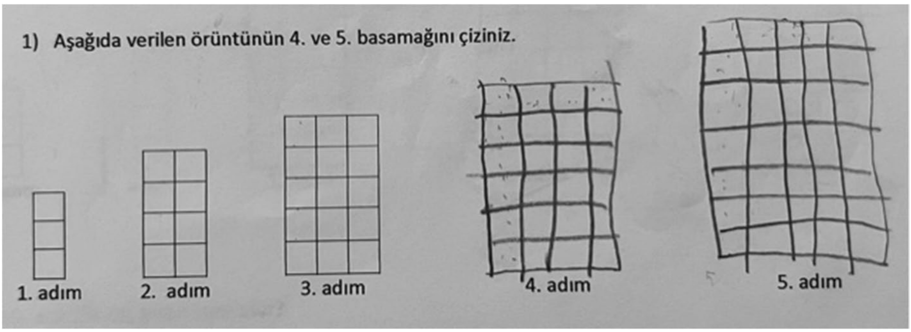
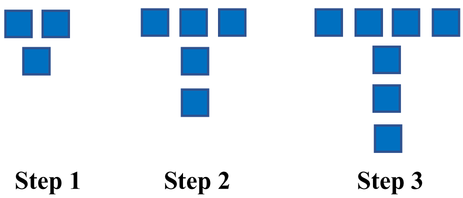


Fig. 4 Near extensive objects drawn by Emir for situation 2

Fig. 5 Situation 1: The Growing Letter T



the teaching episode in which the students shared their strategies for finding the number of tiles in figure 10:

Teacher: How did you guys find the number of tiles in the 10th figure? Who wants to share?
Baha: I did like this, the bottom [tail of the letter T] is always whatever the step number is. The top part is one more.
Teacher: How did you find 21?
Baha: [pointing to the fourth figure] there are 10 at the bottom, 11 at the top, 21.

By abstracting the structure of the letter T from the near figures and connecting this structure to the step number, Baha was able to find a general rule. He used a near particular figure, the fourth figure, as a mediating tool to justify his general rule. In other words, the fourth figure signified the 10th figure. As Fig. 3 indicates, the students referred to the ostensive or non-ostensive concrete representations when they expressed their justifications using verbal representations.

The teacher invited other students to share their reasoning:

Teacher: What was your reasoning, Emir?
Emir: I got 27. (He constructed an ostensive table up to the 10th figure on his worksheet.)
Teacher: How did you find 27?
Emir: I increased by 2 but I think I got wrong.

Arda interfered and counted to the 10th figure by constructing a non-ostensive table (he counted like "6–13; 7–15; 8–17; 9–19; 10–21). The group agreed that there are 21 tiles in the 10th figure. Emir claimed that there should be 210 tiles in the 100th figure as $21 \times 10 = 210$, demonstrating incorrect proportional reasoning. The lesson continued as follows:

Teacher: Let's examine the second and fourth figures. How many tiles are there in the second figure?
Baha: 5
Teacher: How many are there in the fourth figure?
Emir: 9
Teacher: According to your reasoning of multiplying by 2, it should be 10. Are there 10 tiles in the fourth figure?
Baha: So, the rule is 2 times minus 1.
Teacher: We cannot generalize based on one example (explained that a rule should work for all steps) … I want each of you to think about the shape of the 10th figure. Can you imagine it? What will it look like?
Emir: It will be a large T (they laughed).
Teacher: Yes, but how many tiles will there be on each part (tail and top)?
Arda: Oh, I see, the top is one more than the bottom, so this will be 100 and this will be 101 (drew the figure in the air), 201.

Imagining the 10th figure helped Arda notice the geometric configuration of the pattern. The figure he drew in the air signified a generic element of the pattern (intensive object) by indicating a semiotic function between his gesture and the rule "the top is one more than the bottom." Moreover, Arda and Baha noticed that the number of tiles in the tail of the letter T was the same as the step number. This is a new phase of the abstracting process as it not only focuses on similarities and regularities among the extensive objects but also involves relating each extensive object to its order in the sequence. This kind of abstracting helped Arda and Baha to construct the general rule. The teacher asked about the 500th figure to reinforce generalization as it requires using their intensive object in a new case. Arda and Baha were able to find the answer using the intensive object that they generated. As this intensive object is bound to the concrete level, we can claim that at this point of the instructional sequence, both students' mathematical practice is at level 1 of algebraization (Aké et al., 2013). On the other hand, Emir did not abstract the features of the extensive objects yet, so he was not able to generalize. At this point of the instruction, his mathematical practice seemed to be at level 0 of algebraization.

