Educational Studies in Mathematics (2023) 114:129–148
https://doi.org/10.1007/s10649-023-10245-w

# Collaboration between Mathematicians and Mathematics Educators: dialogical inquiry as a methodological tool in Mathematics Education research

**Paul Hernandez-Martinez**[1,2] · **Svitlana Rogovchenko**[3] · **Yuriy Rogovchenko**[4] · **Stephanie Treffert-Thomas**[5]

Accepted: 15 May 2023 / Published online: 26 June 2023


**Abstract**
Collaboration between Mathematicians and Mathematics Educators is crucial in advancing knowledge on the teaching and learning of Mathematics, particularly in advanced Mathematics pedagogy. However, there is a need for the type of collaborations where Mathematicians and Mathematics Educators can find common ground, and the synergy of their expertise results in new, hybrid meanings and understandings that can benefit practice. This paper aims to help researchers from these communities come together by presenting a novel methodology for collaborative inquiry and qualitative data analysis—*dialogical inquiry*—based on Mikhail Bakhtin's theory of dialogism. In this framework, "truth" is conceived as never finalised but always coevolving through dialogue that involves participants with different "voices" and "languages". We describe the three principles that underpin *dialogical inquiry*: motivation, power balance and a process for solving disagreements. These three principles interact together to create a space where critically productive dialogue allows for meanings to coevolve and new, hybrid understandings to emerge. We illustrate our operationalisation of these principles (i.e., the methods of *dialogical inquiry*) in two areas: understanding solutions to linear ordinary differential equations and making meaning of the Bakhtinian concept of superaddressee. We reflect on our use of this methodology in Mathematics Education research and invite the readers to create their own dialogic spaces of collaboration.

**Keywords** Bakhtinian dialogic theory · Dialogism · Dialogical inquiry methodology · University Mathematics Education research · Advanced Mathematics pedagogy

✉ Paul Hernandez-Martinez
phernandezmartinez@swin.edu.au

[1] Department of Mathematics, Swinburne University of Technology, Melbourne, Australia

[2] Swinburne University of Technology, John Street, Hawthorn, VIC 3122, Australia

[3] Department of Engineering, University of Agder, Kristiansand, Norway

[4] Department of Mathematical Sciences, University of Agder, Kristiansand, Norway

[5] School of Science, Loughborough University, Loughborough, UK

## 1 Introduction

In this paper, we present a methodological approach to collaborative inquiry and qualitative data analysis: *dialogical inquiry*. This methodology is based on Mikhail Bakhtin's dialogic theory and principles of dialogism. It originated from a need of our team of two Mathematicians and two Mathematics Educators to collaborate in (advanced) Mathematics Education research. As a team, we found that traditional methodological tools (e.g., thematic or discourse coding/analysis) did not lend themselves to the type of collaborative learning and rich exchange of ideas that were needed, particularly when analysing students' work on *advanced* mathematical concepts. The methodology that we present opened a "dialogical space" where we could discuss ideas, our voices were heard equally, and as a result, shared meanings emerged.

Bakhtinian theory has been used in Mathematics Education research as a lens to interpret students' conceptual development through dialogue (e.g., Barwell, 2016; Kazak et al., 2015) but, to the best of our knowledge, it has never been used as a tool by researchers in this field for their own meaning-making when interpreting data. In other fields (e.g., text analysis), the methodology is only recently beginning to be operationalised and applied (Wells et al., 2021) even though the influence of Bakhtin in the social sciences is well established. Our *aim* is to introduce this novel methodology to Mathematics Education research by describing how we came to use *dialogical inquiry* to (1) analyse data and arrive at shared meanings of the mathematical concepts involved in the research and (2) collaboratively understand the Bakhtinian theoretical and analytical framework that supported the research. In doing so, our contribution resides in answering the following question: What are the principles and general rules of *dialogical inquiry* as a methodology in collaborative Mathematics Education research? In our discussion of *dialogical inquiry*, we share our reflections as researchers on the implications of using this methodology with the purpose of introducing and promoting it as a living theory, that is, where "many of its operational principles, both explicit and implicit, will be of their nature generalisations from practical experience and have as their justifications the results of individual activities and practices" (Hirst, 2012, p. 18). We expect researchers to appreciate the opportunities for interdisciplinary collaboration that this methodology affords by bringing multivoiced dialogue and shared, comprehensive meanings that otherwise could be hard to achieve.

## 2 Collaborative research in Mathematics Education

Collaboration between Mathematicians and Mathematics Educators[1] is crucial in advancing knowledge on the teaching and learning of Mathematics. This is particularly true in advanced Mathematics pedagogy where Mathematicians' intimate knowledge of the Mathematics involved adds important value to the research and is often necessary for conducting such research. In the past two decades, there have been calls for more collaboration between these two communities (Fried & Dreyfus, 2014), and there have been some suggestions as to how to bring their voices together to facilitate reform of pedagogical

[1] We acknowledge that there is no clear-cut distinction between these two groups of professionals. However, for practical reasons, in this paper, we distinguish both groups by their main research expertise *and* activity, one being in the mathematical sciences and the other being in the educational sciences.

practice (Nardi, 2007). The current University Mathematics Education research community acknowledges the need to find new and effective ways to increase collaboration with Mathematicians. For example, researchers from the International Network for Didactic Research in Undergraduate Mathematics (INDRUM) stated that the "importance of collaboration between mathematicians and mathematics educators, through means where both groups have something to say and share, reinforces the need to look for spaces […] where interactions and dialogue conditions are made possible" (Bardini et al., 2021, p. 54). In addition, during the 2022 Conference of the European Society for Research in Mathematics Education (CERME12), the University Mathematics Education Working Group identified as major challenges the establishment of "healthy" collaborations between the Mathematics and Mathematics Education communities and the identification of collaborative models that facilitate research (González-Martín et al., 2022). These statements emphasise the need for collaborative tools and methodologies that can support joint research, and this paper is an attempt at showing how productive collaborations can be achieved.

Wagner (1997) proposed a framework delineating three types of collaborative research between educational researchers and practitioners (in his case, school Mathematics teachers). We adapt this framework to University Mathematics Education research to describe collaborations between Mathematicians and Mathematics Educators, and give examples of these collaborations in the university context:

(1) *Data-extraction agreements*, in which educational researchers, mostly Mathematics Educators, "extract" data from the collaboration with practitioners, who are mainly Mathematicians but can also be other Mathematics Educators. In this type of collaboration, there is an inherent asymmetry in the relationship where neither researchers nor practitioners are expected to share their perspective and expertise. Practitioners facilitate the research by providing access to data but the research and its outcomes are the primary responsibility of the educational researchers. Examples of University Mathematics Education research of this type include Hannah et al. (2011), Mesa et al. (2020) and Viirman and Nardi (2019).

(2) *Clinical partnerships*, in which the focus of the collaboration is on how researchers and practitioners can work together to improve knowledge about educational practice. In University Mathematics Education research, the focus is often on the professional development of lecturers and other practitioners, including pre-service teachers in university education. In this type of collaboration, researchers and practitioners try to develop shared understandings of their separate but complementary activities. Examples of this type of collaboration can be found in studies based on action research or (developmental) design research, such as Doorman et al. (2013), Hernandez-Martinez and Goos (2014) and Konstantinou-Katzi et al. (2013). Examples of collaborative research focused on the development of lecturers or pre-service teachers include Barton et al. (2015) and Goos and Bennison (2018). Other relevant examples in university contexts include Jaworski et al. (2017), Kontorovich (2021) and Thomas (2012).

