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Explorative mathematical argumentation: a theoretical framework for identifying and analysing argumentation processes in early mathematics learning

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Abstract
This paper introduces the term *explorative mathematical argumentation* (EMA), signifying a concept for describing and analysing learners' mathematical argumentation processes. Despite multiple recent empirical evidence for argumentation promoting learning in science education, still little is known about the development of early mathematical argumentation skills and their role within early learning processes. The widely varying use of the term *argumentation* impedes respective research efforts. The concept of explorative mathematical argumentation offers an approach that takes into account the explorative nature of learners' knowledge construction as well as specific aspects of mathematical argumentation. The concept of EMA promotes mathematical argumentation as a separate competence facet within which different forms and tools of reasoning can be deployed. It is suitable for describing and analysing learners' mathematical argumentation processes from an early age onwards, as is illustrated by an exemplary situation with 4- and 5-year-old preschool children. Eventually, methods for identifying and analysing learners' explorative mathematical argumentation processes are presented for discussion.

Keywords Preschool children · Argumentation · Knowledge construction · Early mathematics

1 Introduction

Playing an important role in professional as well as daily life within democratic societies, argumentation is considered "a fundamental tool of reasoning" (Voss & Means, 1991, p. 4) and "a core epistemic practice in the sciences" (Bricker & Bell, 2008, p. 474). Many fields of science state argumentation as a key competence, such as philosophy, jurisprudence, linguistics, mathematics and natural sciences. Hence, *learning to argue* has been subject to recent research in education (e.g., Kuhn & Crowell, 2011; Mercer, 2009; Kuhn & Udell, 2003).

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In addition, the idea of *arguing to learn* is also increasingly to be found in publications in the field of science education (Asterhan & Schwarz, 2016; Andriessen, 2006; Muller Mirza et al., 2009; von Aufschnaiter et al., 2008) and empirical evidence has been proving its benefits for students' learning[1]. Both aspects, learning to argue and arguing to learn, are considered important contributions to educational efforts:

> Argumentation has an increasing importance in education, not only because it is an important competence that has to be learned, but also because argumentation can be used to foster learning in philosophy, history, sciences and mathematics, and in many other domains. (Muller Mirza et al., 2009, p. 1)

Argumentation plays a major role in mathematics education (Schwarz et al., 2010). As Krummheuer puts it, "learning mathematics is *argumentative* learning" (Krummheuer, 2007, p. 62). Despite the internationally unquestioned importance of argumentation as a key competence in mathematics, there is still a lack in research on learners' use of argumentation and its contribution to the early learning process, as well as ways of enhancing argumentation skills in early mathematics education. As Brunner (2019) states, little is known about how young children develop and use argumentation skills in mathematical contexts before entering primary education and what measures can be taken to foster their development (Brunner, 2019, S. 324).

However, various recent research results show that mathematical argumentation can be observed in preschool children (e.g., Böhringer, 2021; Brunner, 2019; Krummheuer, 2018; Lindmeier et al., 2015). Franzén (2015) observed mathematical learning in even younger children, who "use their bodies to develop their mathematic knowledge" (Franzén, 2015, p. 52). Considering mathematical thinking of 4- to 6-year-old children, Krummheuer (2018) puts forward the idea that "[t]he constitutive social condition of the possibility of learning of a mathematical content, concept, or procedure is the participation in a collective argumentation concerning the content, terms, or other procedures", while "[t]he expression of a successful process of learning of a child or a pupil is the increased autonomous participation in such collective argumentation in the process of a current interaction and/or in the following interaction that is thematically imbedded in the actual situation" (Krummheuer, 2018, p. 113). I will come back to this definition of a learning process later. Sfard (2006) claims that from a participationist view "human thinking originates in interpersonal communication" (p. 153) and defines mathematical learning as "individualizing mathematical discourse, that is, as the process of becoming able to have mathematical communication not only with others, but also with oneself." (p. 162).

Based on the understanding that mathematical learning as well as the development of mathematical argumentation can be observed in children at a young age, I want to show that some forms of argumentation are more beneficial for mathematics learning than others. With explorative mathematical argumentation, I would like to introduce a concept suitable for the analysis of young children's mathematical argumentation processes.

In order to do so, the first part of this paper seeks to lay a theoretical foundation for the identification and analysis of learners' and even young children's mathematical argumentation processes, addressing differences between explorative and persuasive argumentation, domain specific relations between argumentation, reasoning and proof, and the benefits and limitations of the Toulmin model in the analysis of argumentation processes. The empirical

[1] For a thorough overview, see Osborne (2010).

example presented in the second part provides an excerpt of a situation of 4- and 5-year-old children engaging in mathematical argumentation and an exemplary application of the concept of explorative mathematical argumentation for the identification and analysis of children's argumentation processes.

2 Theoretical framework

The following paragraphs will lay a theoretical foundation for explorative mathematical argumentation, before the concept is applied in the next section to an exemplary situation of early mathematics.

2.1 Persuasion and exploration — two different forms of argumentation

The widespread use of the term argumentation in various fields, including everyday life, emphasizes its significance as a central interactional communicative practice. However, it also poses the problem that between different contexts (and sometimes even within the same context), the understanding of the term can strongly differ. In the educational context, in mathematics as well as in other domains, the widely used pragma-linguistic approach lays emphasis on the notion of argumentation as a socio-interactive process within a social group. It is thus seen as a communicative social activity (van Eemeren et al., 1996, p. 5; Kurtenbach et al., 2019, p. 27; in the context of science education: Nielsen, 2013, S. 373; in an explicitly mathematics educational context: Schwarzkopf, 2015, p. 32). However, a closer look at this specific form of social activity reveals two different kinds of argumentation, depending on the goals pursued with the activity or the context in which it takes place. Duschl and Osborne (2002), who examine argumentation in science education, detect "a tension between the lay perception of argumentation, as war that seeks to establish a winner, which contrasts with a view of argumentation as a social and collaborative process necessary to solve problems and advance knowledge." (p. 41).

