



How transition students relearn school mathematics to construct multiply quantified statements

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Abstract

Understanding the intricate quantifier relations in the formal definitions of both convergence and continuity is highly relevant for students to use these definitions for mathematical reasoning. However, there has been limited research about how students relearn previous school mathematics for understanding multiply quantified statements. This issue was investigated in a case study in a 5-week teaching unit, located in a year-long transition course, in which students were engaged in defining and proving sequence convergence and local continuity. The paper reports on four substantial changes in the ways students relearn school mathematics for constructing quantified statements: (1) endorse predicate as formal property by replacing metaphors of epsilon strips with narratives about the objects ε , N_ε , and $|a_n - a|$; (2) acknowledge that statements have truth values; (3) recognize that multiply quantified statements are deductively ordered and that the order of its quantifications is relevant; and (4) assemble multiply quantified statements from partial statements that can be investigated separately. These four changes highlight how school mathematics enables student to semantically and pragmatically parse multiply quantified statements and how syntactic considerations emerge from such semantic and pragmatic foundations. Future research should further investigate how to design learning activities that facilitate students' syntactical engagement with quantified statements, for instance, in activities of using formal definitions of limits during proving.

Keywords Quantification · School-university transition · Design research · Commognition · Limits

1 Introduction

In tertiary mathematics, mastering quantification is key for understanding mathematical statements such as formal definitions of limits. Definitions of limits contain multiple quantifications that are nested in intricate ways (Alcock, 2014). Learning the nested quantifier relations in definitions of limits requires students to understand the parts of the definition

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(Oehrtmann et al., 2014) and to become aware of the relations between the parts, which is particularly difficult for students due to the definition's complex dependencies (e.g., Dawkins & Roh, 2020; Durand-Guerrier & Arsac, 2005; Roh & Lee, 2017). Previous research has shown that students can learn these quantifier relations by building on their informal, intuitive understanding of limits and metaphors (Dawkins, 2012; Oehrtmann et al., 2011; Swinyard, 2011). For instance, by giving students a set of epsilon strips with different heights to explore and classify infinite sequences ("epsilon-strip activity"), they can use their intuitive understandings to reinvent the formal definitions of sequence convergence (Przenioslo, 2005).

The transition to tertiary mathematics is often investigated from a tertiary perspective. As such, research has identified many challenges that students face, for example, concerning content learning (Thomas et al., 2015) or life-changing institutional differences (Clark & Lovric, 2008). However, a perspective on extending school mathematics towards fundamental ideas of tertiary mathematics might offer insights into how students can be prepared to learn quantification. This change of perspective allows investigating how school mathematical knowledge is being relearned and revisited (Schüler-Meyer, 2019; Stadler, 2011) while students come to understand quantification. Furthermore, with this new perspective, the limited (Sellers et al., 2021) or informal use of quantification (Dubinsky & Yiparaki, 2000), which can be problematic in a tertiary context, can be conceptualized as a natural starting point to facilitate the use of quantification for making mathematical statements. However, little is known about how transition students can learn to use quantified variables for making multiply quantified statements¹ based on their school knowledge. Therefore, more knowledge is needed about students' learning pathways from school mathematics towards understanding quantification.

This design research study investigates this issue in the context of a transition course in an upper secondary classroom. The study focuses on a teaching unit on defining limits designed to facilitate students in constructing multiply quantified statements. Based on the study, this paper discusses the following question:

How do transition students learn to construct multiply quantified statements, particularly definitions of limits, in a teaching unit on sequence convergence and pointwise continuity that systematically connects to school mathematics?

Sections 2.1 and 2.2 will present previous research on students' learning of quantification, while Section 2.3 presents the discursive framework used here to investigate students' learning processes. Section 3 presents the hypothetical learning trajectory of the teaching unit and its principles. Section 4 outlines the methodology of the study. Section 5 presents the main results to answer the research question.

2 Learning quantification

2.1 Quantifier relations and their intricate dependencies

Let us consider a sequence a_n with values in the set of real numbers and a real number a .

¹ This paper is concerned with multiply quantified statements in the form of $(\forall x)(\exists y) R(x, y)$, where $(\forall x)$ is the *universal quantifier*, $(\exists y)$ is the *existential quantifier*, and $R(x, y)$ is the *predicate* (Dubinsky & Yiparaki, 2000).

(a_n) converges to a if and only if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - a| < \varepsilon$

The definition of sequence convergence is a prototypical example of a multiply quantified statement. In the definition, “for every $\varepsilon > 0$ ” and “for all $n > N$ ” are universal quantifiers, and “there exists an $N \in \mathbb{N}$ ” is the existential quantifier, while the predicate is initiated by the keyword “such that” (“so dass” in German or “es gilt”/“it is valid”). The definition as a whole is a statement. In particular, it is an existential statement typically used to establish the convergence of a sequence or to find its limit.

Constructing this definition requires transition students to understand its multiply quantified nature (Alcock & Simpson, 2005). Understanding the definition as multiply quantified statement is particularly demanding because of the definition’s multiple dependencies of different nature: the N is dependent upon ε (“dependence rule”; Durand-Guerrier & Arsac, 2005), but ε is independent of N (Roh & Lee, 2011). Furthermore, the predicate relates to N in the form of a dynamic process, namely “finding” a suitable N while checking for the validity of the predicate “for all $n > N$, $|a_n - a| < \varepsilon$.” Similarly, the “choice” of ε is a process, and for each “choice” of ε , students would have to repeat the former process of finding an N (Dubinsky et al., 1988).