(3) *Co-learning agreements*, in which the division of labour between researchers and practitioners becomes blurred; both parties are regarded as agents *and* objects of the inquiry. It is expected that the asymmetries between researchers and practitioners are starkly reduced, and that the collaboration is highly reflective, leading to a shared enterprise where learning occurs on both sides. Each team member acquires new knowledge about each other: Mathematicians gain an understanding of educational research and Mathematics Educators advance their mastery of Mathematics. We could not find any

examples of such collaborations in the research literature on University Mathematics Education—perhaps because the field is still very young. But it might also be the case that there are no existing methodologies that facilitate collaborations of this kind, and our paper is an attempt to fill this gap.

We also found in the literature—almost exclusively in the university context—a fourth type of collaboration not described by Wagner's (1997) framework. These collaborations are characterised by a Mathematician *completely* crossing into the field of Mathematics Education research—which becomes their main research activity—and developing into a "broker" between the two fields. Research is carried out between the broker and others who might be Mathematicians interested in educational issues—sometimes seeking to improve their teaching practice—and occasionally other Mathematics Educators. Examples of these collaborations include Bosch et al. (2021), Kondratieva and Winsløw (2018) and Tall and Katz (2014). More recently, Darragh (2022) studied how some Mathematicians are able to broker Mathematics and pedagogy across both communities, highlighting barriers but also learning opportunities that these boundary crossers experienced in their goal to transform practices. We have named these collaborations "brokering partnerships". Despite its importance and indisputable value, this is not the type of collaboration that our paper is about. In *dialogical inquiry*, neither Mathematicians nor Mathematics Educators have the intention (from the outset) to cross *entirely* into the other's field, and there is no broker. In fact, a distinctive characteristic of *dialogical inquiry* is the existence of different "languages", which implies that participants most likely bring with them different perspectives on mathematics and the teaching and learning of the subject. The outcomes of research are the product of the (hybrid) meanings co-produced from the synergy of expertise.

We now describe the theoretical foundations of *dialogical inquiry*.

## 3 Bakhtinian dialogical theory and dialogism

Bakhtinian theory provides the framework for *dialogical inquiry*, the methodology that we describe in this paper. Within contemporary social sciences, the influence of Bakhtin has been called the "dialogic turn" (Aubert & Soler, 2007, p. 523). For Bakhtin, "truth is not born nor is it to be found inside the head of an individual person, it is born between people collectively searching for truth, in the process of their dialogic interaction" (Bakhtin, 1984, p. 11). Truth is always unfinalised, "there can never be neither a first nor a last meaning" (Bakhtin, 1986, p. ix). In this framework, the focus of inquiry shifts "from finding one explanation towards a sensitiveness to the uniqueness and open-endedness of interaction, dialogue and meaning-making" (Bager, 2013, p. 155). The researcher is not a "neutral" onlooker but a constituent part of the research. These ontological and epistemological positions are different from those of, for example, an interpretive (Garrick, 1999) or a critical theory (Matias, 2021) position.

Bakhtin distinguished between "authoritative" and "internally persuasive" discourses. Authoritative discourse is monologic, only one voice is heard, and one point of view is represented. This discourse "demands our unconditional allegiance" (Bakhtin, 1981, p. 343), and ideas remain someone else's. In contrast, internally persuasive discourse is dialogic and acknowledges a multitude of voices (polyphony) with different meanings (heteroglossia) and without assertion of authority. In dialogic interaction, meanings emerge through engagement with historically, culturally, and socially important voices and through addressing and

replying to these voices (Matusov & von Duyke, 2010). An important concept in Bakhtinian theory is that of the superaddressee, "the third" in the dialogue "whose absolutely just responsive understanding is presumed" (Bakhtin, 1986, p. 26). The superaddressee is often interpreted as a rhetorical tool used to warrant someone's argument when trying to persuade others of one's position, such as evoking a trustable source of information. Midgley (2011) contends that addressing similar superaddressees can create connections in dialogue.

In *dialogical inquiry*, researchers aim to open a dialogical space where shared meanings and interpretations about data can emerge. Researchers take their time to genuinely understand the other team members' point of view and negotiate disagreements. As Wells et al. (2021) write, the "dialogue is guided by slowness, the tolerance of uncertainty, embodied reponses, and the coevolution of meaning". We as researchers "suspend our judgement, our propensity for foreclosed inquiry, and our enthusiasm for the early answers that usually seem to present themselves" (Russell & Kelly, 2002, p.10). The dialogic nature of these interactions ensures that a certain balance of power is sustained, given that researchers come from different historical and socio-cultural backgrounds and with different claims to expertise; that is, the aim is to reduce the asymmetries between Mathematicians and Mathematics Educators as in Wagner's (1997) co-learning agreements. Ideas, arguments, and counterarguments are socially and culturally embodied in different ways of speaking (e.g., in mathematical or educational terms) and different ways of acting. Working together in the dialogical space, researchers learn to understand and value each other's perspectives and to construct shared meanings that are the product of "the self-made-up-of-all-the-researchers-on-the-team. Each self requires attention; each is responsible; each depends on the self's own reflexivity and everyone else's reflexivity" (Russell & Kelly, 2002, p. 11). Uncertainty is viewed as a possibility to learn more by reflecting on each other's arguments, examining the literature, going back to theoretical sources and coming back to the discussion with refreshed arguments. The tone is not that of competitive or combative polarised debate, but of a critically productive dialogue (cf. Hyde & Bineham, 2000).

In *dialogical inquiry*, the boundaries between Mathematics and Mathematics Education are blurred. No single voice is granted an excessive power or authority, and the self becomes multivoiced (Aveling et al., 2015). According to Bakhtin (1981), the self arises in and through social relations with others; others are, in this sense, part of the self. Hence, by engaging in *dialogical inquiry*, we expect that, within the dialogical space, Mathematicians will see themselves as Mathematics Educators, and vice versa.

To operationalise this methodology in our collaborative meaning-making process and data analysis, our team set up weekly online meetings to discuss the various aspects of the research, including the Mathematics involved and the theoretical framework. Although we had tentative dates for our milestones, the pace of the inquiry dictated the actual dates, once everyone in the team felt we had sufficient time to reflect and reach common understandings. The process of developing these understandings resembled an ever-growing helix structure; however, sometimes, we needed to return to earlier understandings, revise them and then be able to move forward.

Furthermore, we found the concept of "exploratory talk" (Mercer & Wegerif, 2004) useful in this operationalisation. In exploratory talk,

> partners engage critically but constructively with each other's ideas. Statements and suggestions are offered for joint consideration. These may be challenged and counter-challenged, but challenges are justified and alternative hypotheses are offered. Partners all actively participate, and opinions are sought and considered before decisions are jointly made. (ibid, p. 72)

The culture of exploratory talk is such that "half-baked" thoughts are welcome and considered learning opportunities. In the case of *dialogical inquiry*, a "safe" space is created for asking any question, and nothing is ruled out or pre-determined. In our team's dialogical space, at times, Mathematics Educators experienced mathematical expositions as difficult while Mathematicians experienced didactic-theoretical discussions challenging. These situations required honesty and openness, as well as mutual professional respect and trust.

We also found the technique known as the "steelman" argument useful for ensuring that we understood each other, that our meanings aligned and that there were no hidden conflicts. The "steelman" argument (and its opposite, the "strawman" argument) is used in the debating field. In debate terminology, the "strawman" argument occurs when "one misrepresents an opponent's position in a way that imputes to it implausible commitments, and then refutes the misrepresentation instead of the opponent's actual view" (Talisse & Aikin, 2006, p. 345). The misrepresented argument (the strawman) is then easier to attack. For example, if a debater argues that $CO_2$ emissions should be cut considerably within 20 years to avoid dangerous consequences of climate change, the opponent might use a strawman argument by attacking the debater of being an alarmist that only wants to instil fear by declaring that the world will end in 20 years. Hence, the original argument of the debater is misrepresented—usually on purpose but it might happen inadvertently due to the complexity of an argument—and never addressed nor refuted.

