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Students' geometric understanding of partial derivatives and the locally linear approach

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Abstract

We use Action-Process-Object-Schema (APOS) theory to study students' geometric understanding of partial derivatives of functions of two variables. This study contributes to research on the teaching and learning of differential multivariable calculus and its didactics. This is an important area due to its multiple applications in science, mathematics, engineering, and technology (STEM). The study tests a previously proposed model of mental constructions students may use to understand partial derivatives through a set of activities designed to help students make the conjectured constructions. The model is based on the local linearity of differentiable two-variable functions, and the model-based activities explore the relationship between partial derivatives and tangent plane in different representations. We used semi-structured interviews with eleven students whose teacher used the three-part cycle—Activities designed with the genetic decomposition; collaborative work in small groups and Class discussion; and Exercises for home (ACE)—as pedagogical strategy. The model-based activity set based on local linearity and the ACE strategy helped students construct a geometric understanding of partial derivatives. Results led to reconsider and further refine the model. This study also resulted in improving activity sets and obtaining information on students' construction of second-order and mixed partial derivatives.

Keywords Calculus · Partial derivative · Function of two variables · APOS theory

The notion of change is at the core of understanding what calculus is and its multiple applications to a wide variety of phenomena in science, engineering, and other fields. Most of those phenomena involve multivariable functions, and the study of two-variable functions opens the door to the understanding of multivariable functions. Partial derivatives are used to measure change and commonly appear in multivariate calculus and differential equations instruction. Indeed, partial derivatives measure instantaneous rate of change of a

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two-variable function in the x and y directions and the rate of change in any direction can be expressed in terms of them. Further, the best linear approximation to local change in a two-variable function is also expressed in terms of partial derivatives. They show up in calculus-based optimization problems with more than one variable. They also have prominent applications in economics, as most functions describing economic behavior depend on more than one variable. Partial derivatives appear in thermodynamics, in quantum mechanics, as well as in equations from mathematical physics that express certain physical laws such as Laplace's equation ($\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$) and the wave equation ($\frac{\partial^2 f}{\partial x^2} - \frac{1}{V^2}\frac{\partial^2 f}{\partial t^2} = 0$). The concept of partial derivatives is a prerequisite for many later concepts in multivariable calculus such as tangent planes and linear approximations, directional derivatives and the gradient vector, local and absolute extrema, and Lagrange multipliers (Stewart, 2012). Research is needed to help students understand this basic idea in mathematics, modeling, and applications.

Our study contributes to the study of change in the calculus of two-variable functions, by proposing a model of mental constructions based on the idea of local linearity. This facilitates the connection of partial derivatives to other important concepts of two-variable functions. Our study shows that the model succeeds in helping students understand the different facets of change in these functions.

1 Literature review

Even though the teaching and learning of one-variable functions and their change have been extensively studied (Asiala et al., 1997; Bingolbali et al., 2007; Borji & Martínez-Planell, 2020; Zandieh, 2000), the study of partial derivatives and change in multivariable functions has received scant attention (Martínez-Planell & Trigueros, 2021). Change in multivariate functions falls under the umbrella of covariational and multivariational reasoning (Jones, 2022) where more than two variables relate to each other. In this section, we first consider studies related to one-variable functions. Then, we review the literature regarding covariational and multivariational reasoning. We end by considering research about partial derivatives and the differential multivariable calculus.

1.1 One-variable functions

One-variable function derivatives are the basis for the study of the differential calculus of two-variable functions. It is important to consider these derivatives in the study of change in multivariable calculus because partial derivatives are constructed from one-variable function derivatives and the latter are often at the root of student difficulties (Martínez-Planell et al., 2015; Wangberg et al., 2022; Weber, 2012, 2015). Asiala et al. (1997) proposed that a one-variable function derivative can be constructed by coordinating an analytical interpretation of derivative as the limit of difference quotients with a geometric interpretation as the slope of the tangent line which is the limit of secant lines. These authors also mention coordinating contextual interpretations of derivative but do not provide any detail. Zandieh (2000) proposed a similar framework but considered contextual situations in more detail. The framework consisted of a matrix with process-object layers of ratio, limit, and function as rows, and the contexts of graphical (slope), symbolic (difference quotient), verbal (rate), paradigmatic physical (velocity), and other (contexts) as columns. Regarding the "other contexts" column, Kertil and Gülbagci Dede (2022) reported that prospective secondary school teachers have difficulties interpreting derivatives as a rate of change in real-life contexts. On the other hand, Bingolbali et al. (2007) found that mechanical engineering students prefer seeing derivative as rate of change while mathematics students prefer derivative in its geometric interpretation. Nagle et al. (2013) used a model of 11 conceptualizations of slope to compare students' and instructors' preferences. They found that instructors tended to use geometric (graphical), functional (rate of change), real-world (contextual), and calculus conceptualizations while students referred to procedurally-oriented conceptualizations like behavior indicators (to determine if increasing or decreasing) and parametric coefficients (the m in $y = mx + b$). Weber et al. (2012) proposed thinking about a one-variable function as being made up of "small" line segments (the hypotenuse of right triangles moving along a curve) to underscore the interpretation of the derivative function as the rate of change function. All of this research highlights the importance of the graphical and analytic paths of Asiala et al. (1997) while also attending to context, as underscored by Zandieh (2000), in studying students' understanding of the one-variable function derivative and in bridging different interpretations.

1.2 Covariational and multivariational reasoning

From a mathematical point of view, covariation refers to a phenomenon in which two quantities change simultaneously in a system, and it can be considered in two different ways: (1) the coordination of two quantities is such that they are changing independently of each other (Confrey & Smith, 1994, 1995; Johnson, 2015) and (2) the reasoning about how a quantity changes in response to changes in the other quantity (Carlson et al., 2002; Johnson, 2012). Although there are several definitions of covariational reasoning in the research literature (Carlson et al., 2002; Jones, 2022), the general feature in these definitions is that covariational reasoning requires coordinating the changes in two quantities. Different studies have explored how covariational reasoning is manifested in students' understanding of various topics such as functions (Blanton et al., 2015; Carlson et al., 2002; Confrey & Smith, 1994, 1995; Ferrari-Escolá, et al., 2016; Stalvey & Vidakovic, 2015), rate of change (Thompson, 1994), and coordinate systems (Moore et al., 2013; Paoletti & Moore, 2017).

Research on covariational reasoning has often been restricted to examining two variables changing in tandem (Jones, 2022) but scientific and mathematical contexts often include more than two variables that are related. Jones (2022) built on the construct of covariation by considering cases where more than two explicit variables relate to and change with one another, in what he termed "multivariation." Jones (2022) described three major types of multivariation: independent multivariation (multiple independent input variables individually covary with an output variable, but do not necessarily covary with each other), dependent multivariation (a change in any one variable produces simultaneous changes in all other variables), and nested multivariation (the relationships between variables as having a function composition structure, such as $z(y(x))$. Unlike covariation, research on multivariation is relatively new and is not yet theoretically organized. Functions of two variables (e.g., $z = f(x, y)$) and its partial derivatives (i.e., $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$) are examples of independent multivariation. By examining students' understanding of partial derivatives of two-variable functions and how they are used to model change in two-variable functions, our study contributes to this part of the literature.

