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Combined use of the extended theory of connections and the onto-semiotic approach to analyze mathematical connections by relating the graphs of *f* and *f′*

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Abstract

The literature reports that students have difficulties connecting different meanings, multiple representations of the derivative, and performing reversibility processes between representations of *f* and *f*'. The research goal is to analyze the mathematical connections that university students establish when solving tasks that involve the graphs of *f* and *f*' when the two functions do not have associated symbolic expressions. Seven students from the first year of undergraduate studies in mathematics from a university in southern Mexico participated. For data collection, two tasks involving the graphical context of the derivative were applied. An analysis of the mathematical activity was carried out by the participants with the analysis model proposed by the onto-semiotic approach, and thematic analysis with types of mathematical connections from the extended theory of connections was carried out to infer the connections made in that mathematical activity, which allowed us to consider the reversibility connection between the graphs of *f* and *f*' as the encapsulation of a portion of the mathematical activity. Four students establish the reversibility relationship between the graph of *f* and the graph of *f*'. It has been concluded that some students can establish the reversibility connection between the graphs of *f* and *f*', but the complexity of the mathematical activity that encapsulates the connection explains (by showing everything that the student must do) why some students are not able to establish it.

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1 Introduction

Research in mathematics education has focused on exploring mathematical connections since they are important for students to be able to understand mathematical concepts (Berry & Nyman, 2003; Eli et al., 2011; NCTM, 2000). Likewise, establishing mathematical connections is considered beneficial for students because they relate mathematical concepts, meanings, and representations to each other (intra-mathematical), and they also relate mathematics to real-life phenomena and other subjects (extra-mathematical) (Businskas, 2008; Rodríguez-Nieto et al., 2022b).

At present, the extended theory of connections (ETC) framework is focused especially on intra-mathematical connections. In this theoretical support, mathematical connections are considered "a cognitive process through which a person relates two or more ideas, concepts, definitions, theorems, procedures, representations and meanings with each other, with other disciplines or with real life" (García-García & Dolores-Flores, 2018, p. 229); and one type of included connection is called reversibility: this occurs when a student starts from concept A to obtain concept B and then inverts the process, starting from B until returning to A (García-García & Dolores-Flores, 2021). This type of connection has been analyzed in the graphic representation of a function and its derivative; a key fact is to investigate the properties of graphs that move from the graphical representation of a function g to the graph of the derivative g' and to reverse the process (García-García & Dolores-Flores, 2020). Also, this type of connection has been investigated in different topics, for example for the case of exponential and logarithmic functions (Campo-Meneses & García-García, 2020).

Also, other frameworks report the importance of focusing on the graphic of f, based on the properties of f' and vice versa. Nemirovsky and Rubin (1992) mentioned that it is difficult to relate the function f with its derivative (students graph the derivative f' similar to function f without considering the key aspects that are inferred from its graphs at different intervals). Natsheh and Karsenty (2014) recognized that some students did not sketch the graph of f based on the properties of f', due to limited visual reasoning and procedural-focused learning involving only symbolic or algebraic representations. In Fuentealba et al. (2018a), it was recognized that it is difficult to establish bidirectional relationships or reversible processes where the signs of f' and f'' are linked with f and to also relate monotony and curvature of f with the sign of its first and second derivatives. Ikram et al. (2020) mention that students are competent in solving problems about the derivative when they proceed algorithmically to find f', but they have difficulty interpreting and drawing graphs of f and f'. In this line, García-García and Dolores-Flores (2021) recognized that to graph the derivative of a given function, pre-university students necessarily require the algebraic representation associated with it; otherwise, they would not graph the derivative function.

Focused on connections and particularly on reversibility connection type, in this article, the objective is to analyze the mathematical connections made by university students when solving tasks related to the transit between the graphs of f to f' and vice versa, especially when the graphs of the two functions do not have associated symbolic expressions—as Berry and Nyman (2003) point out, in many cases, it is not possible to have such a symbolic expression.

In our case, we understand the reversibility when a student outlines the graph of *f'* taking the graph of *f* and vice versa, in tasks where there isn't a symbolic expression (neither of the function nor the derivative) and only one of the two graphs.

This directional transition between both graphics is situated into the didactical and mathematical problem proposed by Font (2000), who recognized that there are two fundamental aspects for the teaching and learning of the derivative: (a) difficulties in the understanding of the derivative at a point and on derivative function, especially the definition as the limit of the average rate of the function (and not so much, for example, in the use of derivation rules), and (b) difficulties in understanding how to calculate the derivative of basic functions where the definition of the derivative should be used as a limit (to calculate the derivative functions of trigonometric functions, it is necessary to calculate the derivative of one of them from the limit to obtain, by indirect methods, the derivative of the other functions of the family, and, for example, the same for the family of exponential and logarithmic functions).

Several research studies have reported this first difficulty, which is explained by differing points of view depending on the framework of each one, for example, APOS (Fuentealba et al., 2015, 2018b; Sánchez-Matamoros et al., 2015). On the other hand, researchers with the OSA framework report that the understanding of the notion of derivative at a point and of the derivative function is related to the activation of a complex network of semiotic functions (SFs) that allow us to understand the relationship between *f* and *f'* (Badillo, 2003; Font, 2000; Font & Contreras, 2008). Likewise, within the ETC framework, research has been carried out on necessary connections for a good understanding of the derivative. Concerning to this problem, the reversibility connection between graphical representations of a function and its derivative, on the one hand, requires a certain understanding of the derivative notion and, on the other hand, helps to develop this understanding.

Font (2000) points out that the second aspect mentioned above is related to the fact that two functions are involved in the calculation of the derivative function (*f* and *f'*) and the calculation of *f'* from *f* implies the passage of a representation from *f* to *f'*, but for some functions, a preliminary step should be considered. That is, the calculation of *f'* can be interpreted as a process in which the following must be considered: (1) treatment and/or conversions between different ostensive forms of representing *f* (representations that can be shown directly to another person), (2) the change of an ostensive representation of *f* to an ostensive representation of *f'*, and (3) treatment and/or conversions between different ostensive representations forms of *f'*. Treatments are transformations of representations that happen within the same register and conversions are transformations of representation that consist of changing a register without changing the objects being denoted (Duval, 2006). This process is specified in different techniques for calculating the derivative function in which step 1 and step 3 may not be necessary and, in others, where said steps are essential. In articulation, step 2 can be necessary to relate the graph of *f* with the graph of *f'*. In the same way, the calculation of the antiderivative implies three analogous steps, and in the second step, it may be necessary to go from the graph of *f'* to the graph of *f*.

2 Theoretical framework

In Sections 2.1 and 2.2, we synthesize the two theories considered in this investigation, and in Section 2.3, we synthesize the networking developed between both frameworks.

2.1 Onto-semiotic approach (OSA)

OSA considers that to describe mathematical activity from an institutional and personal point of view, it is essential to have in mind the objects involved in such activities and the semiotic relations between them (Font et al., 2013). Mathematical activity is modeled in terms of practices, the configuration of primary objects, and processes that are activated by practices. Mathematical practice is considered in this theory as a sequence of actions, regulated by institutionally established rules, guided toward a goal (usually solving a problem). In the OSA ontology, the term "object" is used in a broad sense to refer to any entity which is, in some way, involved in mathematical practice and can be identified as a unit. For example, when carrying out and evaluating a problem-solving practice, we can identify the use of different languages (verbal, graphic, symbolic, …). These languages are the ostensive part of a series of definitions, propositions, and procedures that are involved in the argumentation and justification of the solution of the problem. Problems, languages, definitions, propositions, procedures, and arguments are considered objects, specifically as the six mathematical primary objects. Taken together, they form configurations of primary objects. The term configuration is used to designate a heterogeneous set or system of objects that are related to each other. Any configuration of objects can be seen both from a personal and an institutional perspective, which leads to the distinction between cognitive (personal) and epistemic (institutional) configurations of primary objects. The OSA also considers processes, understood as a sequence of practices involving configurations of primary objects.

