Educational Studies in Mathematics (2022) 110:271–290
https://doi.org/10.1007/s10649-021-10123-3

Beyond categories: dynamic qualitative analysis of visuospatial representation in arithmetic

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Accepted: 31 October 2021 /Published online: 5 January 2022

Abstract

Visuospatial representations of numbers and their relationships are widely used in mathematics education. These include drawn images, models constructed with concrete manipulatives, enactive/embodied forms, computer graphics, and more. This paper addresses the analytical limitations and ethical implications of methodologies that use broad categorizations of representations and argues the benefits of dynamic qualitative analysis of arithmetical-representational strategy across multiple semi-independent aspects of display, calculation, and interaction. It proposes an alternative methodological approach combining the structured organization of classification with the detailed nuance of description and describes a systematic but flexible framework for analysing nonstandard visuospatial representations of early arithmetic. This approach is intended for use by researchers or practitioners, for interpretation of multimodal and nonstandard visuospatial representations, and for identification of small differences in learners' developing arithmetical-representational strategies, including changes over time. Application is illustrated using selected data from a microanalytic study of struggling students' multiplication and division in scenario tasks.

Keywords Arithmetical problem-solving · Multimodal data · Multiplicative reasoning · Qualitative methodology · Representation · Inclusion

1 Introduction

Representation is central to mathematical activity, such as for summarizing information, reasoning, offloading memory storage, coordinating results of intermediate calculations, and recording and communicating numerical or functional relationships (Zahner & Corter, 2010). As commonly interpreted in mathematics education research, external representations are any visible or tangible productions that encode, stand for, or embody mathematical ideas or relationships (Goldin, 2018). Their significance in learning is widely acknowledged, and various kinds of representation have attracted the attention of educators, being

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studied through many theoretical lenses and empirical methodologies, particularly since the 1980s (Presmeg, 2006), when Hughes suggested:

> [O]ur understanding of learning and teaching mathematics might well be enhanced if we can identify those images and analogies which are particularly useful in connecting the formal and the concrete - and conversely those which are not. (Hughes, 1986, p.171)

Mathematics pedagogy has been enhanced by now decades of study of imagery, and a recently increasing focus on patterns (Mulligan et al., 2020). However, there have been pervasive assumptions and traditions which are problematic, relating to the ways in which both representations and the learners who use them are described, classified, and, in too many cases, pathologized. Firstly, a great deal of research has contrasted mathematical representations which are considered to be 'formal', 'abstract', or 'standard' with those considered informal, concrete, or nonstandard (chosen terminology varies) as though these were distinct binaries. More generally, the extent of diversity of representational strategies to be found, even within a single mathematical idea or task type, is insufficiently widely acknowledged. Secondly, there are frequently explicit or implicit a priori assumptions regarding hierarchies of representational forms, and thus of those who use them. (This relationship is contextual: when world-class mathematicians such as Maryam Mirzakhani have worked on new material primarily through visuospatial representations, it is commonly considered unusual but brilliant; when children do so in school mathematics, it is commonly interpreted as indicating lower potential (Ben-Yehuda et al., 2005).) This second point has a corollary: we must ask not only which (and whose) representations are valued, and why, but which (and who) are excluded from datasets, or researched as a separate 'atypical' group compartmentalized from the normalized population.

This paper adds to the small but growing body of literature and discussion (e.g., within the representations working group at CERME (Baccaglini-Frank et al., 2019)) that not only acknowledges but appreciates the diversity of both observed representational strategies and the learners who use them, while looking for more flexible, nuanced, and inclusive analytical approaches. In the first half, I address some of the theoretical, methodological, and ethical issues in researching representational activities, particularly of learners to whom mathematics does not come easily, and, in the second, share an example of an alternative framework for qualitative analysis of arithmetical-representational strategies. In Finesilver (2017b), I argued for teachers to observe and value the diversity and creativity in learners and their representations; here I argue for researchers to do so.

The specific research objective addressed in this paper is to construct an analytical framework that can capture the diverse representational strategies developed when students work on multiplication- and division-based problems which they are unable to solve through standard methods.

This paper is not a report of empirical findings, but uses selected data from a wider project to illustrate points of interest. Details of the original research context are provided below, after the theoretical background.

1.1 Example 1: Disappearing vehicles

Before addressing theories and analyses of arithmetical representation, an introductory illustration of the phenomena of interest may be helpful to the reader. Consider the two visuospatial representations shown in Example 1, produced by two dyslexic 13-year-old students during a series of multiplication- and division-based tasks based on scenarios involving vehicles. (Refer to Table 1 for definitions of italicized terms if unfamiliar.)