1.3 Partial derivatives and the differential multivariable calculus

The following studies aim to give a comprehensive panorama of the complexities involved in learning the differential multivariable calculus. Harel (2021) observed that several multivariable calculus textbooks de-emphasize the total differential and treat linearization and approximation superficially. In the case of functions from R^n to R^m, he argues about the importance of linearization to think of the total differential as an approximating linear transformation and uses this idea to treat topics like the general chain rule and implicit differentiation. Martínez-Planell et al. (2015) used local linearity as a unifying idea to develop and interrelate various differential calculus concepts of two-variable functions: plane, partial derivative, tangent plane, directional derivative, and total differential. Their study produced a model of mental constructions (genetic decomposition, GD) that needed to be tested. Like Wangberg and Johnson (2013) and Wangberg et al. (2022), it also emphasized a geometrical approach to partial derivatives where students can imagine the intersection of fundamental planes ($x = c$, $y = c$) and the surface and use their knowledge of the derivative of one-variable functions to examine the resulting curve.

In a different approach, Weber (2012, 2015) used covariational and quantitative thinking to discuss the rate of change in space in different directions. His results showed that students expected a single rate of change for two-variable functions, further stressing the challenge that directional derivatives pose for students. His results also showed students' tendency for an algebraic approach to make sense of rate of change in two-variable functions. Regarding linearization and analogous to the idea of Weber et al. (2012), Tall (1992) explored the local linearity of two-variable functions to propose a geometric interpretation of the total differential of a function at a point. Our proposal in this study builds upon Tall's idea to help students think of two-variable functions in terms of local linearity, that is, as made up of "small" patches of tangent planes that depend on two rates of change (in the x and y directions).

Martínez-Planell et al. (2017) found that students who used the tangent plane to learn directional derivatives could better interpret them graphically and in terms of tangent plane information. McGee and Moore-Russo (2015) discussed how students who explicitly considered the notion of the slope of a line in 3D space evidenced a better understanding of multivariable differential calculus than other students. This result was supported by a study using the locally linear approach for two-variable functions (Martínez-Planell et al., 2022). Other authors (e.g., Thompson et al., 2006, 2012) have reported student difficulty relating physical contexts to mathematics, particularly interpreting mixed partial derivatives in physics. Activities have been designed (Bajracharya et al., 2019; Thompson et al., 2012) to help students understand the physical meaning of keeping variables fixed, partial derivatives, and graphically interpreting mixed partial derivatives, mainly in a thermodynamics context. Moreno-Arotzena et al. (2020) discussed the importance of the use of different representations and conversions between them for gradients of two-variable functions, and Mkhatshwa (2021) studied how covariational reasoning can help students interpret quantities representing partial derivatives in real-world contexts.

2 Theoretical framework

We use APOS theory (Arnon et al., 2014) as our framework. In APOS, an Action is a transformation of a mathematical object that the individual perceives as external. In particular, individuals are not able to justify an Action. A collection of Actions may consist of the rigid application of an explicitly available or memorized procedure. When an Action is repeated and the individual reflects on its meaning, it might be perceived as internal. In this case, we say that the Action has been interiorized into a Process. The internal perception allows the individual to imagine the transformation, omit steps, and anticipate results without explicitly performing the Process. In particular, this allows the individual to generate dynamical imagery of the Process and to coordinate the Process with other Processes to create new ones. Further, as a result of these coordinations, and as discussed in Martínez-Planell and Trigueros (2019), thinking of a mathematical notion as independent of representation is consistent with a Process conception. In relation to covariation, a Process conception of function makes the foundation for covariational reasoning (Carlson et al., 2002). As an individual's Process conception becomes more vigorous, he/she can imagine quantities of the independent and dependent variables as changing continuously. When the individual can perceive the Process as a whole entity and is able to perform Actions on this entity, we say that the Process has been encapsulated into an Object. It also may be de-encapsulated to the Process it came from, as needed in a problem situation. We will not be making explicit use of Schemas in this article since it will suffice to use Actions, Processes, and Objects to propose a model (genetic decomposition) of how students may construct partial derivatives.

The progression from Action to Process and then to Object may appear to an observer as a non-linear progression (Arnon et al., 2014, p. 176). This is because a student may work differently when facing different problem situations involving a specific mathematical notion. Hence, to determine students' understanding of partial derivative as Action, Process, or Object, one needs to consider their overall tendency in different problem situations related to this notion.

3 Genetic decomposition

A genetic decomposition (GD) is a model of how a student may construct a specific mathematical notion. It is expressed in terms of APOS structures and mechanisms. A GD need not be unique. What is important is that it be supported by data obtained from students. A GD serves as a preliminary hypothesis that can be successively refined following experimentation. The GD of the differential multivariable calculus in Martínez-Planell et al. (2017) addressed the topics of plane, partial derivative, tangent plane, total differential, and directional derivative. For convenience, we only describe in what follows the portion of this GD dealing with partial derivatives.

Given a base point in the graph of a two-variable function and a direction (either x or y), Processes of fundamental plane and function of two variables are coordinated by encapsulation of fundamental plane into an Object resulting in a new Process (see Arnon et al., 2014, p. 24). This Process allows to imagine the intersection of the fundamental plane that passes through the given base point and is in the given direction with the graph of the function.

The resulting Process is coordinated with a Process of derivative of function of one variable to envision the base point and intersection curve in the given fundamental plane and the partial derivative of the function at the given point and direction as the limit of slopes of secant lines that approach the tangent line to the intersection curve at the point as the horizontal change gets smaller. The above mental constructions and coordinations are to be made in both analytical and graphical representations with the resulting analytical and graphical Processes being coordinated into a Process of partial derivative at a point.

The need to take a limit permits the encapsulation of the Process of partial derivative at a point into the Object partial derivative at a point.

The coordination of a (pre-requisite) two-variable function Process with the Process resulting from the de-encapsulation of the partial derivative at a point results in a partial derivative as a function Process that allows thinking of the evaluation of a partial derivative at different points as taking points (a, b) as input and producing numbers $f_x(a, b)$ or $f_y(a, b)$ as outputs.

The partial derivative as a function Process is encapsulated into an Object when Actions can be applied to it, for example, to consider higher-order partial derivatives and their graphical representation.

4 First research cycle

Cycles of research and refinement are central in APOS theory research methodology and can be adopted as a method to steadily improve student understanding and create didactical materials to foment conceptual understanding. Each research cycle in APOS, basically comprises conjecturing a genetic decomposition, producing and implementing didactic material to help students do the conjectured constructions, performing semi-structured interviews, and using the results of the interviews to refine the GD and didactical materials. The refined GD can then be the start of an ensuing research cycle. Iterations of the research cycle continue until a genetic decomposition is obtained that reflects very closely the cognition of the particular mathematical concept for students and also serves as a guide for the instruction of the concept. This emphasizes the applicability and practicality of APOS theory in the evolution of didactic materials through the repeated utilization of research cycles and class application of the didactical materials (see also, Martínez-Planell & Trigueros, 2019).