The mathematical objects that intervene in the mathematical practices and those that emerge from them may be considered from the perspective of the following ways of being/existing, which are grouped into facets or dual dimensions (Font & Contreras, 2008; Font et al., 2013): extensive-intensive (intensive objects correspond to those collections or sets of entities, of whatever nature, which are produced either, extensively, by enumerating the elements when this is possible or, intensively, by formulating the rule or property that characterizes the membership of a class or type of objects), expression-content (the objects may be participating as representations or as represented objects), personal-institutional (institutional objects emerge from systems of practices shared within an institution, while personal objects emerge from specific practices from a person), ostensive-non ostensive (something that can be shown directly to another person, versus something that cannot itself be shown directly and must therefore be complemented by another something that can be shown directly), and unitary-systemic (the objects may participate in the mathematical practices as unitary objects or as a system).

Problem-solving is achieved through the articulation of sequences of practices. Such sequences take place over time and are often considered processes. In particular, the use and/or the emergence of the primary objects of the configuration (problems, languages, definitions, propositions, procedures, and arguments) takes place through the respective mathematical processes of communication, problematization, definition, enunciation, elaboration of procedures (algorithmization, routinization, etc.), and argumentation (applying the process–product duality). Meanwhile, the dualities described above give rise to the following processes: institutionalization–personalization, generalization–particularization, analysis/decomposition–synthesis/reification, materialization/concretion–idealization/abstraction, expression/representation–meaning (Font et al., 2013) (see Fig. 1).

This list of processes derived from the typology of primary objects and dual facets used as tools to analyze mathematical activity in OSA, while contemplating some of the

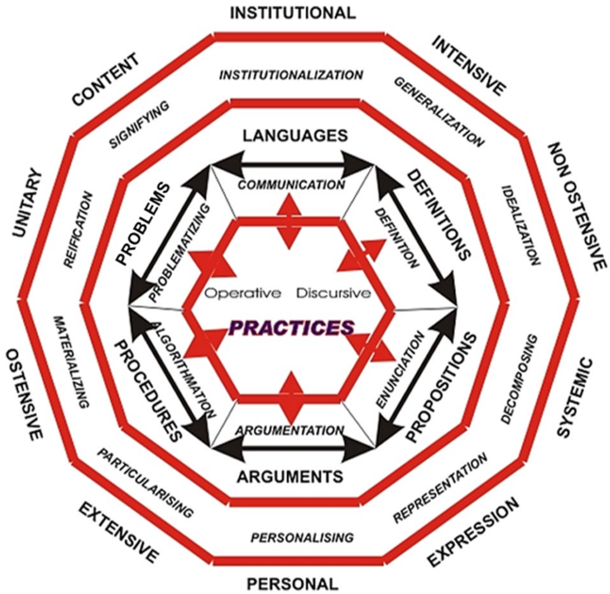


Fig. 1 Onto-semiotic representation of mathematical knowledge (from Font and Contreras (2008))

processes considered important in mathematical activity, is not intended to include all the processes involved in that activity. This is because, among other reasons, some of the most important processes, such as problem-solving and mathematical modeling, are a macro processes (as a set of processes) rather than just mere processes (Godino et al., 2007), since they involve more elementary processes, such as representation, argumentation, idealization, and generalization.

The notion of semiotic function (SF) allows us to relate practices to the objects that are activated (Godino et al., 2007). An SF is a triadic relationship between an antecedent (initial expression/object) and a consequent (final content/object) established by a subject (person or institution) according to a certain criterion or correspondence code (Godino et al., 2007).

The theoretical tools just described allow for analysis of the mathematical activity in which, firstly, temporal analysis of the mathematical practices carried out to solve a certain problem is performed; then, the configuration of primary objects that intervene in those practices is analyzed (which provides information on the elements or parts of this mathematical activity), plotting the SF that interlinks the primary objects which intervene in mathematical practices (e.g., Breda et al., 2021); and finally, analysis in terms of processes is carried out again, to complete the analysis in terms of practices (which provides information on the temporal dynamics of mathematical activity).

2.2 Extended theory of connections in mathematics education

In ETC, two groups of connections are identified: the intra-mathematical and extra-mathematical connections (Dolores-Flores & García-García, 2017). In this work, we only consider the intra-mathematical connections.

1) *Procedural*: these connections are identified when a student uses rules, algorithms, or formulas to solve a mathematical problem. They are of the form A which is a procedure to work with a concept B (García-García & Dolores-Flores, 2021).
2) *Different representations*: they are identified when the subject represents mathematical objects using equivalent (same register) or alternate representations (different registers) (Businskas, 2008).
3) *Feature*: these connections are identified when the student expresses some characteristics of the concepts or describes their properties in terms of other concepts that make them different from or similar to the others (Eli et al., 2011).
4) *Reversibility*: they occur when a student starts from concept A to obtain a concept B and inverts the process, starting from concept B to return to concept A (Adu-Gyamfi et al., 2017; García-García & Dolores-Flores, 2021).
5) *Part-whole*: they occur when logical relationships are established in two ways. The first refers to the generalization relation of form A and is a generalization of B, and B is a particular case of A. The second is that the inclusion relationship is given when a mathematical concept is contained within another (Businskas, 2008).
6) *Meaning*: this mathematical connection is identified when a student attributes a meaning to a mathematical concept or uses it in solving a problem (García-García & Dolores-Flores, 2020).
7) *Implication*: these connections are identified when a concept A leads to another concept B through a logical relationship (Businskas, 2008; Selinski et al., 2014).
8) *Metaphorical*: these connections are understood as the projection of properties, characteristics, etc. of a known domain to structure another lesser-known domain (Rodríguez-Nieto et al., 2022b).

2.3 Networking between extended theory of mathematical connections and the onto-semiotic approach

The networking of theories allows us to explore and understand how different theories can be successfully connected (or not), respecting their conceptual principles and underlying methodological, to understand and detail the complexity of the phenomena involved in the teaching and learning processes of mathematics (Kidron & Bikner-Ahsbahs, 2015; Prediger et al., 2008).

Specifically, Rodríguez-Nieto et al. (2022a) present the networking of the ETC and the OSA. In this paper, the authors respond to the following questions: (1) What is the nature of the mathematical connections from the ETC and OSA points of view? (2) How are the connections of the subjects' productions inferred in both theoretical frameworks? (3) Are there concordances and complementarities between the ETC and OSA of mathematical connections that allow for a more detailed analysis of mathematical connections?

In Rodríguez-Nieto et al. (2022a), the work of articulation to answer the first two research questions is done through the content analysis of central publications of both theories (identifying principles, methods, and paradigmatic research questions). To answer the third question, the typical steps of the theory of networking methodology are followed (Drijvers et al., 2013; Kidron & Bikner-Ahsbahs, 2015; Radford, 2008): (1) selection and description of episodes. (2) Based on the text (written protocol and transcript of the subsequent interview), the mathematical connections were identified using the ETC conceptual references. Simultaneously, (3) the mathematical connections were analyzed using the OSA. One concordance is that the methods used by both theories are content analysis. Now then, the thematic analysis of the ETC uses a typology of mathematical connections established a priori, while the analysis carried out with the OSA uses diverse tools. In this networking, the data were analyzed first in terms of practices, primary object configurations, and SFs that relate to them as proposed by the OSA (as shown in Section 4.2). Finally, parts of the mathematical activity (that is, practices, primary objects, and SFs) were encapsulated as a type of connection proposed in the ETC (as shown in Section 2.2).

Although the level of detail of the two methods of analysis is different, the main conclusion is that both theories complement each other to make a more detailed analysis of the mathematical connections. In particular, the more detailed analysis carried out with the OSA tools visualizes a mathematical connection, metaphorically speaking, like the tip of an iceberg of a conglomerate of practices, processes, primary objects activated in these practices, and the SF that relates them, which enables a thorough analysis that details the structure and function of the connection (as shown in the example in Tables 3 and 4). In this research, a more detailed analysis of the mathematical connections will also be used to analyze the productions of the students.