The opposite of a "strawman" argument is the "steelman" argument, first proposed by philosopher Daniel Dennet. The aim is to understand someone's argument and make it stronger by summarising or explaining the debater's argument in the opponent's own words (heteroglossia). This shows respect for the opponent's position (Friedersdorf, 2017). Once debater and opponent agree that their meanings of the original argument are aligned, they can engage in debating on a common ground. In the previous example, the opponent can rephrase the argument by asking: "Are you saying that there is scientific evidence that if we continue with current levels of $CO_2$ emissions, in about 20 years there will be catastrophic consequences to the Earth's climate?"

Even though *dialogical inquiry* such as ours does not take place within a debate, the concept of the steelman argument is valuable in bringing together and strengthening shared points and meanings, particularly when confronted with complex ideas. Our use of the "steelman" argument technique allowed us as researchers to accentuate differences in meanings and interpretations of the data and to pinpoint disagreements that needed further discussion. The concept of the "steelman" argument is compatible with a Bakhtinian viewpoint of a dialogue between people with different knowledge, expertise and socio-cultural backgrounds. The "steelman" argument can help bring similar superaddressees into the dialogue, facilitating the alignment of meanings. For example, steelmanning an opponent's basic premises or facts about climate change might ease the discussion about its potential future impact on Earth. However, it might not always be possible to agree, and the researchers should be honest in recognising the limits of their own knowledge or ability. In such cases, a need may arise to find ways of overcoming such limitations if feasible within the framework of the research project (e.g., read a book/papers on a particular theoretical framework), or exercise reasonable trust in the expertise of others if this would not impact seriously on future inquiry. In our paper (Hernandez-Martinez et al., 2023), we argue that a superaddressee is a particular type of boundary object that allows participants in a dialogue to cross a boundary into a "third space" (Gutierrez et al., 1999).

The space created by the *dialogical inquiry* methodology allows for negotiation and shared, hybrid interpretations of data and, at the same time, permits Mathematicians and Mathematics Educators to learn more about the practices of each other and reshape new

identities for themselves. In Bakhtin's words, every utterance is "half-ours and half-someone else's" (Bakhtin, 1981, p. 345).

We will now briefly present the background of this work—how we came to work together and developed this methodology—and exemplify our use of *dialogical inquiry* during data analysis in two areas: (1) understanding Mathematics and (2) understanding educational theory. These two areas detail our enactment of the principles of *dialogical inquiry* as a methodology for collaboration where shared meanings emerge.

## 4 *Dialogical inquiry* in collaborative Mathematics Education research

### 4.1 Background to the collaboration

The seeds of our collaboration were planted in 2016, when Yuriy and Svitlana (Mathematicians working in Norway) attended a Mathematics Education meeting. As a practitioner teaching differential equations (DEs) mainly to engineering students, Svitlana realised that problems concerning the understanding of the nature of different types of solutions of DEs are rarely included in the curriculum, which can lead students to have misconceptions in their learning.

The idea of using non-standard problems crystallised after Yuriy and Svitlana attended a conference talk about the use of similar problems in Calculus. It was then that they talked with Stephanie (Mathematics Educator working in the UK) and discussed their ideas about using non-standard problems in their teaching, asking if she would be interested in helping to design the intervention and analyse the data. They felt that a Mathematics Educator could help them design the study, particularly the theoretical and methodological frameworks. Together, they set up the intervention, collected data and presented results at international conferences. They started to realise that attendants were more interested in discussing the tasks they had developed rather than in their educational outcomes and implications. They felt that something else was needed.

At another international conference, Yuriy and Svitlana came into conversation with Paul (Mathematics Educator working in Australia) who was interested in the project. This was the start of our discussions as a group about the appropriate methodology to analyse the data. By this time, the global COVID-19 pandemic restricted international travel and we found ourselves having to communicate exclusively via online methods. We tried traditional methodologies such as coding, which we usually performed separately, coming together to discuss those cases where we disagreed. We soon realised that the rigidity and dichotomic nature of these methodologies did not offer the type of dialogue and reflexivity that we needed, especially when *creating* new meanings. In particular, we were searching for a methodology that allowed us *all* to make sense of the data, align our meanings and learn from each other. Svitlana and Yuriy were not looking for "consultants" to do a job for them, and Stephanie and Paul were not seeking for providers of data to write an academic paper.

We discussed various methodologies for quite a while until we came to realise that there was no formal methodology in Mathematics Education research where dialogue was the analytical tool. This led us to consider Bakhtinian theory as a framework for our work. We should note that this realisation was not immediate, but it progressed as we further engaged in dialogue. Reflexivity was a crucial aspect in looking back at our actions and realising what happened. The methods and principles of *dialogical inquiry* that we show here are

the consolidation of these experiences along with our continuous development in understanding and applying Bakhtinian theory.

We now present examples of our collaborative work to illustrate how we used *dialogical inquiry*. These examples come from the recordings of our weekly meetings. We assigned recordings to each team member so that each had the same amount to listen to and each had recordings from various time periods (i.e., early, middle and towards the end of the collaboration). Once we had located all examples in the recordings, we discussed them and chose two that are representative of the methodology that we are presenting in this paper. In addition, we wrote individual reflections on how the methodology impacted us, and these form the basis of the last part of this section.

### 4.2 Understanding Mathematics—the case of solutions to ordinary differential equations (ODEs)

As a first example, we show our collaborative meaning-making process of the concept of solutions to ODEs.

**Meeting 32** During this meeting, the discussion focused around understanding why the students in our study experienced difficulties in realising the difference between a solution and the general solution to a first-order linear ODE. The inquiry should always have a focus: What is it that we are attempting to understand by engaging in a dialogue? While analysing the students' data, the team looked at the problem from a mathematical perspective, and how the Mathematics concepts involved connect to each other.

At some point in the discussion, Paul brought up his memories of being an undergraduate student and how, when studying second order ODEs, he did not understand at that time the meaning of the general solution and why this solution took a certain form (i.e., the sum of the general solution to the homogeneous equation and a particular solution to the inhomogeneous equation). This prompted Yuriy and Svitlana to discuss this concept in terms of Linear Algebra concepts (e.g., linear independence, space of solutions, linear combinations). We agreed that this was a good way to make meaning of the concept of general solution. However, our data showed that many of our students were not able to link the concepts studied in Linear Algebra and the concepts studied in Calculus (e.g., continuity, differentiability, integrating factor method). From an educator's perspective, Paul suggested a diagram was needed to show the connections between all these concepts (i.e., the map of scientific concepts relating to solutions of ODEs). This diagram could then help us in the analysis of our data and in finding the disconnections that students might be experiencing in their learning. Svitlana created a draft of this diagram for our next meeting.