Most research on student understanding of different mathematical notions that utilizes APOS theory as a framework has been limited to doing single research cycles. The current study shows how a second APOS research cycle may be used to successively improve students' understanding of partial derivatives. The present study builds upon a first research cycle on students' understanding of the differential calculus of two-variable functions reported by Martínez-Planell et al. (2015, 2017) and Trigueros et al. (2018). The first research cycle proposed a preliminary-genetic decomposition (GD) for the differential calculus. Using the preliminary-GD as a guide, they (Martínez-Planell et al., 2015, 2017; Trigueros et al., 2018) developed activities to help students reflect and perform the conjectured constructions. In the first research cycle, instruction used the ACE pedagogical strategy, consisting of students' collaborative work in small groups on the activities with the teacher as a guide (A), whole-class discussion with the teacher (C), and homework exercises with new opportunities for reflection on the concepts involved in the lesson (E). ACE is commonly used in APOS theory to foment the reflection needed for the constructions proposed in a GD (for further details on ACE see Arnon et al., 2014). At the end of the course, in the first research cycle, semi-structured interviews were performed to test the preliminary-GD. It was found that students needed help to generalize the notion of slope to 3D space; some students, for example, still considered that "vertical" is represented by y and the slope as $\frac{\Delta y}{\Delta x}$. Also, the preliminary-GD had posited that a partial derivative Process was constructed through the coordination of a fundamental plane (planes of the form $x = c$, $y = c$) Process and a derivative of one-variable function Process. However, it was observed that some students needed help to construct fundamental plane as a Process (to relate graphical and algebraic representations and place the plane in its corresponding place in space). Others had yet to construct a Process of derivative of one-variable function. Thus, students showed difficulties understanding partial derivatives and their graphical interpretation (Martínez-Planell et al., 2015, 2017; Trigueros et al., 2018). These results led to changes in both the preliminary-GD and the activities. In this article, we test the revised GD (shown above) and corresponding activity sets (described further ahead) in a second research cycle and posed the following research questions: What constructions related to partial derivatives described in the GD are evidenced by students after going through a second research cycle of this notion? Based on this evidence, what further refinement does the GD need?

5 Methodology

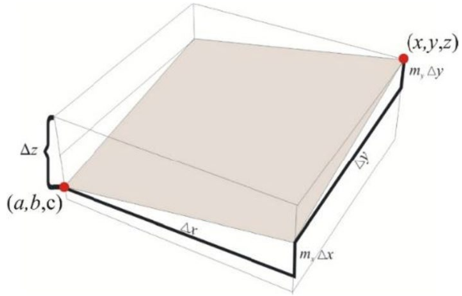
This is an intervention study. While the term intervention is used with various meanings, our definition of it in this article is like its standard use in medicine where an intervention refers to "action taken to improve a situation" (Stevenson & Lindberg, 2012). Hence, *action* is the implementation of student activities with ACE in our second research cycle and *situation* is students' understanding of partial derivatives. To study students' construction of partial derivative after they finished a multivariable course, a sample of 11 students (out of nineteen students) enrolled in an electrical engineering program of a university in Iran was selected from a multivariable calculus course. The course emphasized the construction of two-variable functions according to the GD described previously. The teacher was the first author of this article. He used the ACE cycle in each of the two 2-h online sessions per week to introduce the course topics. Students worked collaboratively online in groups of three or four on activities (see Table 1) designed with the revised-GD resulting from the first research cycle of Martínez-Planell et al. (2017). Students could communicate with their group members during class using the online platform. Each group had to turn in their collective work to be graded. Afterward, the teacher conducted a whole group discussion. To open a discussion, the instructor asked one member of each group to share the final answer of their group and explain it to the entire class. This gave students more opportunities to reflect on their work and allowed the teacher to underline important aspects of the topic. Homework consisted of work on the remaining activities from the set and exercises selected from the textbook (Stewart, 2012).

Classes were organized in online software, which allows instructors and students to share images, voice, photos, and files. The instructor shared his laptop screen, face, and whiteboard. Students' faces or voices were opened to talk and participate in class and online activities whenever needed. Students could take photos of their work and share them with the instructor and other students.

The teacher selected eleven students (out of nineteen students in the course) to be interviewed at the end of the course. The sample consisted of three above-average, five average, and three below-average students, according to their performance in the course, in order to obtain a balanced distribution and to enable researchers to observe a wide range of student responses. These students participated in two 1-h semi-structured interviews, conducted in person, shortly after the semester's end.

As mentioned earlier, the GD resulting from the first research cycle was used to design activities in the ACE cycle and interview instruments. Questions for the first interview were the same as those of the first research cycle (Martínez-Planell et al., 2015). The

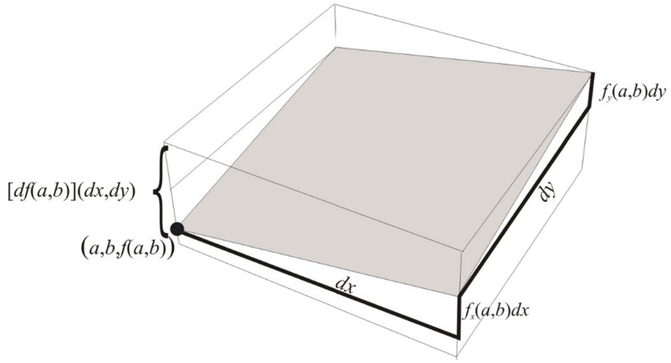
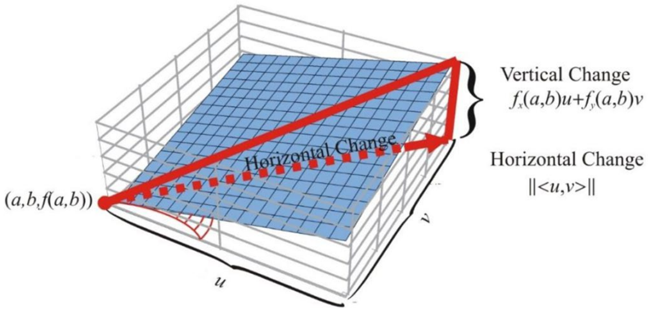
Table 1 Brief description of activities[a]

Activity[b]	Brief description
1	*Three-dimensional cartesian space, fundamental planes, and sections.* This activity aimed to help students construct fundamental planes (of the form $x = c$, $y = c$) as Objects that could be acted upon when constructing partial derivatives.
6	*Slope in a plane.* Because the genetic decomposition is based on the idea of local linearity, the initial activities dealing with the differential calculus strive to help students construct the notion of slopes in the x and y directions in a plane. The activity starts by having students review the geometric interpretation of slopes in two dimensions. Then the activity considers slopes on a plane, in different representations (graphical, tabular, contour diagram, symbolic).
7	*Vertical change and the equation of a plane*. The locally linear approach to the differential calculus is based on the notion of "vertical change on a plane" (below figure), which in turn depends on slopes in the x and y directions. This activity helps construct the notion in different representations.  Vertical change on a plane $\Delta z = m_x \Delta x + m_y \Delta y$
8	*Partial derivatives and tangent planes.* Given a function algebraically and graphically, students are asked for slope of the tangent line at a point, slope of secant lines, to draw them, and comment on the relation. Students relate partial derivatives with tangent planes given in different representations.

second interview dealt with topics not covered in the first cycle interview such as higher-order and mixed partials, and gradient vector. Questions from both interviews directly dealing with partial derivatives are included in Table 2. The interview instruments addressed other topics of the differential calculus, but we will restrict attention to those concerning the construction of partial derivative.