3 Methodology

This research is qualitative (Cohen et al., 2018) and carried out in three phases: (1) the participants were selected; (2) the data was collected through a questionnaire validated by experts, consisting of two tasks about the graphical context of the derivative and the *think aloud method* was implemented; and (3) the data was analyzed using the articulation of two types of analysis to characterize the mathematical activity carried out by the students—first in terms of practices, processes, primary object configurations, and SFs that relate them as proposed by OSA and, finally, parts of the mathematical activity (that is, practices, processes, primary objects, and SFs) were encapsulated as a type of connection proposed in the ETC.

3.1 Participants and context

Seven students (S1–S7) from the first year of undergraduate studies in mathematics from a university in southern Mexico participated. They were selected because they had taken and passed the differential calculus course according to the study plan of Autonomous University of Guerrero (2010). Among the objectives of the course are for the student to master the concept of derivative and their different applications and to be able to use this notion in maximum and minimum application problems.

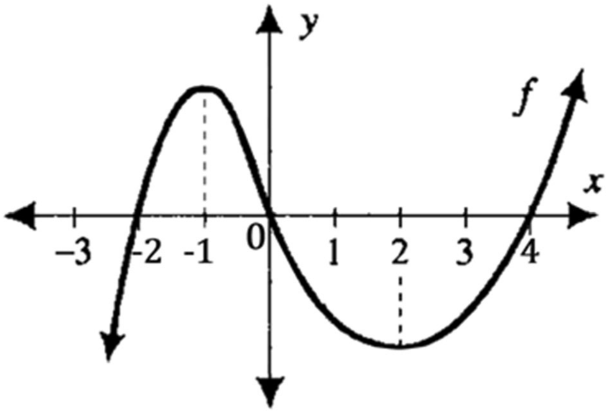


Fig. 2 Graph of *f* (from Leithold (1998))

3.2 Data collection

For data collection, two tasks involving the graphical context of the derivative were applied (Figs. 2 and 3) and the *think aloud method* was used.

3.2.1 Tasks

The objective of Task 1 (T1) was for the students to sketch the graph of the derivative function *f*' from the information provided by the graph of the function *f* (Fig. 2). Task 2 (T2) consisted of sketching the graph of the function *f* a starting point from the graph of the derivative *f*' (Fig. 3). To solve the tasks, it is necessary to know the link between the derivative sign and the intervals where the function increases and decreases, the first derivative test to calculate critical points and its relative extrema, or the second derivative test can be used to find the extremum maximum and minimum and inflection points. In general, the tasks had the purpose of exploring the mathematical connections that students make when solving tasks on the graph of the derivative and doing reversibility processes.

Task 1. Given the graph of the function *f* (see Fig. 2), determine:

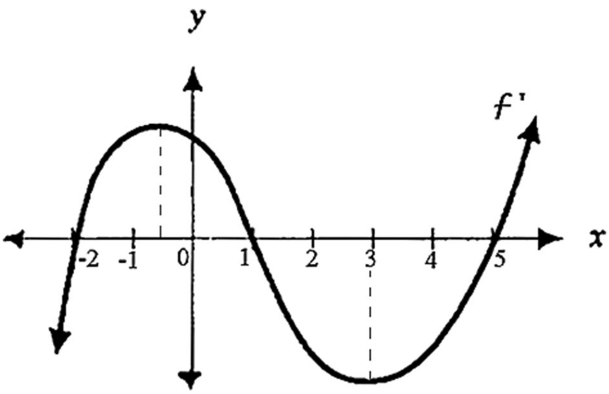


Fig. 3 Graph of *f*' (from Leithold (1998))

a) The intervals in which *f* is increasing or decreasing
b) At what point does the function have a relative maximum or a relative minimum
c) The abscissas of the inflection points of the function
d) At what interval does the graph of *f* is concave up or concave down?
e) Make a possible graph of the derivative function *f*'.

Task 2. Given the graph of the derivative of *f* (see Fig. 3), sketch the possible graph(s) of the function *f*. Argue your answer and also determine:

a) The intervals of growth and decrement of *f*
b) The maximum or minimum values of *f*
c) The inflection points
d) The intervals where *f* is concave up or concave down

3.2.2 Think aloud method

This method consists of asking people to express their thoughts aloud while solving a problem and analyzing the resulting verbal protocols (Eccles & Arsal, 2017; Van Someren et al., 1994). The students were first instructed to read the task and solve it, and then, they were asked to explain aloud everything they did in the process of solving the proposed tasks. In the application of this method, the student was not interrupted or guided; however, when the student did not verbalize his thoughts, they were reminded to speak aloud. The application of the tasks was carried out for 2 h by each student, and to capture and store the information, video recorders were used, and field notes were taken.

3.3 Data analysis

From the transcript of interviews, a temporal narrative was obtained (it is explained mathematically what the subject does when solving the task). Based on it, mathematical practices (Table 1) and processes are described, primary object configuration is built (Table 2), and SFs that relate to them (method for data analysis with OSA tools); finally, parts of the mathematical activity (that is, practices, primary objects, and SFs) were encapsulated as a type of connection proposed in the ETC (Table 3).

Concurrently, with this method of analysis developed in OSA, the data has been analyzed through thematic analysis (Braun & Clarke, 2006) to establish the connections according to the previous categories of connections proposed in the ETC. This type of analysis combines inductive (phases 2 and 3) and deductive methods (phases 4 and 5): (1) familiarizing yourself with your data (transcribe and read the interview);

Table 1 Mathematical practices and the codes inferred from narrative

Practice (Mp)	Description of practice	Code (C)
Mp1	Student S1 read and understood part *a* of the proposed task	
Mp2	They interpreted the derivative as the slope of the line tangent to the curve at a point	C1
Mp3	They identified the critical points on the graph which is where the derivative becomes zero	C2 and C11
Mp4	They determined the intervals of increase $(-\infty, -1)$ and $(2, +\infty)$ and decrease at the interval $(-1,2)$ of the function *f*, from the given graph and considering the sign of the slope of the tangent line	C3, C4, C5, and C9
Mp5	Student S1 read and understood part *b* of the proposed task	
Mp6	They found the maximum point of the function. To do this, they pointed to the point (with abscissa at $x = -1$) on the graph of *f* and stated that, at that point, the graph of the derivative *f'* cuts the *x*-axis is equal to zero	C6
Mp7	They found the minimum point of the function. To do this, they pointed to the point (at $x = 2$) on the graph of *f* and stated that, at that point, the graph of the derivative *f'* cuts the *x*-axis is equal to zero	C7, C8, and C10
Mp8	Student S1 read and understood part *c* of the proposed task	
Mp9	They stated that the point of inflection is $x = 0$ at the origin of the Cartesian coordinate plane since there the function changes from concave upward to concave downward (the student makes gestures with his hands referring to the concavity, see Fig. 4)	C12
Mp10	Student S1 read and understood part *d* of the proposed task	
Mp11	Based on the graph, S1 explains that the function is concave downward at interval $(-\infty,0)$ and concave upward at the interval $(0,+\infty)$	C13
Mp12	Student S1 read and understood part *e* of the proposed task	
Mp13	They drew the graph of the derivative. To do this, they first drew a Cartesian coordinate plane	C14
Mp14	They located the points where the derivative equals zero ($x = -1$ and $x = 2$)	C15
Mp15	They stated that at the interval $(-\infty, -1)$, function *f* is increasing, so the derivative *f'* is positive and they drew it. In addition, they explained that at that interval, the slopes of the lines to the curve are positive and stated that the derivative is the slope of the tangent line to the curve, so the derivative is positive	C16
Mp16	They stated that where the graph of *f* has a maximum point at the abscissa $x = -1$, the derivative becomes zero, and it intersects the *x*-axis	C17
Mp17	They stated that at the interval $(-1,2)$, the function *f* is decreasing, so the slopes of the tangent lines are negative, the derivative is negative, and it must be below the *x*-axis, and they drew it	C18
Mp18	They stated that the function has an inflection point at the abscissa $x = 0$, so the graph of the derivative has a minimum point	C19
Mp19	They stated that the function has a minimum, the slope is zero at the abscissa $x = 2$, so *f'* is equal to zero, *f'* has to cut the *x*-axis at $x = 2$	C20
Mp20	They stated that the function is increasing at the interval $(2, +\infty)$, so it has positive slopes, the graph of *f'* is positive, and it must be above the *x*-axis, and they drew it	C21