**Meeting 33 (1 week later)** Svitlana explained the diagram that she had created (Fig. 1). During this meeting, Paul and Stephanie's mathematical meanings developed through questioning and the use of the steelman technique, as we show now: Paul first tried to reword Svitlana's explanations, which prompted a question: "If I'm understanding correctly, the problem comes from the fact that… so you have the first order linear DE… do you have a homogeneous and an inhomogeneous equation in a first order DE?" Stephanie added: "When you are doing the integrating factor, when you have a perfect integral on one side, in theory there is a homogeneous and a non-homogeneous part, but we never

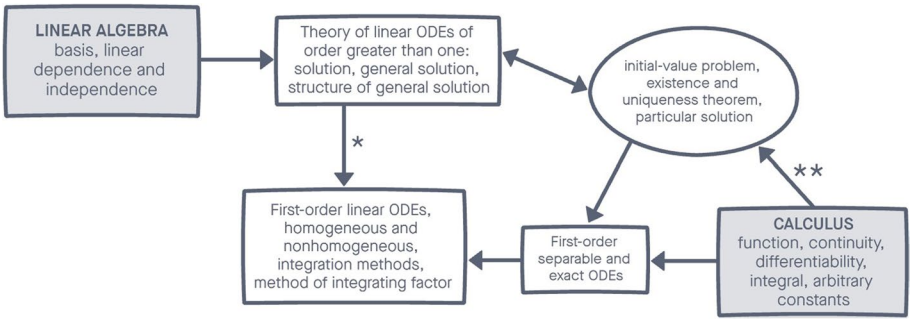


**Fig. 1** The system of scientific concepts relating to linear ODEs

think of it like that." Svitlana agreed, adding that "in DEs, when they [the students] come to this course, it's not explained so good, that the first order equations are part of this general theory and can be treated the same way [as higher order DEs]."

Paul was still dissatisfied with the explanations and asked again: "How would this Linear Algebra approach work with first order DEs?" Svitlana opened a discussion, where the crucial detail for Paul's meaning-making process was the "linear independence" of the only one solution to the homogeneous equation (which still forms a basis), represented by the part of the solution that contains the constant of integration. Stephanie then repeated this explanation with her own example, steelmanning the argument. Finally, Paul also steelmanned the argument, ensuring the alignment of the mathematical meanings from everyone in the team.

Following this discussion, Paul asked Svitlana if she could write a paragraph explaining what she had described so that this piece of writing could then be used in a subsequent meeting to review and refine the diagram she produced. However, the verbal explanation itself was enough to bring out important queries. For example, Paul asked: "Something that is missing still is the link between continuity, differentiability, …." Svitlana realised that "this is a good question. It is good that you see it because I was thinking 'where could we put it?', these are characteristics of the solutions." This method allowed the team to refine the diagram, identify where students' learning disconnections occurred (shown as * and ** in Fig. 1) and further develop our meanings of the concepts. The diagram made it possible to create further connections in dialogue (Midgley, 2011). In Hernandez-Martinez et al. (2022), we report an example of the use of *dialogical inquiry* as qualitative analytical tool, where we use this methodology to analyse student data and further discuss the diagram in Fig. 1.

We can see in this example how researchers with different expertise come together in a safe space, without fear of being judged. This was evidenced by the type of questioning observed mainly from the Mathematics Educators in this case, which some might say are "basic" questions concerning fundamental concepts on differential equations. "Basic" as they were for some of the participants in this team, these questions allowed them to make meaning and also to learn from each other. In this example, not only did the Mathematics Educators expand their understanding of nuanced mathematical aspects but the Mathematicians also learned what is important to look at in educational terms (e.g., examine for missing connections that might explain gaps in students' knowledge and associated learning difficulties).

We can also see that in this synergy of expertise there was a balance of power: everyone contributed to the ideas that eventually crystallised in the diagram, rather than leaving the Mathematics to the Mathematicians and the educational aspects to the Mathematics Educators. For example, Svitlana needed to think as an educational researcher to create a draft of a diagram that could be used to explain students' learning or lack thereof. On the other hand, acting as a Mathematician, Paul contributed to the discussion by identifying missing links between important Mathematics concepts. A willingness to learn and understand the issues at hand by asking repeatedly until satisfied was also shown in this instance, and this was an important characteristic of the inquiry. Also, the Mathematicians did not impose authoritatively their views on mathematical argumentation, instead using an internally persuasive discourse.

### 4.3 Understanding educational theory—the case of the Bakhtinian superaddressee

In our second example, we show our meaning-making process of a theoretical education concept (the superaddressee). The focus of this particular inquiry was to understand what Bakhtin meant by the concept of the superaddressee and how we could use it while analysing our students' dialogic processes of meaning-making.

**Meeting 4** In this meeting, Yuriy talked about what he perceived as a connection between the concept of superaddressee and the notion of warrant in Toulmin's theory (Toulmin, 1958), concerning students' group discussions about ODEs and how our data showed students' attempt to use warrants for their arguments. Yuriy said:

> When we think of the superaddressee, to some extent I thought that it might be related to this like, warrant, what makes the student believe that this is the correct way of answering this question. It is either, 'I saw it in the lecture that [the lecturer] gave', or 'I saw it in the book', or 'This is my calculator', or 'This is the theory and what it says'.

He speculated that for some people, the superaddressee is external to them (as in the previous examples), but he was not sure if the superaddressee could also be considered part of the internal dialogue, or the warrants that students make when in internal conversation with themselves.

Paul argued:

> But it [the superaddressee] has to be different from you, that is why it is a third person […] So if I say, 'This is what my calculator is saying', I am making an argument that comes from me but I'm taking someone [or something, in this case] else to back, or to warrant my interpretation of things.

Yuriy refuted that "the dialogue with the third is somehow inside your head before you come with this [the argument]".

Svitlana proceeded to provide a clarifying argument:

> Yes, but Bakhtin says that every word has been said before […] What you are saying about it [the argument presented in a discussion] is your own interpretation,

> everybody gets information from somewhere, it's not like we are inventing things […] we always need some background, we need something to work on […] I think we should not discuss the question of how it comes to our head because this is a completely different discussion and I don't think it is in the scope of this paper.

The discussion about the superaddressee during this meeting ended by deciding that we would reflect on what was said and look at how the students made use of superaddressees. We revisited the concept one month later in Meeting 8.

**Meeting 8** During this meeting, we discussed students' work and how they chose to use their calculator to plot the functions in one of the non-routine problems they were given. Svitlana confirmed that this problem did not ask them to plot the functions, so the question arose of how they came up with the idea of graphing? Paul suggested that in their quest to make meaning of the situation, they maybe remembered using the calculator in a similar way at school or being told by the teacher to produce a graph whenever they needed to better understand a mathematical situation. Stephanie and Svitlana agreed that this could be a superaddressee, the voice of another (a calculator), a reference to someone or something that is trusted to bring about meaning.

Yuriy brought back his previous argument about the superaddressee being used in internal conversations, with oneself. It was clear to us that this is possible, if we consider an internal conversation as the first and second person (the same person), and the superaddressee as a third person that is brought in to convince oneself about the "correctness" of a thought or internal argument. This can also be seen as a "preparation" for a dialogue with others (i.e., one needs to convince oneself before attempting to convince others).

Even though the dialogue brought light to the discussion, Yuriy asked if we should differentiate between a superaddressee, which "should be a very solid reference", and what is the normal process of making meaning which, according to Bakhtin, always involves other voices. It was decided that we would continue to think about this concept.

**Meeting 14 (6 weeks later from meeting 8)** During this meeting, we reconsidered our dialogue on the superaddressee. Yuriy brought forward a quotation from Bakhtin: "The understanding itself as dialogic element enters the dialogic system and somehow changes its total sense. The one that understands inevitably becomes 'the third' in the dialogue." He argued that, for example, the calculator can be a superaddressee because when trying to convince another, we reference the calculator as something that understands (Mathematics) and produces the graph of the function on which we base our argumentation. We concluded that the superaddressee always appears in argumentation, "in a way that it makes you believe or agree" (Svitlana). The superaddressee is part of the dialogic process involving more than one person always—which is substantially different to the meaning-making process—which can occur as part of self-reflection. In the former, the superaddressee is one that *already* understands, hence cannot be a part of oneself.