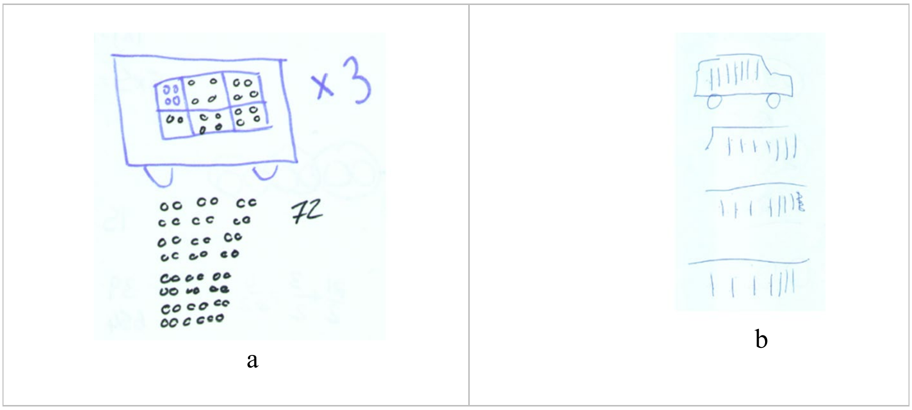

Fig. 1 **a**–**b** Two representations with intra-representation changes to the mathematically functional and non-functional elements depicted (Wendy, George). Changing: resemblance, spatial structuring. Constant: unit-countability (also media, mode)

Figure 1a represents three vans, each of which contains six boxes, each of which contains four bottles; the task was to calculate the total number of bottles. The participant (Wendy) was initially unable to make any attempt at this at all, so was provided with a prompt: the researcher drawing one 'van' container with six 'box' containers in it, one of which had four 'bottles' in it (purple ink). Wendy first filled in 'bottles' for the remaining 'boxes' in the first 'van' (black ink) and then continued to draw all units of the remaining groups-within-groups, but without the 'vans' or 'boxes'. Of particular interest here is that she has replicated the spatial configuration of units in equal groups, but without the use of these containers to delineate these groups. You may also notice she made a minor error (two missing bottles) but still gave the correct total: this may be surprising, considering that enumeration was done by counting all units. This correct count was enabled by the rhythmicity of Wendy's enumeration; with a steady pattern of four numbers being counted verbally and emphasized with a tapping finger moving with regularity between sets, she did not notice that one box actually only contained two visible marks.

Figure 1b was entirely independent work, produced by George to calculate the number of 7-seater taxis required to transport 28 passengers. His first 'taxi' is car-shaped and has wheels that some might call 'decorative', as they function neither as units to be counted nor in organizing the units to be counted. (The role of such elements is addressed below.) Similar to Fig. 1a, in the replications, increasing portions of 'car' are discarded as unnecessary. By the end, his representation is becoming the rows and columns structure of a unit array with the divisor in each row; dot arrays turned out to be a representational form he much favoured in later tasks.

The *inconsistency* of resemblance and spatial structuring is the main point to which I draw attention here. In both cases, the representational strategies were initiated with a quite high level of visual *resemblance* to the scenario as described (i.e., the first 'van' and 'car' are recognizable as such), but then the non-mathematically functional elements disappeared not between one task and the next, but intra-representation, as did the container forms that had been part of the initial *spatial structuring* of the units (visually delineating the equal groups). Meanwhile, other aspects of the representations remained constant: the mode, media, their *unitcountable* nature (i.e., the total quantity represented with one-to-one correspondence), and *enumeration* (rhythmic/grouped counting of units). To simply classify these arithmetical problem-solving representations as ‘pictorial’ rather than ‘abstract’, or ‘concrete’ rather than ‘formal’ (etc.), would miss these details and potentially important changes.

2 Theoretical background

2.1 Representation and arithmetic

It is generally thought that most learners’ arithmetical-representational strategies begin with intuitive actions such as manipulating physical objects and end with standardized symbolic forms. Theorization of this developmental trajectory bears a lasting influence of Piaget’s ‘stage theory’ of progression from sensorimotor to formal (Piaget, 1952), and Bruner’s enactive, iconic, and symbolic (EIS) ‘modes of representation’ (Bruner, 1974).

In their early encounters with quantitative relations involving natural numbers, children become aware of concepts such as conservation of quantity, counting, and additive relations, through interactions with collections of objects. For example, addition as the joining of collections of objects, and subtraction as removing a subset of objects from a collection—in which ordering of objects is unimportant—can be considered basic conceptual ‘grounding metaphors’ (Lakoff & Núñez, 2000). These metaphors, also described in the literature as ‘primitive’, ‘intuitive’ (etc.), underpin various early models for arithmetic, not only counting and additive reasoning but more cognitively complex quantitative relationships, such as ratio and proportion. Metaphors for arithmetic are not the same as representational strategies, but the two are linked; for example, Lakoff and Núñez’s (2000) ‘Arithmetic as Object Collection’ metaphor can be extended from addition and subtraction to multiplication and division, by seeing these as operating with equally sized sets of objects. The increased complexity of multiplicative relationships (compared to additive) is widely considered a significant qualitative change (Greer, 1994; Nunes & Bryant, 1996) but involves struggle for many (Brown et al., 2010). Anghileri (1997) has noted that early models of multiplicative relationships generally seem to involve sets of items organized into equal-sized subsets; many examples of this have been observed. However, the many and varied ways to metaphorize and/or visuospatially represent the nature of one-to-many correspondence deserve closer attention.