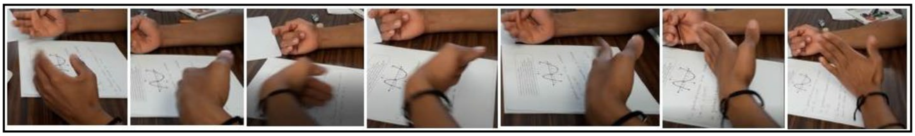


Fig. 4 Gestures of S1 refer to the concavity of *f*

(2) generating initial codes (identification of parts of the transcript that suggest some of the ETC connections); (3) searching for themes (the codes are grouped by themes that are connections); (4) reviewing themes; (5) reviewing and refining mathematical connections (review each code to see if only one type of connection is inferred, or if it is an ambiguous code in which more than one category of connection could be inferred); (6) reports with all types of mathematical connections evidenced in mathematical activity.

Finally, the two analyses carried out are related in such a way that each ETC connection is understood as an encapsulation of a part of the mathematical activity carried out (Table 3). To avoid ambiguities in data interpretation, first, the authors triangulated their analyses, to see if there were concordances and, in cases of discrepancy, agreed to apply a category among the three. Second, the authors used the expert triangulation method (in particular, the cognitive configurations and SFs were triangulated with the authors of the article Breda et al. (2021) since they had done a similar analysis) to reach a consensus with the analysis in Section 3.4 and the analyses that allowed for obtaining the results in Section 4.

3.4 Example of the analysis of a case

This section works as an example of how the answers provided by S1 to Task 1 were analyzed. First, the interviews were transcribed. Second, a narrative is made (considering the transcription of the students' verbalizations during the application of the *think aloud method*) in which it is explained how the students solved the task in mathematical terms, and some codes are also identified (phase 2 "generating initial codes" of the thematic analysis based on the categories of the ETC). Third, from the narrative, mathematical practices are described (third phase). Fourth, the cognitive configuration is constructed by highlighting the identified primary objects (fourth phase). Fifth, the SFs established between the primary objects are shown (fifth phase).

Sixth, the results of phases 3, 4, and 5 from thematic analysis are obtained, which are the mathematical connections in the last column of Table 3. Phases 3, 4, and 5 which are the code of the thematic analysis to identify the connections according to the ETC are not detailed in this analysis because, in a certain way, this would repeat the analyses carried out following the phases from OSA method since they are implicit in them. For example, in Mp4 and Mp6 of Section 3.4.2, the codes C3, C4, C5, C9, and C6 have similarities for the criteria of the first derivative, where we can observe an implication connection (phases 3 and 4 from thematic analysis). Seventh, in Table 3, analyses are integrated, and in this way, each ETC connection is understood as an encapsulation of a part of the mathematical activity carried out.

Table 2 Activated cognitive configuration of primary objects (PO) of S1 for the resolution of Task 1

PO	Description
T	Task (T1)
L	*Verbal*: slope, function, graph, derivative function, derivative at a point, tangent, …
	Symbolic: f, f', intervals: e.g., $(-\infty, -1)$; points: (0, 0); $x=-1$; $x=2$; $x=0$
	Graphic: see Fig. 5
D	Previous concepts/definitions: maximum or minimum, inflection point, critical point, line, graph, tangent, first and second derivative, derivative at a point, … D1: f' is like the slope of the tangent line to the curve at a point D2: A critical point of a function f is a number c in the domain of f such that $f'(c)=0$ or $f'(c)=0$ does not exist D3: Let f continue at c. We call (c, f(c)) an inflection point of the graph of f, if f is concave up on one side of c and concave down on the other side of c
Pr	Propositions: a). (Pr1) the graph of f is increasing at the intervals: $(-\infty, -1)$ and $(2, +\infty)$ b). (Pr2) the graph of f is decreasing at interval: $(-1,2)$ c). (Pr3) the function has a maximum at $x=-1$ and (Pr4) has minimum at $x=2$ d). (Pr5) the inflection point is $x=0$ e). (Pr6) the function is concave downward at the interval of $(-\infty,0)$ and (Pr7) is concave up at $(0,+\infty)$
Pc	Main procedure 1 (Pcp1): Find the intervals of increase and decrease. To perform this procedure, the student used two auxiliary procedures: Pca1.1: They located the critical points (graphically finding the values of "x" where the slope is zero) Pca1.2: Use of the sign of the slope to determine intervals of increase and decrease
	Pcp2: Determine the extremes in the graph of f. To perform this procedure, the student used two auxiliary procedures: Pca2.1: Use the change of increasing to decreasing for determinate one maximum Pca2.2: Use the change of decreasing to increasing for determinate one minimum
	Pcp3: Determine the inflection point of f Pca3.1: Use the change of concavity for determinate the inflection point
	Pcp4: Determine the intervals of concavity of f Pca4.1: Visually determine the intervals where the function is concave down and concave up
	Pcp5: Sketch f' Pca5.1: Make a Cartesian coordinate system Pca5.2: Locate the abscissa points $x=-1$ and $x=2$ which are the intersection points of the derivative with the x-axis Pca5.3: Sketch the graph of f' in parts, considering the increasing intervals, the extremes of the function, and the inflection point

Table 2 (continued)

PO	Description
A	A1: Thesis: The increasing intervals are $(-\infty, -1)$ and $(2, +\infty)$, and the decreasing interval is $(-1,2)$ Reason 1 (R1): If a function is continuous, the critical points separate intervals of increase and/or decrease R2: The critical points are $x = -1$ and $x = 2$ since for these values $f'(x) = 0$ R3: There are three intervals and the sign of f' at each one is always the same. If the sign of the slope is negative, f is decreasing, and if is positive, f is increasing Conclusion: f is increasing at $(-\infty, -1)$ and $(2, +\infty)$, and decreases at $(-1,2)$
	A2: Thesis: The maximum point is at $x = -1$ and the minimum point is at $x = 2$ R1: At $x = -1$ and at $x = 2$, the graph of f' cuts the x-axis, and the slope is 0 R2: If f is increasing and at a point, it becomes decreasing, then there is a maximum, and, if f is decreasing and at a point, it becomes increasing, then is a minimum Conclusion 2: f has a maximum at $x = -1$ and a minimum at $x = 2$
	A3: Thesis: The inflection point is at $x = 0$ R1: At that point, the function changes from concave up to concave down R2: S1 makes gestures with their hands referring to the concavity (Fig. 4) Conclusion 3: At the abscissa $x = 0$ if there is an inflection point
	A4: Thesis: f is concave downward at $(-\infty,0)$ and concave upward at $(0, +\infty)$ R1: f is continuous and has an inflection point at $x = 0$, where it stops being concave downward and becomes concave upward (they gesture with their hand) Conclusion 4: f is concave downward at $(-\infty,0)$ and concave upward at $(0, +\infty)$

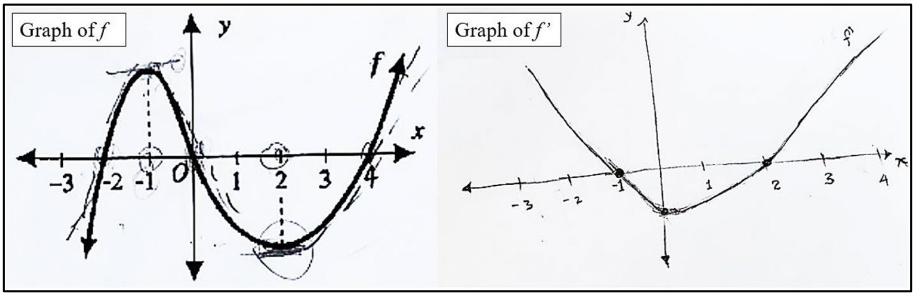


Fig. 5 Graphical representation of f and f'

