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Learning to measure the area of composite shapes

Timothy H. Lehmann[1]

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Abstract
Learning to calculate the area of composite shapes is an important application of area measurement but evidence suggests that many middle school students struggle to calculate the area of even simple composite shapes. In this article, I report the findings of a classroom design study conducted to investigate the collective development of strategies for measuring the area of composite shapes. I used the theory of strategy choice from an emergent perspective to analyze the collective strategy choice of two Year 8 mathematics classes ($n=31$) and the findings revealed that the classes developed a repertoire of strategies to decompose the shapes, measure the area of the constituent shapes, and recompose those areas to calculate total area. The students used their strategies in flexible combinations in response to the varying features of the composite shapes presented and developed justifications to support their emerging adaptive strategy choice.

Keywords Area measurement · Strategy choice · Composite shapes · Decomposition and recomposition

1 Introduction

Learning to calculate the area of composite shapes (shapes composed of two or more basic shapes) is an important application of the area measurement strand of mathematics curricula, primarily because of the immediate use in daily-life and occupational contexts (Diego-Mantecón et al., 2021). It also provides a foundation for the development of future mathematical ideas such as measuring surface area (Battista, 2012) and calculating areas using definite integrals (Jones, 2015). Moreover, the decomposition and recomposition heuristic, often associated with calculating the area of composite shapes, can be applied across domains of mathematics such as geometric reasoning and computational thinking (Pólya, 1945; Schoenfeld, 1985; Spiegel & Ginat, 2017).

Early evidence from large-scale assessments of middle school students' mathematical achievement in the USA and England suggest that many students struggle to calculate the area of even basic composite shapes (Foxman et al., 1980; Hirstein, 1981). More recently

✉ Timothy H. Lehmann
t2.lehmann@qut.edu.au

1 Faculty of Creative Industries, Education and Social Justice, Queensland University of Technology, Brisbane, QLD, Australia

on the 2016 Year 9 annual nationwide test of numeracy in Australia, students were shown a diagram of a composite shape composed of a 3 m × 3 m square appended to a triangle with a height of 2 m. Students were asked to calculate the total area in square meters (12 m^2); however, only a third of students answered the question accurately (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2016). Baturo and Nason (1996) documented preservice mathematics teachers' difficulties in calculating the area of a similar composite shape. Researchers have also found that upper primary-aged students struggle to calculate the area of composite shapes, even when superimposed onto square grids (Tan Sisman & Aksu, 2016; Zacharos, 2006).

Researchers suggest that these long-standing difficulties arise from instruction that emphasizes procedures for calculating area (Patahuddin et al., 2018; Zacharos, 2006) or students' unfamiliarity with the diversity of composite shape configurations (Spiegel & Ginat, 2017). Instead, Spiegel and Ginat (2017) assert that developing students' competence in using the decomposition and recomposition heuristic would enhance their proficiency in measuring the area of composite shapes and potentially avoid the aforementioned difficulties; although there appears to be limited research into instructional approaches that foster such competence. The aim of this study is to address this gap by investigating the collective development of strategies to decompose composite shapes, measure the area of the constituent shapes, and recompose those areas to calculate total area.

2 Review of related literature

Previous researchers and writers of large-scale assessments have used L-shapes to examine students' proficiency in calculating the area of composite shapes and for this reason I will use the L-shape in Fig. 1 to illustrate the findings of the related studies reviewed in this section (ACARA, 2016; Baturo & Nason, 1996; Hirstein, 1981; Moreira-Baltar, 1999). Most existing studies about students' responses to composite shape measurement tasks focus on analyzing students' erroneous strategies and Fig. 1 contains six such erroneous strategies. The first of these is the inappropriate application of the rectangular area formula (Fig. 1a) to calculate the area of a composite shape (Hirstein, 1981; Reinke, 1997; Zacharos, 2006). Second, Zacharos (2006) found a related erroneous strategy dubbed "finishing figures off" (p. 233), in which students add lines to the composite shape to transform it into a rectangle and then calculate the area of the resultant rectangle (Fig. 1b). Third, students calculate perimeter instead of area (Fig. 1c) (Baturo & Nason, 1996; Hirstein, 1981). Fourth, Zacharos (2006) documented a similar erroneous strategy in which students use area = base + height to calculate the area of composite shapes (Fig. 1d). Fifth, students identify a correct area formula for a simpler shape within the composite shape, such as the square in Fig. 1e, but struggle to infer the dimensions needed for the area formula (Baturo & Nason, 1996; Spiegel & Ginat, 2017). Finally, researchers have documented the erroneous strategy in Fig. 1f, in which students enclose the entire shape and measure its area, calculate the area of the non-included shape, but then add these two areas rather than subtract them. As mentioned, researchers attribute the erroneous strategies in Fig. 1a–d to students' lack of conceptual understanding of area and/or instruction that prioritizes the application of area formulae (Patahuddin et al., 2018; Zacharos, 2006), whereas the erroneous strategies in Fig. 1e, f have been attributed to students' lack of experience in reasoning about the geometric configurations of composite

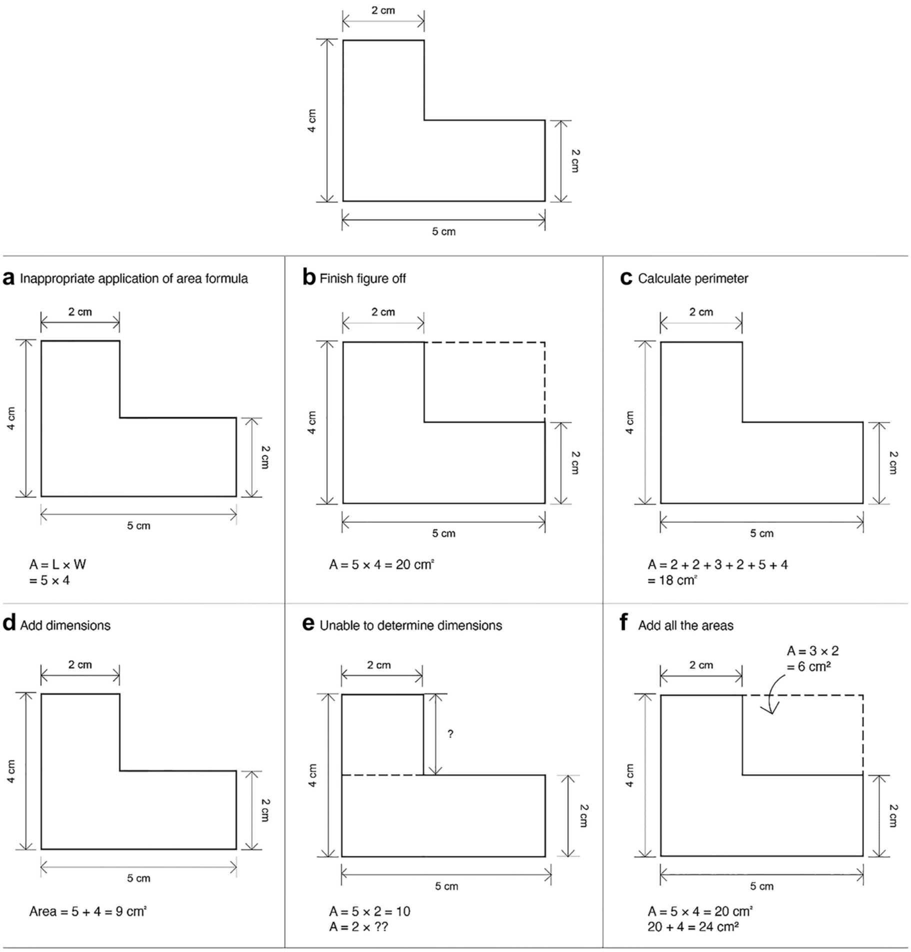


Fig. 1 Erroneous strategies

shapes (Patahuddin et al., 2018; Spiegel & Ginat, 2017). Researchers have also found that students abandon measurement tasks when they conclude that a composite shape has no area (Zacharos, 2006) or the configuration too complicated (Spiegel & Ginat, 2017).

Researchers have also analyzed the strategies used by students who calculated the area of composite shapes correctly and found that these students used the decomposition and recomposition heuristic implicitly. That is, students decomposed a composite shape into constituent shapes, calculated the areas of those constituent shapes, and then recomposed these areas by adding them together (Baturo & Nason, 1996; Hirstein, 1981; Patahuddin et al., 2018; Spiegel & Ginat, 2017; Zacharos, 2006). Just how these students developed competence in decomposition and recomposition is generally not addressed in these studies. Nevertheless, several researchers propose that applying the decomposition and recomposition heuristic to measure the area of composite shapes involves the coordination of three sets of interrelated strategies, as summarized in Fig. 2.