This example again shows the *honesty* and *openness* of the *dialogical inquiry* in bringing about ideas that are "half-baked" that need to then be discussed in order to further develop their meaning. It shows the importance of bringing up details within a dialogue without fear of judgement or pressure to conform or rushing to conclusions. There is tolerance of uncertainty and at the same time willingness to pursue new meanings and better understandings.

It also shows a mechanism for solving disagreements. For example, Yuriy brought back an argument about the superaddressee that was not agreed upon by the group, even though it might have sounded convincing. Reflection and further reading in this case brought illumination to the meaning of what exactly is a superaddressee, until everyone in the group was satisfied that meanings had aligned. This reflection and search for further enlightenment from external sources take time, which means that the inquiry is most often slow, a key characteristic of this methodology. Even when agreement seemed to come immediately, arguments were made on why we thought this was the case. For example, we all agreed very early in our discussions that a calculator could be a superaddressee (Hernandez-Martinez et al., 2023). But we did not take this for granted and gave our reasons why we thought it meaningfully matched Bakhtin's definition of the concept. Only then, did we agree that our meanings on this particular issue seemed to align, but having in mind that "there can never be neither a first nor a last meaning" (Bakhtin, 1986, p. ix). Indeed, one of the reviews to this paper have opened for us a slightly different perspective between the concepts of the superaddressee and that of Toulmin's warrant, which we will continue to discuss and address in future work.

We now present our reflections as researchers on the use of this methodology in collaborative research in advanced Mathematics Education.

### 4.4 Reflexivity in *dialogical inquiry*

One of the distinctive features of *dialogical inquiry* is the promotion of highly reflective collaborations. This reflexivity, in turn, allows researchers to "learn something new about their own and others' practices" (Akkerman & Bakker, 2011, p. 145) and by "making explicit the knowledge and assumptions mobilized in the interpretation of the object" (Hoyles et al., 2007, p. 335). All four of us are practitioners, teaching mathematics at university level; but our backgrounds are different and often we have different perspectives on mathematics and its teaching and learning. However, we all strongly feel that through the use of *dialogical inquiry*, we were continuously "stepping in and out", having a synergistic experience. We learned from each other and from external sources and, with a degree of surprise, we gradually realised that by engaging in this methodology, we also developed new understandings. For example, Svitlana feels that "this kind of discussions is an eye-opener in one's own professional field". Yuriy recounts that he "clashed" on occasions with Svitlana (i.e., examples of embodied responses [Wells et al., 2021]) and

> this led to the rethinking of mathematical details, their better understanding and consequent better explanation. […] Heated 'maths battles' helped me (and others) understand mathematical peculiarities and students' difficulties with maths better thus bringing our discussions to a much higher level witnessed not so often in math ed research.

Stephanie and Paul feel they expanded their understanding of sociocultural theory and, in particular, can now use Bakhtinian theory with more confidence in new research projects. Reflection on our experiences, that is, looking back on what we learned from engaging in collaboration is a key component of *dialogical inquiry*. But reflexivity was also part of the meaning-making process, shown by taking the necessary time to digest what was said in dialogue, by looking for further illumination to bring back different perspectives to the discussion table and by questioning issues until one is satisfied with a certain level of

understanding, "making explicit the knowledge and assumptions mobilized in the interpretation of the object" (Hoyles et al., 2007, p. 335).

Using reflexivity, Stephanie and Paul describe their crossing into Mathematics and how they genuinely learned new things. Stephanie writes that she has

> been seeking collaborations of this nature as I have always felt that Mathematicians should be involved in research into teaching and learning of University Mathematics as their mathematical viewpoint is from a much different and more elevated angle than that of the vast majority of education researchers.

She feels that Svitlana and Yuriy "patiently explained" what she needed to understand and as a consequence her mathematical understanding progressed, albeit slowly. Paul remembers the time when first-order linear DEs were discussed in terms of Linear Algebra, and how he kept questioning until there was an "aha" moment for him. These "aha" or "eureka" moments have been identified in the literature as important human experiences "marked by a profound sense of transformation" (Crowther & Schmidt, 2015, p. 54) that play a significant role in understanding and learning (Liljedahl, 2005). Svitlana writes: "I often had this 'aha' feeling. It came to me on different occasions; sometimes it was about understanding something new from the theory of Mathematics Education but sometimes it concerned Mathematics, and this was very striking for me." Yuriy distinctively remembers how at the beginning of the collaboration he was "overenthusiastic" about suggesting ideas/quotes about educational research, but Paul and Stephanie would not share his enthusiasm. However, by

> reflecting about these situations, I gradually learned how to better build my arguments making them more convincing. Now I intervene with overtly bold suggestions less frequently. I am pleased to hear/see that my growing expertise in maths education is being acknowledged not only in our small group but also by other maths educators.

In fact, Stephanie writes that she "believes that both Yuriy and Svitlana have developed their understanding of educational research to a point where they would be well equipped to carry out such research on their own." It is important to note our genuine surprise at how not only our knowledge of each other's area grew but also how we saw ourselves changing and doing things differently as a result of participating in *dialogical inquiry*.

By reflecting back, we believe that our shared understanding of students' learning of DEs has developed as a result of the collaboration, and that this will have an impact on our own teaching practices. For example, Paul is certain that he will incorporate what was learned through this project in his teaching of DEs, making his mathematical teaching practice more solid. Stephanie writes:

> The topic of DEs was one that I felt comfortable with but have had little opportunity to teach myself. Our research has shown a broken link (or gap) in university level teaching that I am certain I could not have identified myself (without the collaboration). It also made me think about the need to bridge this gap if I were to teach first- or second-year students any DEs.

Svitlana reflects on how she came to understand "how important students' discussions can be for their learning of Mathematics. Introducing students' peer dialogues into Mathematics Education can be very beneficial for the creation of mathematical meanings and development of conceptual understanding by the students."

## 5 Discussion

### 5.1 The methods of dialogical inquiry

One of the characteristics of dialogue is its unpredictable nature, where the flow of communication is constantly adjusted as it evolves (Levy & Murname, 2004). In the previous sections, we exemplified *our* dialogues while inquiring into issues, but we were aware of "how this tiny bit of dialogue could have gone in a hundred other directions" (Sawyer, 2003, p. 4). However, unpredictability makes creativity possible (Boston Change Process Study Group, 2005), and is hence desirable in bringing about new meanings and interpretations in qualitative research. The aim of a dialogical analysis is not to arrive at a "correct" meaning, because truth is "forever testable" (Morson, 2004, p. 319); the aim "is not to recover a singular meaning, but to make sense of the different and ambiguous ways in which a meaning may be experienced" (Sullivan, 2012, p. 14). These different meanings allow us as researchers to have a better understanding of, for example, how our students learn or how a theoretical concept might be applied in research.

Having reflected hard upon this, we concurred that *dialogical inquiry* as a method for Mathematics Education research cannot be defined by a set of specific rules that become rigidly prescriptive and provide a determined sequence of steps for researchers to follow. It is in this sense that the methods of *dialogical inquiry* are unlike those in other methodologies that are based on different ontologies and epistemologies. The methods of *dialogical inquiry* can only be described in general terms. With this in mind, we delineate these general rules in the following paragraphs.

To operationalise *dialogical inquiry*, we took some of the methods of "exploratory talk" (Mercer & Wegerif, 2004), adding dialogic rules and the steelman technique to be able to align our meanings. Hence, dialogue starts with a shared aim, for example, to understand the concept of Bakhtinian superaddressee or to understand the meanings that certain students have concerning solutions to ODEs.

