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Theorizing university mathematics teaching: the Teaching Triad within an Activity Theory perspective

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Abstract
In this paper, we draw on our recent research to inspect again some of the theoretical perspectives we have been using to analyze data and to characterize teaching-learning in university settings. We focus particularly, within a sociocultural perspective on Activity Theory and the construct the "Teaching Triad," seeking to embed the Triad within an Activity Theory perspective. To achieve this, we relate the Teaching Triad with aspects of the sociocultural setting both in and beyond direct interactions in face-to-face teaching. While this is mainly a theoretical paper, an example is taken from observations of teaching in university lectures in a Greek university to show how these theoretical perspectives have provided insights to the institutional and cultural complexities involved and in what ways the Triad construct has evolved.

Keywords University mathematics teaching and learning · Teaching Triad · Activity Theory

1 Introduction to university mathematics teaching

By university mathematics teaching, we refer to the teaching of mathematics which takes place at university level. In our own corpus of work, we are particularly interested in face-to-face teaching in lectures and tutorials in which teachers design their teaching for the benefit of students who attend their sessions (e.g., Jaworski et al., 2017). We are interested in uncovering relationships between teaching and learning within the sociocultural context of university life (e.g., Jaworski & Potari, 2021). This includes the institutional setting as well as the cultures from which teachers and students make sense of the interactions in

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which they engage. In particular, we seek to know more about "what teachers do and think daily, in class and out, as they perform their teaching work" (Speer et al., 2010, p. 99). Our research addresses:

> *What is it that mathematics teachers do and think as they perform their teaching work in a university setting* and how do they include students in their activity?

We see the *work* of *doing* and *thinking* as the basis of teachers' *activity* as we discuss below.

This question takes us into the *didactical* thinking of teachers who consider how best to enable students to think mathematically and develop understandings of mathematical topics. It includes teachers' *pedagogic* thinking in the ways in which they interact with students and use resources to promote students' engagement with mathematics. It also includes the ways in which teachers work within university affordances and constraints, the norms and expectations of university culture and their own educational histories, their views of mathematics and of what it means for students to learn mathematics, and so on.

In our work to date, we have used a number of theoretical perspectives to analyze data from teacher–student interactions in university mathematics teaching. Largely, we have taken a broad sociocultural perspective in which we have aimed to address both micro and macro aspects of teaching. In some of our work, we have more specifically used *Activity Theory* to examine relationships and issues in teaching (e.g., Jaworski & Potari, 2009, 2021; Jaworski et al., 2012; Potari, 2013). In order to analyze teaching, we use a theoretical construct, the *Teaching Triad* (Jaworski, 1994; Potari & Jaworski, 2002), which offers a framework for teaching processes designed to address students' learning of mathematics.

In this paper, our aim is to zoom in on connections and inter-relationships between these areas of theory as they apply in our research into teaching mathematics at the university level. In the setting of a lecture, teacher–student interactions are different from those in a school classroom: addressing students' needs is rather more demanding for the university teacher. Having more than a hundred students in an amphitheater compared to an average of 25 students in a school classroom makes it difficult for the university teacher to diagnose students' needs. Moreover, in the particular setting where our study takes place, the students take written exams at the end of the courses, and they do not usually have class assignments during the course (in contrast to school setting), so the university teachers do not learn about possible difficulties their students face in the course so as to provide help and support their understanding. In order to contextualize the theoretical ideas, we include an example from research into university lecturing with one lecturer that we call L2 (as first introduced in Petropoulou et al., 2020). Our principal research question (above) addresses some key words or terms: *teachers do and think*, *teaching work*, *university setting*, *mathematics, inclusion of students*. The *university setting* is both the place where teaching and learning are formally situated and the ways of thinking about and understanding what it means to engage in learning and teaching processes at this level. What *teachers do and think* within this setting is to an extent prescribed by the institutional norms, values, and expectations. We might call this their "*teaching activity*." However, there is much more to teaching activity than fulfilling institutional demands or expectations. Within their activity, we are interested in how teachers think about what they have to do, and how the doing relates to the thinking. Mathematics figures strongly in this thinking as teachers design activity related to students' making of meaning in mathematics. Since our focus is on the *theories* we are using in relation to the *activity of teaching*, our example focuses on issues in teaching as they are addressed in relation to the mathematical learning of students: the analysis of student activity is beyond our scope in this paper. However, we acknowledge that having data related to students' activity could have enriched our characterization of the teaching activity (e.g., the tensions between the teacher's and the students' goals).