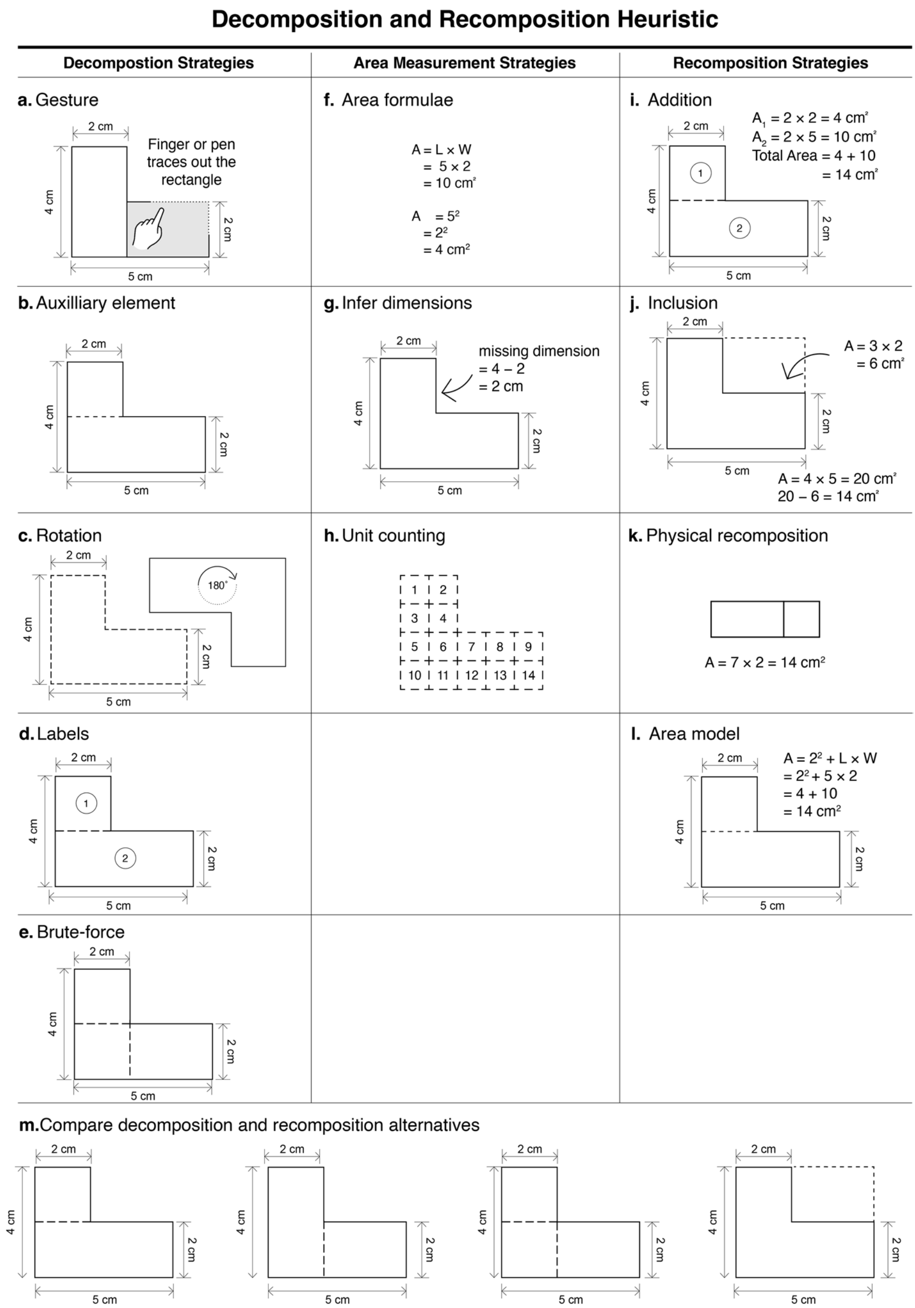


Fig. 2 Strategies underpinning decomposition and recomposition heuristic

Decomposition involves apprehending constituent shapes that form the composite shape using geometric reasoning and then separating those constituent shapes in some way. Decomposition strategies are those that support this process. Patahuddin et al. (2018) identified three such decomposition strategies used by a sample of three junior-secondary students: (1) using gestures such as tracing around the boundary of the shape with their finger or pen (Fig. 2a); (2) adding an auxiliary element (Pólya, 1945), such as a line or shape (Fig. 2b); and (3) using rotation to help visualize the simpler shapes embedded within the composite shape (Fig. 2c). Labels (Fig. 2d) might be used to indicate the virtual separation of those constituent shapes (Hirstein, 1981; Patahuddin et al., 2018).

Having apprehended and separated constituent shapes, the area of those constituent shapes is calculated. This requires a choice in measurement strategy, typically an area formula (Fig. 2f), which involves inferring the relevant dimensions of the constituent shapes (Fig. 2g) and proficiency in computations (Baturo & Nason, 1996; Moreira-Baltar, 1999; Spiegel & Ginat, 2017). The use of area formulae, however, requires an understanding that the procedure yields the number of square units that fill the space inside each shape. Area may be calculated by simply superimposing the shape onto gridlines and counting the number of square units (Fig. 2h) (Moreira-Baltar, 1999).

Finally, researchers suggest several strategies for recomposing the calculated areas, including (1) adding the areas of the constituent shapes (Fig. 2i); (2) enclosing a shape and subtracting the superfluous segments (Fig. 2j); (3) recomposing the shape into a single rectangle (Fig. 2k); and (4) devising a shape-specific area formula (Fig. 2l). Spiegel and Ginat (2017) make the point that composite shapes can often be decomposed in different ways, which might affect the complexity of the recomposition strategy (Fig. 2m). Hence, comparing decomposition and recomposition alternatives is also seen as a desirable strategy (Hirstein, 1981; Patahuddin et al., 2018; Spiegel & Ginat, 2017).

These strategies that underpin the decomposition and recomposition heuristic are largely conjectural, and the ways in which students develop and use these strategies remains unclear. Researchers nevertheless assert that the goal of instruction should be to develop students' competence in the use of such strategies and the ability to apply them flexibly (Hirstein, 1981; Spiegel & Ginat, 2017), which might be achieved by planning tasks "in an orderly fashion, according to suitable measures of challenge" (Spiegel & Ginat, 2017, p. 216). Whether such an approach develops students' competence in measuring the area of composite shapes remains an open question, and the purpose of this study is to address this gap.

3 Theoretical framework

The theory of strategy choice was used in this study since the focus was on the development of students' strategies for measuring the area of composite shapes (Siegler, 1991; Siegler & Lemaire, 1997). Researchers use the theory of strategy choice across domains of mathematics, including measurement, to analyze the emergence and use of students' strategies in response to tasks with varying characteristics (Heinze et al., 2009a, 2009b; Vasilyeva et al., 2013). The analysis of strategy choice comprises four dimensions: strategy repertoire, strategy distribution, strategy effectiveness, and strategy selection (Lemaire & Siegler, 1995; Siegler, 1991; Siegler & Shipley, 1995).

The central premise of strategy choice is that students have access to a repertoire of strategies to solve a problem or task (Siegler, 1991). Students can build this repertoire by acquiring new strategies through their own discovery or invention through problem solving, through their participation in the classroom community, or from a teacher directly (Ellis, 1997; Heinze et al., 2009a, 2009b; Siegler, 1991). As outlined above, a strategy repertoire for calculating the area of composite shapes consists of strategies for decomposition, area measurement, and recomposition (see Fig. 2). Strategy repertoires are needed because composite shapes vary in their complexity (number and type of constituent shapes; type of geometric configuration) and the measurement information available to calculate area.

The utility of strategies varies across problem types in terms of accuracy, efficiency, adaptableness, and level of cognitive demand (Siegler, 1991). Hence, the need for choice between strategies arises because students must consider the utility of strategies in relation to the features of a given composite shape. These choices are reflected in the strategy distribution, which shows the relative frequency with which students use the strategy repertoire across different composite shapes.

Strategy effectiveness refers to the accuracy and speed with which students use strategies. An accurate use of a decomposition strategy results in a correct apprehension of the constituent shapes and an accurate use of a measurement strategy that yields a correct quantification of area. The accuracy of a recomposition strategy requires the student to coordinate the area calculations with the chosen decomposition so that the recomposition corresponds to the original spatial configuration (Spiegel & Ginat, 2017). The speed of strategy use was not considered in the present study because the focus was on students' initial acquisition of strategies.

Finally, strategy selection refers to the *flexibility* and *adaptivity* with which students choose and apply the strategy repertoire (Heinze et al., 2009a, 2009b). Flexibility in strategy use refers to students' proficiency in choosing between multiple strategies, for example, choosing between addition (Fig. 2i) or inclusion (Fig. 2j). Flexibility in relation to composite shapes also refers to the ability to choose between alternative ways of decomposing and recomposing the shape, such as those in Fig. 2m (Patahuddin et al., 2018; Spiegel & Ginat, 2017).

Adaptivity in strategy use refers to students' proficiency in choosing an appropriate strategy for a given task within a given sociocultural context (Heinze et al., 2009a, 2009b). For example, the most appropriate decomposition alternative for the L-shape might be the rectangle and square (Fig. 2m), because the resultant recomposition calculations are less complex (Spiegel & Ginat, 2017). However, the criterion for appropriate choice depends on the sociocultural context (Ellis, 1997; Verschaffel et al., 2009).

Flexibility and adaptivity in the use of strategies can be developed through opportunities to examine multiple methods (Rittle-Johnson et al., 2012), compare worked examples (Newton et al., 2010; Rittle-Johnson & Star, 2007), collaborate with peers (Ellis, 1997; Mercier & Higgins, 2013), or through direct instruction and prompting from the teacher (Star & Rittle-Johnson, 2008). In a classroom setting, students' flexibility and adaptivity is shaped by the classroom community as students accumulate knowledge about the effectiveness of alternative strategies and also learn which strategies are valued by the classroom community (Ellis, 1997; Verschaffel et al., 2009). Given that sociocultural factors can have a significant influence on the development of students' strategy choice, I analyzed students' strategy choice from an emergent perspective (Cobb & Yackel, 1996) and used the four dimensions of strategy choice as classroom mathematical practices that emerge through students' participation in the classroom community (Yackel & Cobb, 1996).

4 Research questions

The aim of this research was to investigate the collective development of strategy choice for measuring the area of composite shapes. The research questions that guided this investigation were:

1. What strategies do students develop and use to calculate the area of composite shapes?
2. How do students develop flexibility and adaptivity in using strategies to calculate the area of composite shapes?

5 Methodology

Mathematics education researchers use classroom design studies to investigate the process of students' learning of a particular domain mathematics in classroom settings (Cobb et al., 2016). This methodology offers researchers ways of testing and refining an instructional intervention designed to develop students' reasoning and the means of support for this development, from a range of interpretive frameworks. For this reason, a classroom design study was conducted to investigate the collective development of students' strategy choice for measuring the area of composite shapes as a set of classroom mathematical practices.

The study took place in two Year 8 mathematics classes at a large independent secondary school in an Australian capital city, with an Index of Community Socio-Educational Advantage (ICSEA) of 1214. The students had a history of above-average mathematical achievement on both school-based and external assessments of mathematics. All students in the two classes ($n = 46$) participated in the instructional intervention; however, I only collected data from the 31 students (mean age = 13.75) who returned the parental consent and student assent forms. The research was conducted by a team of four members, including myself. I was the regular classroom teacher for each of the classes and acted as the teacher for this study.

Classroom design studies are cyclical and involve both macrocycles and microcycles (Cobb et al., 2016). Macrocycles are comprised of three phases: preparation of the instructional intervention, implementation of the intervention in a classroom teaching experiment, and retrospective analysis of the data. Microcycles are also comprised of preparation, implementation, and analysis, but occur daily throughout the macro implementation phase. The study involved three macrocycles, with the outcomes of each iteration informing the next. In this paper, I report on the third and final macrocycle of the project.