Questions I-3a, I-4, and II-7a intend to find out if students can think of partial derivative independently of representation, which would be consistent with a Process conception (Martínez-Planell & Trigueros, 2019): in question I-3a, the information is given by a graphical representation; in I-4 by a tabular representation; and in II-7a by a contour diagram representation of a tangent plane. Also, the GD proposes the construction of a partial derivative Process as coordination of Processes constructed using graphical and analytical

Table 1 (continued)

9	*The total differential and the tangent plane.* Makes use of partial derivatives, always relating back to vertical change on a plane and the tangent plane (figure).  Vertical change on the tangent plane and the total differential
10	*Directional derivative and tangent plane.* Makes use of partial derivatives, always relating back to vertical change on a plane and the tangent plane (figure).  $D_{<u,v>}f(a,b) = \dfrac{f_x(a,b)u + f_y(a,b)v}{\|< u, v >\|}$

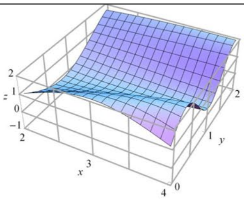
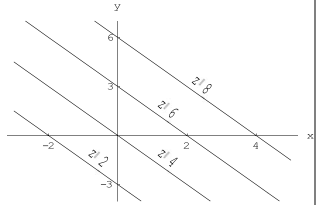
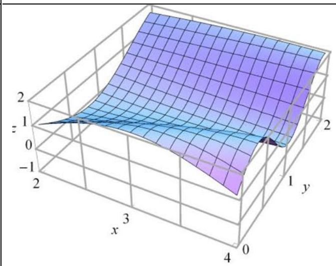
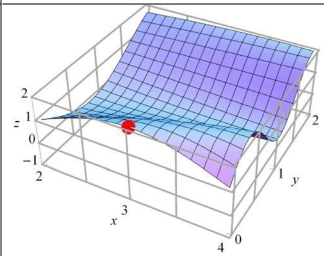
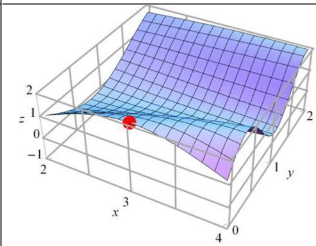
[a]For the full access to the activities, see https://drive.google.com/file/d/1THNw5JNI8QoJwSEAQUhYtB0uKYzvtGAF/view?usp=sharing

[b]Activities 2–5 are not included because they do not deal directly with partial derivatives

representation. Question I-3a can provide information on the graphical construction and question I-4 can give information on students' thinking of a partial derivative at a point as a limit of difference quotients. Questions II-1, II-2, and II-3 deal with higher-order and mixed partial derivatives, which were not included in the activity sets used by students; they were included to obtain information about the construction of an Object of partial derivative at a point and a Process of partial derivative as a function. They may allow determining if students can generate dynamical imagery of partial derivative as a function (consistent with a Process conception) and do Actions on an encapsulated Process of partial derivative at a point (consistent with an Object conception).

Interviews were audio-recorded in Farsi, transcribed, translated into English, and the written work of each of the eleven students in each of the six questions was analyzed

Table 2 Interview questions directly dealing with partial derivatives

<table>
<tr><td>I-3a. Suppose the graph of $z = f(x,y)$ is as follows. State the sign (positive, negative, zero) of $\frac{\partial f}{\partial y}(3.5,0)$ and of $\frac{\partial f}{\partial x}(3.5,0)$. Justify your answer.</td><td></td></tr>
<tr><td>I-4. The following is a table of values of a differentiable function f of two variables. Approximate the value of $\frac{\partial f}{\partial y}(1,2)$ the best you can.</td><td>
<table>
<tr><td>y
x</td><td>1</td><td>2</td><td>2.01</td></tr>
<tr><td>0</td><td>5</td><td>7</td><td>7.04</td></tr>
<tr><td>1</td><td>6</td><td>10</td><td>10.06</td></tr>
<tr><td>2</td><td>8</td><td>14</td><td>14.10</td></tr>
</table>
</td></tr>
<tr><td>II-7a. The following figure shows the contour diagram of the tangent plane of a function $z = f(x,y)$ at the point (0,3). Find $\frac{\partial f}{\partial x}(0,3)$.</td><td></td></tr>
<tr><td>II-1. If $H(t) = \frac{\partial f}{\partial x}(t,0)$ draw how the graph of H may look for $2 \le t \le 4$.</td><td></td></tr>
<tr><td>II-2. State the sign (positive, negative, zero) of $f_{xx}(3,0)$. Justify your answer.</td><td></td></tr>
<tr><td>II-3. State the sign (positive, negative, zero) of $f_{yx}(3,0)$. Justify your answer.</td><td></td></tr>
</table>

independently by each of the three researchers using APOS-based codes. The codes, their definition, examples, and patterns are included as supplementary material. After finishing the individual analysis of all students, we considered the type of mental construction primarily evidenced by each student when solving the different problems dealing with partial derivatives. Students' conceptions were classified as Action, Process, or Object. Then, the coded data were examined for patterns. Finally, the results of the entire analysis were discussed, reanalyzed, and negotiated by the researchers until a consensus (including codes, classification, and patterns) was reached.

6 Results

After completing the data analysis, we found that three students showed evidence of constructing an Object conception of partial derivatives, one student showed the construction of a Process conception, and the other seven students exhibited an Action conception. We will exemplify differences in students' partial derivative constructions by considering the cases of two students (pseudonyms: Sepehr and Mahsa) that clarify what we mean by an Object, Process, or Action conception of partial derivative. These examples illustrate responses from other students who showed evidence that could be considered as having the same type of constructions.

6.1 Object conception

Sepehr was the fourth best in the class in terms of his grades; the instructor considered him an above-average student. Questions I-3a, I-4, and II-7a asked about partial derivatives in graphical, tabular, and contour diagram representations. In question I-3a, Sepehr was given a graph and was asked about the sign of the partial derivatives at a point:

> Sepehr: The slope of the tangent line to the graph f at the point $(3.5, 0)$ in the y direction is like this line (Fig. 1) and has a positive slope so the sign of $f_y(3.5, 0)$ is positive. The tangent line at this point in the x direction is a decreasing line, and its slope is negative therefore the sign of $f_x(3.5, 0)$ is negative.
>
> Question I-4 asked for a partial derivative, $f_y(1, 2)$, of a function given by a table.
>
> Sepehr: I need to find the changes in f over the changes in y near to $y = 2$ when x is fixed at $x = 1$. We have $\frac{10-6}{2-1}$ and also $\frac{10.06-10}{2.01-2}$… the second one is a better approximation because the changes in y from 2 to 2.01 is so small.

