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Is it all about the setting? — A comparison of mathematical modelling with real objects and their representation

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Abstract
Mathematical modelling emphasizes the connection between mathematics and reality — still, tasks are often exclusively introduced inside the classroom. The paper examines the potential of different task settings for mathematical modelling with real objects: outdoors at the real object itself, with photographs and with a 3D model representation. It is the aim of the study to analyze how far the mathematical modelling steps of students solving the tasks differ in comparison to the settings and representations. In a qualitative study, 19 lower secondary school students worked on tasks of all three settings in a Latin square design. Their working processes in the settings are compared with a special focus on the modelling steps Simplifying and Structuring, as well as Mathematizing. The analysis by means of activity diagrams and a qualitative content analysis shows that both steps are particularly relevant when students work with real objects — independent from the three settings. Still, differences in the actual activities could be observed in the students' discussion on the appropriateness of a model and in dealing with inaccuracies at the real object. In addition, the process of data collection shows different procedures depending on the setting which presents each of them as an enrichment for the acquisition of modelling skills.

Keywords Mathematical modelling · Outdoor mathematics · Mathematizing · Data collection · Qualitative content analysis

1 Introduction

Mathematical modelling is characterized through its interplay of reality and mathematics. It offers a way to integrate references to reality into the classroom and shows students where in everyday life their mathematical knowledge can be applied. Nevertheless, empirical studies show that students find it difficult to work on modelling tasks independently. Blum (2015), for example, summarizes the strategy that — if a solution plan is not obvious to them — students "ignore the context, just extract all data from the text and calculate something according to a familiar schema" (p. 79). Whether this strategy still sufficiently

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transmits the basic idea of creating references to reality seems questionable. Even more, the strategy gives rise to the idea of working on mathematical modelling tasks with real objects that provide a task context.

The idea of involving real objects in mathematical modelling leads to the question of how much the way in which a real object is introduced might influence the modelling processes of students. Despite its actual physical presence in reality, a real object could be introduced through different representations and provided artefacts, e.g., newspaper articles, photographs, videos, 3D print replications or combinations. Potentially, the different representations of the real object might lead to differences in the modelling activities of students and motivate a comparison of them. In this article, this is particularly discussed in the context of the modelling steps *Simplifying and Structuring* as well as *Mathematizing*.

To investigate the influence of the modelling setting in more detail, a qualitative study was conducted. It is the aim of the paper to highlight similarities and differences in the different modelling settings in order to provide an orientation for teachers and researchers to choose an appropriate task setting in line with the intended learning aim, i.e., a particular modelling step to be fostered.

2 Settings and representations in mathematical modelling

The idea of mathematical modelling can be described in an idealized way by using the modelling cycle according to Blum and Leiß (2007). A combination of seven modelling steps is used to describe necessary phases in the solution process of a modelling problem (Fig. 1).

Students begin to solve real-world problems by constructing a model of the situation in the real world. Then they translate this model into a mathematical model and switch from the real world to the mathematical world. After that, calculations can be made in the mathematical world, and the mathematical results have to be interpreted and validated with respect to reality (Hartmann & Schukajlow, 2021, p. 155).

This strong emphasis on reality highlights the connection of modelling and in particular the transfer step of mathematizing to the theory of *Realistic Mathematics Education* (Freudenthal, 1973). Even though the purpose of the RME approach is mathematical conceptual learning, a similar connection between reality and mathematics can be seen for

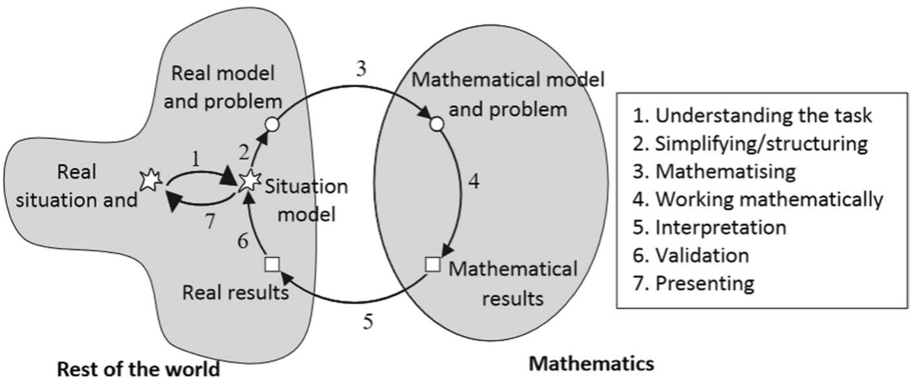

Fig. 1 The modelling cycle according to Blum and Leiß (2007) — reproduced by Kotze (2017, p. 193)

mathematical modelling. Both reality and mathematics are directly linked through the iterative processes of developing and optimizing ideas and strategies to solve the problems with mathematical knowledge (e.g., Buchholtz, 2021; Osawa, 2002). Even though most students do not work on mathematical modelling tasks in the strict organization of the modelling cycle (Borromeo Ferri, 2010), a transfer from reality to mathematics and vice versa is mandatory to solve a modelling task when following the approach of Blum and Leiß (2007).

The modelling task context and setting can differ in its content, visual and material representation. Larsen (2018) highlights the importance of contexts being meaningful and real to the students, or, to speak in the language of Freudenthal (1973), contexts that *beg* to be mathematized. Hereby, a context is mainly related to the topic that is addressed in the modelling task. Still, it is not only the topic itself that characterizes the modelling context, but also *the way* in which a task context is introduced. This point of view is taken up in the article and directly linked to the modelling cycle: In the following, the introduction of the real situation/object is defined as the modelling task *setting*.

Usually, modelling tasks related to a real situation/object are introduced inside the classroom. A textual introduction can be done verbally, via texts and/or newspaper articles (e.g., Eames et al., 2018). The latter is mostly enriched by photographs or pictures (Greefrath et al., 2018). In pictures, usually information is missing which has to be gained through evident assumptions based on the provided material. By doing so, Herget and Torres-Skoumal (2007) assume on a theoretical level that these tasks have a potential to strengthen planning skills apart from pure calculation without context. Hartmann and Schukajlow (2021) state that especially pictures "can be helpful for estimating the missing information and can make the relation between the problem and the real world more obvious" (p. 155). Such *representational* pictures enable to collect data in the sense of mathematizing, e.g., through comparisons of the task object with an object of reference or by reasoned estimations based on previous experiences gained (cf. Greefrath, 2009). On the one hand, Hoogland et al. (2018) point out that pictures, in comparison to a verbal introduction of the task, can support students in solving geometry problems. On the other hand, pictures can be irritating or misleading since (usually) an originally three-dimensional situation and/or object is represented in a two-dimensional photo. This requires considerations concerning the given information in a photo, especially in terms of perspective (Schukajlow, 2013).