In the previous two macrocycles, I conducted classroom teaching experiments in the years preceding this study. The first macrocycle was conducted with one Year 8 group ($n = 13$) at a suburban state high school with an ICSEA of 1055. The findings from this macrocycle revealed that overly complex composite shapes, represented only as figural representations, presented too many complications that inhibited the emergence of strategies for decomposition. The second macrocycle was conducted in a Year 8 class ($n = 16$) at a rural state high school with a ICSEA of 918. The findings from this macrocycle revealed that the use of simple composite shapes, presented as paper cutouts, elicited strategies for decomposition, area measurement, and recomposition. However, the whole-class discussions focused on the strategy repertoires rather than justifications for strategy choice; hence, the emergence of strategy flexibility and adaptivity was limited. The design of the third macrocycle was informed by these salient findings. The three schools are a convenience sample because I was teaching in them at the time. However, the schools had a common goal of improving the area measurement outcomes of their students considering their respective results in the annual nationwide tests of numeracy.

5.1 Preparation phase

In the preparation phase, the envisioned learning trajectory in Table 1 was developed, which formed the basis of the instructional intervention designed to elicit the collective development of strategy choice for calculating the area of composite shapes (Cobb et al., 2016). The first and second columns show the lessons and anticipated collective

Table 1 Envisioned learning trajectory

Lesson	Development of strategy choice	Composite shapes[a]	Anticipated strategies		
			Decomposition	Measurement	Recomposition
1	Develop decomposition and recomposition heuristic Develop repertoire of strategies for decomposition, area measurement, and recomposition		• Physical	• Array • Unit counting • Area formula (rectangle, square) • Determine dimensions from gridlines	• Addition
	Consider alternative decompositions for same shape, including inclusion Justify choice of decomposition alternative	U-shape	• Auxiliary element: inclusion		• Inclusion: subtraction
2	Determine dimensions of shapes, including partial units Develop and/or adapt strategy repertoire for shapes given without gridlines Justify strategy selection for shapes without gridlines	H-shape E-shape	• Physical • auxiliary element	•Determine dimensions with ruler or gridlines on grid paper	• Addition • Inclusion: subtraction
3	Develop and/or adapt strategy repertoire for figural representation of composite rectangles Justify strategy selection for figural representations	Step-shape	•Auxiliary element	• Use given dimensions and infer those not given • Area formulae	• Addition • Inclusion: subtraction
One week break					
4	Develop and/or adapt strategy repertoire for composite shapes that include non-rectangular regions Justify choice of decomposition alternative Justify selection of decomposition, area measurement, and recomposition strategies	House-shape arrow-shape	• Physical • Auxiliary element	• Determine dimensions from gridlines • Area formulae (triangle, rectangle, parallelogram)	• Addition
5	Develop or use strategies to identify trapezoidal regions	Irregular hexagon boot-shape	• Physical • Auxiliary element	• Determine dimensions with ruler or gridlines on grid paper •area formulae (trapezium, parallelogram)	• Addition

Table 1 (continued)

Lesson	Development of strategy choice	Composite shapes[a]	Anticipated strategies: Decomposition	Measurement	Recomposition
6	Develop and/or use strategy repertoire for representations of composite shapes	Octagon	• Auxiliary element	• Use given dimensions or infer those not given • Area formulae (sector/circle, rectangle, trapezium)	• Addition
	Use decomposition strategies to identify trapezoidal and circular shapes Justify strategy selection for figural representations	Floor plan			

[a]Table 2 contains a summary of the complexity and mode of presentation of the composite shapes, which are reproduced in the Appendix

development in strategy choice. The third column contains the means of support for this development, that is, the composite shapes that students were asked to measure. The final three columns contain the anticipated strategies for decomposition, area measurement, and recomposition that might emerge.

The envisioned learning trajectory was based on the premise that collective strategy choice would develop by measuring the area of composite shapes of increasing complexity (Spiegel & Ginat, 2017). The trajectory contains two broad stages: composite rectangles (lessons 1–3) and composite shapes (lessons 4–6). Composite rectangles were posed first because students in the previous macrocycles initially struggled to devise strategies in response to configurations that included two or more different shapes. Composite shapes comprised of quadrilaterals and circular regions seemed to distract the students because the area measurement processes for these shapes was more complex. It was envisioned that students could focus their attention on strategies for decomposition and recomposition in response to composite rectangles because calculating the area of rectangles would be relatively straightforward. Composite shapes were introduced in the second stage because it was expected that students would adapt strategies to measure the area of these shapes.

A primary characteristic of composite shapes is their geometric configuration, which will likely influence students' choice of strategies for decomposition and recomposition (Patahuddin et al., 2018; Spiegel & Ginat, 2017). Spiegel and Ginat (2017) describe three forms of composite shape: concatenated, included, and interleaving. Concatenated shapes (shapes composed of two or more simple shapes joined in their entirety) are the least complex and hence only these forms were used in this study. Secondary characteristics of composite shape tasks include the number of simple shapes (Spiegel & Ginat, 2017) and the types of constituent shapes (Patahuddin et al., 2018), Within each of the two broad categories of composite shapes, the configuration of shapes increased in complexity. The first four columns of Table 2 contain a description of the complexity of the shapes in terms of the number and type of constituent shapes.

The final column in Table 2 describes the mode of presentation for each of the shapes, which refers to the format of the shapes posed to the students. Three modes of presentation were designed for this study: paper cutouts with gridlines; paper cutouts without gridlines; and figural representations with dimensions. The purpose of these three modes of presentation was to elicit different strategies as well as justifications for strategy selection. The final three columns in Table 1 indicate the strategies anticipated from these modes of presentation.

Table 2 Complexity of composite shapes

Class of composite shape	Composite shape	Minimum number of constituent shapes[a]	Types of constituent shapes	Mode of presentation
Composite rectangles • Shapes composed of only rectangles and squares • Shapes can be decomposed in multiple ways to elicit flexibility and adaptivity	L-shape	2	2×2 square 4×2 rectangle	Paper cut-outs • Shapes presented as paper cutouts to offer students the opportunity to develop physical decomposition and recomposition (Moreira-Baltar, 1999) • In previous cycles of design study, students appeared to overcome difficulties in visualizing decomposition by physically cutting up shapes • Multiple copies of composite shapes to allow multiple decompositions/recompositions With gridlines • Composite shapes superimposed onto gridlines to offer students the opportunity to use fundamental measurement strategies such as unit counting or arrays, rather than focus on area formulae (Patahuddin et al., 2018) • Shapes positioned so that the gridlines form the boundaries of the constituent shapes, suggesting multiple alternatives for decomposition • Additional grid paper provided to facilitate physical recomposition or measurement strategies
	T-shape	2	6×2 rectangle 3×2 rectangle	
	U-shape	3 (concatenation)	two 2×2 squares 6×2 rectangle two 4×2 rectangles 2×2 square	
		2 (inclusion)	6×4 rectangle less 2×2 square	
	H-shape	3	two 7×2 rectangles 3×3 square	Paper cut-outs Without gridlines • Composite shapes presented without gridlines so that students were required to develop strategies to visualize constituent shapes • Absence of gridlines also offers students the opportunity to develop area measurement strategy flexibility and adaptivity, including determining the dimensions of the shapes • In previous cycles of design study, students appeared to overcome confusion about dimensions of shapes by measuring the dimensions themselves
	E-shape	4 (concatenation)	11×2.5 rectangle 3.5×3.5 square two 3.5×2 rectangles	
		3 (inclusion)	11×6 rectangle less 3.5×2 less 3.5×1.5 rectangles	
	Step-shape	3	14×13 rectangle 8×4 rectangle 4×3 rectangle	Figural representation • Unscaled diagram of the composite shape with dimensions provided • Students required to infer some dimensions from those provided

Table 2 (continued)

Class of composite shape	Composite shape	Minimum number of constituent shapes[a]	Types of constituent shapes	Mode of presentation
Composite shapes • Shapes composed of different shapes including rectangles, squares, triangles, parallelograms, trapeziums, sector	House-shape	2	Triangle Rectangle Two congruent Right-angled trapeziums	Paper cutouts With gridlines
	Arrow-shape	2	Two congruent parallelograms	
	Irregular hexagon	2	Right-angle trapezium isosceles trapezium	Paper cutouts Without gridlines
	Boot-shape	2	Acute-angle trapezium parallelogram	
	Octagon	3	two congruent isosceles trapeziums rectangle	Figural representation
	Floor plan	3	Semi-circle Rectangle Right-angle trapezium	

[a]The minimum number of constituent shapes was used as a measure of complexity; however, composite shapes can be decomposed into more than the minimum number

5.2 Implementation phase

The implementation phase lasted six 60-min lessons for each of the two classes, in the second term of the academic year. There was a 1-week break (due to a school camp) between the first three lessons about composite rectangles and the second three lessons about composite shapes, which enabled me to conduct a preliminary analysis of the students' responses to the composite rectangles. In the first term, the students and I had established social and sociomathematical norms consistent with a problem-solving approach to learning mathematics (Hiebert & Wearne, 2003; Rasmussen et al., 2003). Specifically, students were accustomed to solving a problem individually or alongside students sitting adjacent to them, discussing their solutions and strategies in these pairs or trios, and then explaining and justifying their solutions and strategies in whole-class discussions. The students were accustomed to learning new concepts or problem-solving strategies through solving problems, and we maintained these social and sociomathematical norms during the classroom teaching experiments.