Sepehr's responses suggest that he could think of partial derivative as independent of representation; this is consistent with a Process conception. By choosing to use y value of 2.01 rather than 1 ("the changes in y from 2 to 2.01 is so small"), Sepehr shows awareness of a partial derivative being the result of a limiting process. It also suggests the possibility that Sepehr can generate the needed dynamical imagery of partial derivative at a point to think of partial derivative as a limit. So, Sepehr might have constructed an analytical Process, as conjectured in the GD. Next, question II-7a asked to compute $f_x(0, 3)$, given a contour diagram of the tangent plane to the function at $(0, 3)$ (Fig. 2). Answering this question

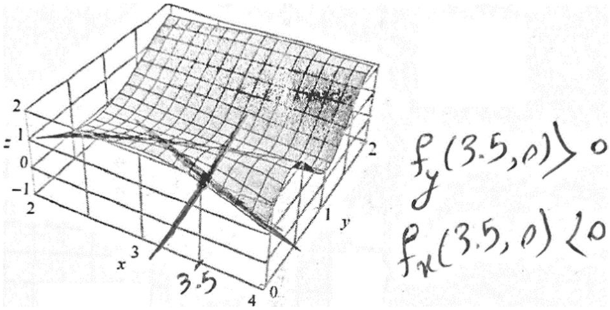


Fig. 1 Sepehr's work in question I-3a

requires considering that the tangent line in the x direction at $(0, 3)$ is part of the tangent plane and extracting the needed information from the contour diagram representation.

> Sepehr: x is 0 and y is 3 and we are on the contour diagram. We can see the changes of the z in the figure. $f_x(0, 3)$ is the changes in the value of z when x changes and y umm does not change.
> Interviewer: Okay, find the value of the partial derivative with respect to x at the point $(0, 3)$.
> Sepehr: If I move from the point $(0, 3)$ to the right side parallel to the x axis and reach the line with $z = 8$ we see z increases 2 units but it's unknown how much value x increases (Fig. 2).
> Interviewer: Do you have a solution for it?
> Sepehr: This contour diagram is a graph of a plane, and we know on a plane all lines in the x direction have the same slope. So, I can move on the x axis and find the slope in the x direction. When x changes from 0 to 4, I can see the value of z changes from 4 to 8 so the slope is $\frac{\Delta z}{\Delta x}$ and this is equal to $\frac{8-4}{4-0}$ so it's 2 [Sepehr mistakenly found $\frac{8-4}{4-0}$ equal to 2].

So, in questions I-3a, I-4, and II-7a, Sepehr showed to have constructed conversion relations between different representations, graphical, tabular, and contour diagrams; this is consistent with the construction of partial derivatives as a Process. Also consistent with a Process conception is Sepehr's dynamic description of the transformations involved in a mathematical notion, as shown in questions II-1, II-2, and II-3. Here, the student demonstrated the possibility of omitting steps and anticipating results without needing to perform the Process explicitly. The internal perception of partial derivative as a Process allows this

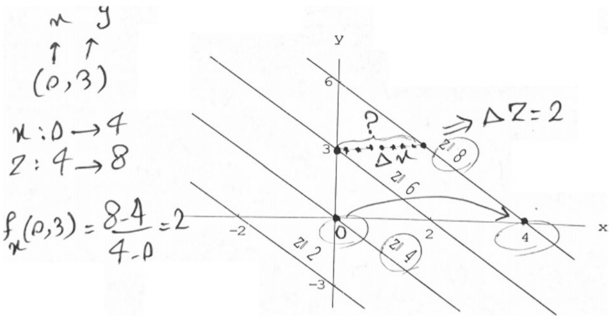


Fig. 2 Sepehr's work in question II-7a

(Arnon et al., 2014). These questions also allow finding evidence of Sepehr performing Actions on an encapsulated partial derivative at a point Process.

Question II-1 required graphing $H(t) = \frac{\partial f}{\partial x}(t, 0)$, on the interval, $2 \leq t \leq 4$, for a graphically given function (Fig. 3).

> Sepehr: $H(t)$ is a function of t that represents the slope of the tangent lines at the point $(t, 0)$ in the x direction… I draw tangent lines in the x direction on the figure. From $(2, 0)$ to $(3, 0)$ their slopes are positive but going to 0 umm because at the point $(3, 0)$ the tangent line in the x direction has a slope of 0 so the curve $H(t)$ is like a decreasing curve in the interval $2 \leq t \leq 3$ and umm $H(3, 0)$ is 0. From $(3, 0)$ to $(4, 0)$ we can see umm the tangent lines are decreasing and their slopes go to negative and umm steepest so the curve $H(t)$ will be again a decreasing curve from $t = 3$ to $t = 4$.

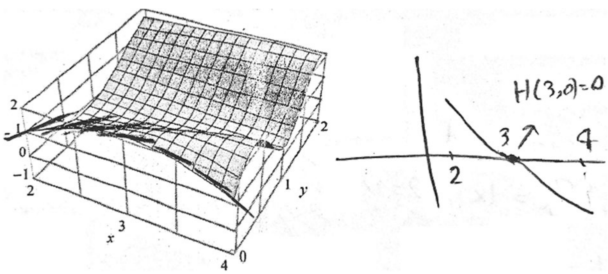
In describing his answer to this question, Sepehr used language that denotes movement, as in "their slopes are positive but going to 0." This evidenced a Process construction; he talked about change in a dynamic fashion, and he showed to be able to coordinate mutually changing quantities, the changing inclination of tangent lines with the decreasing values of slope in the graph of $H(t)$ ("$H(t)$ is like a decreasing curve… the tangent lines are decreasing and their slopes go to negative and umm steepest"). Question II-2 asked to state the sign of $f_{xx}(3, 0)$:

> Sepehr: I think f_{xx} shows changes in f_x in the x direction. Since f_x is the slope of the tangent line in the x direction so umm f_{xx} is like how slopes change at a small neighborhood of the point $(3, 0)$… I draw some tangent lines before the point $(3, 0)$ and umm some tangent lines after this point in the x direction umm like these (Fig. 4). I can see the value of the slope of these lines will be smaller at each step, so f_{xx} at the point $(3, 0)$ is negative.

To do this, he examined how f_x changes as x increases. Sepehr again showed the construction of a second derivative Process. In his response, Sepehr's argument justified that the second derivative should be negative at a concave down curve. Also, he again gave evidence of using dynamical imagery ("the value of the slope of these lines will be smaller at each step") which demonstrates the construction of a Process conception. Further, in this problem, he performed Actions on the encapsulated partial derivative Process to discuss the derivative of the derivative, which gives evidence of the construction of an Object.

Question II-3 asks for the sign of $f_{yx}(3, 0)$. Here, Sepehr confused f_{yx} with $(f_x)_y$, an important distinction in mathematical terms. However, despite this, it is possible to analyze his constructions in terms of the GD. In particular, we can determine if Sepehr is able to do

Fig. 3 Sepehr's work in question II-1



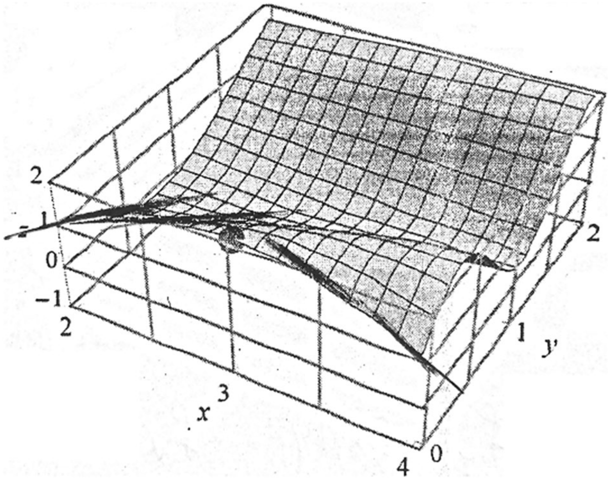


Fig. 4 Sepehr's work in question II-2

the Action of deciding the sign of the partial derivative with respect to y, on a previously constructed Object of partial derivative with respect to x.