2 Introduction to the Teaching Triad

The Teaching Triad (TT) is a theoretical construct developed from earlier research into the teaching of mathematics at secondary school level. It offers a way of characterizing mathematics teaching by acting as a tool for analyzing teaching data from classroom situations (Jaworski, 1994); it has also been used by teachers as a developmental tool (Potari & Jaworski, 2002), by researchers to analyze teachers' attention to mathematics teaching and learning (Ayalon et al., 2021; Paparistodemou et al., 2014) and by teacher educators as a tool for educating teachers (Zaslavsky & Leikin, 2004). At the university level, the TT has been used as an analytical tool to characterize mathematics teaching (Jaworski, 2002; Jaworski et al., 2017; Petropoulou et al., 2020).

The TT comprises three *inter-related* (*not* stand-alone) elements or domains of teaching: Management of Learning (ML), Sensitivity to Students (SS), and Mathematical Challenge (MC). These have been interpreted in terms of the interactions that take place within a classroom setting and, as such, have focused on the *micro* aspects of teaching (i.e., data from classroom interactions related to mathematics teaching and learning), without overt focus on the broader situational and cultural focuses, the *macro* (e.g., the students' and teacher's knowledge and knowing beyond the classroom situation). The following characterization captures earlier visions of the triad from which current conceptions have emerged. We start from the earlier visions (Potari & Jaworski, 2002).

Briefly, *Management of Learning (ML)* described the teacher's role in the constitution of the classroom learning environment by the teacher and students. It included classroom groupings, planning of tasks and activity, use of textbooks and other resources, setting of norms, and so on.

Sensitivity to Students (SS) described the teacher's knowledge of students and attention to their needs, affective, cognitive, and social; the ways in which the teacher interacted with individuals and guided group interactions.

Mathematical Challenge (MC) described the challenges offered to students to engender mathematical thinking and activity; this included tasks set, questions posed, and emphasis on metacognitive processing.

These domains were closely interlinked and interdependent as emphasized in the relational diagram in Fig. 1 where SS and MC can be seen to be linked within ML, within the whole sociocultural setting (the outer rectangle) (Jaworski, 1994). Subsequent research has shown that sensitivity can be experienced as affective, cognitive, or social when it pertains to students' comfort and welfare, to their mental activity, or to their inclusion in the social setting, respectively. We have recognized that a good balance between SS and MC is needed for effective teaching: a lot of SS, but little MC can lead to good teacher-student relations but low mathematical progress (e.g., Potari & Jaworski, 2002); a lot of MC but little SS can result in students feeling stressed or unable to succeed (Jaworski & Potari, 2009). When challenge and sensitivity are well balanced, which varies according to the sociocultural situation, the result is "harmony"—students are suitably challenged and stimulated while supported to achieve.

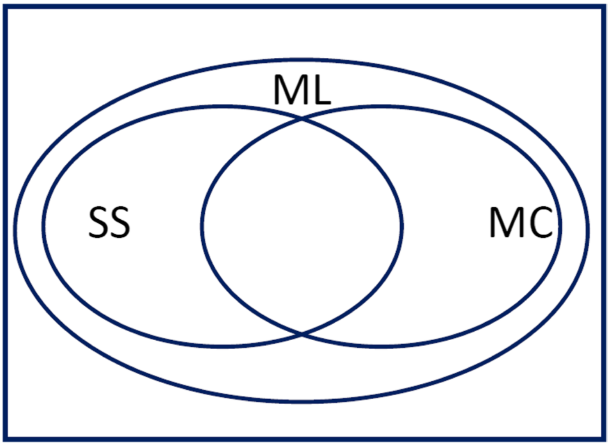

Fig. 1 The Teaching Triad (showing constituent relationships – Jaworski, 1994)

In our approach, we attempt to understand the TT and the relationships of its elements under the lenses of Activity Theory and to illustrate its analytic power to study the teaching and learning of mathematics (here, at university level) linking micro and macro cultural contexts.

3 Teachers' activity—a sociocultural frame

We see *doing* and *thinking* as central parts of human activity. We follow Radford (2021, pp. 50–51) in addressing *activity* as related to *knowledge* and *knowing*. Radford, drawing on Vygotsky, presents knowledge as a cultural phenomenon; "for example mathematical knowledge can only be understood as the historical realization of previous abstract forms of mathematical thinking and action" (p. 48). According to Radford, knowledge is a general entity, a cultural-historical way of doing, thinking, and reflecting, a *potentiality for activity*. The *actualization* or *materialization* of knowledge is what we mean by *knowing*. "Knowing is an *embodiment* of knowledge" (p. 40).

Radford writes (p.51, emphasis in original):

> *Knowing* is the concrete conceptual content through which knowledge is embodied, or materialized, or actualized. However, its conceptual, concrete content appears and can appear only through human activity. This activity actualizes knowledge, brings it to life – like the activity of playing a violin brings musical notes to life. This means that between knowledge and knowing lies activity. In other words, *knowing is the result of a mediation*. There is no such thing as unmediated knowing: all knowing is mediated and what mediates it is an activity.