Moving from the textual picture representation to spatial settings, a replication[1] of the real situation/object can be brought to the classroom. In contrast to the previously introduced modelling settings, it involves three-dimensional material called *artefacts*. Referring to *embodied mathematics* and *multimodality* (Duijzer et al., 2019), the modality of touching the material is added to seeing and possible motor actions in the contexts. Touching, in the sense of an "interaction of the body with objects in their real spatial context" (p. 3), can influence cognitive processes and learning. For example, a study by Bokosmaty et al. (2017) shows an increase in secondary school students' understanding of geometric topics after working with physical manipulatives. Hereby, different material is possible to recreate or rebuild a situation/object. Due to its flexible use (e.g., for various objects) and fast production (Asempapa & Love, 2021), the use of a 3D print model is described, for example. Researchers currently examine the potential of students creating a 3D model using digital technology and printing it (e.g., Anđić et al., 2023). Even without the actual creation, the printed model itself can offer "opportunities to discover properties of their surroundings real-time" (Lavicza et al., 2020, p. 24). In particular, Medina Herrera et al. (2019) described the usage of 3D printed surfaces as manipulatives in mathematics education in terms of spatial visualization and orientation

[1] In the context of the article, only situations/objects that cannot be brought to the classroom are taken into consideration.

skills. As for the photo approach, a suitable object of reference, e.g., a little figure, or information about scale could be given in order to estimate or measure data that help to examine the proportions of the object in reality (cf. Greefrath, 2009).

In contrast to working inside the classroom, a modelling task can be introduced at the real object outside the classroom. In this way, "the participants [...] amplify the possibility of establishing connections between mathematics and reality" (Barbosa & Vale, 2020, p. 48). The students can change their own perspective, interact with the object and focus on details directly at the object. Offering the possibility to gain and reflect these experiences directly at real objects, a link can be drawn to the *Experiential Learning Theory* (Kolb, 1984). In addition, the previous considerations about embodied cognition can be transferred despite the fact that the real object manipulation might be limited. Buchholtz (2017) describes outdoor mathematics as a chance "to create incentives for autonomous mathematising based on real-world problems in a delimited thematic context" (p. 57). In particular, he observes that the choice of a mathematical model can be overwhelming while working at the real object. Moreover, Buchholtz (2021) highlights the data collection as an essential part working at the real object and adds this process in a stage between the modelling steps *Structuring* and *Mathematizing* (see Fig. 2). Table 1 summarizes the similarities and differences in the outlined settings.

As highlighted previously, Freudenthal (1973) mentions the context of a task together with its claim for mathematization, "in the sense of a transition from the lifeworld to the world of symbols" (Buchholtz, 2021, p. 142). It is therefore legitimate to focus on the actual transfers between reality and mathematics. The focus will be on the first transfer between reality and mathematics, namely *Simplifying and Structuring* and *Mathematizing*. The article follows a narrow definition of the two steps by dividing them according to the modelling activities that students conduct. Whereas *Simplifying and Structuring* contains the recognition of characteristics, inaccuracies and the creation of a real model, *Mathematizing* is understood as the transfer of the real model into mathematics by gaining necessary information and data (Blum & Leiß, 2007; Buchholtz, 2021). To investigate the influence of the

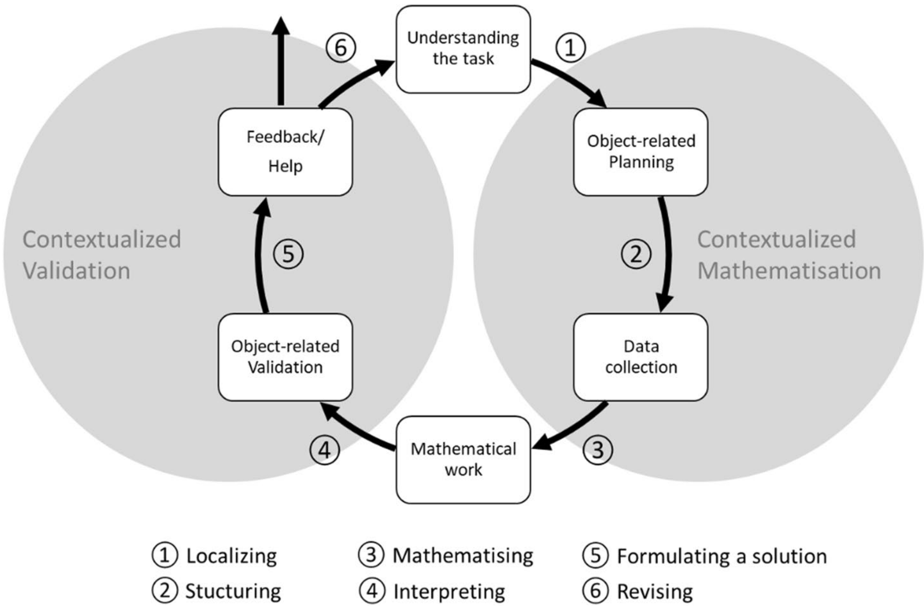

Fig. 2 Adapted modelling cycle for modelling tasks at real objects (Buchholtz, 2021, p. 145)

Table 1 Comparison of the different task settings

Settings	Texts (e.g., Eames et al., 2018)	Pictures (e.g., Herget & Torres-Skoumal, 2007)	3D models (e.g., Medina Herrera et al., 2019)	Real objects (e.g., Buchholtz, 2021)
Usual location (cf. Kolb, 1984)	Indoors	Indoors	Indoors	Outdoors
Representation (cf. Duijzer et al., 2019)	Verbal	Visual (2D)	Visual (3D)	Visual (3D)
Artefacts (cf. Duijzer et al., 2019)	-	-	Artefact (rebuilt)	Artefact (original)
Modality (cf. Duijzer et al., 2019)	Seeing	Seeing	Seeing and Touching (manipulative)	Seeing and Touching
Data collection (cf. Greefrath, 2009)	Extraction, estimating	Estimating, measuring, comparing, scaling	Estimating, measuring, comparing, scaling	Estimating, measuring, comparing

modelling setting in more detail, the next section summarizes the state of the art concerning empirical findings on comparisons of different modelling settings. The retransfer through interpretation and validation will be considered in the concluding discussion.

3 State of the art and research question

Whereas the theoretical section mainly focused on studies that take solely one modelling task setting into account, this section focuses on studies that compare at least two different settings. Following a quantitative approach, Zender et al. (2020) focus on a comparison of working outside at real objects and inside the classroom. In particular, they investigated the learning outcomes when solving geometric tasks outside the classroom involving real objects. They conclude that the outdoor setting can have a positive impact on the learning of mathematics. Still, the results have to be interpreted carefully against the background of a comparison since the students inside the classroom did not solve comparable tasks involving real objects' representations, but followed their usual mathematics classes. Barlovits and Ludwig (2020) take up these findings and compare the learning at real objects with related tasks that use photographs. For both studies, the tasks are not explicitly designed as modelling tasks. Hartmann and Schukajlow (2021) conclude in a comparison of indoor (photograph) and outdoor modelling problems that the setting does not influence interest and emotion on the side of the students — the content of math problems seems to be more relevant.