With respect to the goals pursued in an argumentative action, the German linguist Konrad Ehlich offers a concept of two clearly distinguished forms, one that serves the purpose of collaborative knowledge construction, referred to by the author as *explorative argumentation*, and one that seeks to persuade the interlocutor(s) into adopting the speaker's opinion, termed *persuasive argumentation*. In explorative argumentation, knowledge systems are sought to be collaboratively extended (Ehlich, 2014, p. 47). The (co-)construction of knowledge is often referred to as the objective of the implementation of argumentation in science education (arguing to learn). Thus, Ehlich's concept of explorative argumentation may form a suitable basis for describing learners' mathematical argumentation.

Ehlich states that, despite the special significance of argumentation in natural sciences and mathematics, argumentation analysis has mostly been deployed in the fields of politics and jurisprudence for a long time, which has led to the development of theories and tools that mainly focus on rhetorical means (Ehlich, 2014, p. 46). Rhetorical devices, even applying pressure on the interlocutors by formulating threats, may be considered appropriate in some fields, but they are not considered suitable for problem solving processes in mathematical learning (Rigotti & Morasso, 2009, p. 26). Of course, such devices as appeals to reputation, authority or expert opinion are also found in mathematical scientific discourse (Bricker & Bell, 2008; Inglis & Mejia-Ramos, 2009). Inglis and Mejia-Ramos (2009) report that in case of existing uncertainty about an argument's mathematical status, drawing on an authority figure makes the argument seem more persuasive for mathematics researchers as well as for mathematics students. However, didactical research shows this strategy to be rather impedimental in mathematical learning processes (e.g., Brandt, 2007, p. 1177: “The teachers(sic!) asks, ‘Why is it possible to change the summands in an addition?’, and Marina answers, ‘Because you told us last week!’”). Ball and Bass (2003) describe a difference between “reasoning of justification” in contrast to “reasoning of inquiry” (p. 30), with the latter in particular conducive to the discovery and exploration of new ideas.

Ehlich states that the objective of persuading interlocutors to agree with one’s own opinion emanates from the assumption of knowledge systems in conflict (Ehlich, 2014, p. 44) — an assumption that does not provide a productive basis for the context of mathematical education and limits the means of argumentation analysis, as it “does not focus on the genesis of conclusions, i.e., the individual reasoning process by which people come to believe something, but on their justification, i.e., the communicative process by which people try to convince others of the acceptability of their point of view” (Wagemans, 2019, p. 9). Furthermore, conflict-based situations seem to limit the participants’ production of arguments. When Domberg et al. (2018) compared 5- and 7-year-old children’s argumentation in cooperative and competitive contexts, i.e., either collaboratively trying to win a game by finding the best solution together or competitively trying to win by arguing for their own side, they found that for both age groups, the cooperative context was more motivating for the production of arguments (Domberg et al., 2018, p. 75).

However, the historically developed view of argumentation as a persuasive, rhetorically based instrument used for convincing others based on competing knowledge systems is extended by Ehlich’s introduction of the term explorative argumentation, with the underlying concept of knowledge systems in contrast (Ehlich, 2014, p. 44 ff.). Explorative argumentation seeks to establish convergence between the participants’ knowledge systems and aims at a cooperative development of knowledge:

> Explorative argumentation’s central area of application lies in knowledge gain, which is characterized by the alignment of different conjectures and verbal testing of the range of impact that individual components of preexisting knowledge can cover for the generation of new knowledge. The core is wanting-to-know, understood as a shared, collective task. (Ehlich et al., 2012, p. 71; translation F. R.)
> Das explorative Argumentieren hat einen zentralen Anwendungsbereich in der Gewinnung neuer Erkenntnisse, die durch das Abgleichen unterschiedlicher Erkenntnisvermutungen und das sprachliche Austesten der Reichweite einzelner Teile des bereits vorhandenen Wissens für die Erzeugung des neuen Wissens gekennzeichnet sind. Im Zentrum steht das Wissen-Wollen, das als gemeinschaftliche, als kollektive Aufgabe gesehen wird. (Ehlich et al., 2012, p. 71)

Table 1 gives an overview of the constitutive aspects of explorative and persuasive argumentation.

Unlike persuasive argumentation that has been found in children as young as 2 years old (Muller Mirza et al., 2009) but, as stated above, does not always meet the requirements of argumentation in a mathematical learning context, explorative argumentation requires insight into the fact that another person’s knowledge-related beliefs can differ from one’s own. Thus, the concept seems applicable for the analysis of children’s mathematical argumentation from the age of four onwards, when the progressing development of theory of mind allows such insights (Rakoczy et al., 2007; Wellman, 2014).

Table 1 Explorative versus persuasive argumentation

	Explorative argumentation Interactional types of discourse	Persuasive argumentation
Initial situation	• Knowledge systems in contrast • Interest in convergence	• Knowledge systems in conflict • Emphasis on divergences
Goals	• Collaborative knowledge construction • Extension of knowledge systems	• Interest-based assertion of own position • Convincing and persuading
Tools	• Exploration • Collaborative transformation of the unfamiliar into familiarity	• Rhetorical means (i.e., appealing to authorities)

Data from Ehlich (2014) and Ehlich et al. (2012)

For the purpose of analysing learners' argumentation processes, it seems appropriate to view mathematical argumentation as a form of explorative argumentation, with the aim of collaborative knowledge construction. However, it is necessary to specify the tools deployed in early explorative mathematical argumentation and to extend the concept by the distinctive mathematical aspects of argumentation, which lie in the domain-specific connections with other concepts, such as reasoning and proof. The following considerations seek to contribute to these objectives and lead to a definition of *explorative mathematical argumentation*.