2.2 Language-related issues in learning quantification

Designing a teaching unit to facilitate students in constructing multiply quantified statements requires a linguistic perspective on students’ previous resources. Students’ resources for learning quantified statements are pragmatic and semantic considerations. Semantics refers to the chain of references to mathematical objects that a statement enacts (cf. Sfard, 2008). Pragmatics refers to the individuals’ precedents for the activity of realizing quantification, such as everyday narratives containing quantifier keywords (Lavie et al., 2019). With respect to pragmatics, everyday precedents seem to inform students’ sense-making with quantified statements, while mathematical contexts pose difficulties (Dubinsky & Yiparaki, 2000). Particularly, such everyday precedents of quantification and quantifier keywords can inform students’ understanding of quantified mathematical statements (Cornu, 1991). Furthermore, pragmatic statements that connect to semantics by conveying “semantically interesting” information are more accessible to students (Dawkins & Roh, 2020). Accordingly, the central starting points for transition students to explicitly use quantifier keywords in a mathematical way are the mathematical meanings of statements (semantics) as well as previous everyday and mathematical knowledge, for instance, about logic (pragmatics; Durand-Guerrier & Arsac, 2005).

While pragmatics can act as resource, it can also hinder students’ construction of multiply quantified statements. A particular problem is that learners might not be aware of precedents for quantification, as quantification is often not made explicit in school mathematics. For instance, the following statement “a function is continuous whenever it is differentiable” has a low level of explicitness because it does not explicitly express underlying quantifications, requiring the reader to unpack it (Selden & Selden, 1995, p. 128). Accordingly, transition students might not notice such implicit quantifications (Mesnil, 2017). Thus, learning activities are needed to make such implicit quantifications in familiar mathematics explicit to students.

The construction of quantified statements requires not only the activation of semantic and pragmatic resources, but also learning to syntactically manipulate statements. In

discursive terms, syntax refers to the rules by which a new narrative can be constructed from simpler ones (Sfard, 2008, p. 102). Accordingly, when manipulating a quantified statement, for example, by constructing the contrapositive or negating it, students have to parse this statement from a semantic and pragmatic standpoint, but also from a syntactical standpoint. Particularly, students have to engage with syntactic rules such as changing the order of quantifications (Chellougui, 2009).

In summary, facilitating students' construction of multiply quantified statements such as the above definition of sequence convergence could begin with semantic and pragmatic considerations and informal quantifications. The syntactical dimension of the statement, while very important for capturing its multiple dependencies, is likely unfamiliar for students, so that students need to be able to connect this syntactical dimension to their pragmatic and semantic understanding.

2.3 Learning quantification from a discursive perspective

Students' learning processes with respect to quantification are here framed and investigated in terms of commognition (Sfard, 2008), because it gives insights into the micro level of how students engage with the semantics, pragmatics, and syntax of quantified statements. Commognition conceptualizes learning as the development of discourses (Nardi et al., 2014). Discourses are characterized by their keywords, visual mediators (representations), endorsed narratives, and routines (Sfard, 2008). Routines are patterned activities, where patterns are generated from repetition and adaption of precedents (Lavie et al., 2019). Even though the linguistic notions of semantics, pragmatics, and syntax are not required within this discursive perspective, they provide the necessary connections between commognition and previous research on students' learning of quantification.

Quantified statements can be considered narratives. A narrative is "a series of utterances, spoken or written, that is framed as a description of objects, of relations between objects, or processes with or by objects.... [Narratives are] subject to endorsement or rejection" (Sfard, 2008, p. 300). This description highlights that narratives contain semantic information because they constitute a certain chain of references to objects (in this case, mathematical). The definition of sequence convergence, for example, is a narrative about "convergence" endorsed by the mathematical community. The above-mentioned process of endorsement is called substantiation. During substantiation, discursants ensure that a narrative adheres to mathematical metarules that a group has agreed upon.

New narratives are constructed by adapting and changing previously endorsed narratives. Such previously endorsed narratives are selected based on precedents: situations that are perceived as sufficiently similar to the current situation to warrant the enactment of a particular previously endorsed narrative (Lavie et al., 2019). Accordingly, the search for precedents is guided by pragmatic factors. Two interrelated processes guide the construction and substantiation of new narratives. First, during additive growth, students adapt or combine (school-mathematical) precedent narratives into partly new narratives (Lavie et al., 2019). Second, during metalevel change, students construct narratives that leave behind previous narratives in some substantial way and hence require them to agree upon new metarules for endorsing such a "ground-breaking" narrative (Sfard, 2008, p. 256). For instance, as students may lack precedents for multiply quantified statements (see Section 2.2), it can be expected that during their first attempts at constructing a definition of sequence convergence, they need to endorse a new metarule about the inclusion of quantification (detailed account of intended changes in Section 3.2). Furthermore, metalevel

change can also lead to the endorsement of rules that can be considered syntactic, for example, for how to change the order of quantifications.

3 Reinventing the formal definition of limits for convergence and continuity

This study was part of a year-long extracurricular transition course intended to familiarize upper secondary school students with tertiary mathematics' fundamental ideas. The course was taught in an urban school in Germany in the penultimate year of upper secondary school, within the timeframe of transition to university (Gueudet, 2008). Within this course, a 5-week (five lessons of 90 minutes each) teaching unit was dedicated to defining and proving in the context of limits. This unit was based on Realistic Mathematics Education (RME), because of RME's benefits for developing students' understanding of limits (Dawkins, 2012; Fisher, 2016; Swinyard, 2011; Swinyard & Larsen, 2012). RME provides principles for how the definition of limits can emerge from the organization of phenomena encountered during carefully designed learning activities and contexts (van den Heuvel-Panhuizen, 2019).

3.1 Hypothetical learning trajectory

This study is located in the research program of design research to generate a local theory through iterative design experiments (Prediger et al., 2015). The teaching unit is based on a hypothetical learning trajectory (Clements & Sarama, 2004) that prescribes a developmental trajectory for understanding limits (Cottrill et al., 1996; Swinyard & Larsen, 2012). In particular, this hypothetical learning trajectory, described in Table 1, builds on both research on the learning of limits (this section) and on a priori assumptions about students' previous knowledge and about the effects of specific activities on students' learning (Section 3.2).