Table 3 shows the pattern of instruction that I used to implement the intervention. The students were sitting in rows and engaged in extemporaneous conversations with each other as they

Table 3 Instructional pattern

Phase	Activity	Collective development of strategy choice
1	Students were asked to measure the area of the first composite shape in the series *Note: This phase was repeated for the T-shape in the first lesson*	• Students develop strategy *or* use existing strategy for decomposition, area measurement, or recomposition
2	Whole-class discussion of solutions and strategies	• Flexibility: students examine alternative strategies • Adaptivity: students justify strategy choices
3	Students were asked to measure the area of the second composite shape in the series Students prompted to consider alternative decompositions for U-shape, arrow-shape, irregular hexagon only *Note: This phase was not included in the third lesson because students participated in a one–on–one interview (approximately 10 min) about their strategy choices with teacher/researcher*	• Students develop strategy *or* use existing strategy for decomposition, area measurement, or recomposition • Flexibility/adaptivity: students consider alternative decompositions
4	Whole-class discussion of solutions and strategies	• Flexibility: students examine alternative strategies • Adaptivity: students justify strategies for decomposition, measurement, and recomposition

formulated their responses. Each student formulated their own response, which I collected at the end of the lesson, made copies, and returned the next lesson. I captured the extemporaneous conversations between students with six microphones positioned between pairs or trios of students. I observed the students' mathematical activity, made field notes about students' solutions and strategies, and engaged students in extemporaneous conversations about their strategies, which I captured with a seventh microphone attached to me. All audio recordings were transcribed for analysis.

For the whole-class discussions, I selected students who used differing strategies to explain and justify their solutions to the class. Students also volunteered to present their solution if they believed that their solution differed from those already presented. The presenting students projected their solutions onto a screen using a portable USB document camera, which also recorded their solution, and answered any clarifying questions. An established sociomathematical norm in these classes was that students were expected to explain how a solution or strategy differed and justify why the strategy was selected.

At the end of the third lesson, I conducted a short (maximum 10 min), audio-recorded interview with each student about the strategies they chose to calculate the area of the step-shape. I used this halfway point to document the emerging strategy choice of each individual student, and the analysis of these interviews in the following week informed the design of the second set of lessons about composite shapes.

The team made two substantial adjustments to the envisioned learning trajectory from the ongoing analysis conducted during the implementation phase. First, students were explicitly asked to consider alternative methods of decomposition for the U-shape, arrow-shape, and irregular hexagon because many students chose the first method of decomposition they formulated without considering alternatives. Second, the ongoing analysis suggested that students struggled to identify parallelograms and trapeziums embedded within the house- and arrow-shapes, which was consistent with the previous cycles. We developed the irregular hexagon, boot-shape, and octagon in response to this observation, which offered students additional opportunities to identify these shapes.

5.3 Retrospective analysis

At the end of the classroom teaching experiments, I conducted a retrospective analysis of the copies of students' responses, transcriptions of the audio recordings of extemporaneous and whole-class discussions, document camera footage, and field notes. This retrospective analysis focused on the collective development of strategy choice, according to the four dimensions: strategy repertoire, strategy distribution, strategy effectiveness, and strategy selection.

To determine the collective strategy repertoires that emerged, open and axial coding was used to systematically identify and relate the strategies evident in the students' responses (Corbin & Strauss, 2015). Throughout open coding, I analyzed each response and used memos to record my characterizations of the strategies evident in the response. The axial coding began by grouping strategies into three categories: decomposition, area measurement, and recomposition. The memos were then used to relate the responses within each category and produce strategy codes. For example, "physical decomposition" was used to code all responses in the decomposition category in which students had cut up the paper composite shape. The outcome of this open and axial coding process is the collective strategy repertoires for decomposition, area measurement, and recomposition that emerged in the classes.

Strategy distributions were used in this analysis to document which strategies became normative in each class. A matrix of composite shapes and strategies was set up, and the frequency of each strategy calculated by tallying the number of responses in which each strategy was used. The relative frequency was also calculated by dividing the frequency by the number of students (31). This matrix was then used to document when the strategies emerged, which strategies continued and discontinued to be used across the sequence of shapes, and which strategies became most valued by the classes.

To determine strategy effectiveness, the accuracy of students' responses was assessed by identifying any errors. The errors were analyzed and characterized using memos. These memos were grouped into categories, and the frequency of each error was tallied to determine the frequency of each error category.

Finally, strategy selection was analyzed for flexibility and adaptivity. For flexibility, the strategy distributions outlined above were used to compute a range for the number of decomposition, area measurement, and recomposition strategies used for each composite shape. A range of number of strategies used by each student were also computed.

The three-phase method for documenting collective activity devised by Rasmussen and Stephan (2008) was used to determine the emergent strategy adaptivity. This method facilitated the analysis of the justifications for strategy choice that were accepted by the classes and emerged as classroom mathematical practices. First, I constructed an argumentation log by making notes on the transcriptions about the justifications that students used as I listened to the audio recording and watched the document-camera footage. An argumentation scheme for these justifications was devised using the three core parts of Toulmin's (1969) model: claim, grounds, and warrant.

A claim was a statement made by a student about the use of a strategy that they used to produce their solution or the choice of decomposition alternative, and the grounds was the reason they gave for using the strategy or decomposition alternative. A warrant was a further explanation or clarification given by a student about the connection between strategy selection and the reason.

Second, I analyzed the argumentation log using constant comparison to determine if the justification for strategy selection became a normative way of reasoning about strategy selection (Corbin & Strauss, 2015). A justification was considered to be taken-as-shared if it contained a previously-accepted warrant that was not challenged or when the student did not use a warrant (Rasmussen & Stephan, 2008). Finally, strategy adaptivity as a mathematical practice was established by organizing the taken-as-shared justifications for strategy choice (Rasmussen & Stephan, 2008).

Throughout the data analysis and preparation of this report, peer review was used to minimize the effect of my biases on the data analysis (Confrey & Lachance, 2000). Specifically, I presented my interpretations of students' responses and their justifications to two experienced mathematics education researchers who provided feedback. Revisions were made in line with this feedback and these revisions were discussed with the same two researchers. The results of my analysis, however, should be read considering my subjectivity.

6 Findings

In this section, I report the elements of strategy choice that emerged as classroom mathematical practices over the course of the six lessons. The practices that emerged in each of the two classes were similar, and thus I have combined the findings for both classes throughout, highlighting any differences where applicable.

6.1 Strategy repertoires

Tables 4, 5, and 6 contain the strategy repertoires for decomposition, area measurement, and recomposition respectively that the students developed and/or used to measure the area of the 12 composite shapes. The tables contain a description of each strategy, which also indicates the shape for which the strategy first emerged and a link to the strategies in Fig. 2, and an example response that illustrates the strategy.

The strategy repertoires are presented separately here; however, the students used the strategies in varying combinations across the lessons. Specifically, the students always used a decomposition, area measurement, and recomposition strategy to measure the area of each composite shape. This use of strategy combinations emerged in the first lesson in response to the L-shape and represents the students' reinvention of the general decomposition and recomposition heuristic. There were 20 strategies in total across the three repertoires, whereas there were 13 strategies in Fig. 2. The seven additional strategies are physical decomposition, auxiliary element: inclusion, auxiliary element: analysis, array, decomposition/recomposition, dimensions from gridlines, dimensions from ruler.

Table 4 Strategy repertoire for decomposition

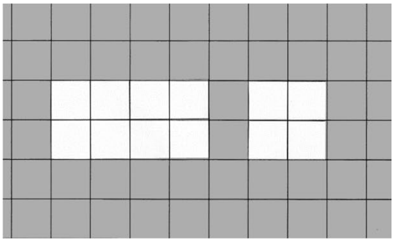
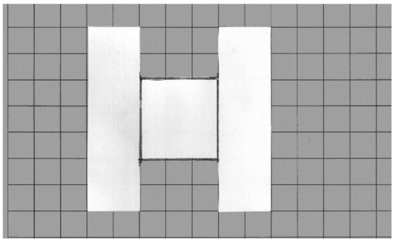
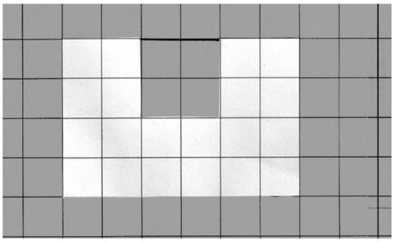
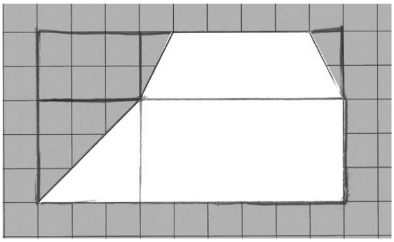
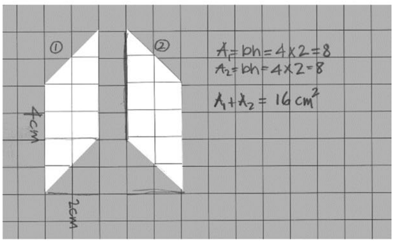
Strategy	Description	Example
Physical decomposition	Student cuts paper shape into constituent shapes and glues onto grid paper First emerged: L-shape	
Auxiliary element: constituent shapes	Student adds line(s) to indicate decomposition First emerged: H-shape (aligns with Fig. 2b)	
Auxiliary element: inclusion	Student adds line(s) to enclose shape First emerged: U-shape	
Auxiliary element: analysis	Student adds line(s) to analyze and determine constituent shapes First emerged: house shape (class 1); irregular hexagon (class 2)	
Rotation	Student rotates shape to analyze and determine constituent shapes or dimensions First emerged: arrow shape (aligns with Fig. 2c) *Note that this strategy was also used in whole-class discussions.*	

Table 4 (continued)

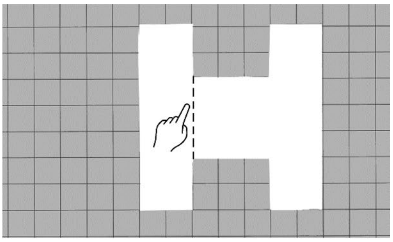
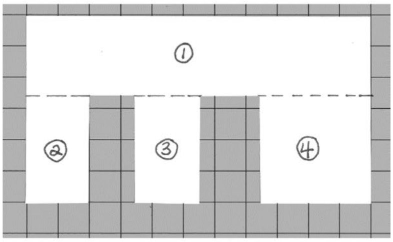
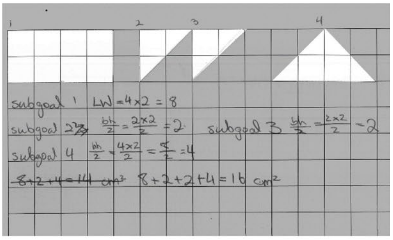
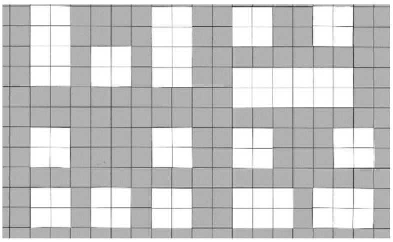
Strategy	Description	Example
Gesture	Student uses finger or pen to trace outline of shape First emerged: H-shape (Only documented in whole-class discussions.) (aligns Fig. 2a)	
Labels	Student labels the constituent shapes to signify decomposition First emerged: E-shape (class 1); H-shape (class 2) (aligns with Fig. 2d) Note: Labels were also used in previous shapes to link physically decomposed shapes to recomposition calculations.	
Brute-force	Student decomposes shape into many atomic shapes First emerged: arrow shape (aligns with Fig. 2e)	 Remington's response
Consider alternatives	Student generates, or describes more than one decomposition First emerged verbally: T-shape First emerged physically: U-shape (aligns with Fig. 2m)	