Then, participants engage in critical but constructive discussion of each other's ideas or interpretations of the data, challenging and counterchallenging them and offering alternative hypotheses. This process requires conscious engagement and committed listeners and talkers. Reflexivity is crucial in this process, requiring the necessary time to digest what has been discussed and to seek further knowledge and possible explanations. New meanings emerge from the active, honest, open-minded and respectful participation of all partners; by constantly seeking to (re)phrase our understanding of the other's meanings (i.e., steelmanning), we consider everyone's thoughts and opinions before decisions are jointly taken.

These methods cannot be understood without reference to its guiding principles, which provide further directions on what is necessary to consider when using *dialogical inquiry* in research.

### 5.2 Principles of *dialogical inquiry* as a methodology

Three main principles of *dialogical inquiry* (Fig. 2) were developed through our reflections and theoretical understandings of Bakhtin's theory:

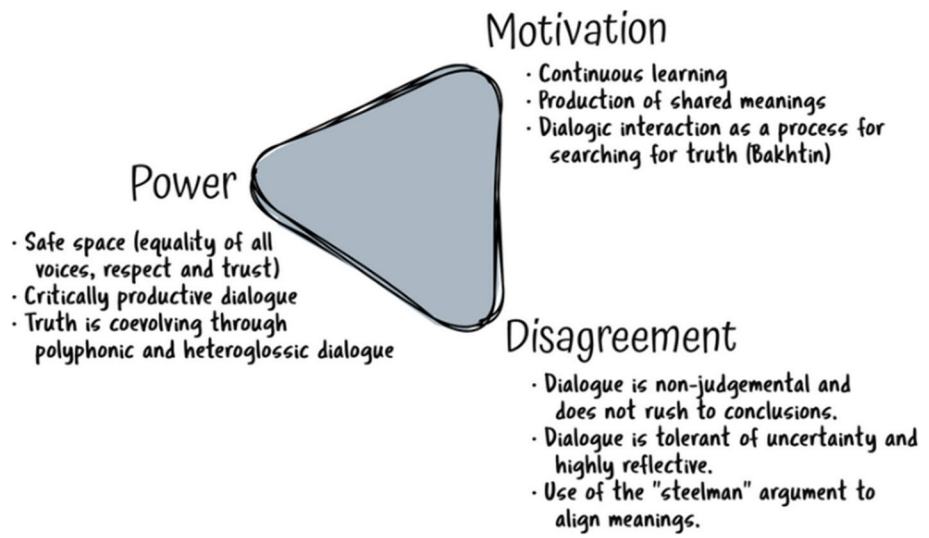


**Fig. 2** Principles of *dialogical inquiry* in Mathematics Education research

(1) *Motivation*. In *dialogical inquiry*, researchers from different subject areas come together not solely to solve a problem, but also to learn and extend their knowledge by engaging with others. They acknowledge that others have expertise that they do not, and that this expertise is required to gain insight into the research problem. However, instead of relying on others to provide expertise, they are interested in a synergy, in continuously learning, understanding and making their own meanings from establishing a critically productive dialogue with others. They are not interested in crossing entirely to another field: Mathematicians do not want to abandon their research in Mathematics and Mathematics Educators are not going to pursue research in Mathematics. A never-ending source of complementary differences is the engine that motivates the collaboration. As a basic tenet of this methodology, researchers agree on the nature of the inquiry: "truth is not born nor is it to be found inside the head of an individual person, it is born between people collectively searching for truth, in the process of their dialogic interaction" (Bakhtin, 1984, p. 11).

(2) *Power balance*. In *dialogical inquiry*, all voices are valued and heard equally and there is no unreasonable claim to power or authority. Each participant can initiate a discussion; others listen and contribute. This balance of power opens a safe space where researchers can discuss all matters, even "half-baked" thoughts (Mercer & Wegerif, 2004). The aim is to negotiate shared meanings through critically productive dialogue rather than a competitive or combative debate. As already stated, researchers conceive "truth" as never finalised but always (co-)evolving through dialogue that is polyphonic and heteroglossic.

(3) A process for solving *disagreement*. For Bakhtin, dialogue is only possible when differences exist. If two voices were to coincide, then the flow of meaning would cease (Bakhtin, 1984). Therefore, any methodology based on Bakhtinian principles must include a process to deal with such differences of meaning, or disagreements. *Dialogical inquiry* takes the form of a dialogue that is slow and non-judgemental and

does not rush to conclusions, tolerant of uncertainty and highly reflective. There is an acknowledgement that arguments and counterarguments are embodied in different ways of speaking and acting but everyone's opinions are valued, heard and reflected upon. Honesty goes hand in hand with professional trust and respect. Researchers acknowledge that for the purposes of reporting and publishing outcomes, there should be "agreement". But this agreement comes from a reflective process. We found particularly useful the use of the "steelman" argument to align our meanings. Even when researchers have an initial immediate agreement on certain issues, it is still expected that reasons for agreement are spelled out. This spelling out ensures nothing is taken for granted, and that there is alignment. This might bring about details that were overlooked or that need further discussion.

These three principles interrelate and influence each other (hence, the "moving" representation of the triangle in Fig. 2). For example, when the power is balanced, disagreements are solved through an internally persuasive discourse rather than an authoritative one (Bakhtin, 1981). Or when the motivation of the inquiry is to learn from one another, researchers see themselves as equals. This rarely happens in collaborations where, for example, Mathematicians are "invited guests", but the analysis of data is almost entirely performed by Mathematics Educators who are experts on the analytical or theoretical frameworks, or in collaborations where postdoctoral researchers are part of the team and, usually, more senior researchers will have a more authoritative voice in the research even though, in theory, all are part of the same team.

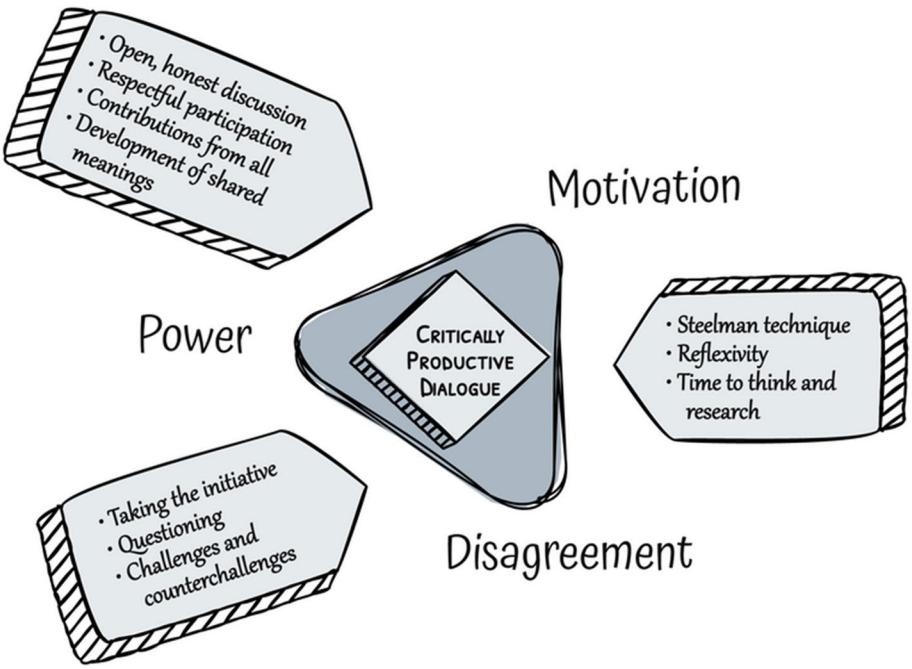


**Fig. 3** *Dialogical inquiry* as a methodology

Researchers who want to use *dialogical inquiry* must make an ontological commitment to its principles and rules. *Dialogical inquiry* is not, for instance, an informal talk between colleagues in the department's corridor. It is a systematic search for "truth" but within a worldview where meanings are never final. Figure 3 represents a full view of *dialogical inquiry* as a methodology, with its general rules and principles. In this figure, the methods are placed near the principles (detailed in Fig. 2) that influence them the most, but all of these methods can be thought of as shaped by all three principles.