> Sepehr: I think f_{yx} is like we first find f_x and then compute $(f_x)_y$. So, I should compare the tangent lines in the x direction when they move in the y direction very close to the point $(3, 0)$ (Fig. 5). The tangent line at the point $(3, 0)$ in the x direction is this horizontal line whose slope is 0. If I move a little bit to the positive direction of the y axis but I'm still at the plane $x = 3$, then the tangent line in the x direction is a line like this which is increasing, and its slope is bigger than the slope of the previous tangent line. So, the change is positive it means f_{yx} at the point $(3, 0)$ is positive.

When Sepehr said, "I should compare the tangent lines in the x direction when they move in the y direction," he demonstrated to be de-encapsulating an Object of partial derivative (f_x) into the Process it came from (computing slopes of tangent lines), in preparation

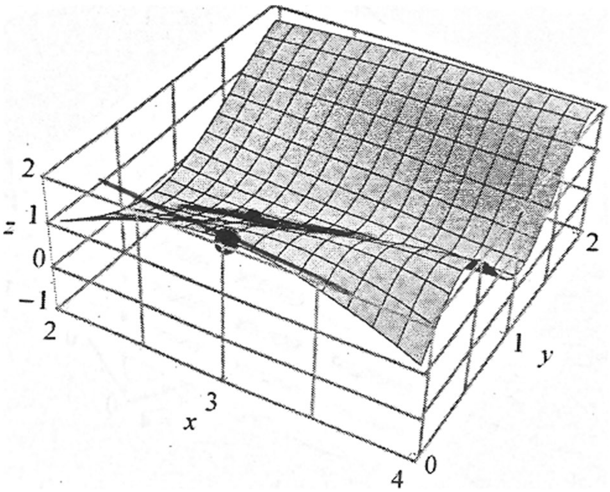


Fig. 5 Sepehr's work in question II-3

of letting these tangent lines "move in the y direction" and doing the Action of examining the resulting slopes.

Overall, Sepehr was able to justify his solutions, generate dynamical imagery, and interrelate different representations of partial derivative. He did this consistently throughout the interviews giving evidence that allowed us to conclude that he had constructed at least a Process conception of partial derivative. Further, he showed to be able to do Actions on constructed Objects to deal with higher-order and mixed partial derivatives. Hence, we concluded that Sepehr demonstrated to have constructed an Object conception of partial derivative of a two-variable function. The only student with a Process (and not Object) conception of partial derivative gave responses similar to those of Sepehr, except on questions II-2 and II-3, where he did not show evidence of doing Actions on an encapsulated Process of partial derivative.

6.2 Action conception

Mahsa was considered an average student by her teacher. She was selected because her responses exemplified the constructions showed by students who evidenced an Action conception of partial derivative. In question I.3a, when asked for the signs of some partial derivatives given a graph, Mahsa responded:

> Mahsa: The point $(3.5, 0)$ is here on the surface [Fig. 6]. In the y direction the surface from the point $(3.5, 0)$ is initially going up in the y direction so $f_y(3.5, 0)$ is positive. The curve at the point $(3.5, 0)$ and in the x direction is going down, it means the slope of the curve or the slope of the tangent line is negative, so the sign of $f_x(3.5, 0)$ is negative.

Mahsa did Actions to identify the point on the surface and draw the directions of change, thus demonstrating she had constructed graphical meaning for partial derivatives. When dealing with question I-4:

> Mahsa: It's like the slope in the y direction. We know the slope for all lines of a plane in the y direction is the same. So, I can use each of these rows. I fix on $x = 0$. When y

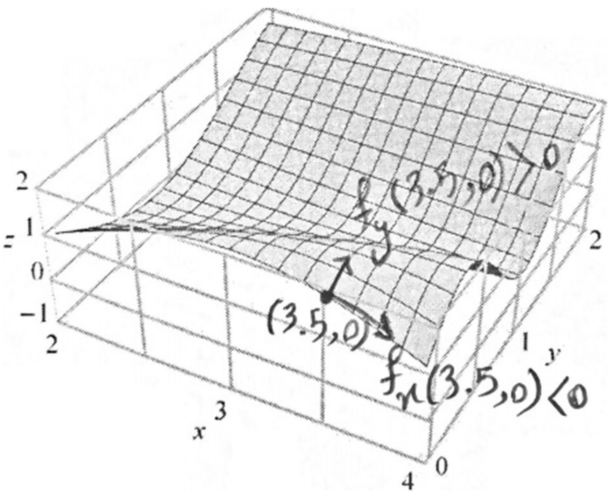


Fig. 6 Mahsa's work in question I-3a

changes from 1 to 2 then z changes from 5 to 7 [Fig. 7]. So, the slope in the y direction is $\frac{7-5}{2-1}$ and we can consider it as an approximation for $f_y(1, 2)$.
Interviewer: This table contains the values of a function f and not a plane.
Mahsa: It doesn't matter. I remember that I could find the slope in the y direction from each row of a table by holding x fixed and umm check the change in z.

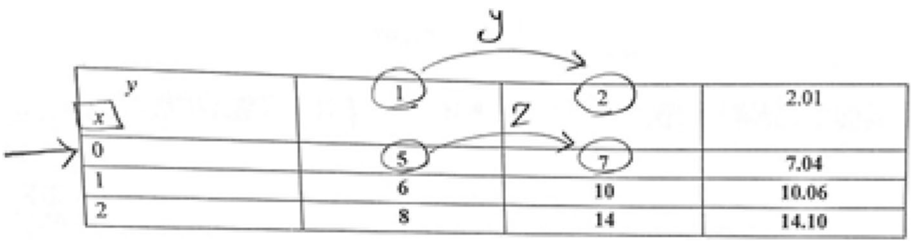
Mahsa's response showed the construction of facts about planes she can remember and referred to the slope of the plane in a particular direction as representing the approximation for the partial derivative in that direction. Mahsa performed Actions to compute the slope in the y direction choosing data from the table. When the interviewer called her attention to the difference between the plane and the function, she maintained her response, disregarding that her computation did not involve the base point and the possibility of obtaining a better approximation for the partial derivative. In question II-7a, examining student's possibilities to interpret partial derivatives when given a contour diagram, Mahsa responded:

Mahsa: It's the slope in the x direction. So, I need to move from any starting point however parallel to the x axis. I prefer to start from the point $(2, 0)$ and go in the x direction. When x changes from 2 to 4 [Fig. 8] umm then z changes from 6 to 8. So f_x is $\frac{8-6}{4-2}$ which is 1.

She did the Actions of choosing a point and reading the change from the contour map to find values to calculate the partial derivatives using the slope of lines. Mahsa gives evidence of constructing an analytical path to a partial derivative through Actions.