The teacher worked one-on-one with Emir starting with re-constructing the third step using two colors, blue tiles for the tail, and red tiles for the top:

> Teacher: Let's try to imagine the shape.
> Emir: It will be a T.
> Teacher: Try to guess the number of tiles at each part. [Silence] …
> Teacher: Let's examine (the figures) in their order. The first figure has 1 tile at the tail and 2 tiles at the top (pointing to the figure). The second figure has…How many tiles are there at the tail of the fourth figure?
> Emir: 4
> Teacher: What about the top?
> Emir: 5
> Teacher: The fifth figure?
> Emir: 5 at the tail, 6 at the top.
> Teacher: 10th?
> Emir: 10 and 11.

Next, Emir was able to calculate the number of tiles for figures 100 and 500, noting that the top is one more than the bottom. In the next question, when the group was asked to explain how to find the number of tiles in any step to another friend, Arda and Baha suggested telling the other student the rule and using the 500th figure as an example. Emir suggested, "First, we should show him the first five figures." It seems like examining the near figures in detail using different colors and relating the figure's parts to each other and to the step number helped Emir abstract the features of the extensive objects and procedures performed on them, an essential process for generalization as indicated in Fig. 3. Once the students observed the structure of the pattern and discerned the similarities in each figure, they discussed the structure of far figures more flexibly using the near figures as a signifier. This is evident in Baha's reasoning above for the 10th figure and Arda's reasoning regarding the 100th figure. Similar reasoning was observed in other patterning situations as well.

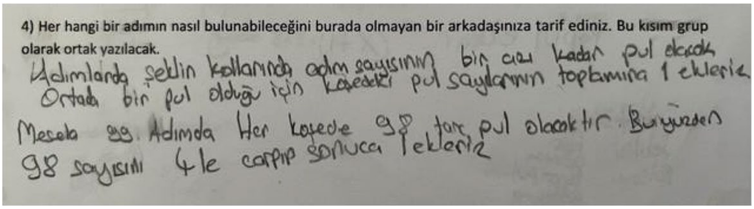
In the first situation, Emir and Arda created a table even though the worksheet did not ask for it (some other situations required completing a table). In situation 2, all students were able to demonstrate abstract thinking based on the pictorial representation of the pattern and generalize a rule. None of the students created a table. In other situations, whenever the students were able to abstract the structure of the pattern, they did not need a table. However, if they failed to abstract the features of the extensive objects, they made an ostensive or non-ostensive table. For example, in situation 3 (see Fig. 1), Arda and Baha were able to engage in the abstraction process and generalize a rule:

> Arda: Ok, the 10th figure, let me give an example first. In the second step, for example, on all corners [arms of the letter X], there is one chip. And there is one chip in the middle. In other words, each corner has one, the number of chips on each corner

is one less than the step number. Therefore, in the 10th step, there will be 9 chips on each corner. Since there are four corners, we multiply 9 by 4, 36, and since there is one chip in the middle we add one more chip, 37.

Emir, on the other hand, created a non-ostensive table and found the number of tiles in the 10th figure by counting on 9. During the group discussion, the students drew a draft picture for the 100th figure (ostensive pictorial representation). They collaboratively described the pattern in writing to a friend who was not there (Fig. 6). This description shows that the group justified their intensive object (signified by the 99th term) by referring to the features noticed in the extensive objects of situation 3 such as each figure has one chip in the middle. Discussions on far figures and features of extensive objects helped Emir notice the general rule of the pattern, transferring personal knowledge to group knowledge. As illustrated in Fig. 3, the progression from the ostensive and non-ostensive concrete/pictorial representation to verbal representation and from there to symbolic representation aligned with students' abstracting and generalizing thinking processes. A draft ostensive pictorial representation of a far figure and verbal representations seemed to facilitate writing symbolic representations. Every time the students found a pattern, they first said it in words and then showed it with symbols.

Students' justifications depended on the type of representation that they used and progressed from being personal to institutional as the teacher scaffolded their algebraic progression. With the pictorial and concrete representations, the students defended their arguments using the near figures as the excerpts above illustrated. At times, they used gestures and mimics to explain their ideas such as drawing a figure in the air by hand to explain its parts and the number of items (e.g., chips) in each part. When working with tables, students initially suggested rules that work for only one pair but in time they defended their rules based on more instances of the pattern such as "I checked it, it works for all pairs [on the table]." (Baha, situation 7).


