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It is probably a pattern: does spontaneous focusing on regularities in preschool predict reasoning about randomness four years later?

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Abstract

The many studies with coin-tossing tasks in literature show that the concept of randomness is challenging for adults as well as children. Systematic errors observed in coin-tossing tasks are often related to the representativeness heuristic, which refers to a mental shortcut that is used to judge randomness by evaluating how well a set of random events represents the typical example for random events we hold in our mind. Representative thinking is explained by our tendency to seek for patterns in our surroundings. In the present study, predictions of coin-tosses of 302 third-graders were explored. Findings suggest that in third grade of elementary school, children make correct as well as different types of erroneous predictions and individual differences exist. Moreover, erroneous predictions that were in line with representative thinking were positively associated with an early spontaneous focus on regularities, which was assessed when they were in second year of preschool. We concluded that previous studies might have underestimated children's reasoning about randomness in coin-tossing contexts and that representative thinking is indeed associated with pattern-based thinking tendencies.

Keywords Mathematical development · Randomness · Representativeness heuristic · Patterns

> What we call chaos is just patterns we haven't recognized. What we call random is just patterns we can't decipher. What we can't understand we call nonsense.
>
> —— Chuck Palahniuk (2011, p. 118)

Kahneman (2011, p. 115) depicts humans as natural pattern seekers who might see patterns or regularities even when, in reality, none are there. Recent research has shown that in some children, a spontaneous focus on regularities such as patterns is already present from a very young age (Wijns et al., 2020). Such a tendency to spontaneously look for or

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create regularities is considered to be an important component of children's mathematical abilities (Clements & Sarama, 2014; Mulligan et al., 2018). However, Kahneman (2011) also suggests that this human tendency might hinder our understanding of randomness. Research exploring the many fallacies related to randomness shows that the comprehension of randomness is troublesome for adults but also for children (e.g., Chiesi & Primi, 2008; Kahneman & Tversky, 1972; Oskarsson et al., 2009). To this day, no attempts have been made to investigate the role of a spontaneous focus on regularities in children's understanding of randomness. However, it has been assumed that judgments about randomness stem from the apparent patterns observed in random sequences (Oskarsson et al., 2009). In the present study, we aimed to explore elementary school children's understanding of randomness and investigate how a spontaneous focus on regularities in preschoolers predicts this understanding of randomness in elementary school 4 years later. First, the representativeness heuristic is explained, as well as some of the related fallacies that are shown to affect our predictions of random events. Second, findings related to the representativeness heuristic in children are described. Third, a spontaneous focus on regularities is explained as well as how it might relate to specific fallacies in children.

1 Fallacies regarding randomness

The difficulties and fallacies regarding the interpretation of randomness and chance have been repeatedly confirmed in the literature (e.g., Bar-Hillel & Wagenaar, 1991; Hirsch and O'Donnell, 2001; Shaughnessy, 1981; Tversky & Kahneman, 1974). Fallacies are systematic errors in reasoning that are often caused by our habit to use heuristics or mental shortcuts to handle complex situations (Tversky & Kahneman, 1974). One such heuristic is called the representativeness heuristic (Kahneman & Tversky, 1972). This heuristic leads us to judge the likelihood of a specific event by comparing it to a typical, thus *representative*, example we have in our minds for this type or category of events. The role of the representativeness heuristic for our interpretation of randomness has been verified in the many coin-tossing experiments reported in literature (e.g., Bar-Hillel & Wagenaar, 1991; Kahneman & Tversky, 1972).

In such experiments, participants have to, for example, judge which head-and-tail sequence is more probable, e.g., HHHHHH vs. HHTHTT (Falk & Konold, 1997). Evidently, the normative answer would be that both sequences are equally likely. However, "representative thinkers" who think in line with the representative heuristic will erroneously predict HHTHTT as being more probable to happen when tossing a coin than HHHHHH. Representative thinkers judge the randomness of a sequence on the basis of two features that are typical for long random sequences, but do not apply to short random sequences (Kahneman & Tversky, 1972). The first feature concerns the *fairness* of the random generator, in this case the coin. Many of us know that a coin has two sides, namely heads, which pictures a person's head, and tails, which is the opposite side and typically pictures a number. Tossing a fair coin a large number of times, heads and tails have an equal chance to appear, so the proportion of heads and tails in the long run will approach the expected value based on the probability of 0.50. This is also known as the Law of Large Numbers (Bernoulli., 1713). In smaller samples, the proportion of heads or tails is more likely to approach extreme values (e.g., HHHHHH) than in larger samples, but people often believe that small samples are representative of large samples (Tversky & Kahneman, 1971). Representative thinkers will also judge HHTHTT as more representative, and thus more likely, for a fair coin or randomness than HHHHHH because the number or the proportion of Hs and Ts in the first sequence is closer to the expected values. The second feature that a representative thinker will habitually take into account is related to the equiprobability of the outcomes in random sequences and concerns the *unpredictability* of the (order of) outcomes (Kahneman & Tversky, 1972). According to Kahneman and Tversky (1972), sequences that are believed to be representative for randomness show irregularity, chaos, and lack of pattern. Representative thinkers will judge sequences that appear too regular (e.g., HHHHHH) as less probable than sequences that look less regular (e.g., HHTHTT; Begg and Edwards, 1999; Falk & Konold, 1997).

Another typical coin-tossing task requires participants to predict the outcome of a toss following a given sequence, for example if a sequence HHHH is presented, participants have to predict what follows: H or T (Matthews, 2013). Representative thinkers are more likely to predict T in this case, because that balances the number of Hs and Ts better (in line with the fairness feature) and breaks the regular appearance of the sequence (in line with the unpredictability feature). This phenomenon is also known as the gambler's fallacy (Mlodinow, 2008, p. 110). However, despite being less common, some people tend to do the exact opposite, predicting H to be tossed, expecting the streak of Hs to continue (Oskarsson et al., 2009). This phenomenon of predicting an outcome that seems *unrepresentative* for randomness is known as the hot-hand fallacy (Gilovich et al., 1985). In adults, the hot-hand fallacy is more commonly observed in experiments concerning sport performance sequences instead of random sequences, while the gambler's fallacy is more present in experiments with random sequences. However, reviewing several studies concerning the judgment of random sequences, Oskarsson et al. (2009) estimated that it was repeatedly found that about 5–10% of participants still tend to predict a run to continue. The representativeness heuristic has been used to explain not only the gambler's fallacy, but also the hot-hand fallacy: a run of identical outcomes is judged as being unrepresentative of randomness and thus it is believed that the outcomes are not independent or caused by a true random generator (Matthews, 2013). For people believing that successive events are positively correlated, it makes sense to predict the outcome that continues the pattern or streak and thus predict the outcome that is, in their minds, unrepresentative of randomness.