Previous research finds that quantifications can be developed from a generic image of ε and N for a convergent function (Pinto & Tall, 2002) or in dynamic graphical representations (Cory & Garofalo, 2011; Mamona-Downs, 2001) by encapsulating the above-mentioned dynamic dependencies (Section 2.1) into a static quantification scheme (Dubinsky et al., 1988). With its focus on defining, the teaching unit rests on the framework of defining as mathematical activity (DMA; Zandieh & Rasmussen, 2010). This framework emphasizes engaging students in actively finding, explaining, and using definitions, as these activities are more beneficial for facilitating understanding than reading definitions (Alcock & Simpson, 2017). DMA conceptualizes defining as a trajectory where, at the beginning, students construct a definition to capture their mathematical activity (*definition-of*), and later, use their definition for their mathematical reasoning (*definition-for*; Zandieh & Rasmussen, 2010). This trajectory facilitates the pragmatic and semantic side of constructing a definition from previous narratives, as they support the connection to previous knowledge and the use of precedents, for instance, about categorizing and about functions (see Section 2.2).

The teaching unit's starting point is the epsilon strip activity (Przenioslo, 2005), where students are given a set of epsilon strips with different heights to explore and classify both convergent and divergent sequences. By recognizing that one has to first choose a strip with a specific height and then has to find a point from which this strip covers all further

Table 1 Overview of the hypothetical learning trajectory

Session	Description of task	Justification
Session 1	<ul style="list-style-type: none"> • Students categorize infinite sequences—both convergent and divergent ones—with the help of epsilon strips • Students perceive that for some sequences, it is possible to find a starting point for any chosen strip from which all further terms can be covered • Students document central aspects of convergence in a treasury (Fig. 1) 	<ul style="list-style-type: none"> • Students need to connect the epsilon strip activity to school mathematical precedents. In this case, the routine of categorizing mathematical objects is a suitable precedent (pragmatics) • Students need to make connections to familiar mathematical objects, e.g., to functions as model for understanding sequences (semantics) • Students need to develop an understanding of the quantified variable ε • Students have to experience metarules of defining, e.g., that definitions are human-made (syntactic) • Students symbolize and name central aspects of convergence, which prepares the reification of epsilon strip activities into objects about which narratives can be constructed (semantic) • Students have to experience metarules of defining, e.g., viability as interpersonal agreement (syntactic) • Students need to shift from a description of activities informed by pragmatic and semantic considerations towards a quantified statement, which requires the inclusion of syntactic considerations • Students need to experience the validity of their definition and the associated discursive developments in order to define sequence convergence • Students need to experience the purpose of a definition as a means to decide whether an object belongs to a category or not in order to further transform the precedent of categorizing into the routine of defining (semantic and syntactic) • The above justifications also apply here • The students need to move from the epsilon strip activity towards generic images, in this case of the $\varepsilon - \delta$-rectangle (semantic and syntactic)
Session 2	<ul style="list-style-type: none"> • Students define sequence convergence in think-pair-share so that they need to rework and revise their definitions 	
Session 3	<ul style="list-style-type: none"> • Students compare their own definitions with a textbook definition to further rework and revise the definition • Students transform the epsilon strip activity into formal steps of proving convergence 	
Sessions 4 and 5	<ul style="list-style-type: none"> • Students replicate this trajectory in activities of reinventing, defining, and proving pointwise continuity. However, defining continuity does not rely on the use of epsilon strips 	

terms of the sequence, the students can develop a *y-first perspective* (Swinyard & Larsen, 2012). By realizing that one strip is a generic example of the possibility to choose any strip, that is, any epsilon, the students can also develop an *arbitrary closeness perspective* (ibid.). Thus, in the epsilon strip activity, the multiply quantified nature of the definition of sequence convergence can be implicitly experienced as a sequence of steps of using strips. In other words, this activity can potentially connect syntactic considerations with pragmatic and semantic considerations rooted in the experiences of the epsilon strip activity (see Section 2.2).

3.2 Transition students' intended learning pathways from a discursive perspective

The learning trajectory design rests on a priori assumptions about students' activities in terms of the justification of the learning activities, as shown in Table 1. As can be seen, because students perceive the task situation of the epsilon strip activity (Session 1) as sufficiently similar to discourses on functions, they endorse rules for activities from these discourses (e.g., movement solely in x and y direction). Additionally, the visual mediators of infinite sequences are sufficiently similar for students to adapt functional narratives. Furthermore, the routine of categorizing mathematical objects is familiar from school mathematics, so students should be able to categorize an infinite sequence into two categories: convergent and divergent. However, the routine of using an epsilon strip likely needs to be scaffolded by the teacher, as it entails new metarules (Section 3.1). In particular, students need to investigate the question of the potentiality of using *any* strip. Otherwise, students could decide that a sequence is convergent based on the use of one strip (Sellers et al., 2021), which would prevent the realization of the universal quantifier at a later point in the learning trajectory. The symbolization of facets of convergence in a treasury (Fig. 1) prepares such an intended quantified use of variables. Furthermore, if students adopt school-mathematical "tends-to" narratives of limits, the teacher needs to intervene and remind students to closely describe the epsilon strip activity instead.

In the defining task (Task 2, Session 2), students are engaged in think-pair-share. In the think phase, individual students use their treasury (Fig. 1) to formulate a description of the epsilon strip activity. In the pair phase, pairs of students compare their descriptions concerning whether they adequately capture the epsilon strip activity. In this phase, the students' definition could consist of partial narratives for each activity step with epsilon strips. These narratives could be connected through keywords that indicate time ("first," "then") or a property ("in order to," "to check whether"). In the share phase, the teacher facilitates the students' comparisons of their definitions with respect to mathematical viability, based on the epsilon strip activity, the graphical representation, and the treasury, with the aim of finding a shared definition. This discussion is informed by students' knowledge of precedents of school textbook definitions, with their problematic lack of explicitness of quantification (see Section 2.2).

Regarding the realization of the existential and universal quantifier, the teacher likely needs to scaffold students to frame the choice of an arbitrary strip as the imaginary activity of trying *any* strip, potentially resulting in the phrases "for any strip with height m " or "for any height m ." Similarly, finding a point from which all subsequent terms are covered leads to the existential quantifier. At this stage, a predicate is a description of terms being in a strip, with school-mathematical precedents for distance and absolute value.