4) Her hangi bir adımın nasıl bulunabileceğini burada olmayan bir arkadaşınıza tarif ediniz. Bu kısım grup olarak ortak yazılacak.

Adımlarda şeklin kollarında adım sayısının bir az kadar pul olacak
Ortada bir pul olduğu için köşedeki pul sayılarının toplamına 1 ekleriz
Mesela 99. Adımda Her köşede 98 tane pul olacaktır. Bu yüzden
98 sayısını 4le çarpıp sonuca 1 ekleriz

4) Explain how you would find the number of chips in any step to another friend. The answer to this question will be written collaboratively.

Students' answer: In each step, each arm of the figure will have chips one less than the step number. Since there is one chip in the middle, we add 1 to the total number of chips at the corners.

For example, 99th Step Each corner will have 98 chips. Therefore we multiply 98 by 4 and then add 1.

Fig. 6 Group's written explanation of the pattern in situation 3

Table 5 Examples of students' work with symbols

Student	Situation 3	Situation 5
Arda and Baha	(x-1).4 + 1 = y	x + 1 + x + 2 = y 2x + 3 = y
Emir	x.4 + 1 = y	x + 1 = y + 1 = z y + z = e

The students' work with symbols in pattern situations showed variety. They were introduced to using letters to write a general formula (institutional knowledge) in situation 1 by the teacher. In the latter situations, they were asked to write a rule using letters after they examined the near and far figures. Baha and Arda tended to convert the verbal representation to the symbolic representation directly in the same order focusing on the verbal and contextual meaning (Godino et al., 2015; Radford & Puig, 2007). For example, in situation 3, they spoke about the rule as "one less than the step number is multiplied by four and then add the one chip in the middle." Both students wrote the rule as $(x\text{-}1).4 + 1 = y$ (in Turkiye, a dot is used to denote multiplication). This type of generalization in functional situations would fall under level 2 of algebraization (Aké et al., 2013). Starting with situation 5, the teacher facilitated operations with symbols. In situation 5, the rule was found to be $x + 1 + x + 2 = y$. When the teacher asked if the rule could be written differently, Arda suggested adding similar terms to obtain 2x + 3 = y. Working with algebra tiles in lesson 3 supported students' addition and subtraction operations with symbols. Later, in some situations, Arda and Baha wrote more institutional known forms of formulas such as 2x + 1 = y instead of x.2 + 1 = y. They were able to operate on symbols independent of their contextual meanings. The two students' practice seemed to progress toward level 3 of algebraization in the context of linear functional relationships. Emir demonstrated a different developmental path. Table 4 compares Emir's work with symbols to those of Baha and Arda. Emir tended to use a different letter for different parts of the figural patterns, failing to abstract similarities and essential features of extensive objects and to relate the step number to the number of tiles/chips in the pattern. For example, in situation 3, he wrote x.4 + 1 = y. This rule related the number of chips in one arm of the letter X to the total number of chips in the letter X and does not involve the step number. As Table 5 shows, his work involved an incorrect use of the equal sign. This seemed to arise from his work in arithmetic as he thought that writing 5 + 3 = 8 + 1 = 9 was mathematically correct. As the lessons continued, at times, Emir correctly used the symbols to represent the formulas of the patterns. For example, in the mid-interview, he correctly wrote x.2 + 2 = y for the chairs around a table question. In question 3 of the final interview, he noticed that all boxes were different representations of the same situation. He showed gradual development throughout the lessons and showed numerical focus in some tasks (level 1 of algebraization) while representing the intensive object with symbols in some other tasks (level 2 of algebraization).

4.2 Representations and functional thinking

As Fig. 3 shows, concrete, pictorial, digital pictorial, and tabular representations typically initiated recursive and covariational thinking. Once students worked with far figures either using non-ostensive pictorial/concrete representations or tables and they expressed their opinions in words (verbal representations), they demonstrated correspondence thinking. An example will be shared from situation 6—Toothpick Squares. In this situation, the students were given the first three steps of the pattern created by toothpicks (Fig. 7) and were

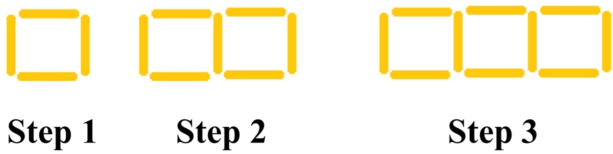


Fig. 7 Situation 6: Toothpick Squares

asked to create the fourth and fifth figures using toothpicks. Then, they were asked to find the number of toothpicks in the 10th and 100th figures. The situation also required them to complete a table as they worked with the near and far figures.