It is also clear from these principles that *dialogical inquiry* differs from research-based models of successful teamwork (cf. Hackman, 2002) in significant ways, while also sharing some important aspects (e.g., trust, respect). In particular, in *dialogical inquiry*, there is no leadership. The principle of "balance of power" means that everyone's expertise is necessary in all tasks, and all are responsible for the outcomes of the project, which belong to the team and not to a particular or leading individual. *Dialogical inquiry* is powerful in translating into long-lasting collaborations where new topics of interest emerge. In contrast, non-dialogical collaborations usually result in short-lived interactions where everyone returns to their own research at the end of the project. Although there are general aims for the research study, within *dialogical inquiry*, the dialogue leads the data analysis and the pace at which the research progresses: it is unconceivable to talk about, for example, "SMART" objectives (i.e., Specific, Measurable, Attainable, Relevant and Time Bound).

## 6 Conclusions

Collaborations between Mathematicians and Mathematics Educators are crucial in advancing knowledge in Mathematics Education research, particularly when it comes to advanced Mathematics where the knowledge and expertise of Mathematicians is necessary. However, there is an absence of collaborations in university Mathematics Education where Mathematicians and Mathematics Educators truly are equal partners, sharing the responsibility for creating common meanings and interpretations. In this paper, we presented a methodology for collaborative inquiry and qualitative data analysis—*dialogical inquiry*—rooted in Bakhtin's theory of dialogism, as a way to promote these partnerships where there is a symbiosis between Mathematicians and Mathematics Educators, without one having to become the other.

We exemplified our operationalisation of the methodology in two areas of inquiry, showing its key characteristics in action. We also provided our reflections on the use of this methodology, and synthetised the general rules and principles that are at the heart of *dialogical inquiry*.

In introducing this methodology to the Mathematics Education research community, we want to encourage more "healthy" collaborations between Mathematicians and Mathematics Educators. At the beginning of this paper, we referred to *dialogical inquiry* as a "living theory" (Hirst, 2012). Hence, we also want to invite readers from these communities to engage with us and others in dialogue, to reflect on the principles of this methodology. We expect that others will refine and enhance it with their own experiences, as they form their own dialogical spaces of collaboration, and engage with others in the always unfinalised search for dialogic collective truth (Bakhtin, 1984, 1986).

**Acknowledgements** The authors gratefully acknowledge a small research award from MatRIC, the Norwegian Centre for Research, Innovation and Coordination of Mathematics Teaching, project number 170803.

**Funding** Open Access funding enabled and organized by CAUL and its Member Institutions

**Data availability** The data that support the findings of this study are available from the corresponding author upon reasonable request.

## Declarations

**Conflict of interest** The authors declare no competing interests.



## References

Akkerman, S. F., & Bakker, A. (2011). Boundary crossing and boundary objects. *Review of Educational Research, 81*(2), 132–169. https://doi.org/10.3102/0034654311404435

Aubert, A., & Soler, M. (2007). Dialogism: The diagotic turn in the social sciences. In J. L. Kincheloe, & R. A. Horn (Eds.), *The Praeger Handbook of Education and psychology* (3 vol., pp. 521–529). Praeger Publishers.

Aveling, E. L., Gillespie, A., & Cornish, F. (2015). A qualitative method for analysing multivoicedness. *Qualitative Research, 15*(6), 670–687. https://doi.org/10.1177/1468794114557991

Bager, A. S. (2013). Dialogue on dialogues: Multi-voiced dialogues (dialogism) as means for the co-production of knowledge in and on leadership communicative practices. *Academic Quarter, 6*, 146–159. https://doi.org/10.5278/ojs.academicquarter.v0i6.2858

Bakhtin, M. (1981). *The Dialogic Imagination: Four essays*. University of Texas Press.

Bakhtin, M. M. (1984). *Problems of Dostoevsky's poetics*. University of Minnesota Press.

Bakhtin, M. M. (1986). The problem of speech genres. In C. Emerson, & M. Holquist (Eds.), *Speech Genres and other late essays* (pp. 60–102). University of Texas Press.

Bardini, C., Bosch, M., Rasmussen, C., & Trigueros, M. (2021). Current interactions between mathematicians and researchers in university mathematics education. In V. Durand-Guerrier, R. Hochmuth, E. Nardi, & C. Winslow (Eds.), *Research and Development in University Mathematics Education. Overview produced by the International Network for Didactic Research in University Mathematics* (pp. 41–58). Routledge.

Barton, B., Oates, G., Paterson, J., & Thomas, M. (2015). A marriage of continuance: Professional development for Mathematics lecturers. *Mathematics Education Research Journal, 27*, 147–164. https://doi.org/10.1007/s13394-014-0134-7

Barwell, R. (2016). Formal and informal mathematical discourses: Bakhtin and Vygotsky, dialogue and dialectic. *Educational Studies in Mathematics, 92*(3), 331–345. https://doi.org/10.1007/s10649-015-9641-z

Bosch, M., Hausberger, T., Hochmuth, R., Kondratieva, M., & Winsløw, C. (2021). External didactic transposition in Undergraduate Mathematics. *International Journal of Research in Undergraduate Mathematics Education, 7*(1), 140–162. https://doi.org/10.1007/s40753-020-00132-7

Boston Change Process Study Group. (2005). The "something more" than interpretation revisited: Sloppiness and co-creativity in the psychoanalytic encounter. *Journal of the American Psychoanalytic Association, 53*(3), 693–729.

Crowther, C., & Schmidt, M. (2015). States of grace: Eureka moments and the recognition of the unthought known. *The Journal of Analytical Psychology, 60*(1), 54–74. https://doi.org/10.1111/1468-5922.12129

Darragh, L. (2022). Brokering across the divide: Perspectives of mathematicians involved in education. *The Journal of Mathematical Behavior, 67*, 100989. https://doi.org/10.1016/j.jmathb.2022.100989

Doorman, M., Drijvers, P., Gravemeijer, K., Boon, P., & Reed, H. (2013). Design research in Mathematics education: The case of an ICT-rich learning arrangement for the concept of function. In T. Plomp, & N. Nieveen (Eds.), *Educational Design Research – Part B: Illustrative cases* (pp. 425–446). SLO.

Fried, M. N., & Dreyfus, T. (Eds.). (2014). *Mathematics and Mathematics Education: Searching for Common Ground*. Springer Science & Business Media.

Friedersdorf, C. (2017). June 26). The highest form of disagreement. The Atlantic. https://www.theatlantic.com/politics/archive/2017/06/the-highest-form-of-disagreement/531597/. Accessed 15 Nov 2022.

Garrick, J. (1999). Doubting the philosophical assumptions of interpretive research. *International Journal of Qualitative Studies in Education, 12*(2), 147–156. https://doi.org/10.1080/095183999236222

González-Martín, A. S., Biza, I., Florensa, I., Gueudet, G., Rasmussen, C., Thoma, A., & Viirman, O. (2022). Introduction to the papers of TWG14: University mathematics education. In J. Hodgen, E. Geraniou, G. Bolondi & F. Ferretti (Eds.), *Proceedings of the 12th Congress of the European Research in Mathematics Education (CERME12)*. ERME.