Mahsa's responses show the construction of partial derivatives on each representation in terms of Actions to describe and compute the slope of the tangent plane in the given direction. But she did not always take the base point into account when equating the slopes of the plane to the partial derivative at the point. Although mentioning their approximation role, in practice, Mahsa showed she needed to construct partial derivative at a point as the best possible approximation to the computation of the limit involved in the definition of one-variable function derivative. Question II-1 asked to graph $H(t) = \frac{\partial f}{\partial x}(t, 0)$, given the graph of f:

Mahsa: I need to draw H respect to t. H is derivative in the x direction from the point $(2, 0)$ to $(4, 0)$. From $t = 2$ to $t = 3$ this curve is increasing.
Interviewer: What do you mean by this curve?
Mahsa: I mean the curve which is the intersection of the plane $y = 0$ with the surface f.
Interviewer: Okay. Draw H.



y / x	1	2	2.01
0	5	7	7.04
1	6	10	10.06
2	8	14	14.10

Fig. 7 Mahsa's work in question I-4

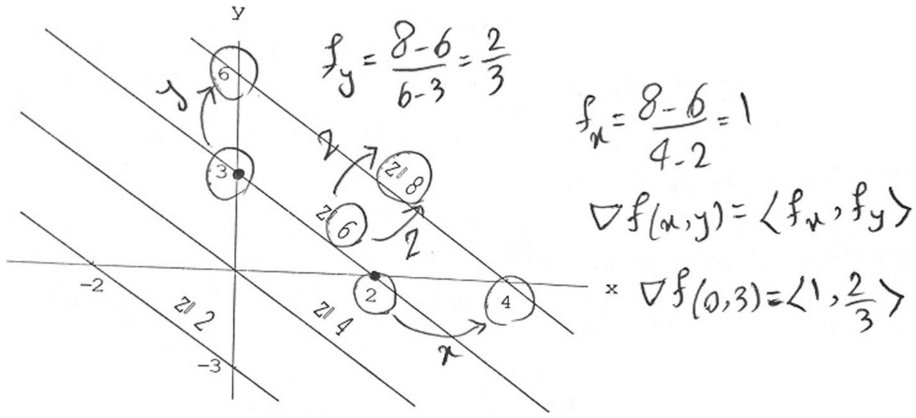


Fig. 8 Mahsa's work in question II-7a

> Mahsa: So, from $t = 2$ to $t = 3$ the graph of H is positive umm it's like this [Fig. 9]. In the interval $t = 3$ to $t = 4$ the intersection curve is decreasing so slopes are negative and it means the values of H are negative so H is below the horizontal axis t umm it's like this.

Mahsa gives evidence of recognizing the intersection curve of the fundamental plane and the surface. She identifies, with ease, where the curve is increasing or decreasing. She also assigns a sign to the derivative and draws a graph for the behavior of the derivative as two independent segments. Mahsa focused just on the sign of the derivative, so she did the Action of drawing segments corresponding to those signs showing she had not constructed the Process of "following" or "imagining" the variation of a continuous curve.

Regarding question II-2, Mahsa only commented about the sign of the second derivative being negative, "Because the concavity is negative, I mean it's downward." Her

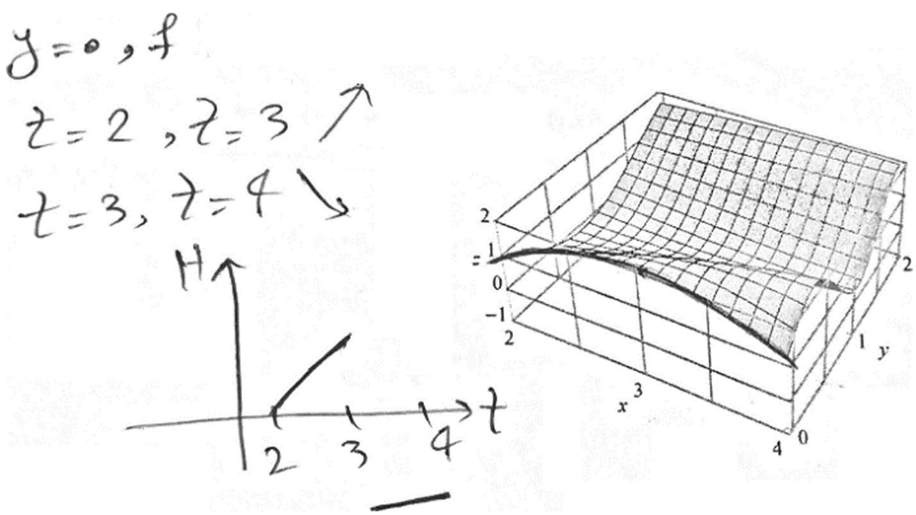


Fig. 9 Mahsa's work in question II-1

response does not give enough information to confirm the use of dynamical imagery in contrast to just considering the overall form of the curve and perhaps applying a memorized fact. In question II-3:

> Mahsa: The derivative respect to y is umm for example tangent lines parallel with the y axis like these lines I mean in the y direction. This one L_1 is decreasing if I consider its slope -2 umm the next line which is L_2 is increasing in the y direction and its slope is for example, 3 [Fig. 10].
> Interviewer: How did you find -2 or 3?
> Mahsa: They are not exact values I only approximate to compare the changes of slopes when I move from left to right in the x axis. From -2 to 3 the change is 5 and is positive so the sign of $f_{yx}(3,0)$ is positive.

Mahsa was able to consider the change in the y direction by approximating the slopes of two lines. She drew them in a neighborhood of the point $(3,0)$. As there is no clear way to calculate the slope, Mahsa focused on the sign of the lines and assigned a number to the slope. As in previous questions she showed being able to work in a graphical context by doing Actions congruent with her definition of derivative as a slope. By doing an Action on the numbers representing the slope, Mahsa found the sign of the change in the x direction to be positive.

Overall, Mahsa's responses gave evidence of an Action conception of partial derivatives. She used the notion of partial derivatives as the slope of the tangent plane in a given direction to perform Actions on different representation of the function, by sketching lines and "calculating" their slopes. Mahsa also mentioned approximations, but the evidence showed she did not use them when needed. Her reliance on using particular computations of slope and possibly memorized procedures rather than general arguments suggests that Mahsa constructed an Action conception of partial derivative. Seven students constructed an Action conception of partial derivative.

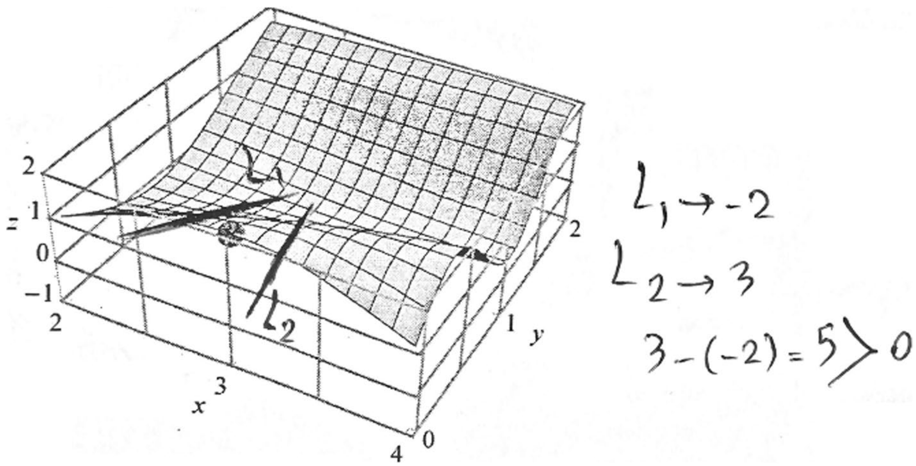


Fig. 10 Mahsa's work in question II-3

6.3 Refined GD

The results of the analysis of all the data from all students led to further refinement of the GD. We observed that students' main difficulty was related to the construction of an Action conception of one-variable function derivative. A Process conception is needed to coordinate with a fundamental plane Process in the construction of partial derivative. To address this situation, the following paragraph was added to the GD:

> Actions of graphically representing secant lines to the intersection curve at the given base point and direction, calculating and symbolizing their slopes, while converting between different representations of secant line and slope, are interiorized into a slope Process where students recognize the slopes in the x and y directions as rates of change of the function in the given point and direction.