As the students were working on the near figures, they quickly noticed that the total number of toothpicks increased by 3, showing recursive thinking. Once the students individually completed the questions for the far figures, the teacher started a group discussion.

> Teacher: Let's share our answers. How did you find the answer for the 10th figure, Emir?
> Arda: I could not find an answer for 100.
> Emir: In each step, the number of toothpicks is 3 times the step number plus 1. Therefore, I multiplied 10 by 3, 30, and added 1, 31.
> Baha: I did like that as well.
> Teacher: How did you find this 3 times plus 1 rule?
> Emir: It works for all of them [pointing to his table].
> Teacher: Did you try it on the table?
> Emir: Yes.
> Teacher: Hmm, I see, you found it by trial and error. What about you, Baha?
> Baha: I did the same. I looked at the table to see if it works for all of them and thought it would.

Emir and Baha were able to find a rule that works for all the pairs on the table. The semiotic function they formed was between the verbal rule "the number of toothpicks is 3 times the step number plus 1" and the number pairs of the table. The structure of the figure was not part of the configuration of objects at this point. After the group discussion, Arda used this rule to find the number of toothpicks for the 100th figure. The students' work with the rule shows evidence of the emergence of correspondence thinking in the whole group as they were relating the step number directly to the number of toothpicks in the figure. Nevertheless, they were not able to justify their formula other than using the trial-and-error strategy. The teacher asked for a justification based on the structure of the figural pattern:

> Teacher: Can you guys think of a reason to justify the 3 times plus 1 rule by examining the shape of the figures as we did in previous tasks?
> Arda: It increases by 3 in each figure.
> Teacher: The number of toothpicks increases by 3 as the step number increases by 1, correct. How can we relate the step number to the number of toothpicks?
> Baha: For example, in the fourth step, there are 13 toothpicks.
> Teacher: Why is that so?
> Emir: There are 3 squares.
> Arda: Oh, the toothpicks go like Cs, in threes. There are [groups of] 3 toothpicks as many times as the step number. [pointing to the fourth figure] Here, there are 4 Cs plus one more [See Fig. 8 for this idea].

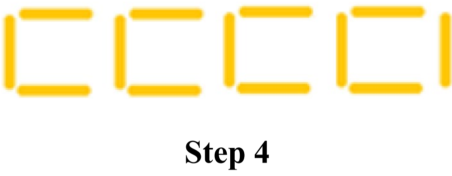


Fig. 8 Arda's reasoning on Toothpick Squares pattern

Emir re-explained Arda's reasoning using the third step as an example. With this elaboration, the students were able to defend the rule using the structure of the figural pattern, extending the configuration of objects they formed for this pattern.

The above episode was a typical illustration of the students' work with patterning situations. The near figures allowed them to experience and manifest recursive and covariational thinking. Thinking about far figures promoted correspondence thinking in two ways. In some situations, they were able to comprehend the structure of the pattern and created a rule of the pattern, showing evidence of abstracting, generalizing, and correspondence thinking. This comprehension seemed to result from non-ostensive concrete/pictorial representations formed by the students. In some other situations, they created a table and derived a rule based on the table using the trial-and-error approach. This latter approach misses the abstracting process and may not be considered algebraic thinking (Radford, 2014b). In the current study, the teacher always asked students to examine the structure of the pattern to promote students' algebraic thinking. In each of these two ways, the verbal representations helped students organize their thinking and translate their ideas into symbolic representations as Fig. 3 presents.

5 Discussion

In this study, we used the lens of the OSA to examine how a group of first-time formal algebra learners used different representations as they engaged in algebraic practices and functional thinking. One primary finding is the link between children's abstraction of the concepts studied as an essential process for generalization. The participating students' abstracting processes included identifying two main features of the extensive objects:

1. Noticing the geometric configuration of the figural pattern and the regularity in the growth of the pattern.
2. Noticing a relationship between the extensive objects and their order in the sequence.

Generalizing a rule for the pattern was supported by abstracting these features, in line with the findings of Radford (2003).

The progression from ostensive concrete, non-ostensive concrete, verbal to symbolic aligned students' abstracting and generalizing practices. Imagining the far figures, sketching a draft of a far figure, and reasoning about the structure of the figural pattern facilitated both abstraction and generalization through helping students construct non-ostensive concrete/pictorial representations (schematic representation). When the students described the structure of these non-ostensive representations and signified them with a draft sketch, they were able to write their symbolic representations. Verbal representations interacted with all representations and preceded symbolic representations.