2.2 Argumentation in mathematics education

While argumentation is a commonly used term in mathematics education, its domain-specific constituting components, as well as the distinction from other constructs such as reasoning and proof, are defined in a variety of ways. Scientific literature on argumentation in mathematics education sometimes states that the prominent role of argumentation is derived from the perception of mathematics as a domain of deductive reasoning within an axiomatic system, and that mathematical argumentation can be seen as an early form of mathematical proof (Schwarzkopf, 2015, p. 31), a first step on the way to a formal deductive procedure.

Though this is undoubtedly true, with respect to learners' (and especially young children's) mathematical development and activities, it is advisable to consider another aspect of mathematics, namely that of mathematics as an empirical science based on observations and experiments (Baker, 2008; Khan, 2015). Both procedures, deductive reasoning within an a priori system and empirical observation of mathematical structures, are constitutional for mathematics and employed by professional mathematicians (Hischer, 2012, p. 39; Baker, 2008, p. 331; Khan, 2015, p. 98). Aberdein (2009) sees mathematical proof as a specific kind of argumentation and states that unlike the product of mathematical proof, its process hardly ever qualifies as strictly deductive (p. 2).

> The interplay between generality and individuality, deduction and construction, logic and imagination — this is the profound essence of live mathematics. Any one or another of these aspects of mathematics can be at the center of a given achievement. In a far reaching development all of them will be involved. (…) In brief, the flight into abstract generality must start from and return again to the concrete and specific. (Courant, 1964, p. 43)

Making discoveries, such as structures and patterns that can be used to form and test hypotheses and draw conclusions, is a typical activity in young learners' engagement in mathematics as well as in professionals'. An unexpected discovery or a provocative statement may lead to an argumentation process, in the course of which hypotheses, alleged conclusions and justifications are being tested for plausibility, which again can lead to the construction of knowledge in the form of new insights into mathematical concepts and coherences between concepts.

We find here a notable correspondence with the concept of creative mathematically founded reasoning (CMR) proposed by Lithner (2008) for the analysis of task solving processes in the mathematics classroom, which will be discussed further below. First, the question arises as to how the relationship between argumentation and reasoning can be described for the purpose of identifying and analysing learners' mathematical argumentation.

Van Eemeren et al. (1996) offer the definition of argumentation as "a verbal and social activity of reason aimed at increasing (or decreasing) the acceptability of a controversial standpoint for the listener or reader, by putting forward a constellation of propositions intended to justify (or refute) the standpoint before a 'rational judge'" (van Eemeren et al., 1996, p. 5). This definition covers several aspects of argumentation, such as:

- The social embedment of the activity,
- Its dialectical nature, and
- The prominent role of reason, reasoning and proof.

All three aspects will be explained in detail below.

2.2.1 Social embedment and the dialectical nature of argumentation

As presented above, educational sciences emphasize the social dimension of argumentation. As education itself is considered a dialogical process (Mercer, 2009, p. 177), both educational approaches, learning to argue and arguing to learn, draw on the idea of argumentation as a collaborative, dialogical activity. The participation of more than one individual is crucial for knowledge construction in argumentation because, as Nielsen states, in dialogical argumentation "the participants [do] not just defend their own claims, but also engage constructively with the argumentation of their peers" (Nielsen, 2013, p. 373). Constructive "dialectical argumentation" (Nielsen, 2013) is not persuasive, but can correspond to the concept of explorative argumentation if knowledge systems stand in contrast and the goal of the activity lies in collaborative knowledge-gain.

> What is basically required is that participants take opposed positions with respect to a view, and act communicatively in order to give reasons for and against the view in a way that is coherent with their positions and a minimal commitment to them. (Baker, 1999)

Argumentation is considered a dialectical activity as it strives to "settle some issue that has two sides" (Walton, 1998, p. 74). In this concept of argumentation as a dialectical activity also lies one basis for a distinction between argumentation and explanation, as "explanation-driven dialogue that is consensual in nature and in which participants do not question or challenge the epistemic status of a knowledge claim is not argumentation"

(Asterhan & Schwarz, 2016). This can be the case when reasoning is requested from a student by their teacher although there is no doubt about the already given answer, or when the teacher explains an issue that goes unchallenged.

In argumentation, a critical position is taken towards others as well as towards oneself, and alleged ideas need to be justified (Rigotti & Morasso, 2009, p. 11). Thus, unlike reasoning in a non-argumentative context, argumentation necessarily involves opposing views, like a counter argument or a surprising discovery that causes a cognitive dissonance. Those opposing standpoints, challenging incidents or explicitly articulated as well as implicitly assumed doubts mark the necessity of bringing on reasons for a standpoint. Again, with mathematical argumentation, different perspectives derive from divergences between knowledge systems or knowledge-based beliefs, not desires.

2.2.2 The role of reasoning in argumentation

Now, what is the role of reasoning in argumentation and how do the two concepts relate to each other? Again, perceptions vary widely. While some authors see argumentation as a tool for or component of reasoning (English, 2004, p. 14; Brunner, 2014, p. 48; Voss & Means, 1991, p. 4), others claim that certain forms of reasoning are being deployed in argumentation (Schwarz & Asterhan, 2010; Walton, 1998, p. 74). Muller Mirza et al. (2009) state that argumentation is a "process that sustains or provokes reasoning and learning" (p. 1). Obviously, there is a close linkage between the concepts of argumentation and reasoning. However, there is no consensus about the nature of that relationship and definitions vary with different objectives. For the purpose of analysing learners' mathematical argumentation, this article draws on a proposition by Schwarz and Asterhan (2010): "Argumentation is not a distinctive form of reasoning; it is an activity that involves reasoning." Cai and Cirillo (2014) attribute a similar role to reasoning in mathematical proof. As the authors put it, "one may certainly attend to reasoning without proving, but it would be hard to conceive of attending to proving without reasoning" (Cai & Cirillo, 2014, p. 139).