Table 5 Strategy repertoire for area measurement

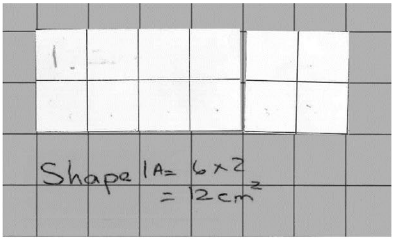
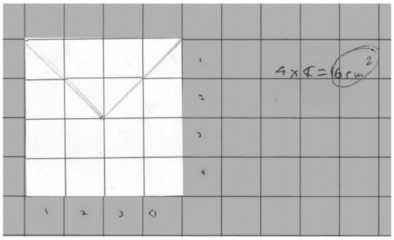
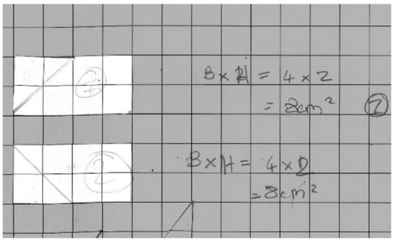
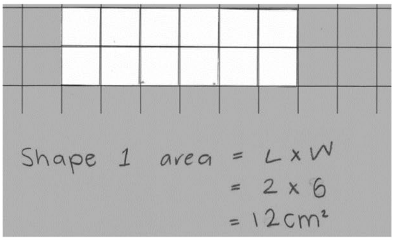
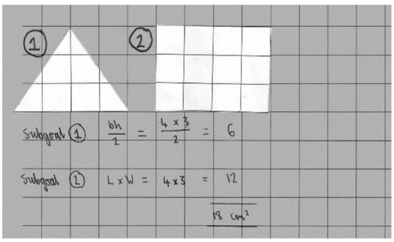
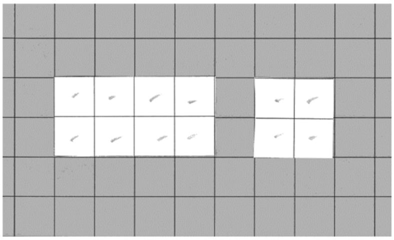
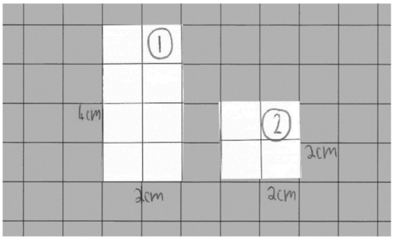
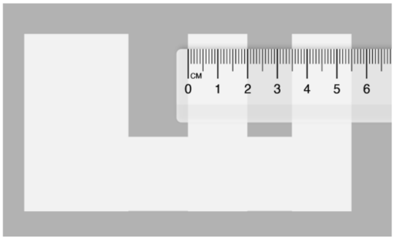
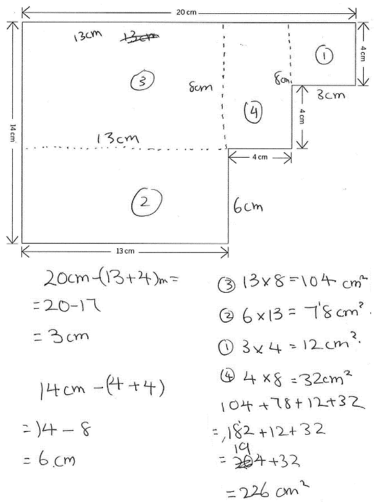
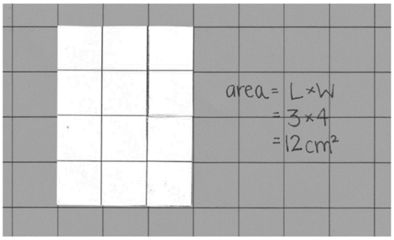
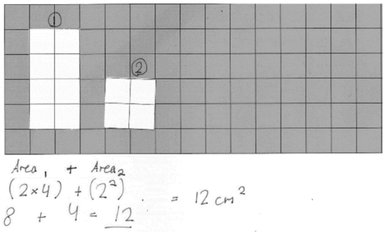
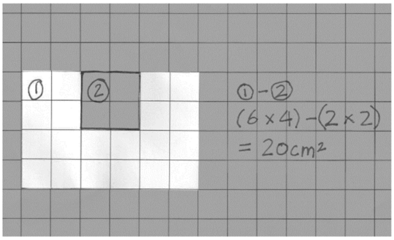
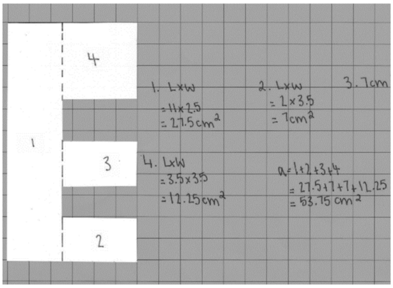
Strategy	Description	Example
Array	Student measures area by counting the number of rows/columns and multiplying it by the number of row/columns first emerged: L-shape	  Brooklyn's response
Decomposition/ recomposition	Student decomposes constituent parallelogram or trapezium and recomposes into a rectangle First emerged: arrow shape	
Area formula	Student measures area by using an area formula $L \times W$ first emerged: L-shape other area formulae first emerged: house-shape (aligns with Fig. 2f)	 

Table 5 (continued)

Strategy	Description	Example
Unit counting	Student verifies measurement by counting units First emerged: L-shape (aligns with Fig. 2h)	
Dimensions from gridlines	Student uses gridlines to determine the length of the dimensions first emerged: L-shape	
Dimensions from ruler	Student uses a ruler to determine the length of the dimensions First emerged: E-shape	
Infer dimensions	Student infers dimensions from the given dimensions First emerged: step-shape	

*The students in this study followed the convention of adding the units of measurement to the final answer typically used in Australian schools, rather than including linear units at each individual step

Table 6 Strategy repertoire for recomposition

Strategy	Description	Example
Physical recomposition	Student recomposes decomposed shape into a single rectangle First emerged: L-shape (aligns with Fig. 2k)	
Area model: addition	Student labels the decomposed shapes (with a number or letter), referred to as *subgoals*, and adds areas of by first constructing an *area model* First emerged: L-shape (aligns with Fig. 2i)	 Frankie's response
Area model: subtraction	Student labels the decomposed shapes and subtracts areas by first constructing an area model (aligns with Fig. 2j)	
Shape-by-shape	Student labels the decomposed shapes, calculates the area of the constituent shapes, and then adds/subtracts the areas First emerged: E-shape (aligns with Fig. 2i)	

6.2 Strategy distribution

Table 7 contains the distribution of decomposition, area measurement, and recomposition strategies respectively. Note that the totals exceed 100% for the shapes in which students used more than one strategy. Furthermore, students' use of the gesture strategy to show decomposition was only documented during whole-class discussions, and hence is not included in the distribution.

Table 7 Distribution of strategies for each composite shape

	Shape	Composite rectangles						Composite shapes					
		L	T	U[a]	H	E[a]	Step[a]	House	Arrow	Irregular hexagon[a]	Boot[a]	Octagon	Floor plan
Decomposition Strategies	Physical decomposition	31 (100%)	31 (100%)	31 (100%)	16 (52%)	6 (19%)	–	30 (97%)	30 (97%)	31 (100%)	31 (100%)	–	–
	Auxiliary element: constituent shapes	–	–	–	10 (32%)	21 (68%)	31 (100%)	1 (3%)	1 (3%)	–	–	31 (100%)	31 (100%)
	Auxiliary element: inclusion	–	–	7 (23%)	5 (16%)	4 (13%)	–	–	–	–	–	–	1 (3%)
	Auxiliary element: analysis	–	–	–	–	–	–	–	–	18 (58%)	13 (42%)	–	–
	Rotation	–	–	–	–	8 (26%)	2 (6%)	–	3 (10%)	13 (42%)	–	–	5 (16%)
	Labels	–	–	–	9 (29%)	31 (100%)	31 (100%)	1 (3%)	1 (3%)	–	–	31 (100%)	31 (100%)
	Brute-force	–	–	–	–	4 (13%)	2 (6%)	–	5 (16%)	–	–	4 (13%)	5 (16%)
	Consider alternatives	–	3 (10%)	31 (100%)	11 (35%)	8 (26%)	12 (39%)	7 (23%)	31 (100%)	31 (100%)	6 (19%)	2 (6%)	–
Measurement strategies	array	12 (39%)	4 (13%)	–	–	–	–	–	4 (13%)	–	–	–	–
	Area formulae	19 (61%)	27 (87%)	31 (100%)	31 (100%)	31 (100%)	31 (100%)	31 (100%)	27 (87%)	31 (100%)	31 (100%)	31 (100%)	31 (100%)
	Dimensions from gridlines and/or ruler	31 (100%)	31 (100%)	31 (100%)	31 (100%)	31 (100%)	–	31 (100%)	27 (87%)	31 (100%)	31 (100%)	–	–
	Infer dimensions	–	–	–	–	–	31 (100%)	–	–	–	–	31 (100%)	31 (100%)
Recomposition strategies	Physical recomposition	12 (39%)	4 (13%)	–	–	–	–	–	11 (35%)	–	–	–	–
	Area model: addition	19 (61%)	27 (87%)	31 (100%)	26 (84%)	17 (55%)	15 (48%)	2 (6%)	1 (3%)	12 (39%)	7 (23%)	4 (13%)	5 (16%)
	Area model: subtraction	–	–	7 (100%)	5 (16%)	4 (13%)		–	–	4 (13%)	–	–	1 (3%)
	Shape-by-shape	–	–			10 (32%)	16 (52%)	29 (94%)	23 (74%)	15 (48%)	24 (77%)	27 (87%)	26 (84%)

[a]The totals exceed 100% because students used multiple methods

This distribution reflects the emergence of strategies in response to the complexity of the shapes and their mode of presentation, mostly in line with the envisioned learning trajectory. First, shapes presented as paper cutouts are associated mainly with physical decomposition and determining dimensions from gridlines/ruler, whereas shapes represented as figures were associated with auxiliary elements and the use of given or inferred dimensions. However, the labels strategy was also used by all students in response to the E-shape and the three shapes presented as figural representations. Second, addition was used as the primary recomposition strategy, as anticipated, but was used in two different ways. The area model: addition strategy was more prevalent for the simpler composite shapes, whereas the shape-by-shape strategy emerged in response to the step-shape and became the dominant recomposition strategy used for the remainder of the shapes. Finally, the use of area formulae for calculating the area of the decomposed shapes was the dominant measurement strategy used across the entire sequence of shapes, although it was anticipated that students would use arrays or unit counting for the shapes superimposed onto the gridlines. Nevertheless, students frequently used unit counting to validate their area calculations, particularly for the paper cutouts without gridlines.