3.4.1 Narrative

This section shows by way of example the beginning and end of the narrative corresponding to the resolution of Task 1, from which the codes (phase 2 from thematic analysis) and mathematical practices are obtained:

Task 1 was proposed to S1 where they were asked to (a) determine the intervals of increase and decrease of function f; S1 understood the questions and implicitly assumed that the derivative is the slope of the tangent line to the curve at a point "the slopes here of the tangents that are formed are negative, so the derivative has to be below the x-axis" (*Code 1, C1*), and then, they answered that the function had critical points in $x = -1$ and $x = 2$ (*C2*) and f is increasing at $(-\infty, -1)$, and then, its derivative has to be positive, that is, the slopes are positive (*C3*), and then it grows again, it decreases, and here, it begins to grow; it is increasing at the other interval $(2, +\infty)$

Table 3 Detailed analysis of the mathematical activity of S1 when solving Task 1

Mp	Processes	Objects	SFs	Mathematical connection
Mp1	Meaning/understanding Problematization	Task (T1)	SF1	
Mp2 (C2)	Problem-solving Enunciation	D1: The derivative is the slope of the tangent line to the curve at a point D2: A critical number of a function f is a number c in the domain of f such that $f'(c)=0$	SF2, SF3, SF4	Meaning
Mp3 (C2, C11)	Problem-solving Representation (verbal, graphic, and symbolic)	Pca1.1: Find graphically the value of "x" in which the slope is equal to zero	SF5, SF6, SF7, SF8	Procedural Different representations
Mp4 (C3, C4, C5, C9)	Problem-solving Representation (verbal, graphic, and symbolic) Argumentation	Pca1.2: Use of the sign of the slope to determine intervals of increase and decrease of the function A1 (Fig. 7): If the function is increasing, the slopes of this tangent line are positive, that is to say, the derivative is positive	SF9, SF10, SF11, SF12	Procedural Different representations Implication
…	…	…	…	…

(*C4*); those are the intervals where the function is increasing. Then, S1 affirms that the graph of *f* is decreasing at the interval (− 1,2), since the slopes of the tangent lines are negative, and the graph of the derivative must be below the *x*-axis (they also drew the graph) (*C5*) (…). Finally, based on the information obtained, S1 responded to part e, explaining the behavior of the graph of *f'* drawn.

3.4.2 Mathematical practices

Next, we describe the mathematical practices (Mp) and the codes (C) inferred from narratives (Table 1).

Next is an analysis of the processes, for example, Mp1 carries an understanding and problematization process. For reasons of space, this analysis is only partially incorporated in Tables 3 and 4.

3.4.3 Cognitive configuration of primary objects of S1 on Task 1 and SF

This section presents the configuration of primary objects evidenced in the sequenced mathematical practices to solve T1 (Table 2).

Based on the mathematical practices and the configuration of primary objects in Table 2, in Fig. 6, the circled numbers linked to the thin blue arrows serve to enumerate some of the SFs that the authors, after triangulation, have inferred that S1 establishes among the different primary objects of the configuration that they use in solving Task 1. For example, to solve part *a* of T1, the student S1 needs to read the problem and must understand that they must find intervals of increase and decrease (SF1). Then, S1 must know that the increase and decrease are related to the sign of the first derivative (SF2) and that the derivative is the slope of the tangent line to the curve at a point (SF3). After this, the student has to relate the slope of the tangent line with the critical point of a function (SF4), and has to relate it with the main procedure Pcp1 (find the intervals of increase and decrease) (SF5) and this procedure with the auxiliary procedure Pca1.1 (the critical points in the given graph are $x = -1$ and $x = 2$ because the derivative is zero) (SF6, SF7, and SF8). Then, the student has to use the procedure Pca1.2 to find that the graph is increasing at $(-\infty, -1)$ and $(2, +\infty)$ and decreasing at $(-1,2)$ (SF9 and SF10). Once they have obtained this result, they consider it true (SF11) and justify it with the argument A1 (SF12). In Fig. 6, the thick blue arrows refer to the propositions being related to the arguments, and these validate and support each statement contained in the propositions and procedures boxes.

3.4.4 Detailed analysis of the mathematical connections in Task 1 based on the integration between ETC and OSA

The last column of Table 3 presents some mathematical connections established by S1 in solving T1. For this, the data has been analyzed through thematic analysis to establish the connections according to the previous categories of connections proposed in the ETC. The rows show the conglomerate of mathematical practices, processes, objects, and SFs that constitute the connection.

Table 4 Detailed analysis of the mathematical activity of S1 when solving Task 2

Mathematical practices (Mp)	Processes	Objects	SFs	Mathematical connection
Mp1	Meaning Understanding Problematization	Task (T2)	SF1	
Mp2	Problem-solving Enunciation	D1: The derivative is the slope of the line tangent to the curve at a point D2: A critical point of a function f is a number c in the domain of f such that $f'(c)=0$	SF2, FS3, SF4	Meaning
Mp3	Problem-solving Representation (verbal, graphic, and symbolic)	Pca1.1: Determine the critical points by finding the values at which f' intersects the abscissa axis	SF5, SF6, SF7, SF8, SF9	Procedural Different representations
Mp4	Meaning Understanding Problematization	T2	SF10	
Mp5 (*) They found that the intervals of decrease of the function are $(-\infty,-2)$ and (1,5) and the intervals of increase are $(-2,1)$ and $(5,+\infty)$. For this, they used the criterion of the first derivative	Problem-solving Representation (verbal, graphic, and symbolic) Argumentation	Pca1.2: Use of the sign of f' to determine intervals of increase and decrease of the function Graphic and symbolic representations (intervals) A1 (Fig. 8): If the derivative is positive at an interval I, then the function is increasing at this interval I. If the derivative is negative at I, then the function is decreasing at I	SF11, SF12, SF13, SF14, SF15	Procedural Different representations Implication
…	…	…	…	…

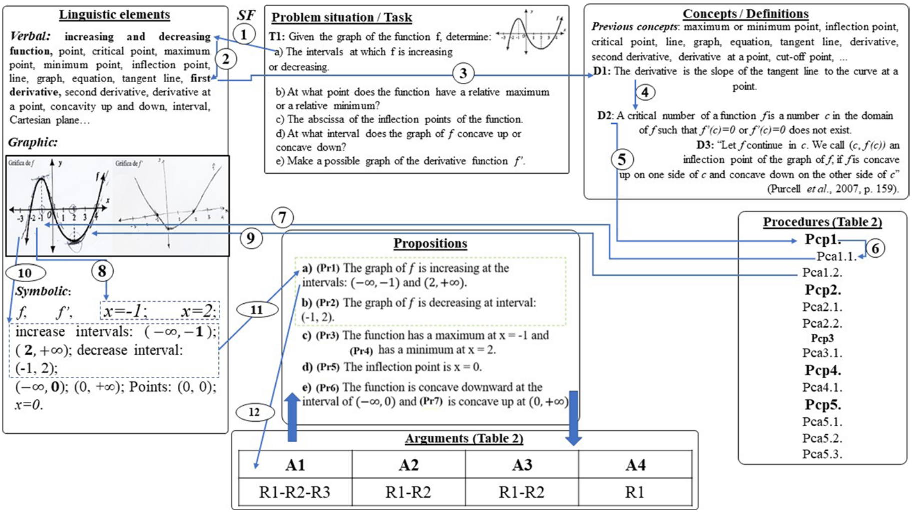


Fig. 6 SFs established with the primary objects by S1 in the resolution of Task 1