A kind of reasoning that fosters mathematical knowledge construction has been described by Lithner (2008), using the term *creative mathematical (or mathematically founded) reasoning*:

> Creative mathematically founded reasoning (CMR) fulfills all of the following criteria.
> 1. Novelty. A new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created.
> 2. Plausibility. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible.
> 3. Mathematical foundation. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning. (Lithner, 2008, p. 266)

Lithner addresses the problem of rote learning interfering with the development of creative problem solving skills in school children's completion of tasks in mathematics classes by contrasting CMR with other task-solving strategies like memorized reasoning and algorithmic reasoning.

While Lithner focuses on formal, school-based learning situations, the concept of EMA aims at describing young learners' early mathematical argumentation processes. However, reasoning plays an important role in explorative mathematical argumentation processes, and CMR provides a beneficial description of the reasoning processes conducted in EMA. What Lithner calls *novelty* is described as *knowledge construction* in explorative mathematical argumentation, referring to a newly discovered concept, a newly formed hypothesis or previously unknown data supporting a claim. It is important to note, though, that EMA, as a dialectic social activity, unlike CMR, necessarily involves collaborative knowledge construction among a group of learners with differing knowledge systems that make reasoning necessary. Thus, reasoning is not the only activity that is conducted in an EMA process, but it is necessarily involved in it. The concept of *anchoring* is quite conducive to the purpose of deciding whether an argumentation process is to be considered mathematical or not (see also Sumpter, 2014).

2.3 Toulmin's argumentation model

The elements of an argumentation that should be mathematically anchored in EMA will be specified in the following section.

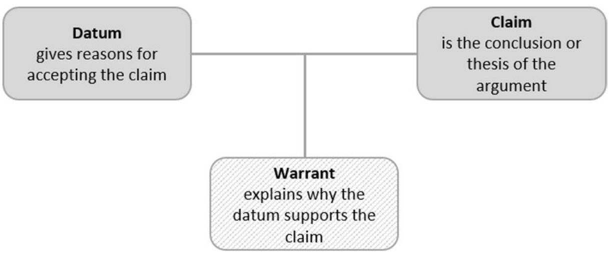
Reasoning can be defined as "a process of thought that yields a conclusion from percepts, thoughts, or assertions" (Johnson-Laird, 1999), or "the line of thought adopted to produce assertions and reach conclusions in task solving" (Lithner, 2008, p. 257). The role of reasoning in argumentation can be clarified by a closer look at the components in an argument and the processes involved in it, as shown by the Toulmin model (Toulmin, 2003). Once referred to by the author as "one of the unforeseen by-products" (Toulmin, 2003, p. viii) of his philosophical book *The Uses of Argument*, Toulmin's model of argumentation has had a formative influence on argumentation-related scientific work throughout all scientific fields to this day. This paper argues that, while the Toulmin model raises several concerns for *analysing* argumentation processes, if extended by the concept of mathematical anchoring, it can be beneficially conducted to *identify* mathematical argumentation.

Toulmin identified different components that constitute an argumentation. The three that are most commonly addressed in didactical research on argumentation are the datum, the warrant and the claim (Fig. 1).

The warrant can in fact remain implicit (Toulmin, 2003, p. 92), but it can still be reconstructed from the argumentation and is considered one of its constitutive components.

However, an argumentation can consist of more than those three components, such as additional data that supports the original datum (Homer-Dixon & Karapin, 1989, p. 392; Toulmin, 2003, p. 218), a backing that gives reasons for accepting the warrant, and attacks that challenge the acceptability of any component of the argument (Homer-Dixon & Karapin, 1989, p. 392). Also, Toulmin mentions the use of rebuttals that state exceptions to the claim and are often combined with the usage of a qualifier for the claim, like

Fig. 1 Model of constitutive components of an argumentation (data from Toulmin, 2003 and Homer-Dixon & Karapin, 1989)



"probably" or "presumably" (Toulmin, 2003, p. 93 ff.), and Homer-Dixon and Karapin (1989) add the component of an attack that can aim at any component of an argument. If qualifiers and rebuttals are used in an argumentation, they form important components of the argumentation and should not be left out of the analysis, as is sometimes the case in didactical research (Inglis et al. 2007).

In the case of explorative argumentation, the attack should also be considered constitutive, as it marks the argumentation's dialectical nature. It may, however, remain implicit and arise from the context in which the argumentation occurs (Fig. 2).

What Homer-Dixon and Karapin (1989) call an "attack" has also been regarded as a specific form of rebuttal (Erduran, 2007, p. 63 ff.). However, this paper suggests viewing the rebuttal as a limitation from within the argument structure that does not challenge the argumentation but specifies the conditions under which it is valid, whereas an attack challenges the validity of the argumentation by questioning one of its components.

Toulmin's argumentation model has been applied in many fields of research. However, several authors point out that the mere application of the model for analysing purposes may not do justice to the complexity of argumentation as a dialogical process (Nielsen, 2013, Erduran, 2007, Gronostay, 2017), as statements can take on different and ambiguous roles throughout an argumentation. What the model does show, however, is that in argumentation, claims are inferred from data or data is used in order to support claims. The process of doing so on a basis of reason is described by the term "reasoning," with different modes of reasoning that can be applied. Thus, for the subject at hand, explorative argumentation