Goos, M., & Bennison, A. (2018). Boundary crossing and brokering between disciplines in pre-service Mathematics teacher education. *Mathematics Education Research Journal, 30*(3), 255–275. https://doi.org/10.1007/s13394-017-0232-4

Gutierrez, K., Baquedano-Lopez, P., & Tejeda, C. (1999). Rethinking diversity: Hybridity and hybrid language practices in the third space. *Mind Culture and Activity, 6*(4), 286–303.

Hackman, J. R. (2002). *Leading teams: Setting the stage for great performances*. Harvard Business Press.

Hannah, J., Stewart, S., & Thomas, M. (2011). Analysing lecturer practice: The role of orientations and goals. *International Journal of Mathematical Education in Science and Technology, 42*(7), 975–984. https://doi.org/10.1080/0020739x.2011.610008

Hernandez-Martinez, P., & Goos, M. (2014). Mathematics at university: Practices, values and participation. In P. Lilhedahl, C. Nicol, S. Oesterie & D. Allan (Eds.), *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education* (Vol.6, p.98). PME.

Hernandez-Martinez, P., Rogovchenko, S., Rogovchenko, Y., & Thomas, S. (2022). *"The theorem says…": Engineering students making meaning of solutions to ODEs* [Manuscript submitted for publication].

Hernandez-Martinez, P., Rogovchenko, S., Rogovchenko, Y., & Thomas, S. (2023). *The concept of super-addressee as a boundary object in mathematics learning* [Manuscript in preparation].

Hirst, P. (2012). *Educational Theory and its Foundation Disciplines (RLE Edu K)*. Routledge.

Hoyles, C., Bakker, A., Kent, P., & Noss, R. (2007). Attributing meanings to representations of data: The case of statistical process control. *Mathematical Thinking and Learning, 9*(4), 331–360. https://doi.org/10.1080/10986060701533326

Hyde, B., & Bineham, J. L. (2000). From debate to dialogue: Toward a pedagogy of nonpolarized public discourse. *Southern Journal of Communication, 65*(2–3), 208–223. https://doi.org/10.1080/10417940009373168

Jaworski, B., Mali, A., & Petropoulou, G. (2017). Critical theorising from studies of undergraduate mathematics teaching for students' meaning making in mathematics. *International Journal of Research in Undergraduate Mathematics Education, 3*(1), 168–197. https://doi.org/10.1007/s40753-016-0044-z

Kazak, S., Wegerif, R., & Fujita, T. (2015). The importance of dialogic processes to conceptual development in mathematics. *Educational Studies in Mathematics, 90*(2), 105–120. https://doi.org/10.1007/s10649-015-9618-y

Kondratieva, M., & Winsløw, C. (2018). Klein's plan B in the early teaching of analysis: Two theoretical cases of exploring mathematical links. *International Journal of Research in Undergraduate Mathematics Education, 4*(1), 119–138. https://doi.org/10.1007/s40753-017-0065-2

Konstantinou-Katzi, P., Tsolaki, E., Meletiou-Mavrotheris, M., & Koutselini, M. (2013). Differentiation of teaching and learning Mathematics: An action research study in tertiary education. *International Journal of Mathematical Education in Science and Technology, 44*(3), 332–349. https://doi.org/10.1080/0020739x.2012.714491

Kontorovich, I. (2021). Minding Mathematicians' discourses in investigations of their feedback on students' proofs: A case study. *Educational Studies in Mathematics, 107*(2), 213–234. https://doi.org/10.1007/s10649-021-10035-2

Levy, F., & Murnane, R. J. (2004). Education and the changing job market. *Educational Leadership, 62*(2), 80.

Liljedahl, P. (2005). Mathematical discovery and affect: The effect of AHA! Experiences on undergraduate Mathematics students. *International Journal of Mathematical Education in Science and Technology, 36*(2–3), 219–234. https://doi.org/10.1080/00207390412331316997

Matias, C. E. (Ed.). (2021). *The handbook of critical theoretical research methods in Education*. Routledge.

Matusov, E., & von Duyke, K. (2010). Bakhtin's notion of the internally persuasive discourse in education: Internal to what? In K. Junefelt & P. Nordin (Eds.), *Proceedings from the Second International*

*Interdisciplinary Conference on perspectives and limits of dialogism in Mikhail Bakhtin*. (pp.174–199). Stockholm University.

Mercer, N., & Wegerif, R. (2004). Is 'exploratory talk' productive talk? In H. Daniels, & A. Edwards (Eds.), *The RoutledgeFalmer Reader in psychology of education* (pp. 67–86). Routledge Falmer.

Mesa, V., Shultz, M., & Jackson, A. (2020). Moving away from lecture in Undergraduate Mathematics: Managing tensions within a coordinated inquiry-based Linear Algebra course. *International Journal of Research in Undergraduate Mathematics Education, 6*(2), 245–278. https://doi.org/10.1007/s40753-019-00109-1

Midgley, W. (2011). Look who's listening: Using the superaddressee for understanding connections in dialogue. In L. Abawi, J. M. Conway, & R. Henderson (Eds.), *Creating connections in teaching and learning* (pp. 153–163). Information Age Publishing.

Morson, G. S. (2004). The process of ideological becoming. In A. F. Ball, & S. W. Freedman (Eds.), *Bakhtinian perspectives on language, literacy, and learning* (pp. 317–331). Cambridge University Press. https://doi.org/10.1017/CBO9780511755002.016

Nardi, E. (2007). *Amongst mathematicians: Teaching and learning mathematics at university level* (3 vol.). Springer Science & Business Media.

Russell, G. M., & Kelly, N. H. (2002). Research as interacting dialogic processes: Implications for reflexivity. *Forum: Qualitative Social Research, 3*(3), 18.

Sawyer, R. K. (2003). *Improvised dialogues: Emergence and creativity in conversation*. Greenwood Publishing Group.

Sullivan, P. (2012). *Qualitative data analysis using a dialogical approach*. SAGE Publications Ltd. https://doi.org/10.4135/9781446268391

Talisse, R., & Aikin, S. F. (2006). Two forms of the straw man. *Argumentation, 20*(3), 345–352.

Tall, D., & Katz, M. (2014). A cognitive analysis of Cauchy's conceptions of function, continuity, limit and infinitesimal, with implications for teaching the Calculus. *Educational Studies in Mathematics, 86*(1), 97–124. https://doi.org/10.1007/s10649-014-9531-9

Thomas, S. (2012). *An Activity Theory analysis of linear algebra teaching within university Mathematics* [Unpublished Doctoral thesis]. Loughborough University.

Toulmin, S. E. (1958). *The uses of argument*. Cambridge University Press.

Viirman, O., & Nardi, E. (2019). Negotiating different disciplinary discourses: Biology students' ritualized and exploratory participation in mathematical modeling activities. *Educational Studies in Mathematics, 101*(2), 233–252. https://doi.org/10.1007/s10649-018-9861-0

Wagner, J. (1997). The unavoidable intervention of Educational Research: A framework for reconsidering researcher-practitioner cooperation. *Educational Researcher, 26*(7), 13–22. https://doi.org/10.3102/0013189x026007013

Wells, R., Barker, S., Boydell, K., Buus, N., Rhodes, P., & River, J. (2021). Dialogical inquiry: Multivocality and the interpretation of text. *Qualitative Research, 21*(4), 498–514. https://doi.org/10.1177/1468794120934409