6.3 Strategy effectiveness

Table 8 shows the types of errors students made and the frequency of errors for each type. In total, there were 16 errors and hence 95.7% of all responses across the two classes were accurate. Seven of the errors (43.75%) were incomplete recomposition; however, it

Table 8 Frequency and types of errors

Error	Description	Example	Frequency
Calculation error	Student formulates an appropriate expression but makes an error in the calculation Example: E-shape	$Area_1 + Area_2 + Area_3 + Area_4$ $A = (11\times2.5) + (2\times3.5) + (2\times3.5) + (3.5\times3.5)$ $= 27.5 + 7 + 7 + 10.5$ $= 52\ cm^2$ $3.5 \times 3.5 = 12.25$	E-shape: 2 (6%) Step-shape: 2 (6%)
Incomplete recomposition	Student decomposes the shape appropriately and accurately calculates areas of constituent shapes but does not recompose the calculated areas Example: E-shape	Rec 1 Area = 11 × 2.5 = 27.5 Rec 3 Area = 3.5 × 3.5 = 12.25 Rec 2 Area = 2 × 3.5 = 7.0 Rec 4 Area = 2 × 3.5 = 7.0 $Total\ area = 53.75\ cm^2$	E-shape: 1 (3%) Step-shape: 2 (6%) Arrow-shape: 3 (10%) Octogon: 1 (3%)
Incorrect inferred dimension	Student uses given diameter to calculate area of semi-circle Example: floor plan	$A^4 = \pi r^2$ $= \pi 8.5^2$ $= 72.25\pi$ $= 226.98$ $A^4 = \frac{226.98}{2}$ $= 113.49\ m^2$ $r = 4.25$	Floor plan: 3 (10%)
Incorrect measured dimension	Student measures dimensions incorrectly Example: E-shape	$Area_1 + Area_2 + Area_3 + Area_4 + Area_5$ $(2\times6) + (1.5\times2) + (2\times6) + (1.5\times2) + (3.5\times6)$ $= 12 + 3 + 12 + 3 + 21$ $= 51\ cm^2$ $Area_2 = 1.5 \times 2.5$ $Area_4 = 2 \times 2.5$	E-shape: 1 (3%) Arrow-shape: 1 (3%)

appeared that students had planned how they would recompose the calculated areas but did not follow through on their plan.

Although there were no errors in identifying the constituent shapes, recognizing parallelograms and/or trapeziums embedded within a composite shape was a difficulty that students encountered persistently. Spiegel and Ginat (2017) document how students overcome this difficulty by using a brute-force approach to decomposition, that is, decomposing the shape into several "'atomic' fragments" (p. 210). Remington's response to the arrow shape, used as an example in Table 4, illustrates this brute-force decomposition in response to the arrow shape. Many students overcame this difficulty by using the auxiliary element: analysis strategy in response to the irregular-hexagon and boot shapes. However, Table 7 shows the frequency of students who used the brute-force in their final response.

6.4 Strategy selection

Strategy flexibility emerged as a mathematical practice in each of the classes, as the classes used multiple strategies for each shape and individual students used multiple strategies across the sequence of shapes (see Table 7). There were three prominent features of this emergent flexibility. First, students used between 3 and 5 decomposition strategies, 3–4 measurement strategies, and 3–4 recomposition strategies. Second, students used at least four different combinations of strategies across the sequence of shapes. Finally, each student chose from alternative decompositions for at least three composite shapes, although some students only used the consider alternatives strategy when prompted.

Table 9 Accepted justifications for strategy selection

Grounds	Description	Warrants
Specifies constituent shapes	The decomposition strategy shows how the shape was decomposed	• Able to visualize constituent shapes
Known method	A decomposition alternative was chosen because it produced shapes for which a known measurement strategy could be used	• Use of measurement strategy substantiates the known method
Adding areas yields total area	An explanation of a recomposition method in relation to the composite shape	• Subsequent calculation yields total area
Verification	The strategy verified a solution	• Area is measured in square units
Easier	Strategy or decomposition alternative selected because it is more efficient	• Fewer constituent shapes • Congruent constituent shapes
Overcome an impasse	Strategy selected because it helped overcome an impasse, particularly in relation to apprehending constituent shapes or area formula	• Identifies constituent shapes • Calculates area accurately
Accuracy	Strategy selected because it ensured correctness of some aspect of the solution	• Determines accurate dimensions from gridlines or ruler • Array or formula is a known method that will yield an accurate answer

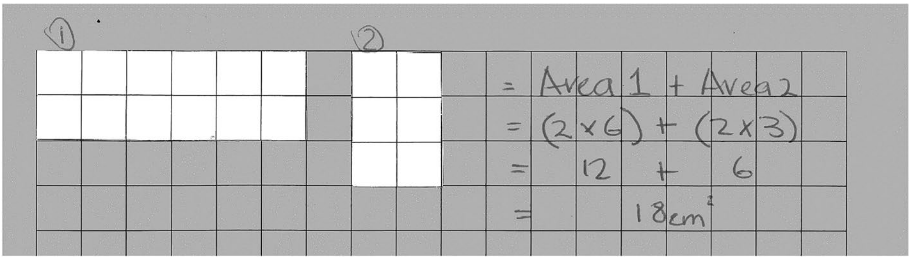


Fig. 3 Hadley's response to T-shape

Strategy adaptivity also emerged as a mathematical practice as the classes negotiated which justifications for strategy choice were appropriate. The students' use of multiple strategies and combinations of strategies was underpinned by seven taken-as-shared justifications for their choice of an appropriate strategy. Table 9 contains a description of these taken-as-shared justifications in terms of the grounds for strategy selection and the warrants used to explain how the strategy was appropriate for achieving a solution. Just as the strategies were used in varying

Table 10 Excerpt of Hadley's solution to T-shape

			Argument
1	Hadley:	Okay so, see these little, like, shape things [points to Fig. 3],	Grounds #1: specifies constituent shapes
2		these are the ones we need	Claim #2: choose decomposition alternative
3		So, we're looking for subgoals,	Claim #3: area model: addition
4		which are area 1 plus area 2	Grounds #3: adding areas yields total area
5		So, this was the first shape we used [points to in Fig. 3]	Grounds #1: specifies constituent shapes
6		And we cut it up	Claim #1: physical decomposition
7		so we could see more practically the two different shapes	Warrant #1: able to visualize constituent shapes
8		So, that's one, that's two	Grounds #1: specifies constituent shapes
9		And then you just, like, write area one, area two	Grounds #3: adding areas yields total area
10		Cause you know how to get area	Grounds #2: known method
11		So, when we look at the area,	
12		two times six plus two times three	Warrant #2: uses area formula
13		And then you just work it out [points to Fig. 3]. It's in c m squared	Warrant #3: calculation yields total area
14	Dawson:	You could count the little square too	Claim #4: unit counting
15	Teacher/ Researcher	Is that how you calculated area?	
16	Dawson:	No, I did it like Frankie	
17		but then I checked it by counting	Grounds #4: verification
18		because its area measurement	Warrant #4: area is measured in square units

combinations to obtain a solution, the justifications for strategy selection were used in combinations. In the remainder of this section, I present four examples to illustrate how students used these justifications.

Several of the taken-as-shared justifications for strategy selection emerged during the first lesson about composite rectangles and were sustained throughout the remainder of the instructional sequence. As students presented their solutions to the class, their justifications for their combination of strategy choices were interweaved throughout their explanation, as exemplified by the excerpt from Hadley's presentation of her solution for the T-shape (see Fig. 3) in Table 10.

This excerpt typifies how students justified their selection of a decomposition strategy (argument #1), choice of decomposition alternative (argument #2), and recomposition strategy (argument #3). The justification for the use of area formulae to calculate was frequently tacit; although this excerpt illustrates how students used unit counting (argument #4) as a strategy for verifying their calculation. Dawson's interjection (line 14) also illustrates the established sociomathematical norm of offering a differing strategy.

The selection of a strategy on the grounds that it was "easier" appeared to be a tacit argument for the efficiency of a strategy or decomposition alternative. The excerpt in Table 11 from an extemporaneous discussion between three students in relation to the arrow-shape illustrates this justification.

This excerpt demonstrates how students used the taken-as-shared justifications for strategy selection in their extemporaneous discussions. Spencer argued that decomposing the arrow shape into two congruent parallelograms was easier than Sterling's suggested brute-force strategy, which she ultimately abandoned after Spencer and Langdon reminded her of the area formula (lines 9 and 12). The excerpt also demonstrates how students considered alternative measurement strategies, that is Spencer considered using the decomposition/recomposition strategy (line 2) and area formula strategy (line 3), but again, the justification for the selection of area formula was not always explicit.