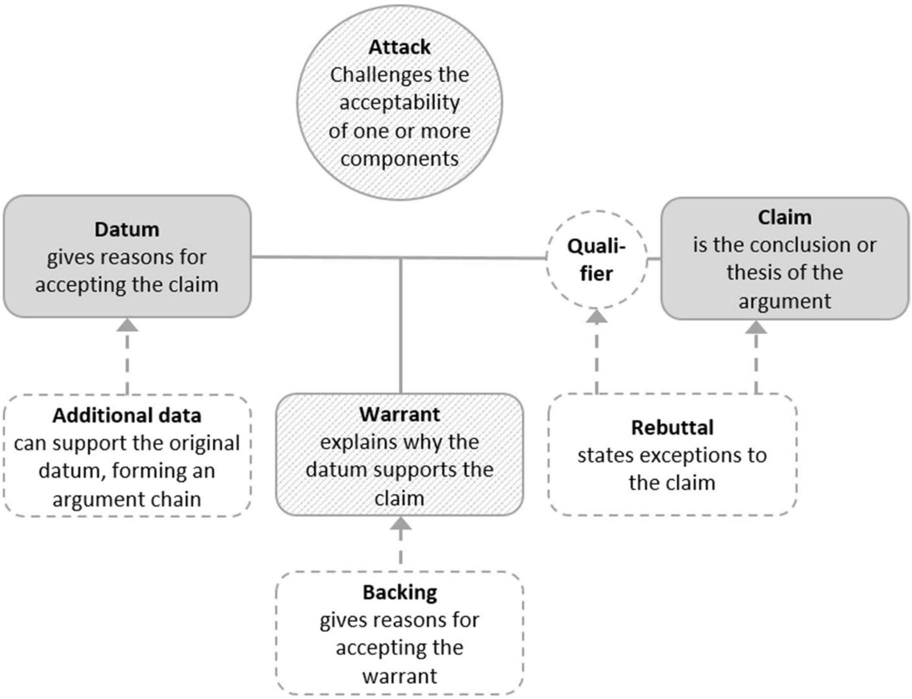


Fig. 2 Extended argumentation model, based on a combination of Toulmin 2003 and Homer-Dixon and Karapin 1989

can be understood as the collaborative process of extending knowledge systems by inferring conclusions from premises and supporting statements by finding reasonable justifications, whereas *reasoning* refers to the mode in which this is done, e.g., deductive, inductive, abductive, or analogical reasoning, and the activities that constitute them.

A short introduction of possible modes of reasoning in mathematical contexts will be given in the following paragraph. These brief statements cannot fully elucidate the complex concepts and their coherences. The purpose pursued here is to show that mathematical argumentation can be considered more than merely formal-deductive proof and to specify some alternatives.

2.4 Modes and tools of mathematical reasoning

The idea of collaborative mathematical knowledge construction plays an important role in the scientific community. Easwaran (2009) describes mathematics as "a social practice, and not a solitary one" (p. 343) and emphasizes the close connection between knowledge and proof, thus constituting the necessity of the transferability of knowledge that, in his view, only *deductive proof* can offer (Easwaran, 2009, p. 343).

Deduction refers to the process of inferring certain conclusions from premises that are held to be true, e.g., by using formal logic. In mathematical proof, axioms and definitions can function as such premises. However, besides deductive reasoning, mathematical argumentation may also include more informal reasoning (e.g., Inglis et al., 2007, p. 6; Brunner, 2014). Fallis (2011) argues that there is a difference between collective epistemic goals within the scientific community which inevitably require the transferability of deductive proof, and individual epistemic goals. Besides the fact that despite the great significance of formal-deductive proof in mathematics, even professional mathematicians use non-deductive methods (Baker, 2009; Steinbring, 2005), it is a well-recognized view in educational discourse that mathematical argumentation and reasoning have to be considered more broadly to suit the field of mathematical learning (e.g., Lithner, 2008, p. 256; Schwarzkopf, 2015, p. 31). It has even been argued that the knowledge generated in deduction cannot actually be considered new knowledge (Meyer, 2014, p. 20; Steinbring, 2005, p. 149), which would lead to the conclusion that deduction does not meet the requirements of explorative reasoning. Lithner (2008) states that mathematical reasoning "is not necessarily based on formal logic, thus not restricted to proof, and may even be incorrect as long as there are some kinds of sensible (to the reasoner) reasons backing it" (Lithner, 2008, p. 257).

In contrast to deduction, *inductive reasoning* uses empirical observation to form conclusions by inferring regularities. An example for inductive reasoning in geometry would be "All equilateral (plane) triangles so far measured have been found to be equiangular. This triangle is equilateral. Therefore, this triangle is equiangular" (Franklin, 2013, p. 14). Gathering examples from empirical observations, finding similarities or patterns among them and generalizing those patterns are typical activities for inductive reasoning.

Another form of reasoning often observed in learners' mathematical argumentation is *abductive reasoning*. In abductive reasoning, an unexpected, confusing observation leads to the forming of explanatory hypotheses of which one is chosen that provides the best explanation for the observation (Lombrozo, 2012, p. 15).

Especially in research on young children's mathematical activities, another form of reasoning that has gained increased attention is *analogical reasoning*, i.e., reasoning with relational patterns (English, 2004, p. 2). Analogies are based on the recognition of relational

Table 2 Initiation, goals, tools and modalities of explorative mathematical argumentation

	Explorative mathematical argumentation
Initial situation	• Mathematical knowledge in contrast • Interest in convergence
Goals	• Collaborative construction of mathematical knowledge • Extension of mathematical knowledge systems
Tools	Mathematically anchored exploration by • Making and communicating discoveries • Questioning or challenging existing knowledge • Finding and using (relational) patterns • Developing and testing hypotheses • Drawing on analogies • Drawing conclusions
Modalities	• Verbal statements • Gestures • Material-based actions

similarities between objects on a structural level (as opposed to object similarities) (Goswami, 2001, p. 438; Rattermann & Gentner, 1998, p. 453). Contrary to the former view that it is a late developing skill, this mode of reasoning has been observed in young children (English, 2004, p. 3). An example of analogical reasoning in an early explorative mathematical argumentation process will be given below.

Reasoning applied in explorative mathematical argumentation is knowledge-oriented and not persuasive, but also not necessarily deductive like in formal-deductive mathematical proof. It may as well be inductive, abductive or analogical reasoning. With learners developing argumentation competencies, instead of assuming that one of those modes of reasoning will be conducted throughout the argumentation, it may be more helpful to investigate argumentation processes with a focus on different aspects of, or tools applied in, those different modes of reasoning, like discovering relational patterns, drawing on analogies, or formulating and testing hypotheses. It is also important to consider that there are different *modalities*, i.e., different communicative measures in which the argumentation process can be carried out, like verbal statements, gestures and material-based actions (Table 2).