According to Kahneman and Tversky (1972), representative thinkers will look for fairness and unpredictability not only in specific sequences of random outcomes, but also in frequency distributions (e.g., 2 H and 4 T in 6 tosses). However, in the case of frequency distributions, the "fairness" and "unpredictability" features might easily conflict. For example, while a frequency distribution of 4 Hs and 4 Ts in 8 tosses meets the "fairness" feature, a representative thinker might judge this distribution as too ordered. Then again, a frequency distribution of 2 Hs and 6 Ts meets the "unpredictability" feature, but violates the "fairness" feature. Therefore, representative thinkers prefer frequency distributions with a slight, but not too big, deviation from a perfect distribution (e.g., 3 Hs and 5 Ts), judging these as more representative for randomness than a perfect distribution (Kahneman & Tversky, 1972). Note that in the case of given frequency distributions, "equally likely" is not the normative answer if the distribution of Hs and Ts differs. In the case of 4 Hs and 4 Ts on the one hand, and 2 Hs and 6 Ts on the other hand, the only normative answer is 4 Hs and 4 Ts as the most likely distribution.

The fallacies and biases reported in literature have behavioral consequences that go beyond judgements in artificial coin-tossing experiments. Teovanović et al. (2021) for example, suggested that fallacies related to the interpretation of randomness predicted how people deal with misinformation. They found that, for example, the gambler's fallacy predicted the tendency to use pseudoscientific practices related to COVID-19, such as consuming honey to prevent infection. Luckily, research shows that not all people are evenly susceptible to fall for such fallacies. Gender, cognitive ability, previous experiences regarding randomness, formal education, context, and also age predicted individual differences in fallacies related to randomness and probability (e.g., Barron & Leider, 2010; Davidson, 1995; Farmer et al., 2017; Matthews, 2013; Morsanyi et al., 2009; Chiesi et al., 2011; Suetens & Tyran, 2012). For example, intellectual abilities are believed to protect people against judgment biases (Kokis et al., 2002) when the context or problem is not too complex (e.g., Kahneman & Frederick, 2002; Stanovich, 1999). However, findings of Primi and Chiesi (2011) suggest that in third-graders, intellectual ability accounts for less than 5% in children's normative reasoning about randomness. Nonetheless, biased thinking related to randomness is already observed in young children. Regarding the representativeness heuristic, it was repeatedly confirmed that it is already at play from a young age, but studies seem inconclusive about the degree and the specific way it manifests itself in young children (e.g., Agnoli, 1991; Davidson, 1995; De Neys & Vanderputte, 2011; Gualtieri and Denison, 2018; Williams & Amir, 1995).

2 Understanding of randomness in young children

Just like adults, children experience randomness and random sequences in daily life (e.g., throwing dice in games). Representative thinking about randomness is believed to be caused by such experiences because they shape the prototype we have for random sequences in our minds (Chiesi & Primi, 2009; Hahn & Warren, 2009; Matthews, 2013; Morsanyi et al., 2009) . Being younger and having less (in)formal experiences with randomness, one might argue that fallacies related to representative thinking are less of a concern in children compared to adults (Bryant & Nunes, 2012). Remarkably, despite having fewer encounters with randomness than adults, representativeness has also been observed in young children (e.g., Bryant & Nunes, 2012; Chiesi & Primi, 2009; Smith, 1998; Williams & Amir, 1995). However, studies with children are more scarce compared to studies with adults (Smith, 1998). Moreover, various studies report different findings; thus, the specific role of age for representative thinking remains unclear (Chiesi & Primi, 2009).

Fischbein and Schnarch (1997), for example, suggested that the tendency to give representativeness-based answers (i.e., answers that represent the "unpredictability" and "fairness" of random phenomena) decreases with age, whereas others found no evolution with age (Afantiti-Lamprianou & Williams, 2003; Chiesi & Primi, 2008; Smith, 1998). Smith (1998) found that about 50% of elementary school children as well as adults preferred the representative sequence. Smith also found that age predicted the number of normative answers, with college students judging given sequences as being "equally likely to happen" more often than elementary school children, of whom only 20% chose the normative answer. This is in line with findings of Fischbein and Schnarch who found that older children estimated given sequences of independent events to be equally likely more often than younger children. However, in Smith, a smaller but still notable number of children also judged "unrepresentative sequences," that is, sequences with a streak or an ordered appearance, as the most probable sequence, which was not a common mistake among college students. Those "unrepresentative answers" were almost absent in all age groups in Fischbein and Schnarch.

In contrast, other studies suggested that these unrepresentative answers were the predominant erroneous choices among young children and that this tendency decreased with age (Bryant et al., 2016; Chiesi & Primi, 2008; Ridgway & Ridgway, 2010). For example, when predicting the next outcome of a toss, most young children predicted the run or the pattern that is present in a given sequence to continue (Chiesi & Primi, 2008; Ridgway & Ridgway, 2010). Several explanations can be used to clarify findings of children continuing runs or patterns in random sequences. First, these children might fail to understand random events, treating a random event as something fixed and ignoring the independence between events (Chiesi & Primi, 2008). Second, some argue that these "unrepresentative predictions" might also be related to the representativeness heuristic: Perhaps children do not consider the given sequence as representative for randomness, concluding that the coin is weighted or unfair (Matthews, 2013). For example, if children receive a run of 5 Hs and therefore believe that the coin is unfair, it makes sense to predict H to be the following outcome. Third, Ridgway and Ridgway (2010) pointed out that there is a strong emphasis on patterning in the school curriculum for young children. This might cause children to overgeneralize knowledge about patterns to other situations. This raises the question of whether this overapplication of regularities to non-regular or random situations might also be driven by a *spontaneous* focus on regularities observed in some children (Wijns et al., 2020).

3 Spontaneous focus on regularities

Within the field of mathematics, regularities are often referred to as patterns or structures. Patterns and structures are believed to play an important role in mathematics (Clements & Sarama, 2014; Mulligan & Mitchelmore, 2009), and some even describe mathematics as the science of patterns (Steen, 1988; Wittmann & Müller, 2007). The development of young children's patterning competencies as part of mathematical development has received increased attention (e.g., Mulligan et al., 2020; Rittle-Johnson et al., 2019; Vanluydt et al., 2021; Wijns et al., 2021; Zippert et al., 2019). However, Mulligan and Mitchelmore (2009) looked beyond patterning abilities, proposing a new construct called "Awareness of Mathematical Patterns and Structures" that comprises children's knowledge of patterns as well as a "tendency to seek and analyze patterns" (p. 38). Based on observations during a patterning intervention, these researchers suggested that such a tendency could support children to generalize their idea of a pattern or regularity toward a broad range of mathematical situations (Papic et al., 2011). Similarly, Clements and Sarama (2014) argued that patterning should be considered as a habit of mind, and that noticing or creating regularities helps one to provide meaning or cohesion to a set of seemingly unrelated elements.