In Session 4 of the teaching unit, the students use their reinvented definition of sequence convergence to construct a multiply quantified definition of local continuity of a function.

Concept	Symbol	Graphical representation
sequence	$(a_n)_{n \in \mathbb{N}}$	
<i>Pendulum sequence</i> PendelFolge konvergente Folge <i>convergent sequence</i>	p_{a_n}	
<i>goal value</i> Zielwert Grenzwert <i>limit</i>	Z a	
<i>distance</i> Abstand	$d_{a_n - z}$ $a_n - a$	
<i>distance value</i> Abstandswert	A $n \in \mathbb{N}$	
<i>maximum distance</i> Maximalabstand	m ϵ	

Fig. 1 Excerpt of the students' treasury for sequence convergence. Red text was added by students in Session 3, after comparing their definition of sequence convergence with a textbook definition. Translations added in italics by author

It is expected that the epsilon strip activity and the defining activity in Session 2 act as precedents such that students combine the graphical representation of strips and the equivalent narrative about arbitrary close neighborhoods from Session 2 into a definition of local continuity. This way, students replicate the above steps of defining sequence convergence for defining local continuity of functions but avoid tends-to metaphors.

4 Methodology

4.1 Data collection in iterative design experiments

This study is an exploratory case study that focuses on the learning processes of one group of students. In line with design research, the data were collected in iterative design experiments. The teaching unit investigated here represents the design's third iteration. It was implemented in a laboratory setting with a group of five upper secondary school students. The groups' activities were filmed from two angles to capture both work in pairs and whole-group work. Hence, the data consists of five 90-min sessions of video material. The design experiments were implemented by two master's students, where one student implemented the second and the other the third iteration (Otto & Heese, 2017). The master's students were experienced university mathematics tutors and acted as teachers to enact the learning trajectory shown above.

The participating students (aged 16–18) were recruited from upper secondary mathematics classrooms at the beginning of the school year. The students self-selected for participating in a transition course. As such, they also volunteered to be part of the teaching unit presented here. The students were highly proficient in their regular mathematics classrooms and also highly motivated. They also declared an interest in picking up mathematics-related university studies.

In this study, tasks with a focus on defining were investigated:

- Session 1, Task 2—compiling a treasury for sequence convergence
- S2T2—defining sequence convergence
- S4T3—compiling a treasury for pointwise continuity
- S4T5—defining pointwise continuity
- S5T2—comparing tertiary textbook definition of continuity with the reinvented definitions of both continuity and sequence convergence

4.2 Qualitative methods of data analysis in a discursive framework

The sensitizing concept for the data analysis was the use of quantifier keywords, for example, “for every” (“für jedes”), “exists” (formal: “*existiert*,” informal: “*gibt*”), and “so that”/“such that” (“so dass,” but also the colloquial “*es gilt*” is possible, which translates as “it is valid”). Discursive theory suggests that keyword use develops from *expressive use* towards *object-driven use* (Sfard, 2008; Table 2). The use of quantifier keywords has to develop towards expressing the intricate quantifier relations of the intended definition's partial narratives, as described above (Section 2.1). Throughout the students' learning pathways in the teaching unit, these keywords should develop from informal everyday use towards using them to establish the intricate quantifier relations between the partial narratives (Sections 2.2 and 3.1). It is assumed that the highest level of object-driven use will indicate metalevel developments, because the flexible use of these keywords suggests that students have endorsed a new (possibly syntactic) rule for how to use them, instead of being guided by pragmatic or semantic factors such as precedents or similar narratives that act as guiding examples.

Using the sensitizing concept of objectified quantifier keyword use, the video material and its transcriptions were analyzed qualitatively by building on an initial

Table 2 Developing quantifier use and illustrative examples from the data (abbreviations explained in Section 4)

Category	Description	Example
<i>Receptive use</i>	Students read a quantifier keyword	Students read a multiply quantified statement, e.g., a given definition
<i>Passive use</i>	Students invoke a routine within the scope of expected communicative actions upon hearing/reading a quantifier keyword	Following the utterance: [Teacher] “Okay, that means for every <i>m</i> greater than 0 there has to exist an <i>m</i> zero, such that...” (S2T2, T320), the students invoke the routine of substantiating a new predicate narrative
<i>Routine-driven use</i>	Students actively use a quantifier keyword, but only in specific routines and/or specific discursive sequences	The existential quantifier “there exists an <i>A(m)</i> ” is used several times for initiating the routine of substantiating a predicate narrative: [Lawrence] “ <i>there exists an A m, whom [sic.] contains all following elements</i> ” (S2T2, T353)
<i>Phrase-driven use</i>	Students actively use a quantifier keyword without a specific routine but embedded within a certain phrase	[Lawrence] “No idea. So one can take <i>every m</i> which is bigger than 0.” (S2T2, T220)
<i>Object-driven use</i>	Students use quantifier keywords flexibly to denote logical or hierarchical relationships, also in reverse order	The universal quantifier “every <i>m</i> ” is used as a phrase, that is, as a constant element in changing narratives The students can use synonyms for quantifier keywords, such as “it is valid” for the predicate or “this <i>A(m)</i> ” for “there exists an <i>A(m)</i> ”

turn-by-turn analysis for segmenting the material. Subsequently, all segments were identified in which one of the *quantifier keywords* or their informal/everyday synonyms were used. These segments were analyzed further to identify *partial narratives* and quantifier relations, resulting in a task-specific overview of the developments of each of the three quantifier keywords towards their object-driven use. Episodes of object-driven uses were further analyzed to identify metalevel changes, using the analytical instruments of commognition (Sfard, 2008), such as additive growth, objectification, and narratological analyses in terms of Lavie et al. (2019).