Still, the results are product-oriented, focusing on the actual outcome of learning outside the classroom. It remains unclear *how* students work and learn in different modelling settings. In a first qualitative attempt, differences in *Simplifying*, *Mathematizing* and *Validating* could be observed for modelling at real objects and modelling by means of a picture (Ludwig & Jablonski, 2021). The students outside the classroom, on the one hand, discussed their choice of a mathematical model and the available data more intensely since they were able to view it from different perspectives. Inside the classroom, on the other hand, the students estimated more frequently and argued more intensely on their assumptions based on the given picture. The indoor photograph setting involved a two-dimensional representation of a real object with only one photograph. Thus, it remains unclear whether multiple photos and/or a three-dimensional representation allowing to touch an artefact could influence the comparing observations. In addition, these results have to be discussed carefully in terms of the study's small sample.

It is the aim of the article to focus in more detail on the actual modelling activities that happen in the different settings involving representations of a real object. Incorporating the hypotheses from previously conducted research concerning differences in the modelling steps, the first research question is formulated:

[RQ1] Which similarities and differences based on different modelling settings can be observed in the modelling steps Simplifying and Structuring as well as Mathematizing?

The scope of the article is limited to modelling tasks that involve a geometric object from reality. With this focus, the modelling settings *Photographs*, *3D Print* and *Real Objects* are of particular relevance. A verbal setting is not taken into further consideration since it does not appear to be suitable for geometric questions and comparable to the other settings. From the theoretical examination concerning the particularity of working at real objects being the only setting that happens primarily outside the classroom, a second research question is formulated:

[RQ2] What characterizes the modelling activities that happen outside the classroom while working at a real object?

4 Methodology

To answer the question, a qualitative study was chosen for two reasons. Firstly, the study follows an explorative approach. Secondly, the questions ask for observations on a qualitative level since they are process-oriented in terms of similarities and differences in the modelling steps. The study was conducted in 2022.

4.1 The sample

The sample of the study consists of 19 students in Grades 6–8 (12–14 years old). All of them visited an enrichment program which supports potentially mathematically gifted and interested students in an extracurricular format. The students were chosen from this program since it can be expected that they were interested in *new* tasks and learning formats and bring the necessary resilience to solve a modelling task with not all data being obviously provided. This is important to guarantee a reliable comparison, since the students should be able to perform a complete modelling process in all settings. Still, the potential positive selection has to be taken into consideration in the interpretation and discussion of the results.

4.2 The tasks

The students were divided into six groups of three or four members each (groups A–E). During a 90-min session, each group solved three tasks which focus on three different real objects (Fig. 3):

Fig. 3 Task objects (from left to right): *Body of Knowledge*, *Stone* and *Rotazione Sculpture*

(i) The *Body of Knowledge* which is a big sculpture of a sitting person: the task is to estimate the sculpture's height if it was standing;
(ii) A *Stone* of which the students should determine the volume[2];
(iii) The *Rotazione Sculpture* where the task is to determine its surface area.

For each of the objects, three different settings were defined. In the *real object* setting, the students solved the task outdoors at the site of the object. For the data collection process, they were equipped with a folding ruler. Indoors, the students worked similar to a usual group work setting in classrooms, namely working at tables with a representation of the real-world object. Two different indoor settings were distinguished: the *photo* setting, in which the students solved the task by means of a set of photographs presenting different perspectives and a person as an object of reference (Fig. 4 left), and the *3D print* setting, in which the students worked on the task with a 3D print of the object and a LEGO figure as an object of reference (Fig. 4 right). In both cases, the students were allowed to take measurements with a ruler.

For the *photo* setting, a series of pictures of the real object was taken together with a person of reference being 1.75 m tall (Appendix Fig. 9). The 3D print models were created in two different ways. For the *Body of Knowledge* and the *Stone*, a scan app for 3D models was used. Since the original *Body of Knowledge* sculpture is 8 m tall, it was not possible to scan it. Therefore, a person sitting in the same position as the *Body of Knowledge* was scanned and adapted to the original. The *Stone* was directly scanned. For the *Rotazione Sculpture*, a 3D model was offered by the artist. Before printing, the three models were scaled, assuming that the LEGO figure represented a person of height 1.75 m.

Independent from the setting, the tasks are related to geometry, whereby the *Body of Knowledge* is about length and proportion, the *Stone* focuses on volume and the *Rotazione Sculpture* is about area. From their formulation, a strong connection to the modelling, especially the *Simplifying and Structuring* as well as *Mathematizing* process, can be seen. On the one hand, each task can be solved in different ways, depending on the chosen model. In the example of the *Stone*, the students have to face irregularities at the stone's surface and might idealize it as being

[2] The students should not determine the mass of the stone since it was expected to bring more difficulty to the mathematical work which is not the focus of the analysis.

Fig. 4 Representations of the stone: photo (left) and 3D print (right)

completely even. In addition, the students have to decide which shape might describe the stone adequately. It might be possible to choose a cuboid, cylinder, prism or a compound body as a real model. Still, it becomes obvious that none of them describes the stone's shape perfectly — the students therefore necessarily have to simplify the object (Ludwig & Jablonski, 2021). Therefore, they have to discuss which simplifications are necessary and also appropriate in that they do not influence the result in a significant way. Similar considerations can be made for the *Rotazione Sculpture* and the *Body of Knowledge*. On the other hand, the mathematizing, i.e., the data collection process, is mandatory in all three settings since no data are given. For example, while collecting the data of the stone, they have to face its irregularities again since the height varies greatly and the students have to decide which value (e.g., highest, lowest, mean, difference) suits best.

4.3 The data collection

Each group solved three tasks, involving the three different objects and settings. The objects and representations were arranged according to the Latin square design aiming at a systematic variation (Field & Hole, 2002; Table 2). This process should reduce possible effects from one setting to another in terms of the tasks' order. In addition, it ensured that every group experienced every task formulation and object representation.