3 Application of the framework

Based on the above, explorative mathematical argumentation is a collaborative process of inferring conclusions (claims) from premises (data) and supporting statements by finding reasonable justifications (data, warrants and backings), e.g., by formulating and testing hypotheses or drawing on analogies, whereby the process leads to the construction of mathematical knowledge and, within the argumentation structure, data, warrants and claims are mathematically anchored, i.e., they refer to mathematical concepts.

3.1 An example of explorative mathematical argumentation

The following sequence that took place in a mathematical workshop for kindergarteners serves as an example of explorative mathematical reasoning. A group of seven children, one 5-year-old (C1) and six 4-year-olds (C2-C7), had filled six egg cartons with plastic eggs. Each carton could hold six eggs, but the cartons were filled with amounts from one to six and put in an ascending order. As a dialectical component, the kindergarten teacher training student (S) introduced a hand puppet that kept disarranging the order of the egg cartons. After every change, the children discussed what was wrong and why. The situation was designed to address the cardinal and ordinal aspect of number and to establish a connection between them. The situation is suitable to illustrate certain aspects of early explorative mathematical argumentation.

In this first part of the sequence, the children (C1 to C7) solved several tasks by counting up to the missing carton from one onward. When the carton with five eggs was identified as missing, C1 immediately said:	
C1: Because there 4. (Places hand over all 4 eggs in the fourth carton)	Weil da vier.
S: You said 'Because there four'. (Places hand over all 4 eggs in the fourth carton) Did you see that without having to count?	Du hast gesagt 'Weil da vier'. Hast du das gesehen, ohne dass du zählen musstest?
C1: (Looks at S, nodds)	
S: Explain that to me. That's interesting to me, why you said 'Because here four'. (Shortly places hand over all 4 eggs in the fourth carton) Explain it to me.	Erklär mir das mal. Das interessiert mich jetzt, warum du gesagt hast: ‚Weil hier vier'. Erklär mir das mal.
C1: (Looks away) Because there was four. (hand waving; imprecise, swinging pointing gesture towards the carton with 4 eggs)	Weil da vier war.
S: And after four…	Und nach der vier…
Children: Comes 5!	Kommt fünf!
In the following sequence, two cartons are swapped. In the beginning of this sequence, the presented order of the amounts of eggs in the cartons is 1, 2, 3, 5, 4, 6.	
C 1: (Touches the carton with 4 eggs and the carton with 5 eggs and looks at the student)	
C 2: It swapped them!	Sie hat das umgetauscht!
Other children: (laugh)	
C 1: (Interchanges the carton with 4 eggs and the carton with 5 eggs) This cheeky mouse.	Diese freche Maus.
S: Ah, okay.	Ah, okay.
C 3: This cheeky mouse, right?	Diese freche Maus, stimmt's?
S: Why did you swap these two now? (Points on the two cartons)	Warum hast du die zwei jetzt getauscht?
C1: Because after 4 (lays one hand on all four eggs in the fourth carton) comes 5 (lays the same hand on the five eggs in the fifth carton, looks at student).	Weil doch nach vier fünf kommt.
S: Okay. (Nods, then interchanges the two cartons again with the children looking) Had it… would it also, like this… Had it been okay this way too, with the 6? Does 4 come before 6?	Hätte es… wäre das so rum auch… Hätte es so rum gepasst mit der 6? Kommt vor der 6 die 4?
Children: No.	Nein.
C4: (shakes head no)	
C1: That doesn't fit. (interchanges the cartons)	Das passt doch nicht.
S: No? You shake your head no. Doesn't the 4 come before the 6?	Nein? Du schüttelst den Kopf. Kommt vor der 6 nicht die 4?

C4: No. (laughs)	Nee.
S: What comes before the six?	Was kommt vor der Sechs?
Children: The 5!/5!	Die 5!/5!
S: Ah, well done!	Ach, super!
Now, explorative mathematical reasoning sets in:	
C1: I AM 5. (holds up five fingers)	Ich BIN 5.
S: You are 5? (holds up five fingers) Who else is 5?	Du bist 5? Wer ist noch 5?
C4: I am 4. (holds up four fingers) S: (holds up four fingers)	Ich bin 4.
C5: I am also 4. (holds up four fingers)	Ich bin auch 4.
C6: I am also 4. (holds up four fingers)	Ich bin auch 4.
S: So who is older, him (points to C1) or you (points to the other children)?	Und wer ist dann älter, er oder ihr?
C1: Me. (points to himself)	Ich.
S: Why are you older?	Warum bist du älter?
C1: Hands up everyone who is 5. (puts hand up)	Wer alles 5 ist, der streckt.
S: And when you are 5, you are older than those who are 4?	Und wenn du 5 bist, dann bist du älter als die, die 4 sind?
C1: (nods)	
S: Why?	Warum?
C3: Because 5 is more.	Weil 5 mehr ist.
S: Ah… because 5 is more than 4?	Ah… weil 5 mehr ist als 4?
C1: Yes, look: This is only two (shortly lifts the two eggs in the bottom row of the 4-eggs-carton up and puts them back in) plus two (does the same with the two eggs in the upper row, then turns to the 5-eggs-carton), THERE are two (shortly lifts the two eggs in the bottom row of the 5-eggs-carton up and puts them back in) plus THREE (does the same with the three eggs in the upper row).	Ja, schau: Das sind nur 2 plus 2, DA sind 2 plus 3.

The example will be further examined in the following paragraph in order to give an idea on how EMA processes can be identified and analysed, taking into account the above considerations.

3.2 Suggestions for identifying and analysing early explorative mathematical argumentation

In the above sequence, the children engage in an interactional process of justifying the claim that a 5-year-old is older than a 4-year-old. As a first step, the Toulmin model can be applied to identify constituting components of an argumentation process (Fig. 3).