Children might differ in the extent to which they spontaneously look for or create regularities, and these differences might be related to their mathematical development. However, despite the fact that several researchers have pointed toward the important role of spontaneous focus on regularities (Clements & Sarama, 2014; Mulligan & Mitchelmore, 2009; Papic et al., 2011), only two attempts were made to measure such a spontaneous focus (Sharir et al., 2015; Wijns et al., 2020). Sharir et al. (2015) developed three tasks to measure children's Recognition of Mathematical Structures (ROMS). Two tasks required children to describe a picture, either a picture with geometric shapes or a picture of a real-world situation. In the third task, children had to copy the behavior of an experimenter who put colored discs into a box. Each task evaluated children's recognition of three different structures, namely quantities (e.g., OOO), mathematical patterns (e.g., OO OO OO), and arithmetic series (e.g., O OO OOO). Individual differences in children's spontaneous recognition of these mathematical structures were found, and these were positively correlated with general mathematical performance. Whether this task truly measures a spontaneous focus toward structures or regularities has, however, been criticized, given that the structures were prominently present in the tasks and there was little else to describe in the two picture tasks or to copy in the behavioral task (Wijns et al., 2020).

Similar to the idea of a spontaneous focus on regularity, Wijns et al. (2020) aimed to measure preschoolers' spontaneous focusing on patterns. Spontaneous mathematical focusing tendencies have received a lot of research attention over the past two decades, with researchers investigating children's spontaneous focus on several mathematical elements such as numerosities (Hannula & Lehtinen, 2005), number symbols (Rathé et al., 2019), and quantitative relations (Degrande et al., 2017; McMullen et al., 2014). Driven by previous observations of children spontaneously creating patterns during free play (Fox, 2005; Garrick et al., 2005; Seo & Ginsburg, 2004) and in accordance with the task criteria set up by Hannula and Lehtinen (2005) to assess a spontaneous focus on numerosities, Wijns et al. (2020) developed the tower task. In this task, children had to create a tower with a set of blocks in three colors. More than one-third of the 4-year-olds in their study created a tower in which the colors were organized as a pattern, 14% sorted the blocks on color, and others made a (seemingly) random creation. Children who created a pattern had better patterning, spatial, and numerical skills than children with unstructured creations. Children who sorted the blocks on color had strong patterning and spatial skills, and this accounted for their relatively high numerical skills.

In general, looking for regularities in mathematical situations seems helpful, as mathematics is often described as "a science of patterns" (Steen, 1988). Recognizing the structure or pattern is the first step in transferring this structure to new situations. Consider, for example, the base-ten structure of our number words: Children who understand this will be able to count verbally up to one hundred once they are able to count up to twenty and they know the decades. Once they know that 145 is "one hundred and forty-five," they might even be able to count up to a thousand. In contrast, regularities in situations that involve randomness or probability are less evident compared to other mathematical domains, as there is always a factor of uncertainty present. Batanero and Serrano (1999) showed that high-school students tend to judge sequences of heads and tails as random when a regular pattern seems absent and the frequency of Hs and Ts are close together. Thus, observing (non-existent) regularities in given random sequences might affect our judgment about the actual randomness of the situation (Kahneman, 2011). Driven by the idea that children differ in their spontaneous focus on regularities, the present study investigated whether this focus on regularities affects their predictions of random situations.

3.1 Current study

Two research questions were described for the present study:

RQ1: How do children perform on randomness tasks in different coin-tossing contexts? Findings about children's performance in coin-tossing experiments remain inconsistent. Therefore, our first aim is to investigate children's understanding about randomness taking into account different coin-tossing contexts. More specifically, we will explore children's correct reasoning, as well as two non-normative tendencies, namely (1) to prefer or continue regularity in random sequences, that is, giving the answer that is unrepresentative of randomness; and (2) to give the answer that is representative of randomness and in line with the representativeness heuristic. Moreover, cognitive ability is believed to foster normative reasoning, or in other words protect against heuristic reasoning (e.g., Primi & Chiesi, 2011; Kokis et al., 2002). Therefore, we will explore how cognitive ability is related to normative and both non-normative answering tendencies in coin-tossing contexts.

RQ2: Is there a relation between children's tendency to spontaneously focus on regularities and their interpretation of random situations?

Prior research has also shown individual differences in children's spontaneous focus on regularities, and these were positively related to children's mathematical abilities (Sharir et al., 2015; Wijns et al., 2020). In line with Kahneman (2011), we argue that such a spontaneous focus might also impact one's interpretation of randomness. Kahneman (2011) suggests that the representativeness heuristic stems from pattern-based thinking which translates into predictions that seem representative for randomness. However, Ridgway and Ridgway (2010) found that children often predict unrepresentative outcomes, such as streaks or patterns in given coin-tossing sequences. Therefore, our second aim is to investigate whether individual differences in children's spontaneous focus on regularities are related to reasoning about randomness. More specifically, we expect spontaneously focusing on regularities to be positively related to the tendency to give "representative" and "unrepresentative" answers in coin-tossing experiments. We will control whether a possible association holds when controlling for cognitive ability, as it is believed to be related to spontaneous focusing on regularities as well as normative and heuristic reasoning (e.g., Kokis et al., 2002; Wijns et al., 2020).

The present study is part of the same, larger research project as studies presented in Vanluydt et al. (2021) and Wijns et al. (2020, 2021). The research project followed the development of several mathematical competencies. Consequently, the focus and time points differed across studies. Moreover, regarding children's understanding of randomness, a measure was specifically designed for the present study, and related data are not described elsewhere.

4 Method

4.1 Participants

This study is part of a large-scale study on the development of young children's early mathematical competencies (https://ppw.kuleuven.be/o_en_o/CIPenT/WisCo/development-and-stimulation-of-childrens-core-mathematical-competencies). Seventeen schools participated in a 4-year longitudinal study and were selected based on governmental data to represent a diverse sample in terms of socioeconomic status (SES). In Flanders, Belgium, 3 years of preschool are organized and most children attend preschool (>95%). All children in their second year of preschool during the 2016–2017 school year were invited to participate, and an active informed consent was received for 410 children. Several children left the sample over the course of the 4 years because they changed schools. Full data was available for 302 participants (148 boys). In terms of their spontaneous focus on regularities, the proportion of children in the final sample (52%) was similar to the initial sample (51%). The educational level of the mothers, as indicated via a parent questionnaire, was used as a proxy for SES. Mothers of participating children were distributed across the low (=no education, primary education or lower secondary level education, 10%), below-average (=upper secondary level education, 25%), above-average (=professional bachelor degree, 22%) and high (=academic bachelor's degree, master's degree or doctoral degree, 38%) SES categories, respectively. For 5% of the sample, data on SES was not available. The present study took place in Belgium, Europe, where the majority of citizens are Caucasian, and collecting data on ethnicity is not a usual practice. At the first time point, children had a mean age of 4 years and 10 months (SD=3.43 months, range 4 years 2 months–5 years 4 months).