While the students solved the tasks, they were accompanied by a project assistant who filmed the students' interactions and solving processes. In addition, the assistant provided the material and task formulations to the students and guided them from the classroom to

Table 2 Latin square design to vary the task objects and settings

Groups	Body of Knowledge	Stone	Rotazione Sculpture
A and D	Photo	3D print	Real object
C and F	Real object	Photo	3D print
B and E	3D print	Real object	Photo

Table 3 Duration of solution processes for the tasks in each setting

Setting	Median	Minimum	Maximum
Real object	13:40	06:20	19:30
Photo	14:30	08:40	25:40
3D print	13:25	10:10	28:40

the real object and back. The setting ompares the narrative in real time walk which was adapted by Buchholtz et al. (2020) to the setting of outdoor mathematics tasks. In the case of this study, it was extended to all settings. From the data collection, in total, 18 video-recorded solution processes with a total length of 195 min as well as the written solutions of each group exist. Table 3 gives an overview of the median and the minimal and maximal length of a solution process sorted by the three different settings:

4.4 The data analysis

The filmed solution processes were transcribed from the video files and deductively coded (Appendix Table 9). Six out of the seven modelling steps by Blum and Leiß (2007) were taken into consideration. The modelling step of *Presentation* was excluded during the coding process since it was not relevant in the data. In terms of the analysis' reliability, the coding was partly done by three independent coders and their accordance is quantified and interpreted with Fleiss' Kappa. For 39 selected video sequences, the three coders reached $\kappa = 0.73$ which can be interpreted as a substantial and good reliability. The results of the coding process were visualized by means of activity diagrams (Ärlebäck & Albarracín, 2019). Figure 5 shows the activity diagrams for the three tasks solved by group B.

For each of the solution processes, the diagram contains information about the task setting and the total length of the solution process divided into minutes. The modelling steps by Blum and Leiß (2007) are represented through different colors. For example, group B solved the *Stone* task in the *real object* setting and started to understand the task. They continued with structuring and simplifying the object and switched between this and the mathematizing step multiple times for about 6 min. After a short activity in interpreting and validating, they come back to structuring and simplifying before the mathematical work. Similarly, the solution processes can be interpreted for the *photo* and *3D print* settings. Appendices Figs. 10 and 11 present the activity diagrams for all groups.

Secondly, the students' activities in the modelling steps *Simplifying and Structuring* as well as *Mathematizing* were taken into consideration in more detail. Therefore, a qualitative content analysis according to Mayring (2000) was carried out. In this second step,

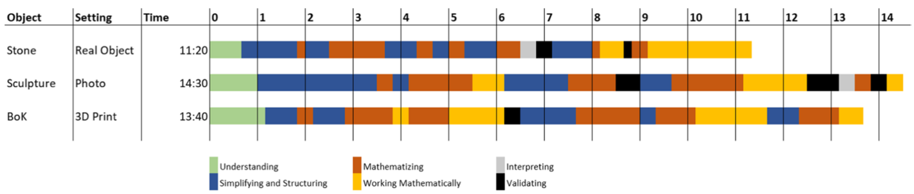

Fig. 5 Visualization of group B's solution processes in terms of modelling steps

the qualitative content analysis started inductively from the video material. The categories identified through this process are presented in the results section. The coding was done by two independent coders, reaching $\kappa = 0.77$ for 18 sequences that were assigned to the category *Simplifying and Structuring* and $\kappa = 0.74$ for 17 sequences in the category *Mathematizing*. Finally, the chosen models, collected data and obtained results were analyzed by means of the groups' written solutions and combined with the previous observations.

5 Results

5.1 Overview of the modelling steps

Based on the activity diagrams of the six groups, the analysis is firstly focused on the duration of the modelling steps. To quantify the visual data, each modelling step's duration was analyzed in the frame of the whole solution process. For each group, the duration of a modelling step was calculated as a percentage since the duration of the solution processes varied between individual groups (Table 3). Table 4 summarizes the mean values for each modelling step and setting.

A comparison of the different settings shows similar values in the steps *Understanding*, *Mathematizing*, *Working Mathematically*, *Interpreting* and *Validating* in terms of the steps' duration in the whole solution process. The most relevant differences can be observed in the modelling step *Structuring and Simplifying* for which the students spend more time on the *real object* (on average 37% of the solution process) and *photo* setting (35%) compared to the *3D print* setting (30%). Hence, for all three settings, this modelling step is most relevant in terms of duration. It is followed or equivalent to the step of *Mathematizing*.

On the one hand, this overview emphasizes the relevance of the steps *Structuring and Simplifying* and *Mathematizing* while modelling with real objects. The steps are most relevant independent from the actual setting which can be concluded as a similarity of the three settings. On the other hand, the results raise the hypothesis that potential differences might not be of a quantitative character in terms of durations, but of a process-oriented character. With the quantitative relevance of the steps *Structuring and Simplifying* and *Mathematizing* in the work with real objects, the processes of the groups related to these steps are compared in the following.

Table 4 Mean percentage for the modelling steps

Modelling Step	Real Object	Photo	3D Print
Understanding	5 %	4 %	6 %
Structuring & Simplifying	37 %	35 %	30 %
Mathematizing	27 %	28 %	30 %
Working Mathematically	22 %	23 %	21 %
Interpreting	1 %	2 %	2 %
Validating	8 %	10 %	11 %

5.2 Process-oriented analysis

The results from the qualitative content analysis start with the modelling step *Structuring and Simplifying*. Table 5 gives an overview of the categories in terms of the students' activities. Each category is illustrated by a definition and the number of groups (out of six) that actually did the activity during the solution process in the settings real object (R), photo (P) and 3D print (3D).

Inaccuracies: By analyzing the identified categories in *Structuring and Simplifying*, it is possible to describe qualitative differences of the students' work concerning the task setting. Starting with the analysis of inaccuracies, this activity seems particularly relevant in the real object and photo setting. For example, a resulting discussion can be observed related to the *Stone* task between two students:

S1: I'd say that's a cuboid with one corner cut off.
S2: Let's walk around first, then we can take a good look at it.
(*Stone//Real Object*)

In the *real object* setting, the students have the possibility to change their own perspective in relation to the object since they are able to walk around it. Already in this quote, the students claimed to work as precisely as possible since being outdoors at the real object offers a wide range of possibilities. This is also relevant for the analysis of inaccuracies in more detail, e.g., the uneven shape of the stone. In the example of the group presented in Fig. 6 (left), the statement "The surface is fully uneven. Maybe we have to subtract something" goes along with actually touching the real object.

Also in the *photo* setting, handling with inaccuracies can be observed, still the discussions' focus differs: While working with photos inside the classroom, the students were aware of not being on site of the object. As a result, they accepted that it is necessary to work with inaccuracies. The difference to the handling with inaccuracies in the real object setting can be reflected by contrasting the following quote from a group working on the *Stone* task inside with photos:

S1: This is where the stone gets thinner, isn't it?
S2: It could be, but I'm not sure.
S1: If we were outside now, we could look at it.
(*Stone//Photo*)

This awareness goes even beyond changing perspective since another student explicates this for the analysis of the stone's shape as well by stating: "If we were on site of the stone, we could see how even this side was". The analysis of inaccuracies working with 3D objects does not seem to be relevant to all groups as it was only observed in three out of six groups. As for the *real object* setting, the students can view the object from different perspectives, but in contrast to this, they do not change their own perspective, but rotate or manipulate the artefact as in Fig. 6.