The datum (5 is more than 4), its backing additional data (4 is only 2 plus 2, 5 is 2 plus 3), the implicit warrant (the number signifying the larger amount also signifies the higher age) and the claim (the 5-year-old boy is older than the 4-year-old children) are mathematically anchored, which meets another requirement of explorative mathematical argumentation. The third constituting aspect, the collaborative extension of mathematical knowledge, lies in the application of different aspects of numbers and of the principle of covariance, as will be presented further below. First, I want to show the abovementioned limitation in the use of the Toulmin model for the *analysis* of argumentation processes.

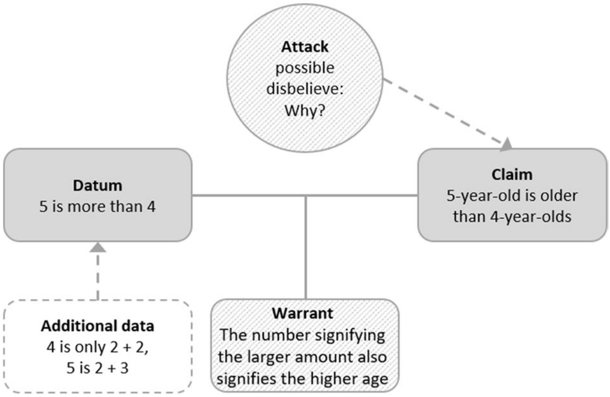


Fig. 3 Argumentation structure "5-year-old is older than 4-year-olds"

Although, as shown, the model can be used to identify an argumentation process, there are some problems attached to it. For example, in the argumentation structure illustrated above, it is questionable if the child's reference to the amount of eggs in the subsets of the two quantities should be treated as additional data supporting the original data (Fig. 3) or rather as a datum giving reason for the claim that 5 is more than 4 (Fig. 4).

As an alternative, the argumentation process can be structured by a combination of the applied tools and the modalities of communication they are applied to. Table 3 presents a general analysing scheme that will be filled and concretized below.

To give an example of the use of the scheme, it is now filled with a short sequence at the end of the transcript (Table 4).

After the cardinal and ordinal aspect were originally addressed by the student initiating this situation (five eggs; five comes after four), the selected sequence starts with the statement "I AM five.", accompanied by the gesture of holding up five fingers. Analogies are drawn between different aspects of the number 5: the ordinal aspect (5 comes before 6) and the 5 years of age (measurement aspect), represented by the gesture of holding up five fingers (cardinal aspect). The child's age is of high personal relevance (the verbal utterance "Me" is accompanied by a pointing gesture), thus questioning the fact that he is older than the other children creates a need for plausible reasoning. However, after another child verbally claims that "five is more", it is not the five fingers that are subject of the material-based justification of the claim, but now the eggs in the carton serve as a representation of the ages of 4 and 5 years. By using the carton's array structure for dividing each of the two quantities into two subsets and slightly lifting up the subsets in question, the child claims that while one subset of both quantities has the same cardinality, the second one differs. So, detouring via the measurement aspect of age, which usually has high personal relevance for

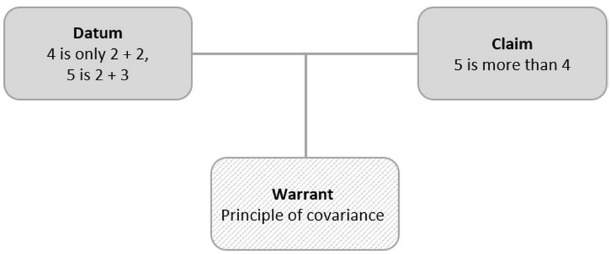


Fig. 4 Argumentation structure "5 is more than 4"

Table 3 Proposed analysing scheme for explorative mathematical argumentation

Tools	Modalities		
	Verbal statements	Gestures	Material-based actions
Making discoveries			
Communicating discoveries			
Challenging existing knowledge			
Finding (relational) patterns			
Using (relational) patterns			
Developing hypotheses			
Testing hypotheses			
Drawing on analogies			
Drawing conclusions			

Table 4 Exemplary application of the analysing scheme

Tools	Modalities		
	Verbal statements	Gestures	Material-based actions
Drawing on analogies	C1: I AM 5.	C1: holds up five fingers	
Communicating discoveries	Me. [I am older.]	C1: points to himself	
Challenging existing knowledge	S: Why are you older?		
Drawing on analogies	C3: Because 5 is more.		
Using (relational) patterns	C1: Yes, look: This is only two plus two, THERE are two plus THREE.		C1: Shortly lifts the two eggs in the bottom row of the 4-eggs-carton up and puts them back in, does the same with the two eggs in the upper row, then turns to the 5-eggs-carton. Shortly lifts the two eggs in the bottom row of the 5-eggs-carton up and puts them back in, does the same with the three eggs in the upper row.

children, and drawing analogies between different aspects of number lead to the application of the mathematical principal of covariance as a justification of the claim that the 5-year-old is older than the 4-year-olds.

3.3 Observable (co-)construction of mathematical knowledge

The given situation allows an observation of processes of how mathematical knowledge — here, knowledge about number — is being conducted and worked on by a group of children. To illustrate C1's mathematical development within the situation and its social embeddedness, I will apply the "comprehensive 6 level model for describing, explaining and predicting the development of key numerical concepts and arithmetic skills from age 4 to 8" proposed by Fritz et al. 2013 (p. 38).