So, from this standpoint, teachers' doing and thinking comprise their teaching-learning *activity* within their institutional setting within society and draw on their cultural-historical knowledge (including knowledge in mathematics) within their own lives within society. Now, according to Leont'ev (1981), "*Activity* is the non-additive, molar unit of life … it is not a reaction, or aggregate of reactions, but a system with its own structure, its own internal transformations, and its own development" (p. 46, our emphasis). Thus, we see here teachers' *knowing*, what teachers *do*, to be part of their teaching-learning *activity* and what they *think* to be a reflection within this same activity. Furthermore, teachers' *knowing*, their *teaching work*, their doing and thinking, refers to mathematics teaching activity within the sociocultural frame that includes the university setting and all its socio-historical-cultural appurtenances.

4 Activity Theory

4.1 Three layers of Activity Theory

Leont'ev makes the following point "in a society, humans do not simply find external conditions to which they must adapt their activity. Rather these social conditions bear with them the motives and goals of their activity, its means and modes" (Leont'ev, 1981, pp. 47–48). Here, we focus particularly on Leont'ev's three layers of human action which constitute *Activity*. The outer, or top layer is labelled "Activity" which according to Leont'ev (1981) is *always motivated*, although the motive (or *object*, see below) might not be explicit. Within Activity, the second layer consists of the "actions" of humans engaging in Activity. *Actions* are *goal-directed*, such that the goals are always explicit or conscious and relate to the motive of the activity. In the third layer, actions include "operations," which depend on the "conditions" within which actions take place. In earlier research, we have used Leont'ev's layers to explain issues and tensions which have emerged from analyses between teachers' and students' perspectives on mathematics teaching and learning (Jaworski & Potari, 2009; Jaworski et al., 2012).

Teaching always presupposes the intention to create learning (Pring, 2004). If we think of the *activity* of a university teacher, teaching mathematics within a university setting, subject to all the forces within which the activity takes place, we might think of the *motive* (or *object*) of this activity to be the mathematical learning of students participating within the complexities of this setting. *Actions* for us here are the *teaching actions* which take place as the teacher engages in the teaching process in relation to the mathematics which is the focus of teaching. Such actions are *goal* directed and relate to ways in which the teacher thinks about her teaching and acts in relation to her students. Thus, teacher intentions and theoretical perspectives form *goals*, and didactical and pedagogic processes form *actions* in this activity setting. The *operations* within this role, with which the teacher engages, are closely related to the practicalities of the role, for example, setting exams, creating pages in the course's electronic platform, assessing students' work. These operations must take place within the affordances and constraints of the university system which impose *conditions* on the operations.

In our analysis of activity with relation to a teacher whose teaching we have studied, to access the knowledge the teacher brings to his interactions and the knowing that we see in the teaching activity and settings, we link the three levels of Leont'ev with elements of the TT seeking to show how the TT provides insights beyond the classroom setting, as we discuss below. This leads us to interpret the teacher's activity, considering tensions and implied contradictions related to the elements of the TT and to their relationships.

4.2 Tensions and contradictions

Teachers' doing and thinking as components of the activity of university mathematics teaching are formed through tensions and contradictions emerging in the context of the activity. Radford (2021) makes explicit this complex relationship: "Thinking in general and mathematical thinking in particular, embeds and reflects society's various processes and expresses intrinsic societal tensions and contradictions" (p.167). Radford (2021) shows, through an example of the development of mathematical thinking in Ancient Greece, the contradiction as a dialectical category that shapes mathematical thinking. He distinguishes two forms of mathematical thinking: those developed by practical geometers and calculators (the practitioners) and those developed by theorematic mathematicians (the theorists). He shows how these forms of thinking aligned with the central contradictions of Greek society that were experienced differently by the practitioners (usually slaves) and the theorists (the elite). His dialectical position is that one form of thinking could not exist without the other: "To say that they are dialectically (over) produced means that one is the effect and the condition of the other" (Radford, 2021, p. 175). These subjective stances of mathematical thinking developed by different groups of participants are both embedded in, and at the same time manifesting, cultural-historical societal objective contradictions (e.g., freedom vs. slavery). Leont'ev (2009) also considers that the activity of a subject develops not only through reflection but also "… as a process containing in itself those internal, impelling contradictions, dichotomies, and transformations that give birth to the psyche, which is the indispensable moment of its own movement of activity, its development" (p.38). These contradictions are present in the common work of the teacher and the students and are of emotional, conceptual, or epistemological character manifesting dialectical tensions or contradictions (Radford, 2022; Stouraitis et al., 2017). Instead of *contradictions*, Stouraitis and his colleagues use the term *dialectical oppositions* to refer to "the unity of different aspects of mathematical concepts or how concepts are used and transformed in teaching" (Stouraitis et al., 2017, p. 207). Taking a dialectic perspective in contradictions leads us to handle these opposite poles in a unifying way understanding that one cannot exist without the other as in the example of mathematical thinking in Ancient Greece mentioned above.