Table 5 Inductive categories from the qualitative content analysis in *Structuring and Simplifying*

Category	Definition	R	P	3D
Inaccuracies	Students analyze inaccuracies (e.g., uneven surfaces or irregular shapes) at the object and discuss their relevance for the solution process. Furthermore, this category involves statements on the real object being different to idealized geometrical bodies	6	6	3
Appropriateness	Students discuss the appropriateness of a real model to describe the real object and find a basis for the mathematizing	6	3	4
Availability	Students discuss which data would be available to solve the task and adapt their model accordingly	5	1	0

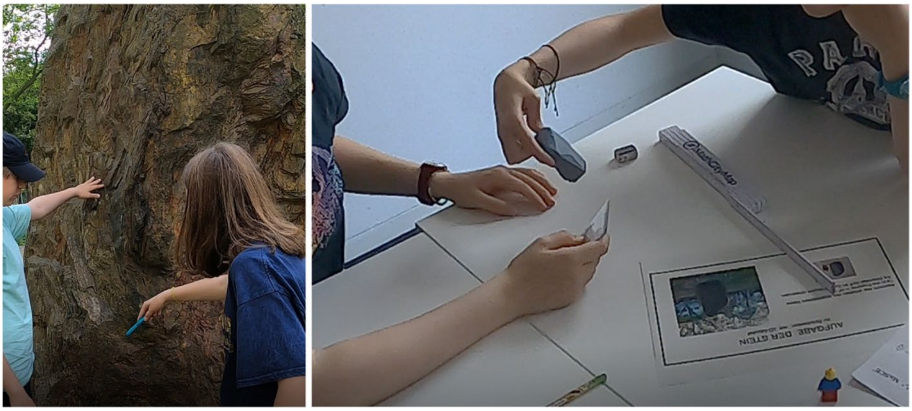

Fig. 6 Analysis of the stone's surface and shape at the real object (left) and in the 3D setting (right)

Appropriateness An activity that happens in relation to the work with inaccuracies is the discussion of a real model. In line with the previous observations, the students outside discuss the appropriateness of the real model most intensely in comparison to the students in the indoor setting. It might be a connection to the already described statement to work as precisely as possible and therefore choose the most appropriate model as we can observe in the following statement:

S1: What about the missing piece on this edge? We have to subtract that.
S2: Do you think it is important?
S1: Sure. […] I think that it would give us a wrong result.
(*Stone//Real Object*)

In contrast, this discussion is missing in three out of six groups working with photos. Since the groups regard inaccuracies as something unavoidable with this representation, the appropriateness of the real model might not be the main focus. Working with the 3D representation, the discussion about an appropriate model is observable for four groups, e.g., "I don't know if we can just say that this is a cuboid. If, then we should say that this is two parts, one above and one below".

Availability Finally, the focus is on the discussion of the necessary data and if they are available. Again, the relevance of this activity seems to be highest in the *real object* setting with regard to being on site of the object and being able to actually take direct measurements. In addition, it can be possible that existing data are not directly measurable. For the *photo* and *3D print* settings, this activity seems of less or no relevance. Especially with the 3D representation, the students are usually able to identify the data by comparing it to the LEGO figure and in the *photo* setting they usually estimate the necessary missing data.

The analysis continues with the modelling step *Mathematizing*. Analogue to the previous step, Table 6 gives an overview of the categories.

Assumptions At first sight, the settings seem quite similar in terms of the numbers of groups that did the identified activities. The most obvious difference between the settings happens with regard to making assumptions. Whereas this step does not seem relevant in

Table 6 Inductive categories from the qualitative content analysis in *Mathematizing*

Category	Definition	R	P	3D
Assumptions	Based on the previous step, students make assumptions in order to narrow down the real model to a mathematical model	1	6	6
Choice of data	Students decide which data are necessary to complete the mathematical model and how they can be collected	6	6	6
Data collection	Students collect the data through measurements, comparisons, estimations, …	6	6	6

the work with the real object — measured by the number of groups doing the activity — it is particularly relevant when solving the tasks with photos or 3D prints. Here, the students obviously have to make assumptions. Since the subsequent process of data choice and collection is provided in both settings by an object of reference, most of the assumptions deal with an estimation of the reference object's size in reality, e.g., "The person might correspond 1.8 m in reality". As the *real object* setting allows direct measurements, the students do not need this sort of assumptions. In the photo setting, the students tend to make further assumptions about perspective. Since it is the only two-dimensional representation of an originally three-dimensional problem, considerations as follow can be found frequently: "It [the height of the stone] might be more since the person stands in front of the object".

Choice of data Both the choice and collection of data are particularly relevant in the three task settings. For the *real object* setting, it is interesting to observe that the groups tend to continuously follow the idea from their dealing with inaccuracies and appropriateness since they discuss more intensely how the data can be collected in the most exact way, e.g., "We should not measure only one height [of the stone]. We should measure the big and the small height and then divide it by two". The question which data are necessary at all is the focus of the *photo* and *3D print* setting instead — an observation that is in line with the assumptions of accepting not being on site of the object, made in the modelling step of *Simplifying and Structuring*.

Data collection When it comes to the actual data collection, students tend to use different methods dependent on the representation. Table 7 gives a comparative overview of the different strategies, again giving the number of groups that used them in a particular setting. In order to deepen the analysis of the data collection, the procedures are highlighted for each setting afterwards.

Table 7 Use of strategies of data collection in the mathematizing step

Data collection	Definition	R	P	3D
Comparison with available objects	Students use an available object for comparison that does not allow any direct measurements	0	0	6
Comparison with participating persons	Students involve a participating person to gain necessary data, e.g., the body size	3	2	2
Estimating	Students approximate a value without direct measurement	3	5	0
Measuring	Students use a (folding) ruler to directly measure a needed value	6	6	6
Recourse to basic concepts	Students use (common) knowledge to determine a needed value, e.g., the mean body size of a person	0	6	6

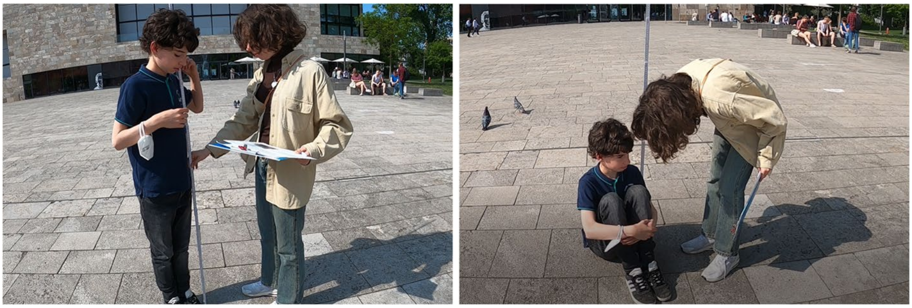

Fig. 7 Data collection for the *Body of Knowledge* task using a comparison with a participant

Real object Measuring is particularly relevant with all three settings. In the *real object* setting, it is enriched by doing estimations, especially for the tasks *Body of Knowledge* and *Rotazione* since these objects are too high to directly measure all data. As a way of estimating, three outdoor groups compared a participating person while standing and sitting (Fig. 7) with the proportions of the *Body of Knowledge*.