The auxiliary element strategy first emerged as an alternative for physically cutting up the shapes and was then used by all students to specify the constituent shapes for the

Table 11 Excerpt of extemporaneous discussion about arrow-shape

			Argument
1	Spencer:	You've got two same parallelograms	Claim #1: choose decomposition alternative
2		And then we'll make them squares [rectangles]	
3		Or we can just do base time height	
4	Sterling:	Or you can just do this:	Claim #2: brute-force strategy
5		triangles, that, and two triangles	Warrant #2: identifies constituent shapes
6	Spencer:	I know but that's three, four shapes. We only need two	Warrant #1: fewer constituent shapes
7		Sterling, just make your life easier	Grounds #1: easier
8	Sterling:	What do we…?	Grounds #2: overcome an impasse
9	Spencer:	Area equals perpendicular height by base	
10	Sterling:	We don't know how to do this	Grounds #2: overcome an impasse
11	Spencer:	Yeah, we do	
12	Langdon:	You do the perpendicular height times the base	
13	Sterling:	Oh yes!	

Table 12 Excerpt of Parker's solution to boot-shape

			Argument
1	Parker:	If you cut it along here [refers to boot-shape]	Claim #1: choose physical decomposition
2		you get shape 1 parallelogram and 2, trapezium [see Fig. 4]	Grounds #1: specifies constituent shapes
3	Teacher/ Researcher:	Tell us how you found those shapes	
4	Parker:	It was hard to see these ones	Grounds #2: overcome an impasse
5		So, I drew a rectangle around it [see Fig. 4]	Claim 2#: choose auxiliary element: analysis
6		and thought this one was a triangle	
7	Students:	[laugh]	
8	Parker:	I know right?!	
9		But then I thought if the line went all the way across [gestures on Fig. 4]	
10		it would be a trapezium, and then parallelogram	Warrant #2: identifies constituent shapes

composite shapes drawn as diagrams with dimensions. However, the students also began using the auxiliary element to support their attempts to find alternative methods of decomposition. The excerpt in Table 12 from Parker's presentation of her solution to the boot-shape (see Fig. 4) illustrates the justification used for the selection of this strategy.

The first justification for strategy selection (argument #1) occurred in lines 1–2, when Parker specified the constituent shapes (line 2) as grounds for her use of the physical decomposition strategy (line 1), although the warrant for these grounds was taken-as-shared by this stage of the instructional sequence. Before identifying the parallelogram and trapezium, Parker explained that she had used the auxiliary element: analysis strategy (argument #2) on the grounds that she had reached an impasse (line 4). Parker explained how drawing around the shape (line 5) and visualizing a line through the shape (line 9) helped her identify the constituent shapes (line 10).

Finally, accuracy emerged as grounds for area measurement and recomposition strategy selection. The excerpt in Table 13 illustrates how Vivian used this justification for his solution in Fig. 5.

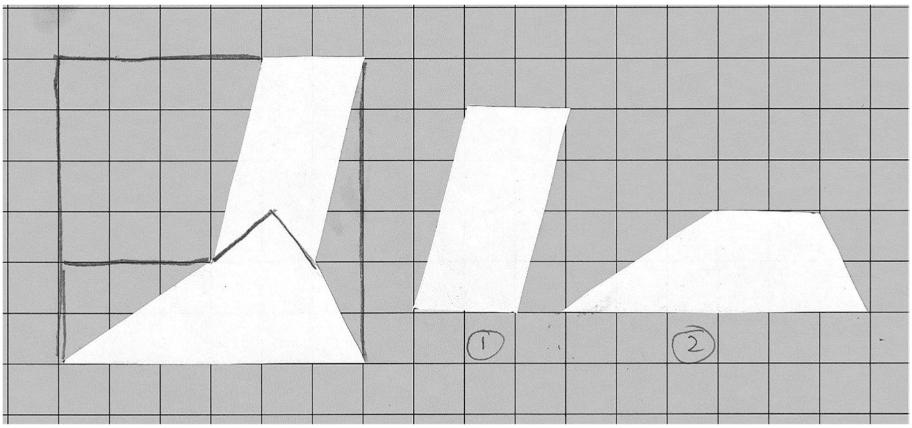


Fig. 4 Parker's Response to Boot-shape

Table 13 Excerpt of Vivian's solution to E-shape

			Argument
1	Vivian:	I thought about making this five shapes [gestures on Fig. 5],	
2		or four shapes [gestures on Fig. 5],	
3		But I ended up going with one shape minus these two	Claim #1: area model: subtraction
4	Students:	Whoa!!	
5	Teacher/ Researcher:	Why did you choose that one Vivian?	
6	Vivian:	I think this way is easier	Grounds #1: easier
7		because you've got only three rectangles	Warrant #1: fewer constituent shapes
8		But you have to be careful	
9		That's why I stuck it down on the grid like this [gestures on Fig. 5]	Claim #2: choose determine dimensions from gridlines
10		so that I could get the right measure-ments	Grounds #2: accuracy
11		See this one [gestures to "A" on Fig. 5]	
12		its 1.5 width and 3.5 length,	Warrant #2: accurate dimensions from gridlines
13		which you can see from the gridlines	

In this excerpt, Vivian chose the determine dimensions from gridlines strategy (argument #2) on the grounds that it would yield accurate dimensions (line 10), which he substantiated in lines 11–13. Students also used similar justifications for the use of a ruler, as well as the shape-by-shape recomposition strategy for more complex shapes, such as the E-shape. The excerpt is also another example of how students chose a recomposition strategy on the grounds that it was easier (argument #1), as well as choosing between alternative decompositions (lines 1–2).

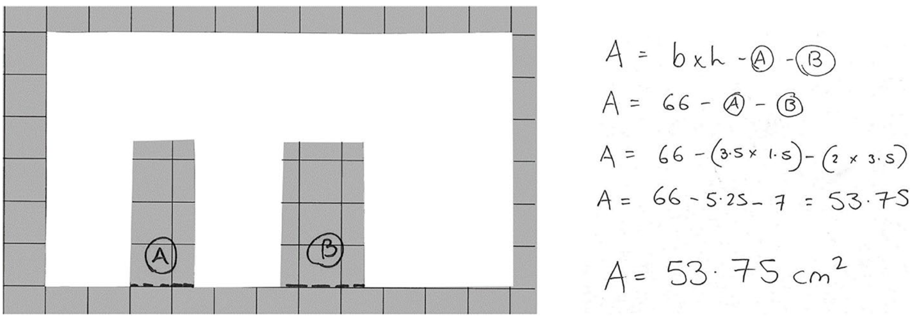


Fig. 5 Vivian's response to E-shape

7 Discussion

Strategy choice emerged as a set of collective mathematical practices in each of the two classes. In this section, I discuss each of the elements to strategy choice that emerged.

The strategy repertoires developed by the classes in this study align with most of the strategies documented in the existing studies synthesized in Fig. 1. This finding consolidates previous research; however, the analysis of students' strategy choice in terms of decomposition, area measurement, and recomposition used in this study extends the literature in three substantial directions. First, existing studies document either single instances in which students used some of these strategies, whereas the students in this study used several different strategies across a sequence of shapes (Patahuddin et al., 2018; Spiegel & Ginat, 2017; Zacharos, 2006). This use of multiple strategies, however, emerged in response to a set of composite shapes with features different from those presented in previous studies. This finding suggests that, in practice, students' proficiency in the use of multiple strategies for measuring the area of composite shapes might be developed through opportunities to measure the area of composite shapes presented as paper cutouts, with and without gridlines, before attempting figural representations. This finding is consistent with Spiegel and Ginat's (2017) proposition that students' competence in using the decomposition and recomposition heuristic develops through such an instructional plan. Second, existing studies that document students' use of similar strategies to those documented in this study do not account for their origin or development. In contrast, the classes in this study collectively developed their own strategy repertoires in response to measurement tasks presented as problems that required the students to devise or use existing strategies. Third, existing studies either focus broadly on students' use of the decomposition and recomposition heuristic (Spiegel & Ginat, 2017; Zacharos, 2006) or focus narrowly on strategies for decomposition (Patahuddin et al., 2018). In contrast, the focus of the present study was on the three sets of strategies for decomposition, area measurement, and recomposition. The findings from this perspective highlight that measuring the area of composite shapes involves choosing strategies from each of the three repertoires and coordinating their use in varying combinations in response to the features of the composite shapes. This suggests that in practice, teachers might consider the students' proficiency in measuring the area of composite shapes in terms of decomposition, area measurement, and recomposition when planning instructional tasks.

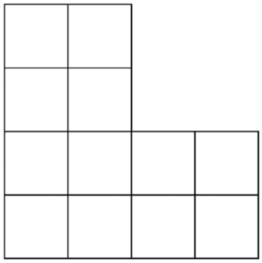
The students' use of the strategy repertoires was highly effective, and very few of the errors synthesized in Fig. 1 emerged in this study. Previous studies have documented errors in response to tests or task-based interviews whereas the findings of this study emerged in a classroom setting in which students were able to discuss their solutions, which may account for the students' overall accuracy. Having said that, previous studies are unclear about the instructional experiences of their participant students (Patahuddin et al., 2018; Spiegel & Ginat, 2017). The students in this study had extensive experience in quantifying area in square units (cf. Zacharos, 2006) but less experience in 2D geometric reasoning. This may account for the difficulties they encountered in relation to identifying trapezoidal regions as opposed to the measurement errors documented in previous studies. This suggests that competence in the use of the decomposition and recomposition strategies is underpinned by both measurement and geometric reasoning, although more research into this connection is needed.