5 Embedding the TT into the sociocultural perspective

Briefly, we see teaching as a process of mediation between students and mathematics: not a simplistic relationship but one with several dimensions which the TT serves to accentuate (as discussed above). Our shift from the perspectives of classroom interaction to our taking account of a wider setting (micro to macro) was a first step in moving to a more sociocultural frame in which we recognized the cultural antecedents of teacher or students' perspectives on a mathematical task and its demands. Our shift to Activity Theory has been to gain greater insight into these contrasting perspectives and their influence on teacher's goals and actions.

Thus, in order to characterize a teacher's ML socioculturally, we need to rethink it from an activity perspective. Following Radford (2021), we see, as examples of the teacher's "knowing," her theoretical perspectives transposed into tasks generating activity for students and concomitantly, through SS, addressing students' own "knowing." Radford, however, does not see teacher and students separately but as engaging in activity jointly: "The interaction of the teacher and the students appears as a *dance of consciousness* trying to grasp something together" (Radford, 2021, p. 22, our emphasis). In Jaworski and Potari (2009), we write of a mathematics class where teacher and students struggle to engage together, in which students resist a teacher's carefully designed tasks. The "dance of consciousness" here reveals contradictory elements of the sociocultural setting in which serious cultural differences impede fruitful activity. In this activity, both students and teacher struggle to address each other's perspectives on the classroom situation and both learn from the experience.

Achieving *harmony* between sensitivity and challenge is far from straightforward. In Leont'ev's AT layer of *actions* and *goals*, actions are designed overtly to achieve goals as part of the motive/object of activity. We see that the complexity of the teaching-learning context imposes a multiplicity of challenges, contradicting the material process from action to achievement of goal. As we have shown in our analyses of classroom situations, experienced and dedicated teachers, and indeed their students, have to face such contradictions as a central part of the teaching work. We address the relationships between tensions and contradictions further below.

6 Data and analysis: introducing mathematics teacher L2

The mathematics teacher that we call L2 is an experienced research mathematician and university lecturer, with many research publications and prizes, in a Greek university mathematics department. We draw on data from a wider, qualitative study of university mathematics teaching which consisted of observations of teaching and interviews with four lecturers, who were interested to participate in the study (Petropoulou et al., 2020). L2 was asked to participate in this wider study because of his, known to us, popularity with students. The data were collected by the third author of this paper. To exemplify theoretical aspects above in a sociocultural perspective, we include a narrative developed from in-depth analyses of data from close observation of L2's lectures (19 1-h observations) and from interviews which took the form of intense discussions, between L2 and the research team, right after each lecture (7 reflective discussions lasting on average 45 min each). The observations were not structured in the sense of having specific points to address, but their scope was to identify critical issues relating to L2's teaching actions. Discussing these issues with L2 in the after-class meetings allowed us to go into depth on his activity, revealing his perspectives on working with his students in lectures and his provision of resources to help them make sense of mathematics beyond the lectures (some extracts are numbered as E1, E2 etc., for ease of reference in supporting the analytical claims).

The topic in focus is Calculus, a compulsory first-year course (with no compulsory attendance) in which the emphasis is mainly on proofs. According to L2, who keeps statistical records about the progress of students' studies, first-year Calculus is one of the most difficult courses in this department and many students fail at the final exams. This failure leads them to graduate after 6.5 years of studies on average, instead of 4 years, which is the normal duration of studies for a mathematics degree in Greece. L2 is aware of this problem and, as we show in the narrative below, he takes it into account together with other problems of teaching mathematics at university level as well, such as the variance in students' attendance. In the analysis, we also unfold his conception of mathematics as well as his perspectives on institutional demands and expectations such as the demand to set the questions for course examination together with colleagues or the expectation of results in scientific research rather than results in teaching performance.

Following this narrative, we analyze it from theoretical perspectives presented above. In particular, we relate sociocultural perspectives of activity involving teacher's reflections and dialectical contradictions to the TT as we use it to characterize L2's teaching.

6.1 Narrative

University amphitheatres, even an hour before the start of L2's lectures on Calculus I or II, are crowded with students who are there early just to ensure a seat. When a researcher pays attention to what students say, she can hear students praising his lectures—how he writes everything on the board and how "analytical" he is in his teaching (i.e., providing repetitions, detailed explanations, and steps). Students consider that "attending and keeping notes" from his lectures on a regular basis as well as "studying the notes" L2 uploads on an official website (including the basic theory, a set of carefully chosen and solved exercises, past exam questions, and lecture notes for students who do not attend) are "enough" for them to pass the final examination of the term, "even to get a good grade." L2 is very focused on students' success with this final examination and he wants to help students have a smooth transition from high school to university thinking. He organizes extra lessons, he keeps statistical information about students' success in Calculus courses, he has initiated a change in how his department offers compulsory courses such as Calculus from once to twice a year, and he attempts to persuade his colleagues to use course exam questions accessible to the majority of students (e.g., with graded steps in their formulation).