Photo In the *photo* setting, all groups assumed the average size of a person for the object of reference in the pictures. Most groups, in addition, used estimations besides taking measurements, e.g., in terms of perspective or in particular with respect to the struts of the *Rotazione Sculpture*. For the *Body of Knowledge*, the groups followed the same strategy described in Fig. 7 for the real object setting.

3D print In the solution of a group in Fig. 8, for example, the students utilized a comparable concept of a recourse to basic concepts in the 3D setting. Hereby, they formulate an assumption on the average size of a person (in the example: 1.75 m) and transfer it to the provided LEGO figure. All groups continued to compare the necessary data with the available object. Therefore, all their considerations happened in terms of scale. Again, the *Body of Knowledge*

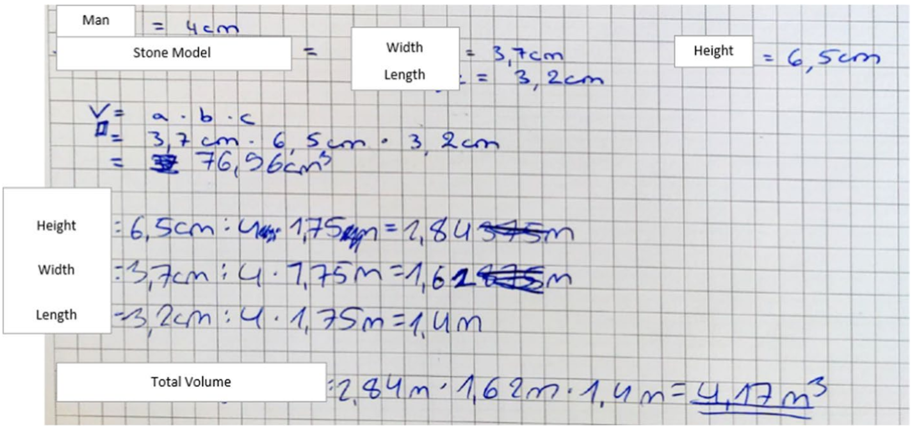

Fig. 8 Example solution of a group solving the *Stone* task inside with 3D model

task made the students compare the proportions of the sculpture with one of the participant's proportions.

5.3 Product-oriented analysis

Finally, the models and results of the groups are taken into consideration. For the *Body of Knowledge*, the task was to determine the height of the sculpture if standing. There are two different models that can be observed, whereby no preferences concerning the task setting can be assumed:

- Idea of *proportion*: comparison of the height of a person sitting and standing;
- Model of *composed lengths*: addition of the individual lengths.

For the volume of the *Stone*, the students mainly used a *cuboid* to approximate its shape. They used medium lengths, especially to balance the differences of the stone in its height, but also its length and width. Only one group — working with a 3D model — decided to divide the stone into a cuboid and an additional prism with a triangular base — using the approach of a *composed body*.

Finally, the *Rotazione Sculpture*'s surface was determined using the following models:

- Disassembly into *triangles* or *rectangles*: disassembly of the surface into several smaller areas;
- Model of *cylinder*: description of the sculpture as a cylinder.

Table 8 gives an overview of the results that the groups produced in relation to the representation and object. To compare the quality of the results, the table includes a reference to an interval that was created in advance. It was independently done by the author and teams of university and doctoral students in mathematics education, using a variety of models in the different task settings.

For the *Body of Knowledge* task, it appears that all groups underestimated the actual sculpture's height. A problem was the estimation of the actual height of the sitting sculpture. All groups underestimated the height of 8 m. Moreover, the sculpture sits as if on a soft surface, i.e., part of it appears to be below the ground. This can be observed in all representations when the sculpture is viewed from the side. Nevertheless, it was neglected by all groups. Likewise, the groups related the sculpture to only one group

Table 8 Produced results in comparison to the task setting and object

	Body of Knowledge	Stone	Sculpture
Real object	17.4 m	6.4 m^3	29.8 m^2
	15.4 m	6.2 m^3	21.6 m^2
Photo	11.5 m	2.6 m^3	14.1 m^2
	10.5 m	3.1 m^3	11.0 m^2
3D print	16.9 m	4.2 m^3	21.6 m^2
	17.0 m	5.3 m^3	23.8 m^2
Solution interval	[18.0–22.0 m]	[5–9 m^3]	[17–26 m^2]

member. With regard to the interval, the groups working in the *real object* and *3D print* setting reached the results closest to the interval. Since the models used were particularly similar, the difficulties in the *photo* setting might come from perspective. Concerning the *Stone* task, the groups working with real objects receive results that are inside the interval. Also, the 3D print groups are (closely to) inside the interval, whereby the photo groups underestimate the stone's volume. This last observation can also be made for the *Rotazione Sculpture* task. In contrast to the previous observation, there is one group working at the real object that overestimated the result in comparison to the interval, whereas the 3D print groups seem to have produced the most precise results. For all objects, the groups using the photograph representations reached the most divergent, underestimated results in comparison to the predefined interval.

6 Discussion

With the analysis of modelling in the settings *real object*, *photo* and *3D print*, the research questions of the study can be answered. First of all, the focus is on the question of similarities and differences based on different modelling settings in the steps *Simplifying and Structuring* as well as *Mathematizing*.

Starting from a quantitative overview, it can be stated that the three settings resemble each other with a focus on the steps *Structuring and Simplifying* and *Mathematizing*. Both steps are — measured from the amount of time that the students spend on them — most relevant when working with real objects independent from the setting. The small sample does not allow any quantitative comparison between the groups. Still in the setting of this study, it seems that *Structuring and Simplifying* was mostly discussed when the groups worked with real objects, whereby *Mathematizing* reaches the highest percentage in the *3D print* setting. For the *real object* and *photo* setting, this observation is in line with the previous research from Ludwig and Jablonski (2021) and extends it to the *3D print* setting. With a qualitative focus on *Structuring and Simplifying* in the settings, the following findings can be summarized:

- Real object: The students have more possibilities to change perspective. Therefore, they make use of more intense discussions about the real model, the relevance of inaccuracies and the availability of data. It is particularly interesting that the students claim to work as precisely as possible since they are on site of the object, being aware of the possibilities they have.
- Photo: This setting forces the students to discuss inaccuracies, too, but from a different perspective. The students are aware of not being on site and accept that inaccuracies cannot be avoided. It also affects the considerations for the appropriateness of the mathematical model which is partly relevant in the discussions. Since the students have to make assumptions and do estimations, there are no discussions about available data.
- 3D print: The students do not focus inaccuracies in the same amount as in the other settings — hypothetically because the actual 3D print already simplified the real object. The

discussions about an appropriate model resemble the outdoor setting — still in this setting, no discussion about available data can be found since the 3D representation usually allows to determine all data.