In Flanders, little attention is given to repeating patterns in preschools. There is one developmental goal (which is not considered as a target, but more a guideline for preschool teachers) related to patterning, which states that “Children can extend a series of objects that follows a pattern.” A previous study has suggested that on average, Flemish preschool teachers report doing a patterning activity less than once a month (Torbeyns et al., 2021) . Also, the concept of probability or randomness receives little attention in the elementary school curricula, and typically none in the first years. The concept of probability is mentioned in only one of the National Educational Standards for elementary school and reads: “Recognizing in examples that fractions can be understood as the representation of a probability” (Vlaams Ministerie van Onderwijs en Vorming, 2010). Therefore, we expected participating third graders to have received little to no instruction on the topic of probability or randomness.

4.2 Procedure

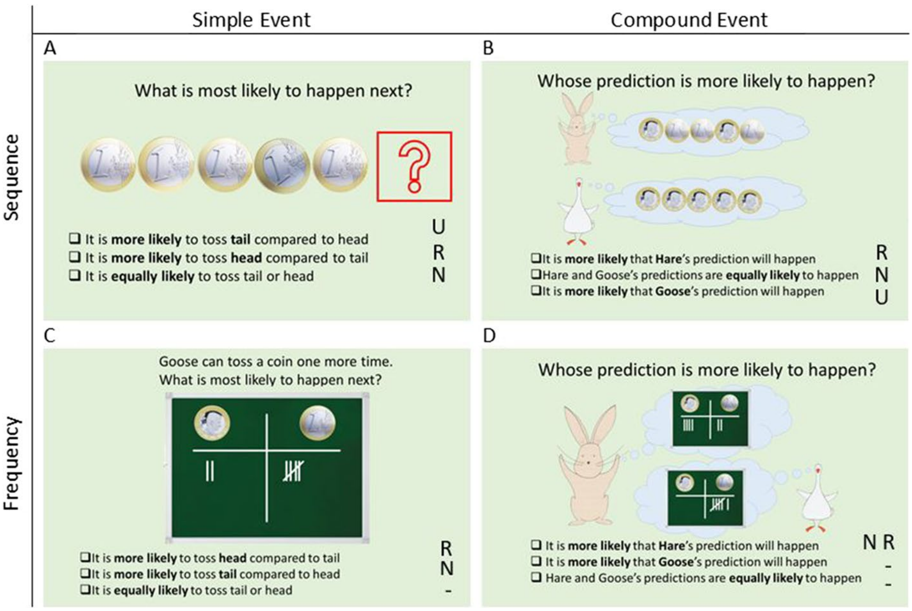
As part of the large-scale longitudinal study, tasks on several mathematical competencies were administered in the spring of 2017, 2018, 2019, and 2021 and in the fall of 2017, 2018, 2019, and 2020. Children were at the end of the second year of preschool (spring 2017) at the first time point of data collection and at the end of the third year of elementary school at the ninth and final time point (spring 2021). For the present study, we include the spontaneous focus on regularities task that was administered at the first time point (spring 2017), the intellectual ability task that was administered at the eighth time point (fall 2020), and the task on understanding of randomness that was administered at the final time point (spring 2021). Tasks administered on other time points involved other mathematical competencies and were not relevant for the present study.

5 Materials

5.1 Understanding of randomness

To measure understanding of randomness, a fifteen-item questionnaire was developed, which we will further refer to as the Heuristask. Children were introduced to a goose and a hare who were playing a head-and-tail game. Children first saw a coin and were shown which side was head (i.e., the side that pictured a head) and which side was tail (i.e., the side that pictured the number one). The Heuristask consisted of four categories of items. In the first category (A) children had to predict the outcome of a future coin toss after seeing a past sequence of coin tosses. In the second category (B), children had to judge out of two random sequences of coin tosses, which one was more probable to happen. In the third and fourth category, the order or sequence of outcomes was not given. In category (C), children had to predict the outcome of a future coin toss after receiving a frequency distribution of past coin tosses. In the fourth category (D), children had to judge out of two frequency distributions of coin tosses, which one was more probable to happen. So, depending on the category of items, children had to predict a simple event or a compound event, based on either given sequences or frequency distributions. For each item, children were told to indicate the correct answer out of three alternatives and that only one alternative was correct (see Fig. 1). All materials regarding the understanding of randomness are publicly available in the original language (Dutch) (see: https://osf.io/y9nr4/?view_only=aad01d50eb50446494b3cb69b293feb7). A rudimentary English translation can be obtained from the authors upon request; however, one should keep in mind that the translated version of the task was not validated.

Table 1 shows the structure of the instrument, with an overview of the items for every category along with the answer alternatives and the coding of these alternatives. Within categories A and B, there were three types of answers, that is, the normative answer, the representative answer, and the unrepresentative answer. The normative answer is the correct answer (see Fig. 1 for examples). The representative answer is in line with the features of fairness and unpredictability related to representative thinking about randomness (see Fig. 1 for examples). The unrepresentative answer refers to the alternative that violates these features of fairness and unpredictability and reflects a clear pattern or structure randomness (see Fig. 1 for examples). In category C, two out of three items consisted of one normative answer, one representative answer, and one answer with no underlying reasoning relevant for the present research questions. The third item within this category included one normative answer and two representative answers that are each representative according to a different feature of representativeness; one alternative would meet the feature of fairness, while the other alternative would meet the feature of unpredictability. Category D was the only category in which "equally likely" was not the normative answer. The high number of items in which "equally likely" was the normative answer might induce a response bias in children, thinking "equally likely" is always the correct response. To temper the possibility for this response bias to occur, three additional items were included in category D compared to the number of items within the other categories.



Note. The letters "N", "R", "U", were not presented to children and serve to indicate to the reader of the manuscript the different types of responses that could be given. "N" stands for the normative or correct response. "R" stands for the response that would be predicted by the representativeness heuristic. "U" stands for the unrepresentative response in which regularities are predicted or continued. "-" means the response had no relevancy for the aims of the study. Note that in category D, the normative or correct answer would also be predicted by the representative heuristic, thus in this category, it is impossible to distinguish normative from representative answers.