As a lecture begins, one can see L2 speaking and writing everything on the board in a clear neat way while the students copy from the board. L2 rarely discusses with them and he maintains a lively pace in his lecturing. He uses colloquial language for many comments and appeals to the students by using "you" singular. As he says himself: "It is as though I have a student in front of me and try to draw her attention to what I am doing – I do the same for all of them" (E1). The attendees are very quiet even though they are many (over 200); the communication between them and L2 is mainly based on their facial cues, according to the observing researcher. He himself says: "In some occasions, I don't ask anything during the whole lesson" and in the case that a student asks a question: "I respond quickly to the student who will get what I am saying, I give some more time to the student who really needs help, and I am even rude to the student who offers an irrelevant contribution just to show off to his peers i.e. he wastes my time" (E2). However, L2 attempts to make the content relevant to the students by creating an atmosphere in which they would experience an air of relief from the difficulty of advanced mathematics: he reminds students of prior knowledge they are expected to know (and it seems to him that they don't); he provides detailed steps in relation to a process (e.g., a proof); he emphasizes delicate mathematical points in the content taught; and he provides tools (e.g., graphs, counterexamples) to better explain how students should think about the particular content.

L2's knowledge of how students come to learn the advanced mathematical content seems to come from prioritizing, at this stage of their studies, their familiarity with the content rather than a deep understanding of it; he said,

> Some things in mathematics cannot be discussed … You know, there is a saying 'you cannot really understand mathematics, you can only get used to it'. This is true! This is the case with the compulsory courses in the first years. Later, in their studies, their understanding will develop. (E3)

Beyond his being a research mathematician and a university teacher, L2 has spoken of personal feelings and ideology: his emotions, personal needs, personal history (e.g., as a student "who liked mathematics and was good at it" himself). For example, he considers teaching of the same importance to him as his research in mathematics, even if teaching occupies him at the expense of research. The reason has to do with a personal "sense of balance":

> When you do research in mathematics and you produce nothing for six months or one year, you feel disappointed. At least with teaching, you can say to yourself that you have done something important! You have done 70 h of lessons and 200 students passed Calculus! This gives you a sense of balance; a sense that you did something this term. (E4)

Attending to the way in which L2 and his students work to produce mathematical meaning together, we see him drawing on his own nature as a person:

> I don't discuss that much in the classroom. There are colleagues who have it in their nature to discuss before they prove a theorem … But, in a 200-person audience, if you begin a discussion with 2 or 3 students, first, these students probably will be the strongest ones and the others will feel bad, and, second, nothing will remain on the board. For this reason, I have resulted in not discussing, but probably it is in my nature as well. That is, not much discussion, let's say. (E5)

L2's personal story as a student who "did not attend the lessons but studied for them on my own" makes him aware of the students of the same mentality:

> If you [as a teacher] think of all students that attend your lessons, then you think of about half of them. You ignore all the students that do not attend - for example because they have to work - but they do study on their own. You have to take these students also into account. (E6)

L2 is engaged in different activities (research, teaching, professional, personal) from which he often experiences tensions. For example, he says: "I go on with a too fast pace and this has to do not only with the students and the content but also with other reasons such as my being tired! I have noticed that I am faster [in teaching a topic] when I am tired. This is, I am bored to negotiate [this topic] so I move on" (E7). Another tension with regard to his research and teaching activities stopped him from preparing a new book (preparing was taking place during the summer) in order to prepare his lessons for the next academic year because he "could not do both" the way he wanted. Preparing for and succeeding in the final examinations are important aspects of students' university culture. Students need to have success in the final examinations in order to proceed smoothly with their studies. This culture is taken into account by L2. The practical conditions and circumstances to meet his students' needs are in his conscious awareness:

> We want as many students as possible to start their studies smoothly. In this department, given the enormous number of students, the new students face difficulties in adjusting. Also, concerning the difficulty of the subject, our aim for the average student would be to have passed all the compulsory courses, which should take two years, at least during the first three years of study. I believe this is feasible. (E8)

Being successful in the exams is related to the difficulty of the exam questions. L2 collaborates with his colleagues to prepare the exam questions but he often experiences tensions:

> I tried once to split one question into sub questions and in this way almost all the students could respond. Otherwise nobody could solve it … However, some of my

colleagues think that, by this splitting, the initial question has been trivialised. If the questions are too difficult, it is not fair for the students. (E9)

7 Illuminating theories through the activity of L2

7.1 L2 thinking and knowing

We interpret the thinking of L2, as a teacher of mathematics, through his reflections which involve *real* connections and interactions with his students, colleagues, and others. The production of knowledge cannot exist or arise without L2's *activity*. Where knowledge is concerned, we see this including mathematical knowledge (which we regard as highly developed) and didactic knowledge (which is perhaps not so highly developed).