In the *Mathematizing* step, the following conclusions can be drawn:

- Real object: Despite a general emphasis on the data collection (Buchholtz, 2021), the setting is dominated by taking measurements enriched through comparing and estimating, depending on the actual measurability of the object and sizes.
- Photo: The students have to work with a two-dimensional representation of an originally three-dimensional object by means of an object of reference — a circumstance that leads to a combination of estimation, measuring and recourse to basic concepts. The step in this setting is primarily characterized by assumptions and discussions on perspective.
- 3D print: This setting allows not only to measure, compare and use basic concepts, but requires a comparison with available objects. The primary focus of the discussion is about scale and the transfer from the representation to reality.

Taking a product-oriented perspective, the students' models are mostly similar in all settings. In some cases, as for the *Stone* or *Rotazione Sculpture*, they might be more precise in the *real object* or *3D print* setting in comparison to the work with photos, also in comparison to the previously defined solution interval. The *photo* setting leads to underestimating results, possibly since the students have to take both scale and perspective into account. Hereby, a connection to Schukajlow's (2013) research on problems with perspective when working with photos can be drawn. This raises the meaningfulness of the work with real objects and 3D prints in terms of realistic results. It could be observed that the results reached at the real objects are most precise in those cases, where the students can measure all lengths directly at the real object, e.g., the *Stone*. As soon as the height is too high to measure, the 3D print allows more precise results, as for the *Body of Knowledge* and the *Rotazione Sculpture*. This result enriches the previous comparison of working with real objects outdoors and photos since in these settings, the outdoor results were in general better than the results gathered inside (Ludwig & Jablonski, 2021).

Coming back to the second question with a focus on the particularities of the modelling activities that happen outside the classroom while working at a real object, the following statements can be formulated:

- The work with real objects seems to set a particular focus on the *Structuring and Simplifying* step. It is characterized by considerations concerning *inaccuracies*, *appropriateness* and *availability*.
- In addition, this work seems to emphasize the process of data collection as part of *Mathematizing* in a different manner than the other settings. It is the only representation that allows direct measurements at the real object without considerations of scale and perspective. The setting leads to the most precise results for the data collection if data can be directly measured.

Hence, mathematical modeling at real objects can be seen as a chance to enrich the modelling that happens usually inside the classroom with, for example, photos and 3D prints. What is an appropriate model? Which inaccuracies are of relevance, which are not? How to deal with unavailable data? It should not be seen as an alternative to the other modelling settings, but as an enrichment. The students' statements *Let's walk around first, then we can take a good look at it* in the real object setting and *If we were outside now, we could look at it* in the photo setting can be highlighted in particular. Even though it was not explicitly anticipated in the assignment, the students considered the appropriateness and limitations of the existing setting in comparison to other settings. For example, the students would like to go outside and look at the real object more closely for a detailed analysis when they were working with photographs instead. Here, the potential to enable authentic modeling by combining and making the different settings available is evident.

As an observation, it is obvious that students modelling with a photo are aware that this situation comes from reality, but they are not able to work on site. This observation raises the question whether their awareness could influence the students' perception of reality in these tasks. In addition, it would be interesting to face possible influences of the different settings, e.g., *Does the setting (sitting at tables vs. walking around) influence the quality of results and their documentation?* Furthermore, a quantitative approach could extend the results concerning the possible positive selection in this study. Even though different grades, schools and task formulations were involved, all participating students have a mathematical interest and/or are mathematically gifted. Therefore, it is unclear whether the results can be generalized to more heterogeneous groups. Furthermore, it should be taken into consideration that in a heterogeneous group, not all students would reach a result — independent from its quality. Therefore, it is of particular interest to see which support might be necessary in order to allow independent mathematical modelling in the settings and take advantage of combining different modelling settings. Finally, since the study's focus is on the steps *Structuring and Simplifying* and *Mathematizing*, how the settings influence subsequent modelling steps remains particularly open. From the activity diagrams, it becomes obvious that validation does not amount to a huge part of the students' modelling activities in general. Nevertheless, it is possible to formulate hypotheses about similarities and differences in validation based on the observations made in the different settings especially the emphasis on the category *Appropriateness* while working at the real object which shows a basis for intermediate evaluations. This could lead to the fact that validation is influenced by appropriateness or at least referred to it. In further research activities, it would be of particular interest to focus on these considerations in more detail.