Fig. 1 Example items for each of the categories

Table 1 Overview of the items and answer alternatives of the Heuristask for each item type (category), with order number of appearance(Nr), composition of the sequences or frequencies of heads (H) and tails (T) (Item), possible answer alternative (alternatives), number of respondents for each alternative (N), and coding of relevant alternatives (Coding)

					Coding		
Category	Nr	Item	Alternatives	*N*	Normative	Representative	Unrepresentative
A Sequence simple	4	T-T-T-T-T-?	Both H T	131 48 123	x	x	x
	7	T-H–H-T-H–H-T-?	Both H T	122 155 25	x	x	x
	9	T-T-H-T-T-H-T-?	Both H T	88 33 181	x	x	x
B Sequence compound	1	1) H–H-H–H-H 2) H-T-T-H–H	Both 1) 2)	68 68 166	x	x	x
	10	1) H-T-T-H-T-T-H-T-T 2) T-H–H-H-T-T-H-T-H	Both 1) 2)	168 72 62	x	x	x
	14	1) H–H-T-H–H-T-H–H-T 2) T-H-T-T-H-T-H–H-H	Both 1) 2)	136 91 75	x	x	x
C Frequency simple	3	2H, 5 T Next = ?	Both H T	141 55 106	x	x	NA
	6	5H, 0 T Next = ?	Both H T	126 107 69	x	x	NA
	12	4H, 3 T Next = ?	Both H T	158 58 86	x	x (I) x (F)	NA
D Frequency compound	2	1) 4H, 0 T 2) 2H, 2 T	Both 1) 2)	57 40 205	x	x	NA
	8	1) 0H, 6 T 2) 4H, 2 T	Both 1) 2)	97 43 162	x	x	NA
	11	1) 5H, 0 T 2) 2H, 3 T	Both 1) 2)	79 42 181	x	x	NA
	15	1) 3H, 3 T 2) 0H, 6 T	Both 1) 2)	88 180 34	x	x	NA
	5	1) 4H, 4 T 2) 2H, 6 T	Both 1) 2)	107 147 48	x	x (F) x (I)	NA
	13	1) 2H, 4 T 2) 3H, 3 T	Both 1) 2)	133 61 108	x	x (I) x (F)	NA

Four out of six items in category D included one normative answer which coincided with the representative answer and two other alternatives trivial for the aims of the present study. The remaining two items included one normative answer which was also representative according to the "fairness feature" underlying representative thinking. A second erroneous representative answer met the "irregularity feature" underlying representative thinking and the third erroneous alternative was trivial for the aims of the present study. All children received the items in the same order (item numbers in Table 1 reflect the order of the items).

For analyses, three main variables were constructed. A weighted total accuracy score was calculated representing the accuracy on the overall task by, first, dividing the number of normative answers within each item category by the total number of items within that category. The sum of these individual subscores was then divided by four (the total number of categories), resulting in an accuracy score ranging from 0 to 1 (15 items; $\alpha = 0.65$). This procedure was repeated for the total representativeness score (now dividing the number of representative items within each item category by the total number of items within that category and subsequently dividing the sum of these four subscores by four). This resulted in a representativeness score ranging from 0 to 1 (15 items; $\alpha = 0.75$). Regarding the unrepresentative answers which reflected a tendency to continue or prefer regularity, an unrepresentative score was constructed based on the total number of answers in the first two categories of items that reflected this tendency divided by the total number of relevant items, resulting in an unrepresentativeness score ranging from 0 to 1 (6 items; $\alpha = 0.68$).

6 Spontaneous focus on regularities

The task and data regarding children's spontaneous focus on regularities was previously presented in Wijns et al. (2020) . Children received 15 blocks in three different colors (five blocks per color), and were asked to create a tower that went straight up, using all the blocks provided. The experimenter took a picture of the constructed tower, and these pictures were used to categorize the child's response. Wijns et al. (2020) used three categories in their study, namely pattern (i.e., at least one sequence of two full units of a repeating pattern and the start of a third one, e.g., ABABA), sorting (i.e., all block are sorted on color, AAAAABBBBBCCCCC), and random (i.e., all constructions that did not fit the definition of pattern or sorting). A perfect inter-rater reliability was found based on two independent raters scoring 10% of the constructions in the Wijns et al. (2020) study. In the present study, we were interested in children's spontaneous focus on regularities in a broader sense. Thus, both the pattern category and the sorting category were considered as regular arrangements of the colored blocks. A dichotomous variable was created to represent presence (score of 1; i.e., pattern or sorting) or absence (score of 0; i.e., random) of a spontaneous focus on regularities.

7 Intellectual ability

The Raven Standard Progressive Matrices Test (Raven, 2003) was included as a measure of intellectual ability. This test consists of five sets of 12 multiple-choice items ($\alpha = 0.88$), in which a black and white picture with a missing element is shown, and children have to select the missing element out of 6 to 8 options. Children received 1 point for each correct answer, leading to a maximum score of 60.

8 Results

8.1 Research question 1: children's reasoning about randomness

Table 1 represents the number of children that selected each alternative for each item. Out of 15 items, children had a mean accuracy score of 7.02 (SD = 2.99, range = [0;14]). The representative alternative was chosen 5.86 times on average (SD = 3.11, range = [0;15]). The unrepresentative alternative was chosen, on average, 2.28 times (SD = 1.73, range = [0;6]) out of the 6 items in which this was possible (i.e., all items with a given sequence). Frequency distributions for all raw total scores on the Heuristask are shown in Fig. 2. The frequency distribution of the raw accuracy score on the Heuristask is symmetrical and bell-shaped. The frequency distributions of the representative and unrepresentative total scores are more skewed to the right compared to the accuracy scores. Moreover, the distributions show that a considerable number of children never gave any of these types of responses. Regarding intellectual ability, children had a mean score of 34.29 (SD = 7.86, range = [11;52]).