The situation aims at activating concepts found on levels II and III of the model: On level II, *Mental number line*, children can identify preceding and succeeding ordinal numbers. Thus, "[c]hildren can now correctly answer the question: 'which number is larger, 4 or 5?'" (Fritz et al., 2013, p. 45). The concept of *Cardinality and Decomposability*, reached on level III, enables children to compare numbers and quantities through the number of elements: "4 is less than 5 because the quantity 4 consists of fewer elements than the quantity 5" (Fritz et al., 2013, p. 46). Both concepts are activated by the children throughout the situation. First, the focus lays on the ordinal aspect of preceding and succeeding numbers ("Because after 4 comes 5."). After the claim that the 5-year-old is older than the 4-year-olds is questioned, C3 offers the cardinal justification "Because five is more". C1 picks up this idea and, employing the material at hand, activates concepts assigned to level IV, *Class inclusion and Embeddedness*: "In understanding the part-part-whole concept, it becomes possible for the children (…) to carry out solution procedures based on derived facts (compensation and covariance)" (Fritz et al., 2013, p. 48).

Furthermore, a profound development in C1's engagement in the mathematical argumentation can be observed, corresponding to the abovementioned definition of a successful learning process by Krummheuer as "the increased autonomous participation in such collective argumentation in the process of a current interaction and/or in the following interaction that is thematically imbedded in the actual situation" (Krummheuer, 2018). After a mere repetition of the statement "Because there four", respectively, "Because there was four" in the first sequence, C1 produces a complete ordinal statement "Because after 4 comes 5.", which may have been prompted by the training student's approach of offering the unfinished sentence "And after four…" earlier in the situation. But what stands out much more is the argumentation C1 conducts when their being the oldest child of the group is questioned. After C3 gives the cardinal reason that "5 is more", C1 spontaneously declares "Yes, look: This is only two plus two, THERE are two plus THREE." Despite the principle of covariance remaining implicit, giving this datum to justify the claim and visualizing the idea by lifting subsets of the eggs up from the carton without even being asked to back C3's statement marks C1's increased autonomy in the argumentation. In addition, Sfard (2006) offers an outlook on the continuing learning process stating that, from a participationist point of view, learning mathematical problem solving is "a gradual transition from being able to take a part in collective implementation of a given type of task to becoming capable of implementing such tasks in their entirety and on one's own accord." (p. 157).

4 Discussion

While this exemplary analysis only involves a short sequence of explorative mathematical argumentation, applying the analysing scheme to various situations may show patterns in the interactions between learners and learning guides or correlations between tools and modalities, thus providing ideas on how to analyse and foster mathematical argumentation processes.

As the above example shows, explorative mathematical argumentation can be observed and encouraged in initiated learning situations with learners as young as 4 and 5 years of age. Toulmin's (2003) model of argumentation provides a useful approach to identifying and structurally analysing argumentation processes. However, many researchers have pointed out that the understanding of argumentation in the Toulmin model is that of a product rather than a process (Gronostay, 2017; Nielsen, 2013). As both aspects have to be considered, the structure of the argument *and* the process of argumentation (Kuhn & Udell, 2003; Duschl & Osborne, 2002, p. 41), the Toulmin model may not suffice for a thorough analysis, especially when applied in a reduced form, generally leaving out modal qualifiers and rebuttals.

This paper suggests that in order to *identify* explorative mathematical argumentation in learners' communicative actions, the Toulmin model can be applied and complemented by Ehlich's definition of explorative argumentation and by the aspect of anchoring from Lithner's concept of creative mathematical reasoning:

- Is there at least a claim to be found, a datum that gives reasons for accepting the claim, and an explicit or implicit warrant that explains why the datum supports the claim?
- Is one of the components challenged by an attack (such as a surprising discovery or another person's disbelief)?
- Do we also find modal qualifiers and rebuttals that complete the argumentation structure?
- Are the three main components (claim, warrant and datum) mathematically anchored?
- Does the collaborative process lead to the (co-)construction of mathematical knowledge?
- Are justifications given to support a claim in an explorative manner, as opposed to persuasion (e.g., referring to authorities)?

It is important to note the fact that in children's argumentation, assertions often remain implicit, and it is of high importance that the researcher is able to recover these implicit components (Rocci et al., 2020). Also, to meet the requirements of researching learners' mathematical argumentation processes, the use of gesture and material-based actions should be taken into account as well as verbal utterances (Walkington et al., 2014; Krummheuer, 2010, p. 4), both in the process of identifying and in that of analysing mathematical argumentation processes. Thus, videography ought to be the means of choice for data collection and analysis.

In order to *analyse* the communicative interactions and the tools of reasoning applied in the process of collaborative knowledge construction, the presented analysing scheme can contribute to a thorough analysis of an explorative mathematical argumentation's process, structure and content. The choice of method will depend on the specific research interest. Questions arising from the theoretical exploration of explorative mathematical argumentation could approach modes and tools of reasoning, deployment of language, gestures and material-based actions, interactive processes among learners as well as between learners and learning guides, and the way mathematical knowledge is (co-)constructed in the argumentation process. The article introduced explorative mathematical argumentation as a theoretical concept for identifying and analysing mathematical argumentation in learning processes. Explorative mathematical argumentation signifies a process of collaborative knowledge construction that involves inferring conclusions from premises and supporting statements by finding reasonable justifications within a mathematical context, which means that data, warrants and conclusions are mathematically anchored. Reasoning refers to the tools applied in the process, like forming and testing hypotheses and drawing on analogies.

Modalities refer to what communicative measures are taken to carry out the argumentation process. Gestures and material-based actions have to be considered as well as verbal utterances.

A thorough analysis of learners' explorative mathematical argumentation processes will have to consider structural aspects of the argumentation, the tools of reasoning and the modalities of communication used within the process, as well as the underlying social and content-related context. As shown, the proposed framework can contribute to research on mathematical argumentation in young children. As for practice, the framework implicates a high potential of argumentation processes among groups of children concerning collaborative mathematical knowledge construction. Instructors fostering children's mathematical development may profit from identifying different kinds of argumentation in order to plan and accompany learning processes. According to the theoretically and empirically founded model of professional knowledge and skills for early mathematics education by Gasteiger and Benz (2018), kindergarten teachers' explicit knowledge about explorative mathematical argumentation as well as their competence to observe and perceive these situations in children's everyday activities will affect the ability to design, implement and evaluate respective learning opportunities.

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