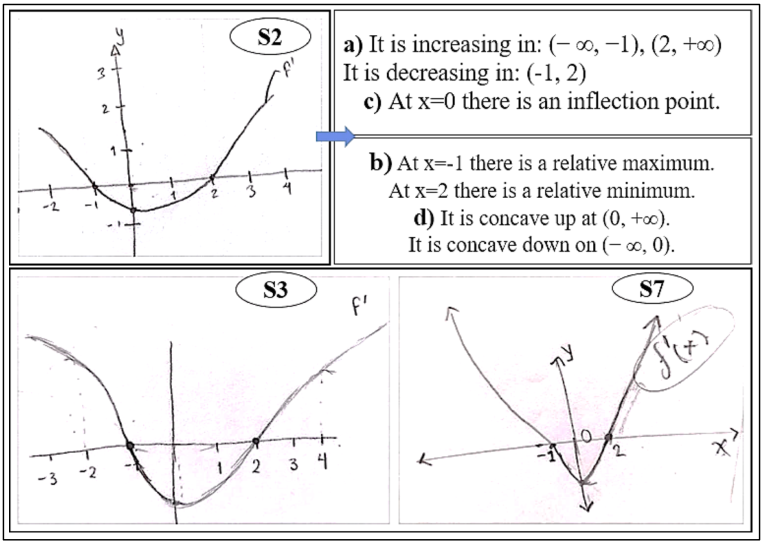
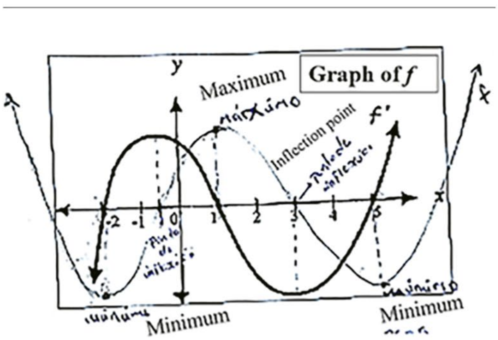


Fig. 7 Graphical representation of *f'*

4 Results

In Sections 4.1 and 4.2, we report on the overall findings of the seven students and include excerpts from some cases to illustrate the findings (in Section 3, we explain the case of S1 in T1 as an example). In Section 4.3, we show a synthesis of students that established connections of reversibility type and those who did not. The incorrect answers and their possible explanation are treated in Section 4.3, where the reversibility connection is analyzed from a systematic point of view for T1 and T2.



a) (Pr1) $f'(x)$ becomes zero at $x = -2$, $x = 1$, $x = 5$

→ -2,1, 5 are critical points of $f(x)$

b) The increasing and decreasing intervals are:

$(-\infty, -2)$, $(-2, 1)$, $(1, 5)$, $(5, +\infty)$

$(-\infty, -2)$ the function is decreasing (Pr2)

$(-2, 1)$ the function is increasing (Pr3)

$(1, 5)$ the function is decreasing (Pr4)

$(5, +\infty)$ the function is increasing (Pr5)

c) At the critical point -2 there is a minimum (Pr6)

At the critical point 1 there is a maximum (Pr7)

At the critical point 5 there is a minimum (Pr8)

d) The inflection points are $x = -1/2$, $x = 3$ (Pr9)

The function is concave upward at interval $(-\infty, -1/2)$ (Pr10)

The function is concave downward at interval $(-1/2, 3)$ (Pr11)

The function is concave upward at interval $(3, +\infty)$ (Pr12)

Fig. 8 Propositions considered by S1 in sketching the graph of *f*

4.1 Task 1

Concerning the other students, it was evident that S2, S3, and S7 made mathematical connections similar to S1, which allowed them to solve T1 correctly (Fig. 7). For example, to sketch the graph of *f'* S2 uses the criteria of the first derivative associated with the growth and decrease of the function in the intervals indicated in response to the question (a). S7 proceeds in the same way as S2 and uses gestures to represent the concavity of the function *f*. S3 responded to the researcher's questions as follows: "What I considered from *f* to obtain *f '*, was information to know where the maxima and minima were, as well as the increasing and decreasing intervals, because between *f* and *f '* there is a relationship that if my function *f* is increasing then my derivative is going to be positive and if the function *f* is decreasing then the derivative is negative. In the maximums and minimums, I am taking into account, or I am guided by, the slope of the tangent line, at that point its slope is zero, and this is how I relate it (…)."

The students that answered T1 incorrectly (S4, S5, and S6) also answered T2 incorrectly. For this reason, the incorrect answers and their possible explanation are treated in Section 4.3, where the reversibility connection is analyzed from a systematic point of view for T1 and T2.

4.2 Task 2

S1 managed to establish mathematical connections to solve T2. We do not explain the same detailed analysis made with T1. We will limit ourselves to presenting (a) the graph that S1 drew; (b) a part of their written production that shows the justification for how they made the graph, in particular, the part where they made explicit the propositions of the cognitive configuration they used (Fig. 8); (c) the result of the cognitive configuration and the SFs established between the primary objects of this configuration (Fig. 8); and (d) a part of Table 3 where each ETC connection is understood as an encapsulation of a part of the mathematical activity carried out.

Figure 8 shows in detail the propositions used by S1 to sketch the graph of *f* from the graph of *f'* considering growth and decrease intervals, critical points, maximum, minimum, inflection point, and the analysis of the concavity of the function.

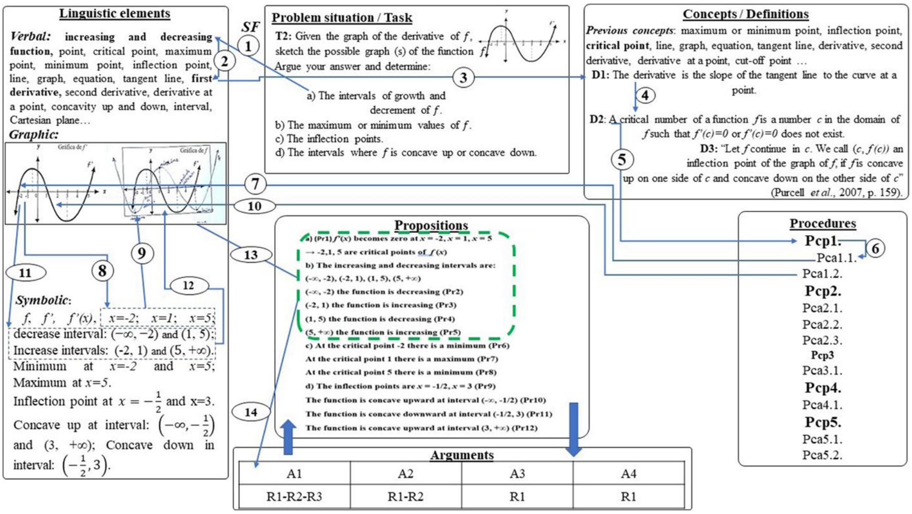


Fig. 9 SFs with the primary objects established by S1 in the resolution of Task 2

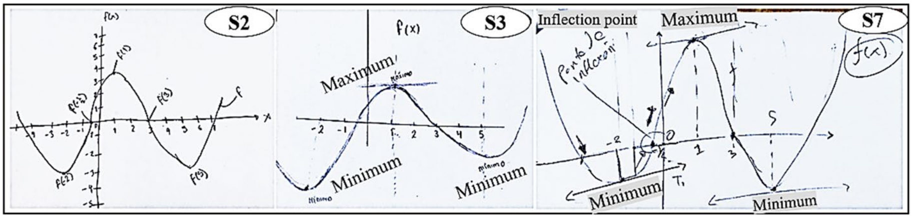


Fig. 10 Graph of *f* from the information of *f’*

Figure 8 shows a part of the primary objects of the cognitive configuration presented in Fig. 9. In turn, Fig. 9 shows the SFs that relate to the primary objects of the configuration. For example, as in Fig. 6, to solve part *a* of T1, the student must understand that they have to find intervals of increase and decrease (SF1), and then, they must know that the increase and decrease are related to the sign of the first derivative (SF2) and that the derivative is the slope of the tangent line (SF3).