Also, in keeping with the work of Zandieh (2000), Kertil and Gülbagci Dede (2022), and Mkhatshwa (2021), on one-variable function derivative, and the lack of consideration of contextual situations in the previous GD, the first paragraph now explicitly refers to representations other than the graphical. The refined GD now starts with:

> Given a base point (a, b) in a two-variable function in a numerical, graphical, verbal or analytical representation and a direction (either x or y), Processes of fundamental plane and function of two variables are coordinated into an intersection curve Process where it is possible to imagine the curve resulting from the intersection of the fundamental plane that passes through the given base point on the function and that is in the given direction in each of the considered representations.

7 Discussion

One important result of this study is that the intervention helped three of the interviewed students to construct an Object conception of partial derivatives in this course. Other studies using APOS theory have reported that this construction is difficult to achieve in a one-semester course (Arnon et al., 2014). The designed activities paired with the collaborative work in the classroom made this important achievement possible, suggesting that the activities fostered students' reflection. However, most interviewed students evidenced an Action conception of partial derivatives indicating that there are either constructions that need to be considered in the refinement of the GD or that there is a need for more activities to foster reflection on the necessary constructions.

Our GD of partial derivative does not explicitly address the construction of a slope in 3D Process, although this can be challenging for students (Moore-Russo et al., 2011). Nevertheless, our results show that using the tangent plane to start the construction of change of two-variable functions, as Tall (1992) suggested, helps students construct the 3D notion of slope (Martínez-Planell et al., 2022). The explicit consideration of slopes in 3D has been argued to improve students' understanding of derivative in multivariable calculus (McGee & Moore-Russo, 2015). Results show that it is possible for students to interiorize the notion of slope from Actions of computing slopes of tangent planes and secant lines in given base points and x and y directions. In this study, evidence showed that students recognized partial derivatives as slopes of tangent planes and the possibility to approximate them using slopes of secant lines, with some students also recognizing the slope of a tangent line as a limit of slopes of secant lines. This also underlines the importance of understanding the role of one-variable function derivative in the definition of partial derivatives, as suggested by the GD and as reported in other studies (Wangberg et al., 2022; Weber, 2012). Moreover, all interviewed students demonstrated the construction of the curve resulting from the intersection of fundamental plane and two-variable function, as also suggested by the GD.

Another important result of this study is that students showed they could interpret partial derivatives in different representations, as suggested by Moreno-Arotzena et al. (2020) and Wangberg et al. (2022). We consider that activities used throughout the course contributed to this construction and students' understanding of partial derivatives. In particular, ten of the eleven interviewed students demonstrated that they could interpret partial derivatives in different representations when addressing the graphical representation in question I-3a and the tangent plane contour diagram representation in question II-7a, and seven demonstrated that interpretation by using the tabular representation in question I-4. Moreover, students' responses to questions II-1, II-2, and II-3, dealing with higher-order and mixed partial derivatives, show how students who evidenced an Action conception of partial derivatives were constrained in their geometric interpretation (Thompson et al., 2006, 2012), and the rich dynamical imagery and understanding shown by students who demonstrated a Process or an Object conception of partial derivative.

In terms of the GD, all students in this cycle showed the coordination of fundamental plane and two-variable function Processes and were able to imagine the curve resulting from the intersection of the fundamental plane that passes through the given base point on the function and that is in the given direction in different representations. Most encapsulated this Process into an Object. Also, students demonstrated the construction of the slope in 3D Process and the construction of at least an Action of partial derivative at a point. Students who showed the construction of a Process or Object conception of partial derivatives showed all the constructions in the GD. We consider that work with the activities and the teaching strategy promoted students' reflection on the role of variables and slope in 3D space and the construction of a graphical understanding of partial derivative. The improved understanding of slope and its use in the differential calculus of two-variable functions is further supported by Martínez-Planell et al. (2022).

Students need to make sense of the "two-change problem" (Weber, 2015) and understand the total differential as a linear approximation to a function in a way that can be generalized to a notion of derivative of functions from R^n to R^m (Harel, 2021; Weber, 2015). The "locally linear" approach used in this experience, as reflected in the GD of Martínez-Planell et al. (2017) and the activity sets tested in the present study, contributes to such an understanding.

While analyzing the interviews of students showing an Action conception of partial derivatives, we found that most of them did not evidence the coordination of the Process related to the intersection curve and the derivative of one-variable function. This is related with the possibility of envisioning partial derivatives as the limit of secant lines that approach the tangent line to the curve at the given point. This may be due to their difficulties with the construction of the derivative of a one-variable function described in the literature (e.g., Asiala et al., 1997; Zandieh, 2000). So, we decided to include more detail about the constructions needed in this respect in the refined GD of partial derivative.

The activity set must be revised according to the refined GD to help more students understand partial derivatives as Processes or Objects. In particular, although our study did not include anything dealing with the contextual interpretation of partial derivative, other studies (Bingolbali et al., 2007; Kertil & Gülbagci Dede, 2022; Mkhatshwa, 2021) suggest the value of explicitly considering those situations in helping students make sense of this concept. We coincide and consider the need to include this type of problem in the revised activity sets to diversify the opportunities to reflect on partial derivatives.

8 Conclusion

This study contributes to a better understanding of how students construct the notion of partial derivative and underscores the usefulness of the "locally linear" approach to the differential calculus of two-variable functions. It shows how research-based activity sets, constructed according to a GD together with the pedagogical strategy (ACE cycle), can help students construct an understanding of partial derivative as a Process or Object. It thus stresses the potential of a GD to inform the design of teaching materials and to guide instruction. The refined GD is an important result of the study. Another contribution of this study is showing how a succession of research cycles can be used to progressively help students enrich their understanding, in this case, of partial derivative.

The study has some limitations. Teaching the course online can be considered as a limitation because the role of the teacher when guiding students' work through the ACE cycle (Trigueros & Oktaç, 2019) was a new experience for all involved. However, this is not a problem for this study because one can only expect the ACE pedagogical intervention to improve with face-to-face students' group discussions and whole-course teacher-guided interventions. The lack of problem situations investigating students' understanding of partial derivatives in contextual situations was another limitation. In this regard, the GD was refined to consider several representations other than the graphical and the activity set must be revised accordingly. Finally, other limitations are the small sample size of the study (11 out of 19 students were interviewed) and that all students were enrolled in electrical engineering thus results may differ when a richer and larger sample is used. However, the purpose of the study is not to generalize but to illustrate how a research-based GD and activities can promote students' understanding of partial derivatives. The other issues can be examined in future research.

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Data availability The datasets generated and analyzed during the current study are available from the corresponding author upon reasonable request.

Declarations

Conflict of interest The authors declare no competing interests.

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