The narrative above, emerging from analyses of L2's teaching and his own words in conversation and interview, speaks of L2 as an individual with his own reflections, a human *being* with human characteristics (including speaking quickly when tired, E7, his being not of a discursive nature, E5), a research mathematician and a mathematics teacher. Roth and Radford (2011, p.2) speak of "a culturally and historically evolved form of reflection"—"a complex that includes subjects and the symbolic and material reality that surrounds them.." Our observations of L2 and his teaching start to paint this surrounding (cultural) complexity, while L2's own words provide insight to his ways of thinking and knowing, emerging through his own experiences, his perspectives on students, on mathematics, and on the system and community with which he works. Radford (2021) recognizes this dialectic relationship between the individual (L2 in our case) and culture: "… through our engagement in social practices we affect culture. In turn, culture affects us: it affects both our ways of knowing and thinking as well as our ways of becoming, (p. 157–158). We see, throughout our analysis of L2 data, that his identity in activity is culturally formed.

We continue by addressing L2's activity. His actions and goals are motivated by his priorities for his students to be successful in the course examinations and complete their studies smoothly (E8). L2's actions are embedded within his historical past, being "good at maths," winning awards. He sees students who are perhaps not so good at mathematics having to work harder than he did himself, being hindered by their need to work to support themselves (E6). His actions include using colloquial language, drawing students' attention to what he is writing, seeking to include all students: these all point to his goal, as one of his colleagues had remarked, of *wanting all students to learn* (see excerpts E1, E8 in the narrative above). His teaching approach can be seen as a traditional lecture style, common in Greek university culture: he avoids discussion, responds to genuine questions, but abhors students who show off (E2).

L2 has care for all students, including those whose own circumstances impede their opportunity to attend the lectures (E6), to the extent that, at the expense of his own research time (E4), he takes time to provide resources on-line and succeeds in changing the system to allow examination within a reasonable timescale. This indicates L2's societal consciousness that, as Roth and Radford argue, "presents itself in the form of ideologies" (Roth & Radford, 2011, p.5). L2's ideological positioning about teaching is that it must have "a practical result for the masses" as expressed in the narrative. It is clear from students' observed actions that very many of them value highly what L2 offers; thus, their image of the motive of the teaching promoted by L2 fits with their own expectations and motives.

7.2 L2: teaching actions and Teaching Triad

Management of learning L2's management has a high degree of provision of tools/resources for students' learning. He addresses students via a website accessible to all students which is constantly enriched by him with updated lecture notes and information about the course (e.g., about extra lessons). His provision appeals to students, even those who do not attend. He manages lecture time (e.g., he limits interaction with individual students) to ensure time for repeating basic methods and solving more exercises. His clear board work portrays a step-by-step navigation into the advanced mathematical content. This close guidance to students seems to attract them and make them feel safe with respect to their performance in the final exams.

Sensitivity to students L2's overt "caring" for the students, according to his own perspectives on caring, suggests a high level of sensitivity in the *affective* domain. He wants students to be able to succeed—*within the system*. His perspective seems to be one of doing all that he can to provide for what students need in order to pass the examinations. There is little emphasis on seeking students' conceptual understanding. We see this kind of caring to be in the affective domain because he makes students feel comfortable with what they have to do and secure in the resources for doing it. His "take" on *cognitive* sensitivity may be seen as enabling their conceptions to the extent of being prepared for the examination, perhaps without further need for in-depth comprehension (see for example E3 and E4). On the other hand, he demonstrates social sensitivity as he cares for the students who do not attend the lectures for many reasons (e.g., they have to work for a living). Also, his attracting of large numbers of students to his lectures creates a community in which his goals, portrayed through his actions, become a part of the social milieu, including institutional norms, values, and expectations, and therefore his consciousness of his students.

Mathematical challenge We might see mathematical challenge as implicit in the content of the calculus course—we are told that most students find the course exceptionally challenging. In other words, they find the mathematics *hard*. Thus, L2's goal here seems to be to reduce the challenge through the tools and resources he provides. Despite his own high level of success with mathematics, L2 can be seen to show empathy for students who struggle. Even though the students do not show the quality of mathematical abstraction that he himself has experienced, his sensitivity to his students drives him to ease their struggle. We suggest that, alongside a deep knowledge of mathematics and his empathy for the struggling students, L2 believes it is too difficult for students to engage with high cognitive challenge. Hence, he supports them through affective rather than cognitive sensitivity, resulting in a low degree of mathematical challenge. This means that students should not be prepared to deal with exam questions that require a higher degree of cognitive engagement, at the first year of their studies, and hence the need for exam questions to be modified, contrary to his colleagues' wishes and exam culture.