5 Results

This study finds four metalevel developments towards the construction of multiply quantified definitions of limits over the five sessions of the teaching unit. Notably, these metalevel changes occurred during the construction and interpretation of definitions of limits (S2T2, S5T2), but not during the other tasks (see Section 4.1). The most likely reason for this phenomenon is that during the construction and interpretation of definitions, students were actively using quantifications for establishing relations between partial narratives under the guidance of the teacher. During the defining activities in task S4T5, students did not discuss quantifier relations, as they reproduced a school-mathematical definition of continuity by using precedents rooted in school-mathematical tends-to narratives.

The four metalevel changes are illustrated in Fig. 2. The accompanying discursive developments that lead to and enable these changes are indicated by arrows. The foregrounded arrows will be discussed in the following sections. Extended arrows indicate that the respective development continues to enable later metalevel changes.

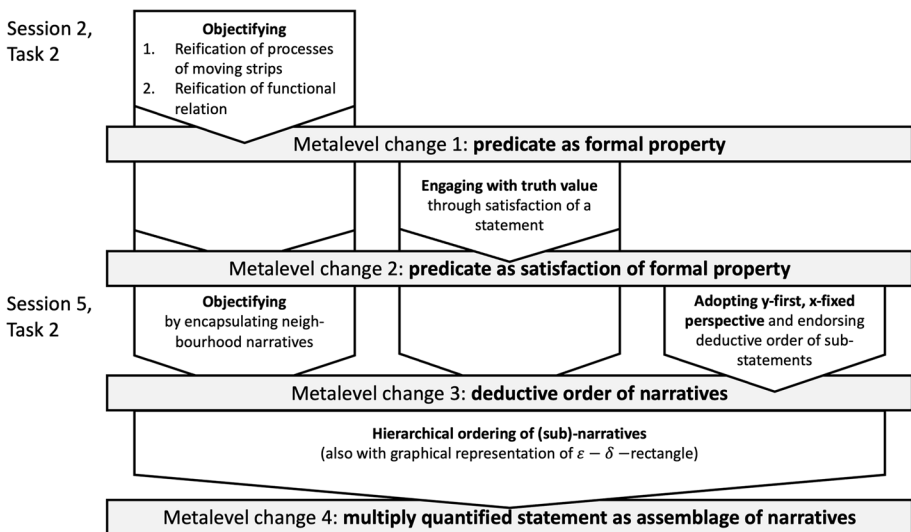


Fig. 2 Metalevel changes and their underlying discursive developments

5.1 Metalevel change 1: embrace formalization to replace metaphors of epsilon strips with narratives about objects

Before the following episode, the students discovered that, for an infinite sequence to be convergent, there is a certain point $A(m)$ [N_ϵ] after which all following terms are “in” a selected epsilon strip. $A(m)$ is the result of a discussion where students determined that the “distance value” A is dependent upon (the choice of) m [ϵ] (see Fig. 1), in a functional sense. In that discussion, students also agreed that one has to take a *y-first perspective*, where one first has to choose a strip with a specific height (the y -value m) to find a corresponding A .

- 353 Lawrence there exists an A m , whom [*sic!*] contains all following elements
[Dennis writes for a second time: “there exists an $A(m)$,” Fig. 3]
- 354 Teacher Where should they then be contained in?
- 355 Lawrence Ah, for the, the #
- 356 Leif # smaller than the maximum value
- 357 Lawrence # the, smaller than or equal to m . [...]
- 359 Lawrence ehm, exists an A m whose d is smaller than or equal to m .
indicates that a student takes over a turn.

At the beginning of the episode, Lawrence substantiates the following predicate narrative: “whom [*sic!*] contains all following elements” (T353). This narrative is informal, rooted in the epsilon strip activity and the use of epsilon strips as containers (“container-schema”; Lakoff & Núñez, 2000).

Initiated by the teacher’s request to reframe this informal container narrative (T354), the students endorse the formal narrative “ d smaller than or equal to m ” (T355–T359). This new narrative has been endorsed previously because it was found to be equivalent to the former informal container narrative (T330, not shown here). This move towards formalism suggests the students begin to adopt metarules that favor symbolism and signifying relevant aspects of convergence through symbols. Therefore, metarules of formalization lead to the realization of the *predicate as formal property* “ d smaller than or equal to m .”

Formalization is here enabled by two previous semantic additive growths of the discourse. First, $A(m)$ has been reified into a function object before this episode. Following this previous reification, $A(m)$ seems here to be framed initially as a domain that contains all subsequent terms (“whom contains”), towards $A(m)$ as a functional object with certain properties (“whose”) towards $A(m)$ as a point (“at which,” “from this $A(m)$ it is true”). Hence, there are two complementary school-mathematical notions of $A(m)$ that are endorsed here as being complementary, namely, *$A(m)$ -as-object* and *$A(m)$ -as-point*, while the notion of a domain seems to be dropped in the following. Even though the functional perspective on *$A(m)$ -as-object* can be problematic (Section 2.1), it is here a beneficial interim notion of N_ϵ that helps the students to construct a formal predicate.

Für jedes m echt größer 0 existiert ein $A(m)$, bzw. d ist kleiner als $A(m)$

Translation: For every m greater than 0 exists an $A(m)$, ~~there exists an $A(m)$~~

Fig. 3 Partly developed definition at Turn 305

Second, the processes of selecting and moving strips have been reified into objects. As a result, d and m are not only aspects of the epsilon strip activity (as captured in the treasury, Fig. 1) but also objects in their own right (“abstract discursive object”; Sfard, 2008, pp. 172f). In particular, the distance d is realized as having specific properties, in this case being smaller than m . It can be speculated that school-mathematical narratives for variables, where variables are used to signify objects and have certain properties, are a necessary precondition for this reification.

In sum, the reification of $A(m)$, m , and d into objects throughout S2T2 allows the students to talk about these objects’ properties and how they relate to each other. The treasury provides the necessary foundation for these objectifications, as the treasury helps students to use shared symbolism and terminology in their construction of narratives about the epsilon strip activity. Furthermore, in commognitive terms, the treasury supports the development of $A(m)$, m , and d into abstract objects, which allows leaving behind the experiential notion of primary objects (“objects that are perceptually accessible”; Sfard, 2008, p. 173). Consequently, the students can replace informal talk about the processes of choosing m or finding $A(m)$ with more formal neighborhood narratives, enabling the adoption of the metarule of formality.