Answer alternatives that were predicted by correct reasoning, representative thinking, or the tendency to prefer/continue regularity were cross marked in the table above. Answer alternatives that were not predicted by any of these were not cross marked. Unrepresentative answers were non applicable (NA) in category C and D. For items in category D, the normative or correct item coincided with a representative answer. For example, in Item 2, the second frequency distribution (2H, 2 T) is more probable compared to the first (4H, 0 T), but it is also predicted by the features of irregularity (I) and fairness (F) that are related to the representativeness heuristic. For some items in category C and D, two alternatives correspond to representative thinking because one of the alternatives meets the feature of irregularity (I) and the other meets the feature of fairness (F). For example, in considering Item 5, the feature of fairness would predict a child to choose the first frequency distribution (4H, 4 T) because this better represents the fairness of a coin. However, the second frequency distribution (2H, 6 T) looks less ordered and would be predicted

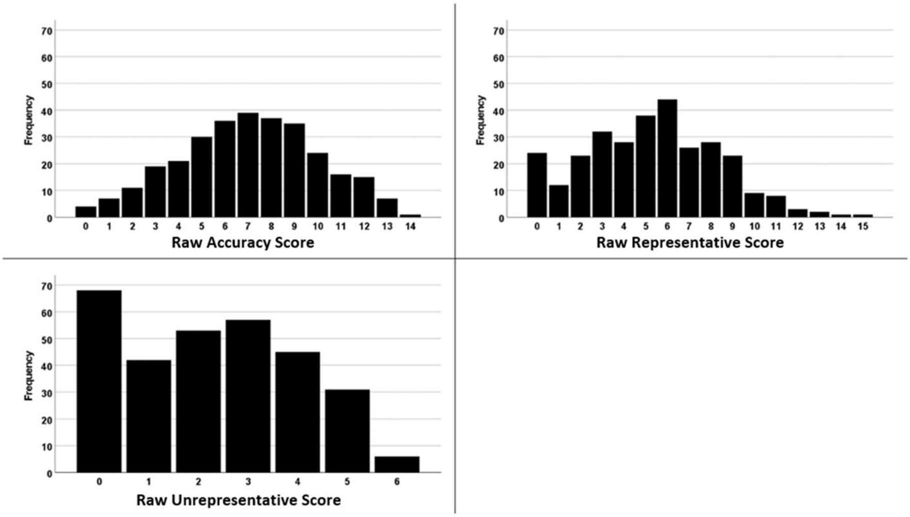


Fig. 2 Frequency distributions for children's raw total scores derived from the Heuristask ($N = 302$)

Table 2 Zero-order correlations for performances on the Heuristask with intellectual ability

Variable	Intellectual ability
1. Accuracy Score	0.16*
2. Representative Score	0.06
3. Unrepresentative Score	−0.12*

*$p < 0.05$, **$p < 0.001$

by the irregularity feature that characterizes random situations according to representative thinkers.

Table 2 shows correlations between intellectual ability and the weighted scores of the children's Accuracy Score, Representative Score, and Unrepresentative Score on the Heuristask. A small positive correlation was found between intellectual ability and the Accuracy Score on the Heuristask. No evidence was found for intellectual ability being related to the Representative Score. A small negative correlation was found between intellectual ability and the Unrepresentative Score.

8.2 Research question 2: association between spontaneous focus on regularities and interpretation of randomness

Overall, 158 children (52%) showed a spontaneous focus on regularities at age 4, whereas 144 (48%) did not. Table 3 presents the descriptive statistics for the main variables for children with and without a spontaneous focus on regularities as well as the results of the *t* tests. No differences were observed regarding the Accuracy Score on the Heuristask between these two groups of children, nor regarding their Unrepresentative Score. Small but statistically significant differences between groups were found related to the Representative Score, with children with a spontaneous focus on regularities providing more answers in line with representative thinking about randomness. There were also small but statistically significant group differences for intellectual ability, with children with a spontaneous focus on regularities showing higher intellectual ability scores. We verified whether results held when controlling for intellectual ability using ANOVA-analysis. Again, no significant group differences were found for their Accuracy Score, $F(1, 299) = 0.01$, $p = 0.94$, $\eta^2_{partial} = 0.00$, or Unrepresentative Score, $F(1,299) = 0.25$, $p = 0.67$, $\eta^2_{partial} = 0.00$. The differences regarding the Representative Score remained small but statistically significant, $F(1, 299) = 5.88$, $p = 0.02$, $\eta^2_{partial} = 0.02$. During the review process, we additionally ran requested analyses while controlling for gender and SES. Inclusion of these variables did not impact our findings.

Table 3 Means and (SD) for two groups, t-statistics and effect sizes for *t* tests

	No spontaneous focus on regularities	Spontaneous focus on regularities	$t(300)$	Cohen's d
Accuracy score	0.45 (0.24)	0.45 (0.22)	−0.27	−0.03
Representative score	0.31 (0.18)	0.36 (0.20)	−2.54*	−0.29
Unrepresentative score	0.39 (0.29)	0.37 (0.28)	0.73	0.08
Intellectual ability	33.26 (7.87)	35.23 (7.76)	−2.20*	−0.25

*$p < 0.05$

9 Discussion

9.1 Children's reasoning about randomness

Our first aim was to investigate children's overall performance in coin-tossing contexts. The Heuristask joined several coin-tossing contexts in one task, allowing for a more nuanced picture on children's abilities to reason about randomness to be painted compared to previous studies. Our study showed that third-graders frequently selected the correct (normative) response (see Table 1), suggesting that earlier studies might have underestimated third-graders' reasoning about randomness in coin-tossing contexts. Furthermore, we found with third-graders that several types of erroneous responses occur when they are asked to reason about coin-tossing events: responses that are predicted by the representativeness heuristic (referred to as representative answers) or by the tendency to predict regularity (to continue) in random events (referred to as unrepresentative answers). However, the distribution of scores suggests that strong differences exist between and within children with some being fairly consistent in giving one type of response and others giving several types of responses (see Fig. 2).

First, we evaluated the extent to which children provided normative answers. Some studies suggested that children in third or fourth grade of elementary school rarely judge independent events to be equally likely to happen when this is the normative answer (Ridgway & Ridgway, 2010; Smith, 1998). Chiesi and Primi (2008) claimed that third-graders primarily base predictions of random events on past outcomes, which conflicts with normative reasoning. However, the present study found third-graders, on average, to select the normative answer in about 47% of items. Moreover, for 11 out 15 items of the Heuristask, the normative answer also seemed to be the predominant answer for third-graders (see Table 1). Perhaps these previous studies might have underestimated young children's abilities to grasp the independence of events, because they focused on just one specific context.

Regarding the representative answers, Ridgway and Ridgway (2010) found that third-graders rarely chose the representative answer, that is, the answer that appeared representative for random events according to the fairness or unpredictability features. On the contrary, Smith (1998) found that the representative answer was predominant in fourth-graders. In the present study, third-graders, on average, selected the representative answer in about 40% of items. Across almost all items, the representative answer was the least popular alternative for third-graders, *except* for the items in which the representative and normative answer coincided (see Table 1).