This brings us back to the words of Radford quoted above (2021) "… through our engagement in social practices we affect culture. In turn, culture affects us: it affects both our ways of knowing and thinking as well as our ways of becoming" (p. 157–158). L2 affects culture in university mathematics through his practices and perspectives, and culture affects him through the common activity in university mathematics and hence the perspectives of his colleagues. We see a need for the TT to recognise the importance of culture in its three domains; we address this through a consideration of tensions and contradictions in L2's teaching activity.

7.3 L2: tensions and contradictions

The TT, through its constructs of management, sensitivity, and challenge, has addressed potential tensions in the teaching process. For example, considerable affective sensitivity with little cognitive challenge can result in a comfortable learning environment but little high-level mathematical engagement (e.g., Potari & Jaworski, 2002). On the other hand, a high degree of challenge without corresponding sensitivity can leave students frustrated or resistant to the teaching (e.g., Jaworski & Potari, 2009). We have shown (in previous publications) that when we analyze a classroom situation, it is important to consider the *macro* elements alongside the *micro*, so, we look not only at what we see to happen, but also at the cultural contexts that underpin the activity observed. Here, we look at L2's activity in facilitating students' mathematical activity in their course for success in the exam. We see that his activity embodies considerable affective sensitivity but correspondingly little cognitive sensitivity and a low degree of mathematical challenge (MC). Yet we might expect that, as a high achieving mathematician himself, he might emphasize MC more highly. For example, high-level research in mathematics shows a high degree of *inquiry* into processes and resulting theorems (Burton, 2004; Singh, 1998). However, we see little focus on inquiry in the joint teacher-student activity.

We recognize L2's empathy for students, many of whom struggle with their mathematics for a wide variety of reasons. One of his goals is to reduce this struggle and hence enable students' success. It is clear from students' comments during the observations that this approach is greatly appreciated. His actions in achieving this goal (highly time consuming for teaching and detracting from his research time) come into tension with a factor in university culture supported by his colleagues, that of the nature of the exam demanding students' high(er) cognitive activity beyond what the course has provided (see E9). The tension here lies between L2's sensitivity to students and local culture's emphasis on the degree of mathematical cognition required for the exam. To submit to cultural demands means a contradiction for L2. His careful and extensive preparations for students will be to no avail if the exam questions are not modified. However, as a mathematician within the mathematical culture of his department, he understands also the perspectives of his colleagues. To reduce the challenge of the exam goes against the culture of which he is a part.

The tension here for L2 is obvious. He belongs (physically and culturally) to the mathematics department, knowing the ways in which courses are created and examined. His sincere care for his students leads to a contradiction that relates to an emphasis on procedures and problem-solving steps rather than on inquiry into mathematical meanings, on the formality of mathematics involving concepts, relations, and unfamiliar problems that are often expected in the exams. As a result, the students cannot reach the required cognitive level to deal effectively with the exam questions. The alternative is to persuade his colleagues to change the exam questions, which is anathema for them. This contradiction can be characterized as epistemological in nature, what Stouraitis et al. (2017) call *object versus process*, and it has a dialectical character. By recognizing these "opposite" goals of the mathematical activity in his teaching/research, the lecturer might offer a higher level of mathematical challenge to his students where conceptual and procedural aspects of mathematics could coexist. L2 does not seem to consider such possibility for first-year students in the given conditions as expressed above (see E3).

The relationship between MC and SS and the emphasis on affective and social sensitivity is also deeply culturally embedded. L2 perceives the goal of his teaching activity to be for the students to succeed in the course examinations, to complete their studies on time, rather than making changes to modes of teaching or the nature of the exam, both of these firmly embedded in the culture of the department. Rather, L2 seems willing to achieve his goal by sacrificing the development of students' deep understanding of mathematics (see E8). A contradiction here, students' *success versus deep understanding*, indicates dialectical tensions between how L2 experiences mathematics as a mathematician and the institutional and social cultures in which the students and he live. L2 reduces the challenge in order to meet the students' current needs and realities. By projecting to the future, he recognizes that students will have other opportunities, during their studies and later, to develop deeper mathematical understandings, while denying them the opportunity to start to develop deeper mathematical processes now. Also, getting the degree late has other cultural connotations: late access to the workplace, families continued financial support for the studies and often the dropout of some students. Reflecting on these aspects of social and institutional culture helps us to understand L2's goals and actions. We also recognize that the contradiction, success versus deep understanding, is embedded in and at the same time manifesting societal contradictions like students' quick access to workplace vs. continuous support from family.