5.2 Metalevel change 2: acknowledge that statements have truth values

The discursive developments *A(m)-as-point* and *predicate as formal property* prepare a second metalevel change, namely, students’ transition towards investigating the truth value of statements. This second metalevel change becomes evident a few turns after Episode 1:

- | | |
|--------------|--|
| 367 Leif | Yes, but then we cannot write, “there exists a distance value” [3 sec.]. Can’t we? |
| 368 Lawrence | For me it did make sense, but |
| 369 Teacher | Well, what Lawrence attempted to say was that to the right of this distance value, there we need to think how we can describe this to the right. That to the right of this distance value all terms of the sequence are within this strip. |
| 370 Leif | Suggestion: There exists a distance value, an $A(m)$, [laughing], at which every d is smaller than or equal to m . |
| 371 Ludwig | Yes, if this here is valid [points at what Dennis has written down; Fig. 3] |
| 372 Lawrence | Yes, that’s what I already said. |
| 373 Leif | Ok, convinced. I was with the distance value a bit... |
| 374 Lawrence | [strikes second “there exists an $A(m)$ ” through; Fig. 3. Adds: “from which on”] |
| 375 Lawrence | [to teacher] can I now write: from this A m [onwards] it is valid d is smaller than or equal to m [2 sec] or from which on is valid? |

Based on the teacher’s request for a new description (T369), the students frame the *predicate as formal property* in terms of truth, which highlights a central metalevel change towards *predicate as the satisfaction of a formal property*. Ludwig initiates talk about truth in terms of satisfaction of a narrative: “Yes, if this here is valid [gilt]” (T371), referring deictically to the partial narrative “for every $m > 0$ exists an $A(m)$ ” (see Fig. 3). Later, Lawrence adopts the keywords “it is valid [‘gilt’]” (T375) to write down the final definition as

a description of $A(m)$ -as-point. The keyword “it is valid [‘gilt’]” highlights that the *predicate as formal property* is now framed in terms of satisfaction (Tarski, 1944). Satisfaction means that the arbitrary discursive objects $A(m)$, d , and m can be potentially replaced with a certain constellation of experiential objects in the epsilon strip activity, which would allow evaluating the truth value of the associated predicate narrative that connects $A(m)$, d , and m . Through the idea of satisfaction, truth has become a property of the narrative.

The nature of the satisfaction that students investigate suggests that the metalevel change *predicate as the satisfaction of a formal property* is facilitated through pragmatic discursive developments where talk about experiences is replaced with talk about narratives. Starting in Episode 1, the students have shifted from experiential narratives about the epsilon strip activity (container narrative; Section 5.1) towards a formal predicate. The students determine the satisfaction of this predicate pragmatically based on it being consistent with already endorsed narratives. This shift towards consistency of narratives is evidenced by Ludwig’s deictic language and the students’ negotiations of different phrasings (T367, 370, 375). It is enabled by reifying $A(m)$, m , and d into abstract discursive objects, as described in Episode 1. Consistency with previous narratives can be seen as a form of deductive order, which is a feature of university mathematics. Notably, Tall (1991) identifies the shift away from experiential narratives towards deductively organized narratives as a significant step towards university mathematics.

The *predicate as the satisfaction of a formal property* “it is valid that d is smaller than or equal to m ” (T375f) does not evolve into a quantified statement in the form of $\exists N \in \mathbb{N}, P(N)$. While the students connect this predicate to the existential quantifier with different keywords (“whom,” T353; “whose,” T359; “at which,” T370; “it is valid,” T375), the use of the existential quantifier keyword “there exists” remains a static element in a routine. Particularly, the students use of “there exists” acts as a jumping-off point for the routine of constructing the predicate (routine-driven use; see T353, T359, T367, T370). This use is evidenced by the fact that the existential quantifier is used in the same way as it has been written down originally as “there exists an $A(m)$ ” (Fig. 3). Accordingly, the use of “there exists” does not fulfill the function of quantification here. Therefore, students do not yet realize a quantified statement in S2T2.

5.3 Metalevel change 3: recognizing that multiply quantified statements are deductively ordered, and that the order of their quantifications is relevant

In Session 5, Task 2, students compare a textbook definition of pointwise continuity with the familiar formal definition of sequence convergence as realized in Session 2. The textbook definition of local continuity given to the students was phrased as follows:

A function f is continuous in $x_0 \in \mathbb{R}$ if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x - x_0| < \delta$ then it is true that $|f(x) - f(x_0)| < \varepsilon$

334 Leif No, ε cannot tend to zero, that’s why it is written the other way around. If for every delta [1 sec.] x minus x_0 is smaller than δ , then this is valid [colloquial “gilt”] [2 sec.] $f(x)$. And the δ can become, like, zero at the jump point, but the ε does not take every value, then not every ε exists [colloquial “gibt”] [2 sec.] I will do it with the middle line again, but in principle, if one takes ε over the concrete length or not, it does not matter. [draws as shown in Fig. 4]. Then epsilon can, hence, only ehm#