We postulate that the students formed semiotic functions among ostensive, non-ostensive, verbal, tabular, and symbolic representations of the figural patterns that they worked with, forming a configuration of objects. Aligned with previous research (Font et al., 2007; Montiel et al., 2009; Radford, 2014a, b), we found that the more semiotic functions formed by the students, the deeper understanding and algebraic thinking they developed about the patterns that they examined. The current research highlights the significance of verbal representations from a new perspective as we found that they are central to the semiotic functions formed by the students. As the students worked with different representations and engaged in abstraction and generalization processes, they continuously used verbal representations to make sense of the concepts they were learning. This finding might explain students' success with verbal representations in previous studies (Panasuk, 2010; Wilkie, 2016).

The second finding is that the symbolic representations used by the students showed variety. Two students' practices progressed from level 1 of algebraization to level 3 of algebraization (Aké et al., 2013). Operating on symbols using algebraic thinking requires students to work with unknown quantities analytically and act on variables represented by symbols as mathematical objects (Aké et al., 2013; Radford, 2014b). While two of the participants successfully used the symbols, the third student, Emir, had some challenges in using the symbolic representations flexibly. Part of his struggle seemed to stem from earlier work in arithmetic such as the incorrect use of the equal sign. Students' misconceptions in arithmetic influence their performance in algebra (Stacey & MacGregor, 1997; Warren, 2003). Emir also failed to show correspondence thinking in his use of symbols in earlier situations. He needed more opportunities to form richer semiotic functions among different representations in order to develop an institutional form of using symbols. It took more time for Emir to abstract and generalize the functional relations but got better as the research progressed, showing level 2 of algebraization in some of the situations.

The third main finding is that working with near figures promoted recursive and covariational thinking while examining the far figures usually resulted in correspondence thinking. Tables promoted trial-and-error thinking, while thinking about the structure of the figural patterns promoted analytical thinking. While we think that trial and error is an important problem-solving strategy in teaching and learning mathematics (Butts, 1985), we propose that teachers scaffold this approach by requiring their students to explain the reasons behind the formulas in completing algebraic tasks to promote analyticity (Radford, 2014b). In the current study, the participating students examined the structure of figural patterns and defended their formulas. It could be interesting to study whether examining the structure of numerical growth as given in Fig. 9 promotes correspondence thinking as well as abstracting and generalizing.

Fig. 9 A table highlighting the structure of the numerical pattern

Step Number	**Number of Tiles**
1	1=1
2	5=1+4
3	9=1+4+4
4	13=1+4+4+4

6 Implications for the school curriculum and suggestion for future research

Drawing on the OSA, this study explored the role of different representations on a group of students' algebraic practices and functional thinking processes as they engaged in algebraic situations. The implications of this study for teaching and learning mathematics are multiple:

- First, teachers should encourage students to connect different representations to each other as the more semiotic functions established by the learners, the richer understanding they tend to build.
- Second, the abstraction process helped students reach a generalization. Accordingly, teachers may ask probing questions such as how near figures are similar to or different from each other to help students abstract the features of these terms. Then, visualizing the far figures (e.g., imagining and describing the 100th figure, making a sketch of the 100th figure) may trigger the generalization phase. As most of the figural pattern tasks in the current literature miss these scaffolding questions (e.g., Mielicki et al., 2021; El Mouhayar, 2018; Wilkie, 2020), we suggest that teachers include such questions in their instruction.
- Third, verbal representations were at the center of interactions among different representations. Teachers may use this finding and scaffold students' use of various representations by asking them to explain their thinking in written and spoken language. Particularly, verbalizing the features of the extensive objects (their geometric configuration and procedures to obtain the next figure) and relating them to their order in the sequence were found to be helpful.
- Finally, different representations promoted different types of functional thinking. Non-ostensive pictorial representations and ostensive tables seemed to promote correspondence thinking, while ostensive pictorial representations and ostensive/non-ostensive tables seemed to promote recursive and covariational thinking. To promote analyticity, teachers should help students explore the structural relationships in the patterns and help them build correspondence thinking in relation to algebraic thinking. Students should be encouraged to justify any correspondence relationships that they deduce based on different strategies including trial and error.

One of the contributions of this study to mathematics education research is interpreting students' abstracting and generalizing processes using the OSA framework. As the current study included introductory lessons to the concepts of growing patterns and variables, future research examining students' abstracting and generalizing processes over a longer period may further illuminate the interaction between these key algebraic practices.

Data availability The datasets generated during and/or analyzed during the current study are available from the corresponding author upon reasonable request.

Declarations

Conflict of interest The authors declare no competing interests.

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