The last column of Table 4 presents some mathematical connections established by S1 in solving the Task 2 and the row shows the conglomerate of mathematical practices, processes, objects, and SFs that constitute the mathematical connection.

Students S2, S3, and S7 also managed to establish mathematical connections to solve T2. Next, evidence of the graphs of *f* that the students made based on the information of the derivative graph is shown (Fig. 10).

In addition, extracts from the written productions of S2, S3, and S7 are presented (Fig. 11) as evidence that they correctly answered the question using the sign of the derivative to determine the increase or decrease of the function.

The students that answered T1 incorrectly (S4, S5, and S6) also answered T2 incorrectly. For this reason, the incorrect answers and their possible explanation are discussed in detail in Section 4.3, when the reversibility connection is analyzed jointly for Tasks 1 and 2.

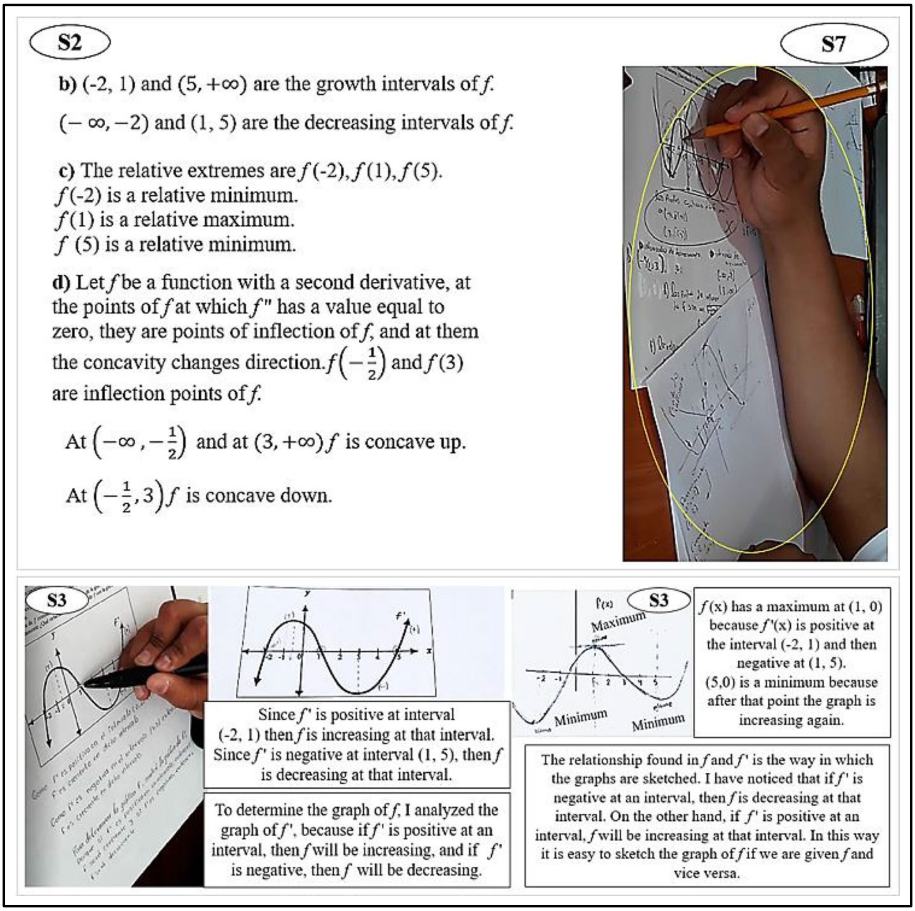


Fig. 11 S2, S3, and S7 explain how to draw out *f* graph

4.3 Establishing the mathematical connections of reversibility

In the resolution of T1 and T2, mathematical connections of meaning, part-whole, different representations, procedural, and implication were evidenced. Particularly, the mathematical connections of the implication type established by S1, S2, S3, and S7 when they solved the tasks (Tables 3 and 4) are the foundation of the mathematical connection of the reversibility type, since they are bidirectional logical relations made to graph *f*' based on the information in *f*, or graph *f* based on the information in *f*', as shown in Fig. 12.

However, S4, S6, and S7 did not establish the reversibility connection because it did not establish the implication connection that is necessary to successfully solve T1.

In Fig. 13 and written production of student S4 to solve T1, it is observed that they carried out (among other) the following mathematical practice (Mp9 and Mp11) where a wrong connection is observed: Mp2. In part *a*, they determined the intervals of increase $(-\infty,-1)$ and $(2,+\infty)$ and decrease at the interval $(-1,2)$ of the function *f*, from the given graph, but did not consider the sign of the slope of the tangent line (…): Mp9. In part *d*, S4 states that the concave intervals of *f* are concave up at interval $(-2,0)$ and

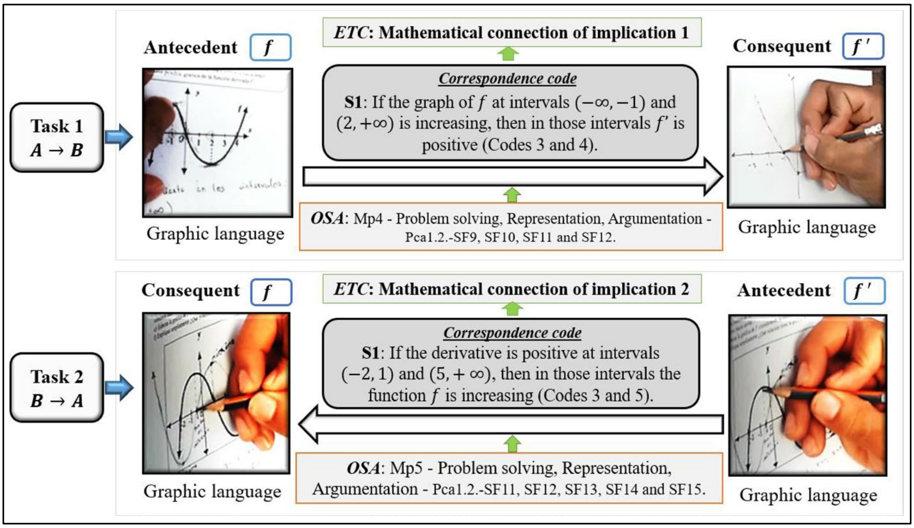


Fig. 12 Structure of the mathematical connection of reversibility

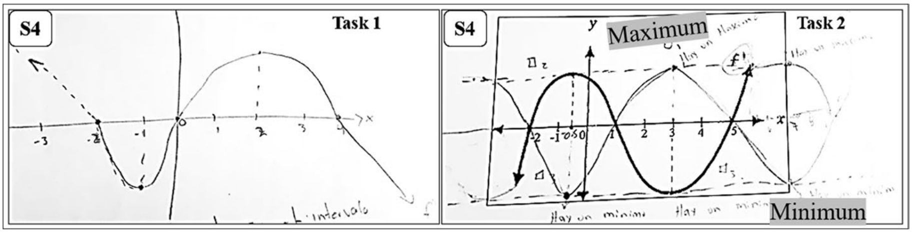


Fig. 13 Sketches of the graphs of *f* and *f'* made by S4

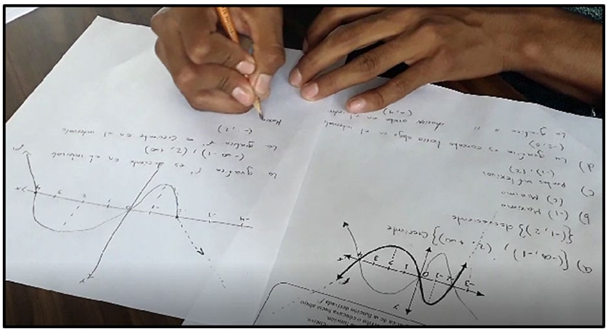
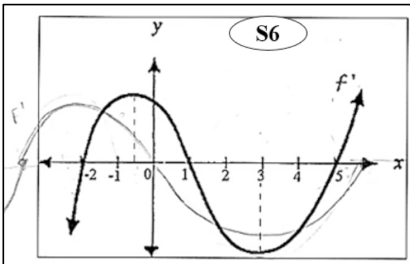