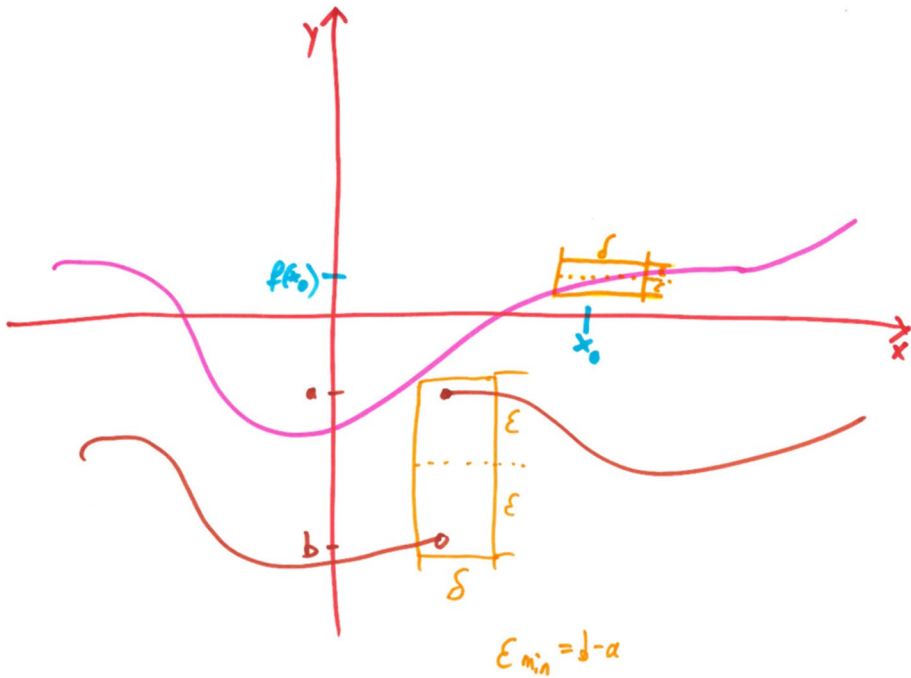


Fig. 4 Leif’s drawing of a discontinuous function

- 335 Dennis #till a certain line, and then it is not for all.
 - 336 Leif Yes (...) ϵ minimal equals # [writes $\epsilon_{min} = b - a$].
 - 337 Dennis #two, three or so.
 - 338 Ludwig Yes, that is written like that in the definition.
 - 339 Leif And because of that, for every epsilon that exists, I mean, not every ϵ , bigger than 0 [exists]. Where the definition says, if for every $\epsilon > 0$.
 - 340 Dennis And if it is not fulfilled, it is not continuous.
- # indicates that a student takes over a turn.

The episode illustrates a third substantial change in the metalevel development of the discourse on the formal definition of limits, namely, the syntactic *recognition of the deductively ordered nature* of the definition. This metalevel change is evidenced by Leif’s utterance that indicates a hierarchy: “that’s why it is written the other way around” (T334). While investigating a function with jump discontinuity, Leif finds that δ has to be “chosen” in order to decide whether an epsilon can be found (“if x minus x_0 is smaller than δ , then this is valid”; T334, T339). Similarly hierarchical, Leif connects the existential and universal quantifiers: “And the δ can become, like, zero at the jump point, but the ϵ does not take every value, then not every ϵ exists” (T334).

The change towards *recognizing deductive order* is closely related to the students’ substantiation of relevant quantified sub-statements. In the graphical representation, the students realize $|f(x) - f(x_0)|$ in terms of an epsilon strip, and argue that “I mean, not every ϵ , bigger than 0 [exists]” (T339). Consequently, the students investigate the correctness of the

implicitly present, singly quantified statement “For every ε , it is true that $|f(x) - f(x_0)| < \varepsilon$,” as suggested by Leif’s utterance “not every ε exists” (T334) or Dennis’ utterance “then it is not for all” (T335). Accordingly, the students seem to “carve out” this statement from the definition, allowing them to investigate its truth value separately. Notably, as in Episode 2, this statement’s truth value is determined pragmatically and semantically in terms of satisfaction based on both experiential narratives in the graphical representation as shown in Episode 2 and formal narratives *about* narratives (Section 5.2). For instance, Leif translates Dennis’ experiential narrative in T335 about neighborhoods into a formal narrative about ε (T335f; also evident within T334).

In contrast to the previous *predicate as the satisfaction of a formal property* in S2T2, quantification now has a crucial role, leading to the *singly quantified statement* “For every ε , it is valid that $|f(x) - f(x_0)| < \varepsilon$.” The students’ starting point for substantiating this *singly quantified statement* is the investigation of a counterexample, during which the students keep the existential quantifier narrative static. In particular, the students assume the existence of δ as a precondition that is met: “If for every delta (\cdot) x minus x_0 is smaller than δ , then this is valid” (T334). Accordingly, students assume an *x-first perspective*, because δ is the foundation for the further investigation of the *singly quantified statement*. By focusing on the singly quantified statement, particularly the predicate “ $|f(x) - f(x_0)| < \varepsilon$,” and by using the δ -narrative as starting point, they change their perspective towards a *y-first, x-fixed perspective*. This coordination of the two perspectives is likely enabled by drawings of epsilon-delta rectangles and the given textbook definition, that is, by pragmatic considerations. These considerations allow Leif to use deictic language to refer to previous narratives and to coordinate the narratives’ hierarchical order.

The discursive developments in this episode build on and extend the previous meta-level changes from S2T2. First, concerning the first metalevel change of formalization, the students continue the objectification of central elements (Episode 1; Fig. 2, left side). For instance, $|f(x) - f(x_0)|$ is treated as an object with properties, and the students conceptualize distances in the form of neighborhoods. Second, the students connect experiential narratives with formal narratives that reference previous narratives (Episode 2; Fig. 2, middle), as highlighted above for Leif and Dennis (T335f). Third, truth values are determined through the satisfaction of narratives, in this case of quantified statements (Episode 2).

5.4 Metalevel change 4: realize a multiply quantified statement as an assemblage of partial narratives.

Concerning metalevel change, the episode above highlights a fourth metalevel change of students adopting the notion that a *multiply quantified statement is an assemblage of partial narratives*. Students achieve the multiple quantifications of the textbook definition by adopting the metarule that a quantified statement is endorsable if it adequately describes the epsilon-delta rectangle in both the *x-first* or *y-first x-fixed perspective*.

The metalevel change towards *multiply quantified statement as the assemblage of partial narratives* is enabled by all previous discursive developments. *Formalization* prepares talk about properties of formal objects such as ε , and the focus on the *truth value* of narratives prepares the investigation of singly quantified statements, which in turn allows students to “carve out” partial statements from a multiply quantified statement and to investigate them separately.