It has been suggested that representative thinking related to random situations increases with age because as we get older, the number of personal experiences we have with randomness increases and those experiences shape the prototype we hold in our minds for randomness (Bryant & Nunes, 2012; Chiesi & Primi, 2009). Compared to findings of Ridgway and Ridgway (2010) and Smith (1998), our findings might raise the question as to whether a shift takes place in children's reasoning about randomness between third and fourth grade of elementary school. Following this line of reasoning, our sample might have been composed of children who had or had not yet made this shift, while in Ridgway and Ridgway (2010), none or only few children, and in Smith (1998), all children had experienced that shift. However, it seems a bit farfetched to expect children to suddenly have a high number of experiences with randomness between third and fourth grade of elementary school in order to explain the differences in prevalence of representative errors across the different studies.

A more plausible explanation for the lower prevalence of erroneous representative responses found in the present sample might again be that young children actually have a relatively correct understanding of randomness that was underestimated in previous studies. Still, about 8% of children never chose the representative alternative, even when the representative answer coincided with the normative answer. Perhaps this group of children had little experience with randomness, which would explain why, on the one hand, they do not understand randomness but also, on the other hand, why they are not susceptible to representative thinking about randomness.

Regarding the unrepresentative answers, the Heuristask included six items across two item types (i.e., sequence simple and sequence compound) that allowed to assess children's tendency to predict regularities in given coin-tossing sequences (see Table 1). Previous studies used only one type of item, asking children to predict the next event following a given sequence (i.e., our sequence simple items). For example, Chiesi and Primi (2008) and Ridgway and Ridgway (2010) found that third-graders are more susceptible to predict regularity in a random sequences, which is seen as unrepresentative for randomness, than two outcomes being equally likely (i.e., the normative answer) or the regularity to discontinue (i.e., the representative answer). More specifically, Ridgway and Ridgway found that the proportion of third-graders that predicted that a given streak or pattern of coin-tosses would continue exceeded 77% in every item. In the present study, the proportion of unrepresentative answers in each item was lower, ranging between 23 and 60% (see Table 1). Moreover, in four out of the six items in which it was possible to choose an unrepresentative answer, the normative alternative was predominantly chosen (see Table 1), which might again be an indication that children's understanding of randomness might be underestimated in Ridgway and Ridgway (2010).

Moreover, the distribution of the unrepresentative scores in the present study (see Fig. 2) indicates that more than half of the third-graders never or rarely opted for the unrepresentative answer (never, once, or twice out of six items). Items in which children had to judge which of two given sequences of coin-tosses is more probable (i.e., sequence compound items) were not included in the studies of Chiesi and Primi (2008) or Ridgway and Ridgway (2010). Our findings (see Table 1) indicate that unrepresentative answers are less prevalent in this category of items compared to items in which sequences are given and children have to predict the next outcome (i.e., sequence simple items). Thus, perhaps, the inclusion of this additional category of items in the Heuristask explains why unrepresentative answers were less commonly observed in the present study compared to previous studies.

Regarding intellectual ability, Primi and Chiesi (2011) found that it accounted for less than 5% of third-graders' normative reasoning about randomness. Although in the present study no causal claims can be made, the small positive correlation between performance on the Raven Progressive Matrices and the Heuristask (see Table 2) supports the idea that intellectual ability might protect against erroneous reasoning about randomness (Kahneman & Frederick, 2002; Stanovich, 1999). However perhaps other aspects, such as relevant knowledge about probability and heuristics, or the experiences one has had with randomness, are more important (Primi & Chiesi, 2011). For example, Williams and Connolly (2006) investigated gambling behavior in a sample of university students and found that instruction about probability improved their resistance to gambling fallacies compared to a control group of university students.

Our study furthermore hints that the degree to which normative or heuristic reasoning about randomness is expressed differs depending on the type of items that are used. Unrepresentative answers were more frequently given in items where children had to predict the next event when given a sequence of events containing a streak or a pattern, compared to items in which children had to choose which of two given sequences was more probable (see Table 1). Remarkably, this context was equivalent to the tasks used in Ridgway and

Ridgway (2010), who found a strong tendency in third-graders to continue a given pattern or streak in predictions. Perhaps predicting the next outcome of a coin-toss is a more familiar context for children, enabling illusions of control or beliefs in having a hot-hand. This fits the idea of Oskarsson et al. (2009) that specific task characteristics might impact our judgment of random events, which might explain the differing findings across studies with children of the same age. Nonetheless, the Heuristask was developed to grasp children's overall reasoning about randomness in coin-tossing contexts. In order to minimize the burden for participating children, the number of items within each category was limited. Three out of four categories of items of the Heuristask consisted of only three items (see Table 1), which does not allow us to make strong claims about the role of different task or item characteristics. Future studies could, for example, double the number of items in each category to investigate whether specific item characteristics correlate with children's reasoning about randomness.

Future studies could also focus on using coin-tossing context to investigate other biases in children. Our findings, for example, show that the "equally-likely"-answer was frequently chosen when children had to choose the most probable frequency distribution with one out of two given distributions being more probable. This suggests that the equiprobability bias, which is the tendency to believe that equal probabilities for all events exist when random processes are involved, might also be present in young children. In the Heuristask, the equiprobable answer and the normative answer corresponded in 9 out of 15 items and thus, the equiprobability bias might have distorted our findings regarding children's normative reasoning about randomness. In a previous study, Chiesi and Primi (2008) found that third-graders never opted for the equiprobable alternative when it did *not* correspond to the normative answer. However, post-hoc analysis of the present study indicated that 5% of third-graders still consistently chose the "equally likely"-answer across all items. In one of these items (13), the item in which the difference in probabilities between the two sets of events happening was smallest compared to other items within the same category, the equiprobable answer was even the predominant choice for children. Perhaps the equiprobable answer was chosen in this item by "equiprobable thinkers" as well as by children who were generally able to correctly estimate the most likely set of events, but for whom the small difference in probabilities in Item 13 was too fine-grained.

9.2 Association between spontaneous focus on regularities and reasoning about randomness

Regarding our second aim, the present study was the first to explore the association between children's early spontaneous focus on regularities and later reasoning about randomness. Fallacies related to representative thinking are often explained by our natural tendency for pattern-based reasoning (Kahneman, 2011). Wijns et al. (2020) suggested that already in preschool, children differ in their tendency to focus spontaneously on patterns or regularities and that these differences are positively related to mathematical and spatial abilities. Our findings suggest that children with a spontaneous focus on regularity also have higher intellectual ability, which might support the idea of a spontaneous focus on regularities being beneficial. However, our findings also nuance this one-sided positive view. No evidence was found for preschoolers' tendency to spontaneously create regularities to predict the number of normative answers given four years later, but it did predict the number of representative answers in coin-tossing contexts, above and beyond intellectual ability. Thus, a spontaneous focus on regularities seems to be related to heuristic thinking, which often leads to systematic errors. This supports claims that representative thinking is related to our natural tendency to seek for patterns. One explanation for this relation is that children with a strong awareness and preference for regularities might be more sensitive to recognizing contexts in which regularities are absent, such as with randomness. Perhaps they more easily overgeneralize this knowledge about randomness to small samples of simple events, not taking into account the independence of events.