Fig. 14 Written production of S4 solving Task 1

concave down at interval (0,4) (wrong connection): Mp11. In part *e*, they drew the graph of the derivative making a wrong connection; see Fig. 14 (Task 1) and the following excerpt from the transcript.



a) The critical numbers are: ?
b) The growth intervals are [−∞, 0]. The decreasing intervals [0, ∞].
c) The relative extremes are 5 and -2.
d) The inflection points are:

c) The intervals where *f* is concave up are [4, 5]. The intervals where f is concave down are [2, 3].
d) The turning points (4, 5, 2).
f) Explain extensively. What relationship does the graph of the function *f* have with the graph of *f'*?
Answer by **S6:**
The *f* is the function, the *f'* is the derivative, the relationship between these is that when both grow, the other will also grow, their inflection points will be the same, both graphs will have the same concavity.

Fig. 15 Written production of S6

S4: I have the Cartesian plane if f is concave downward, for its derivative f' it has to be concave upward, then in this case, because the points ($x = -2$ and $x = 4$), for the graph of f was from $(-\infty, -1)$ increasing, but for f' it will be decreasing in $(-\infty, -1)$ and f' increases from $(2, +\infty)$.

I: What helped you build the graph of f'? (*the interviewer made the question for to the incorrect answer of S4*).

S4: Being able to see the graph of f, I observed that where it was concave it had a maximum point for the derivative, the graph was the inverse if it was maximum for f, for the other ($f1$) it was minimum. If I follow the same procedure, the graph of f' is increasing from $(-1, 2)$ but for the graph of f was decreasing and the maximum of f' is at $x = 2$ and the minimum at $x = -1$. In this case, the inflection points do not change, because it is the same behavior. Also, f' is concave upward at $(-2, 0)$ and concave downward at $(0, 4)$ (Fig. 14).

These incorrect implication connections (e.g., if the graph of *f* is concave upward at an interval, then the graph of *f'* is concave downward at that same interval) cause S4 not to perform the other mathematical practices that are key or necessary for the correct resolution of the task (see mathematical practices in the resolution of S1). Given this situation, an explanation for this difficulty is that the complexity of the mathematical activity necessary to establish the connections that allow for finding the graph of *f* or *f'* may be higher than the mathematical activity that the student can perform (in this case S4), which leads them to stop carrying out some practices, to stop establishing some SFs, etc., and therefore, to stop establishing a certain mathematical connection. Each of these connections that S4 did not make is the main cause of different difficulties in drawing the graph of *f'* from the graph of *f* or vice versa; some of them have been indicated by other researchers (Berry & Nyman, 2003; Fuentealba et al., 2018a; Ikram et al., 2020; Natsheh & Karsenty, 2014; Ubuz, 2007). Then, S5 solved similar to S4 and obtained one graph practically equal to the graph of S4.

In the case of S6, it is shown that they draw the graph of *f* following the same performance as the graph of *f'* (Fig. 15).

5 Discussion and final considerations

In this article, we study the mathematical connections established by university students when solving tasks that involve the graphs of *f* and *f'*, without symbolic expressions. We conclude that the most significant mathematical connection for solving these tasks is the reversibility connection. The results indicate that some students can do bidirectional processes where they link the signs of *f'* with *f* and relate the monotony and curvature of *f* with the sign of its first derivative, while other students are not able to do so. This is a result consistent with other research that has shown that these processes become difficult for some students (Fuentealba et al., 2018a). It must be highlighted that, to make the mentioned connection possible, students must establish other mathematical connections, particularly the implication connection (they are needed to solve the tasks). In other words, reversibility is the mathematical connection specifically used to solve tasks that involve the graphs of *f* and *f'*, although the other types of connections are present.

Metaphorically, this mathematical connection can be understood as the visible part of an iceberg, while the underwater, non-visible part is a vast network of SFs that are at its base. The non-establishment of some of those SFs sheds light on the reasons why the students do not establish the desired connection (for this reason, there is a wide spectrum of possible causes of why the students fail to establish a determined connection).

Since the students who have not solved T1 are the same as those who have not solved T2, it can be concluded that both processes present the same difficulty for students. This is one possible conclusion but there are others, for example, that the successful students were successful in learning differential calculus and the unsuccessful students were unsuccessful in learning differential calculus; calculus courses are notorious for failing many students and they end up being confused on the whole course, not just specific processes. The conclusion that both processes present the same difficulty for students contradicts the results of Ikram et al. (2020) who state that it is not a trivial task for many students to sketch the graph of the function when given the graph of the derivative, even though they can find the graph of the derivative when given the graph of the function. However, the part of the mathematical activity that encapsulates the connection of implication in each task helps to explain the significant difficulty that T2 presents relative to T1.

This way of characterizing the reversibility connection (as the visible part of an iceberg) is consistent with the one used in the study of the understanding of exponential and logarithmic functions (Campo-Meneses & García-García, 2020). In some cases, the role of reversibility in these function pairs has been investigated with categories from other theoretical frameworks—for example, Ikram et al. (2020) use APOS to study the function's case and its inverse. A possible line of research is to study whether the characterization made here of this connection applies to other mathematical contents where it is relevant, as is the case of the relationship between a function and its inverse, the power function, the root function, etc.

The difficulties observed in this research when students solve tasks that involve the graphs of *f* and *f'* are similar to those reported in different investigations (Fuentealba et al., 2018a; Nemirovsky & Rubin, 1992). Now, this article explains that the reason why some students did not solve the task was that they did not establish some of the SFs that they had to establish between the conglomerate of practices, primary objects, and processes that are encapsulated by the notion of reversibility connection.

In the research by García-García and Dolores-Flores (2021), the students made reversibility connections in a graphic environment, but it was necessary to use the algebraic symbolic representation to graph, while, in the results of this research, the students used qualitative criteria of the functions to relate *f* to *f'* and vice versa. The fact of presenting the reversibility connection as an encapsulation of a portion of complex mathematical activity makes it possible to specify in detail the reason why reversibility is not established and justifies that in these cases the use of the symbolic expression is key to establishing it.

While different authors have pointed out the importance of the reversibility connection and, more generally, reversible reasoning for mathematical understanding (Ikram et al., 2020; Sangwin & Jones, 2017), this research shows that the relevance of this type of reversibility connection goes beyond its role in understanding the derivative as it may be a necessary step in alternative techniques for calculating the derivative and the antiderivative.

In addition to the result that the most significant mathematical connection for solving tasks that involve the graphs of *f* and *f1*, without symbolic expressions, is the reversibility connection, this integrated view provides other relevant results. On one hand, it provides results about the complexity of the mathematical activity necessary to establish the connections that allow for finding the graph of *f* or *f1*; in particular, it allows for explaining how the complexity of the mathematical activity that encapsulates the connection explains why the connection cannot be established (by showing everything that the student must do). On other hand, the relevance of this paper is that it contributes, together with other works, to illustrating how networking between two theories of different levels can be applied in a specific situation—in this case, a general theory for the analysis of mathematical activity (OSA) and a theoretical framework for the analysis of the specific mathematical activity of connection (ETC). This approach results in an integrated proposal of the two theories for the analysis of the mathematical activity of interest to the specific theory, in this case, the connection process required for solving tasks that involve the graphs of *f* and *f1*, without symbolic expressions. This type of networking between OSA and theories that make specific analyses of mathematical activity has also been carried out with other processes such as modeling (Ledezma et al., 2022).

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Data availability Anyone who makes a reasonable request to the first author of the article will be provided with the data that supports the results of the study.

Declarations

Conflict of interest The authors declare no competing interests.

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