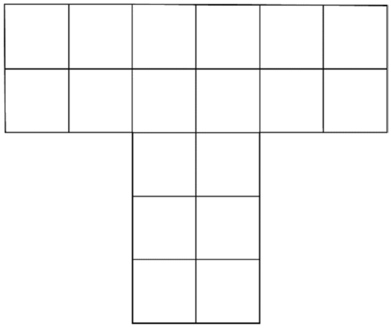
The strategy repertoires, distributions, and effectiveness that emerged from the instructional intervention extend the existing literature in some new directions, but the findings about strategy selection perhaps make the most significant contribution to our understanding about how students develop competence in measuring the area of composite shapes. The classes in this study made substantial progress toward strategy adaptivity by choosing appropriate strategies to decompose, measure, and recompose the composite shapes. A strategy was considered appropriate by the classes through their acceptance of justifications for the use of a strategy in response to the features of the shapes, consistent with the theory of strategy choice (Verschaffel et al., 2009). Although researchers stress the importance of strategy flexibility and adaptivity in developing students' competence in the decomposition and recomposition heuristic, there appears to be very little evidence of how this emerges (Moreira-Baltar, 1999; Spiegel & Ginat, 2017). For example, Patahuddin et al. (2018) provide an example of one student generating two alternatives for decomposing and recomposing a composite shape but they do not address which alternative the student would select. The students in this study also generated decomposition and recomposition alternatives, and considered alterative measurement strategies, but made choices between these alternatives based on the taken-as-share justifications. These justifications were not necessarily used in the same way by each student, and therefore represent only one possible explanation for the strategy choices of individual students. Nonetheless, strategy selection emerged as a mathematical practice out of opportunities to justify choices in the whole-discussions and extemporaneous conversations, and make sense of the reasons that other students used for their strategy selections (Ellis, 1997; Verschaffel et al., 2009).

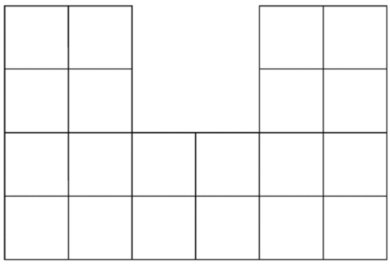
Aside from this immediate instructional implication for composite shapes, the focus on strategy choice as a classroom mathematical practice has possible wider implications. Flexible and adaptive strategy use is a hallmark of strategic competence in mathematics (Heinze et al., 2009a, b). To support the development of this aspect of students' strategic competence in classroom settings, teachers might foster strategy flexibility and adaptivity as classroom mathematical practices, negotiated through the acceptance of taken-as-shared justifications for strategy selection, across domains of mathematics.

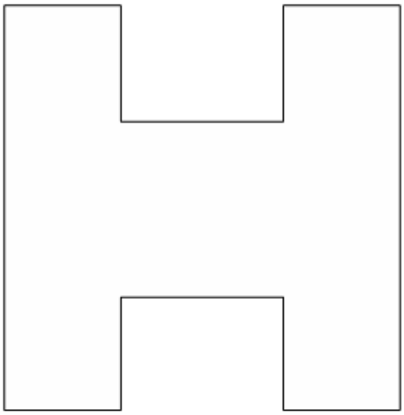
There are several limitations of this study that should be considered in interpreting the findings. First, this study was limited to an examination of simple concatenated shapes comprised of no more than four simple shapes. Further research might examine students' strategic competence in relation to more complex shapes, especially those involving interleaving areas (Spiegel & Ginat, 2017). Second, the students' proficiency in measuring the area of composite shapes was limited to an analysis of their strategy choices, rather than on their geometric reasoning (cf. Patahuddin et al., 2018) or measurement reasoning (cf. Zacharos, 2006), and further research might focus on these forms of reasoning about composite shapes. Third, the analysis and coding of the data was completed by me, which potentially diminishes the reliability of the findings. Two experienced mathematics education researchers evaluated my analysis at each phase and, between the two of them, checked the analysis of all the data. Nevertheless, using peer review to ensure reliability is a limitation of the methodology. Finally, the generalizability of the findings is limited because the study was conducted in two classes of Year 8 students at the same school. However, the instructional intervention was developed in response to the findings of two teaching experiments conducted in classes at schools with vastly different characteristics. Having said that, the purpose of classroom design studies is not to demonstrate the effectiveness of an instructional design, but rather to identify the features of the classroom environment that support the emergence of students' reasoning (Cobb et al., 2016). Hence, the focus on strategy choice embedded in the instructional design, especially the students' explanation of their strategy choices, is an enduring finding with substantial practical utility.

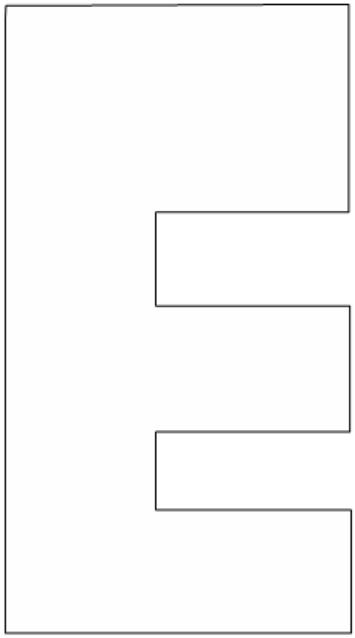
Appendix

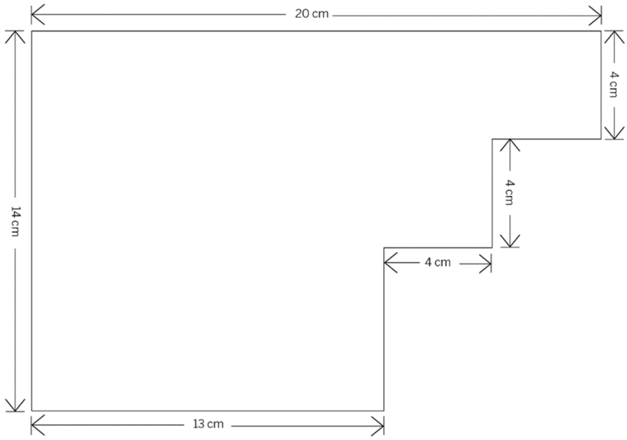








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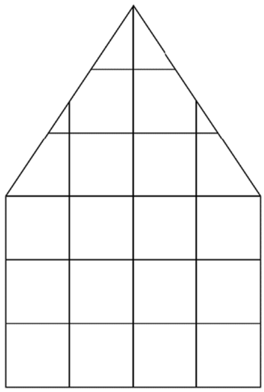

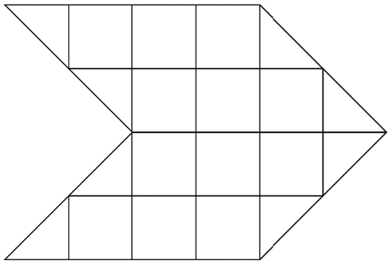

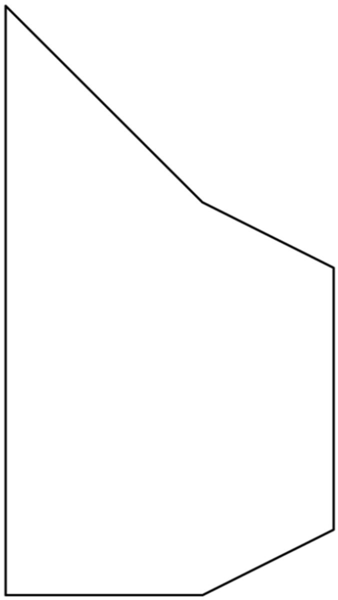

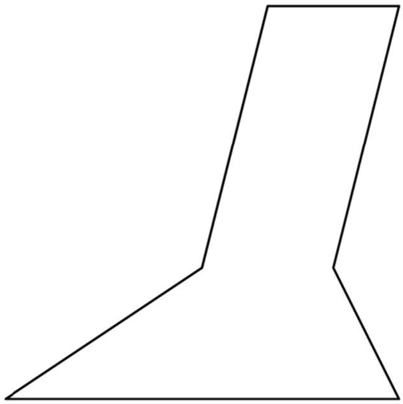

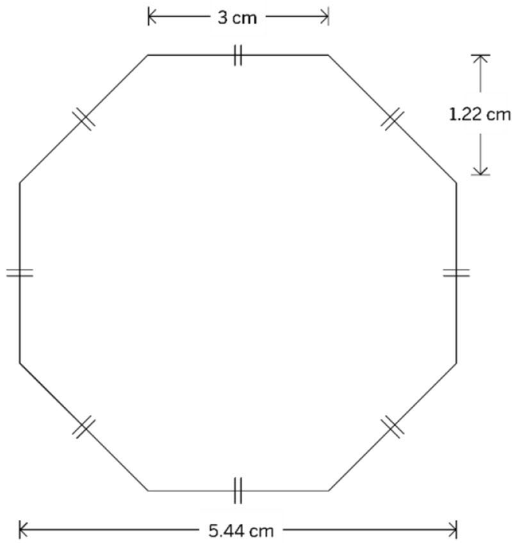
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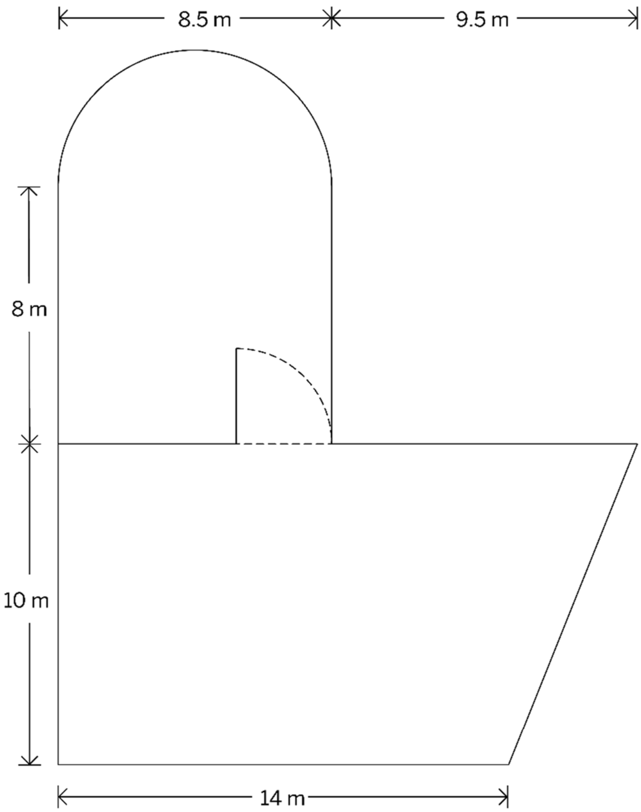

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Data availability The data used in this study are not available to the public due to the constraints imposed by the Ethics Approval process at Queensland University of Technology.

Code availability Not applicable.

Declarations

Ethics approval The data used in this study was collected after the Queensland University of Technology Human Research Ethics Committee approved the project. I obtained informed consent from the student participants and their parents.

Conflict of interest The author declares no conflict of interest.

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