Remarkably, no evidence was found for the tendency to spontaneously create regularities in young children to predict the number of unrepresentative answers four years later, that is, predicting that a streak or pattern is more probable (sequence compound) or will continue (sequence simple). Ridgway and Ridgway (2010) suggested that young children's predictions of a streak or pattern in random sequences might be caused by the misapplication of skills taught in algebra class and the strong focus on patterns in the curriculum. First, our findings do not support the idea that children with a strong spontaneous focus on regularities are more susceptible to predict regularities in random situations. However, the question of whether or how a curriculum that focuses strongly on patterns affects reasoning about randomness was not addressed in the present study. Future research could investigate whether such a preplanned focus in the curriculum can strengthen children's focus on patterns in their environment, and whether this might, in turn, affect reasoning about randomness. Second, the extent to which algebra, or even mere patterning skills play a role for reasoning about randomness remains unexplored. Individual differences in preschoolers' patterning abilities have been reported repeatedly (e.g., Rittle-Johnson et al., 2019; Wijns et al., 2021). It is conceivable that the third-graders in the present study, both with and without a spontaneous focus on regularities, all possessed the basic skills to recognize and extend the simple patterns in the coin-tossing sequences of the Heuristask. To investigate to which extent patterning skills are related to reasoning about randomness, future research could opt to administer a patterning ability task and the Heuristask when children are younger and large individual differences are expected in children's patterning ability.

It is, however, important to acknowledge that the tower task that was used in this study measured children's tendency to spontaneously *create* patterns rather than their tendency to *look for* patterns. The assumption in the present study is therefore that both spontaneous behaviors rely on the same tendency to spontaneously focus on regularities, which was shown to be related to children's reliance on the representativeness heuristic in the Heuristask. Future research might develop a measure that taps more into children's tendency to look for patterns when investigating the association with their understanding of randomness.

9.3 Strengths and limitations

The present study comes with strengths and limitations. First, our findings suggest that the unique contribution of children's spontaneous focus on regularities for representative thinking is small. This might raise the question whether this small variance is even meaningful. However, one must take into account that, following Wijns et al. (2020), the spontaneous focus on regularities was assessed using a one-item measure (the tower task). Future studies could focus on developing a more elaborate measure, allowing for more variation between children. Nonetheless, using multiple-items might hinder the measure's capacity to grasp the spontaneous aspect of a spontaneous focus on regularities. Moreover, in the present study, an interval of 4 years between the dependent and independent variable was present. In 4 years, a lot happens around and within young children and, thus, the effect found in the present study might be biased downwards (Taris & Kompier, 2003).

In between these 4 years, children might have received instruction on patterning, which perhaps unconsciously also affected their spontaneous focus on patterns. Moreover, children's understanding of randomness might also be affected by instruction. It has been suggested that children are primarily exposed to a curriculum that fosters deterministic ways of thinking (Falk et al., 1980), contributing to the idea of the world as being perfectly predictable and determined. Such a deterministic view might in turn impede children's understanding of randomness. Future studies could opt to measure a spontaneous focus on regularities and reasoning about randomness simultaneously at multiple time points. This might give a better understanding of the relation between a spontaneous focus on regularities and the understanding of randomness at different ages. However, one has to be aware that it is difficult to account for the possible role of instruction when all participants in the study have been exposed to the same curriculum.

Regarding the measure for reasoning about randomness, one overarching context frequently used in research, namely a coin-tossing context, was used. Future studies could focus on improving generalizability to other probabilistic contexts. Moreover, the goal of the present study was to grasp the overall ability of children's reasoning about randomness in coin-tossing contexts. Therefore, four categories of items were jointly used in one instrument. Future studies could focus on extending the number of items within these categories, allowing for more reliable conclusions about the effect of certain task characteristics. Patterns and streaks were also used interchangeably to present regularity in random events. While children might continue streaks because of the hot-hand fallacy, this does not explain the continuation of patterns in random events. Future research could explore whether different mechanisms affect reasoning about randomness in the context of streaks compared to patterns. In three items, the features of unpredictability and fairness related to the representativeness heuristic conflicted and each feature predicted a different response. Three items is too few to form conclusions about the role of each of these features for children's representative thinking tendencies. Adding more of these conflicting items could lead to a better understanding of whether one of the features is more dominant in young children, which might be beneficial to develop interventions. Future research might also choose to question children's judgements on the Heuristask using interviews. Interviews in which children explain their reasoning might help in confirming some of the potential explanations for the findings in the present study. For example, children's verbal argumentation might clarify whether they chose the equiprobable item in Category A, B, and C, because they understand the independence of events or because of the equiprobability bias.

Our findings were based on correlational analysis and, thus, causality cannot be inferred. Moreover, we are aware that the sample in our study only consisted of children from the same region and birth year, who were primarily Caucasian and had highly educated parents. Thus, findings of the present study cannot be generalized to the entire population. However, the predictive value of an early spontaneous focus on regularities for later representative thinking supports the idea that humans have a tendency to seek for patterns that impacts our judgment and behavior (Kahneman, 2011), and that this tendency is already present from an early age. Further exploration of whether fallacies regarding randomness can be traced back to unlearned behavior already present in preschool could open avenues for early detection and intervention. Tarr and Jones (1997), for example, showed that from merely experiencing randomness resulting in sequences that are not "representative" (e.g., streaks or patterns in coin-tosses), older students learned that small samples do not necessarily reflect the parent population. Future studies could investigate whether deliberately exposing children to randomness from a young age impacts reasoning about randomness and whether this differs for children with and without a spontaneous focus on regularities.

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Data availability The datasets generated during and/or analyzed during the current study are not publicly available due to the fact that research related to the data is still in progress but are available from the corresponding author on reasonable request.

Code availability SPSS Syntax of the present study is available from the corresponding author, upon reasonable request.

Declarations

Ethics approval The study was approved by the social and societal ethics committee of KU Leuven (G-2016 07 591).

Consent to participate Informed consent was obtained from parents of all participants.

Consent for publication It is affirmed that the authors named above are the sole authors of the present manuscript, that this manuscript has not been published elsewhere and that this manuscript will not be submitted elsewhere until the Journal's editorial process is completed.

Conflict of interest The authors declare no conflict of interest.

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