Appendix 1

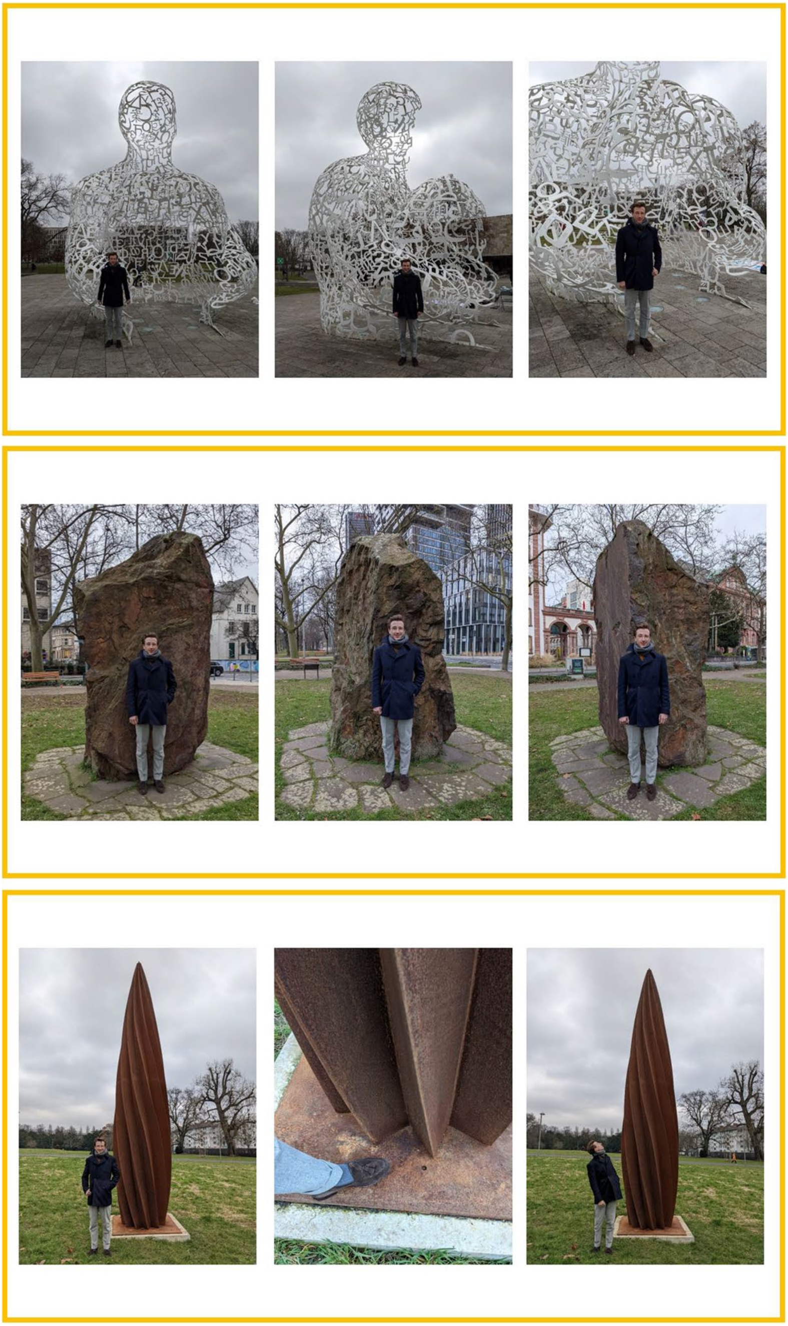

Fig. 9 Task material for photo setting

Appendix 2

Table 9 Overview deductive categories (cf. Blum & Leiß, 2007; Buchholtz, 2021)

Modelling step	Definition	Indicators
Understanding	Students read and understand the task	- Reading the task instruction - Identifying the object and provided material
Simplifying and Structuring	Students recognize the object, analyze its size and shape, work with inaccuracies and create a real model	- Analyzing the object (e.g., seeing, touching if possible) - Identifying relevant characteristics (e.g., shape, inaccuracies) - Proposing similarities with a geometric body in order to build a real model
Mathematizing	Students transfer the real model into mathematics by collecting the data needed (e.g., measuring) and put them into a meaningful relationship	- Narrowing down the real model to a mathematical model - Collecting data (e.g., measuring, calculating, estimating)
Working Mathematically	Students work on the problem mathematically by means of the collected data, their mathematical knowledge and calculations	- Calculating with the collected data - Using mathematical formula
Interpreting	Students interpret their mathematical solution and transfer it (back) to reality	- Explaining what the mathematical result means
Validating	Students check their solution in reality and decide whether the result is reasonable	- Assessing the reached results in relation to the task object

Appendix 3

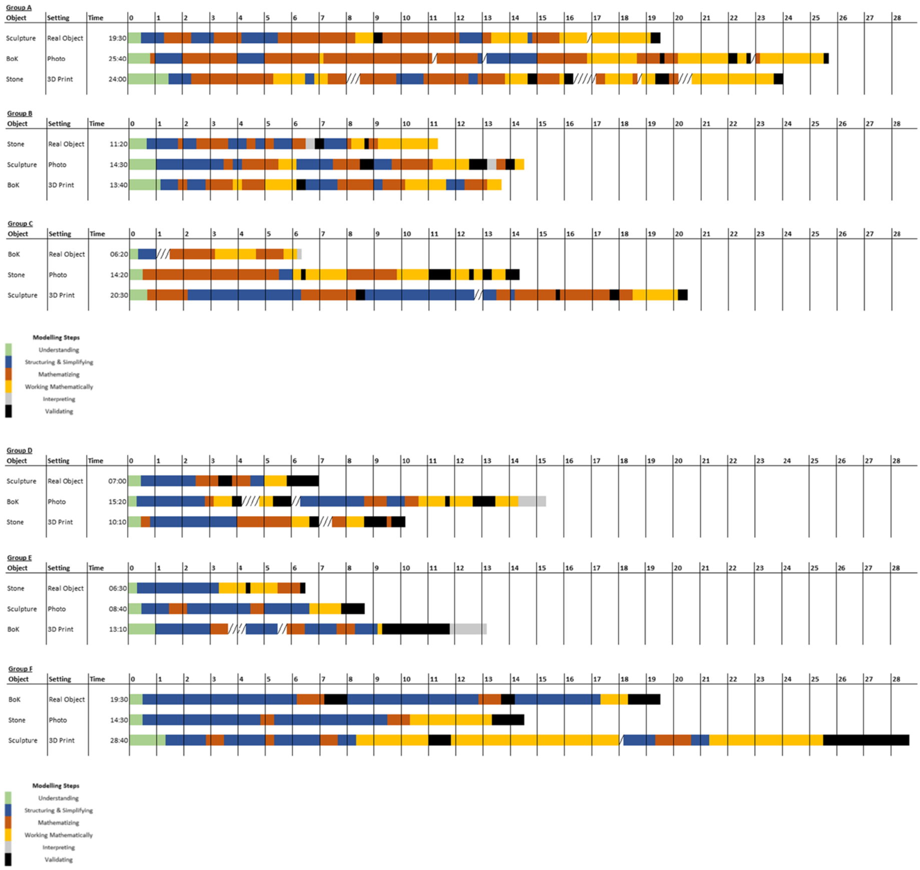

Fig. 10 Activity diagrams sorted by groups

Appendix 4

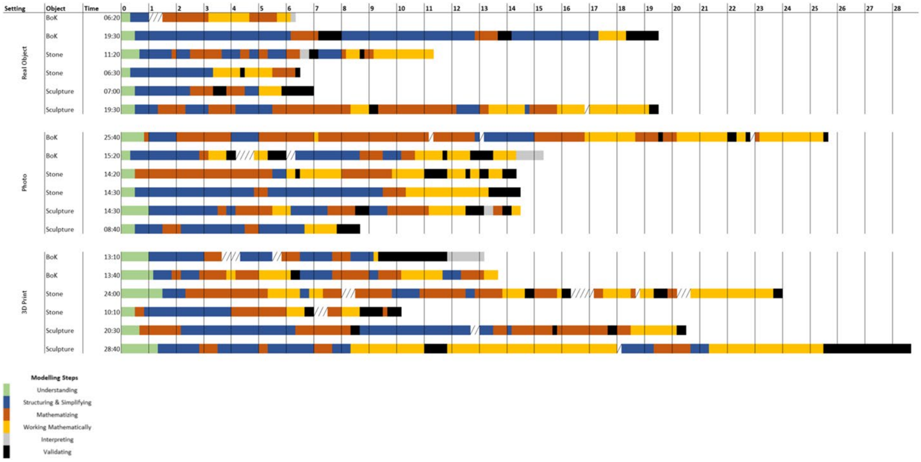

Fig. 11 Activity diagrams sorted by setting and object

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Declarations

Competing interests The author declares no competing interests.

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