L2's social sensitivity to students comes from his ideology: as he had expressed many times in interviews, everyone has the right to learn and proceed. On the other hand, he needs to take into account the rules that the various communities involved in university mathematics teaching have established (e.g., a high number of attendees, the advanced content, relations with colleagues, customs of the students' communities, students working for a living). The contradiction that seems to underlie this tension is of a cultural nature and can be expressed as *a desired future outcome* (the students to get their degree in time and subsequently have good working opportunities) *versus the current reality* (institutional and social priorities and rules). The societal nature of L2 as a human being is realized in his consciousness about students' priorities. He attempts to handle this contradiction in his teaching by developing on-line resources (lecture notes, solved exercises, past exam questions), undertaking extra teaching hours for practicing exercises, and writing clearly and in every detail what he says on the board to allow students to keep notes that they can use in their own study.

7.4 Implications for the Teaching Triad and its use

The dialectics of the contradictions that we expose relate in some way to issues in both SS and MC within ML. We have recognized L2's activity in teaching as demonstrating little opportunity for students to engage with mathematics as inquirers. Where the object of teaching is for students' success in their course, i.e., getting an appropriate grade on the examination, while the teaching process encourages learning of procedures and formalities, we see tensions that are clearly felt by L2.

The contradiction arises when the teaching goal is addressed through affective means, and the resulting deficiency by a change to the object of the process: by this, we mean the changing of exam questions to fit with the affective procedural approach. It is important for us to acknowledge that there is nothing wrong with mathematical procedures and formalities, but a problem lies in becoming familiar only with the "what" of them rather than the "why." We are aware of many ways in which the "why" can be addressed—for example, inquiry-based learning is one of them. However, a change in the exam questions, reducing the challenge of the mathematics in the exam, does not address the contradiction, it rather embeds it.

As teachers and teacher educators ourselves, as well as researchers, we understand and feel for L2. Through this research and our object of using Activity Theory to make the TT more discerning of issues in teaching, that is using the TT to address tensions and reveal contradictions in teaching processes, we are brought up against central questions/issues that promote tensions and contradictions. We recognize issues related to the cultural-historical nature of mathematics and what it means to learn mathematics. We ask, where does mathematical insight come from and how does this relate to mathematical achievement? We are concerned about the creation of mathematical activity for students. We want to nurture in students a love of mathematics that goes beyond the standard exercises and procedures that can be learned without addressing important "why" questions. In all of this, the activity of teaching with *affective* (only) concern for students, begs all the questions about the culture of mathematics and processes in its teaching.

These considerations suggest a focus not only on appropriate teaching and on the didactics of teaching, but also on the cultural nature of mathematics in relation to its teaching. Within the TT, the nature of mathematics needs to be embedded at the teaching activity level. Thus, when tensions arise and contradictions result, their recognition within the TT can lead us more knowingly towards the tensions we need to address such as a procedural teaching approach versus the style of the exam and its content. This is a dialectical situation. On one side of our coin, there are all the questions about object, process, activity in and growth of mathematical knowledge (the theoretical side) and on the other side are the basics of mathematics and its development in relation with task design, student thinking and mathematical reproduction (the operational side).

8 In conclusion

The Teaching Triad addresses teachers' actions in relation to ML, MC, and SS in the classroom context. Embedding the actions in the teaching activity lifts the TT up to the sociocultural terrain. In the previous section, moving from the TT to the tensions and contradictions is a way of placing the TT in the context of the activity, where culture is a key medium. In addition, the way that L2 formulates his actions and goals provides us with insights as to why certain relations in the TT exist. In this concluding section, we attempt to address these "why" questions illustrating a new perspective on the TT.

For example, the low MC that characterized L2's teaching in his attempt to make the content accessible to all the students (SS) can be interpreted through the contradictions of object vs. process and success vs. deep understanding. Moving to the contradictions, we understand better, on the one hand, the source of the contradiction (e.g., different goals in research and teaching activity in the first contradiction, or the goal of the teaching activity per se in the second), and on other hand whether L2 attempts or not the synthesis of the two poles of the contradiction. The fact that L2 does not consider these two in a dialectical way, prioritizing one over the other, may be explained by the cultural environment in which his teaching takes place. Balancing MC and SS can be achieved if L2 addresses dialectically the two contradictions. However, L2 does synthesize the *desired future outcomes* with *the current reality* in his ML, indicating social sensitivity to students' current and future needs.

In our analysis, we demonstrated how the TT can be transformed from a theoretical and analytical tool capturing classroom interactions to a tool addressing the (cultural) activity of teaching and the underlying contradictions. Lifting the relations among the elements of the TT to the level of activity brings to the front the dialectical contradictions involved in the activity and explains teachers' thinking within the cultural, institutional, and social settings. Our theorizing and the analytical process contributes to the efforts in mathematics education research to study mathematics teaching addressing both micro and macro issues considering the complex network of mathematical practices and the culture(s) within which they have meaning. So, while management of learning, sensitivity to students, and mathematical challenge keep their original meanings, they take on enhanced meanings related to activity: to mathematics, to growth of (mathematical) knowledge, and to teacher and students' (mathematical) activity.

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Declarations

Competing interests The authors declare no competing interests.

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