6 Summary and discussion

This paper contributes to knowledge about how students learn to understand and construct multiply quantified statements. It offers three major contributions:

First, it identified *four substantial changes* in the ways in which students need to relearn school mathematics when they engage with the construction of multiply quantified statements while defining limits in tertiary mathematics: students have to (1) endorse a predicate as formal property by replacing metaphors of epsilon strips with narratives about the objects ε , N_ε , and $|a_n - a|$; (2) acknowledge that statements have truth values; (3) recognize that multiply quantified statements are deductively ordered and that the order of their quantifications is relevant; and (4) assemble a multiply quantified statement from partial statements that can be carved out from a multiply quantified statement and can be investigated separately. These substantial changes extend previous knowledge on genetic decompositions for learning limits (Cottrill et al., 1996) and learning quantifier relations (Dubinsky et al., 1988; Dubinsky & Yiparaki, 2000) by highlighting how the norms (Yackel & Cobb, 1996) or metarules (Sfard, 2008) of what counts as acceptable mathematics need to change throughout learning formal definitions of limits.

Second, the findings emphasize the roles of language, particularly of semantics, pragmatics, and syntax in students' learning to construct multiply quantified statements, confirming previous research on the topic (Dawkins & Roh, 2020; Durand-Guerrier & Arsac, 2005). Particularly, with respect to semantics (see its commognitive definition in Section 2.2), this paper was able to show that students' constructions of definitions were enabled by the experiences in the epsilon strip activity and especially, the chain of references to familiar school-mathematical objects of functions and variables that this activity elicited. These semantic considerations informed students' attempts at making multiply quantified statements—initially in the form of talk about concrete material actions and later as talk about the epsilon- N /delta rectangle both as graphical representation and as a signifier of abstract objects ε , δ , N_ε .

In addition, concerning pragmatics, it was found that students relied heavily on school mathematical precedents to frame the quantified nature of ε and N_ε/δ . Naturally, the above-described semantic chain of references to familiar school-mathematical objects also requires students to have knowledge about these objects. Prototypically, the precedent narrative about the function object shows how precedents can act as interim narratives on the students' pathway towards the intended definitions of limits. Previous research has suggested that students need to relearn school mathematics in substantial ways (Stadler, 2011; Thomas et al., 2015). Yet, this proof of existence of the relevance of interim narratives highlights the importance of school mathematics in understanding the formal definitions of limits. This finding points towards the need for research to further investigate the potential of carefully connecting school mathematics with tertiary mathematics to enhance students' learning. However, it also needs to be noted that interim narratives can be a double-edged sword: as an interim narrative, the school-mathematical narrative about functions enables relevant metalevel changes. Yet, in the long run, this interim notion might turn out to be problematic, as ε and N_ε/δ are not functionally related (see Roh, 2010).

Furthermore, this paper was able to provide empirical evidence that pragmatics plays a relevant role in making true statements. Previous research has found no evidence that pragmatics has an impact on making true statements, even though it has been hypothesized (Dawkins & Roh, 2020). This paper presents evidence that students' previous everyday notions of truth—in terms of satisfaction of a property—are a crucial prerequisite to

constructing quantified statements in terms of their truth value. Thus, the presented findings extend previous knowledge and suggest that the previously hypothesized double role of pragmatics for (a) making true statements and (b) making “statements that are worth saying” (Dawkins & Roh, 2020) does indeed need to be considered when designing learning opportunities.

With respect to syntax, which was conceptualized here as the rules by which a new narrative can be constructed from simpler ones (Sfard, 2008, p. 102), it is evident that the construct of metalevel change allows identifying the emergence of formal syntactical rules. The metalevel changes found (Fig. 2) illustrate how students adopt new metarules for how to construct new narratives and how these narratives relate to one another, leading to the construction of quantified statements. These metarules are heavily informed by semantic and pragmatic factors, extending previous research on the topic (Dawkins & Roh, 2020; Durand-Guerrier, 2003; Durand-Guerrier & Arzac, 2005). For instance, the investigation of a singly quantified statement is enabled by the idea of satisfaction of properties (Tarski, 1944), which is informed by students’ pragmatic considerations of everyday precedents. Beyond semantics and pragmatics, the findings in this paper support the hypothesis that metarules are also tightly intertwined with syntactical rules, as evidenced by the third substantial change of deductive order where sub-statements are coordinated through syntactic rules (Section 5.3). Yet, even though this paper finds some evidence that metarules could relate to syntactical rules, further research is needed to understand whether and how syntactical rules develop from, or together with, metarules. For instance, activities of defining should allow students to engage with rules of formal logic (Chorlay, 2019), but from the perspective of discourse development, it is unclear if students also adopt new metarules while doing so. Possibly, to investigate this issue, activities where students use previously constructed quantified statements *for* their mathematical reasoning (*definition-for*, Zandieh & Rasmussen, 2010, p. 58) could give students opportunities to engage with mathematical statements in a more syntactical way (Durand-Guerrier, 2003). By adopting commognitive theory, such a study could add substantial insights into how syntactical rules emerge from students’ pragmatic and semantic considerations.

Finally, and thirdly, this study contributes to commognitive theory. It exemplifies how commognition can inform the design of learning activities that facilitate specific discursive developments (Schüler-Meyer, 2020; Viirman & Nardi, 2021). Furthermore, commognitive theory, and particularly the notion of metalevel change, has shown their potential for studying the language resources that students use in learning to engage with (multiply quantified) mathematical statements. Accordingly, future commognitive research could use the presented construct of metalevel change to design learning activities that facilitate students’ learning of syntactical rules for interpreting and constructing mathematical statements. More generally, while commognitive theory is suitable to design a learning trajectory, the here identified metalevel changes were not predicted in the a priori analysis. Even though this finding is not surprising, it nevertheless strengthens calls for local instruction theories as basis and as result of design research (Prediger et al., 2015) also for commognitive theory. In fact, the *four substantial changes* above could support such a local commognitive theory for constructing multiply quantified statements, as they could enable designing learning activities that specifically facilitate these changes.

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Declarations

Ethics approval The author states that the research complies with ethical standards.

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