The visuospatial representations made by children as they explore mathematical relationships are diverse, creative, and sometimes unexpected (Deliyianni et al., 2009; Worthington & Carruthers, 2003), compared to those for whom the material has become familiar and routine. Learners representing their mathematical thinking in not-yet-routinized ways can be a particularly valuable source of data for researchers, analysis of which can provide a window into early mathematical understanding and reasoning, in particular the awareness of patterns and structures which is fundamental to mathematical development (Mulligan et al., 2020).

However, where linear sequences of stages are theorized, certain representational characteristics can become associated, explicitly or implicitly, with particular ages or curricular stages—even if, like Bruner’s EIS, they were not intended this way. Although theoretical perspectives have broadened to move away from traditional notions of mathematical reasoning as 'abstract' and 'disembodied, to contemporary views that it is 'embodied' and 'imaginative' (English, 2013), or involves dynamic and recursive cycling between representational modes (e.g., Mason, 1980; Pirie & Kieren, 1994), there is still a widespread view in both research and practice that the less that arithmetical representations superficially resemble formal symbolic notation, the lower the level of arithmetical thinking demonstrated. In classrooms, learners' observed representational strategies may be judged immature: it is not uncommon to hear rigid views like 'By [X] age, children should no longer…' expressed by teaching staff. Meanwhile in research, overly categorical, linear, and/or age-linked interpretations of representational trajectories also have several significant methodological and ethical issues; furthermore, I will argue below that the effects of this may both stigmatize and pedagogically disadvantage 'atypical' learners.

2.2 Diversity of learners' representational strategies

Firstly and most obviously, while the application of broad stage models to arithmetical representations may have some use in population-level description, it is well established that individual development is 'messier' and should not be treated as progressing linearly through neat stages. At a given time, individual children and adults may use a variety of strategies for a given arithmetic problem (Baroody & Tiilikainen, 2003; Dowker, 2005)—considered by some to reflect an 'overlapping waves' model (e.g., Opfer & Siegler, 2007; van der Ven et al., 2012). Problem-solvers may use 'backup strategies' (e.g., Gonzalez & Espinel, 2002; Ostad, 1997; Zhang et al., 2013) or vary their strategies adaptively to meet the situational demands (Siegler, 1988); this flexibility is valuable. It is not always stated explicitly that such strategic variability may apply not only to the calculations carried out, but to their representational preferences.

Secondly, the representative strategies employed by learners are, unsurprisingly, strongly influenced by the implicit and explicit expectations of the classroom. For example, Deliyianni et al.'s (2009) comparison of children in pre-school and the first year of school describes 'visual creativeness' turning to 'obedience to the didactical contract rules' (p.108) regarding representational forms. This is unsurprising in educational cultures in which formal symbolic notations are perceived as 'the almost sole desired and valued outcomes of mathematics learning' (Karsenty et al., 2007), and Mason (1980) addresses some of the educational problems that result from using symbols without due consideration. While few would argue that symbolic notation and standard calculation procedures are not highly useful mathematical tools, an education system that rushes learners into exclusively symbolic notation can result in the performance of conceptually empty 'maths-like behaviours' made of fragments of poorly understood symbolic representation, and the unfortunate acceptance (or even misguided encouragement) of this by teachers (Finesilver, 2017b). In fact, analysis of individuals' visuospatial representational strategies would ideally be considered not only in terms of individual cognitive development, but in relation to the educational contexts in which they learn. In practice, this is a considerable challenge, and may not be possible in smaller studies, but one can at least avoid treating representational choices as though they were intrinsic to an individual mind and independent of classroom cultures and norms.

There is a third issue with overly linear categorical stage models that is surprisingly absent from much of the discourse. Where there exists a perception of what 'typical' development or progression looks like, a contrasting group is thus implicitly created of all learners with 'atypical' development (often interpreted as those who have failed to progress as expected according to local norms). Some studies—particularly those using quantitative methods—have contrasted the strategies of participant groups predefined on such bases, e.g., 'normal'/'mathematically disabled' in Ostad (1997), 'high'/'low achievers' (Gray et al., 2000; Karsenty et al., 2007), 'highly-' or 'less-able' (Mulligan et al., 2020), and 'typically achieving students'/'poor performance'/'arithmetic learning disabilities' (Gonzalez & Espinel, 2002). While this is sometimes done with good intentions—such as better understanding difficulties in order better to support learning—there is a danger of considerable oversimplification in dividing up student populations in this way, and problematizing those who do not or cannot comply with narrow culture- and context-dependent expectations. As noted by Lewis (2014), 'disability' is overwhelmingly conceptualized in this literature as individual cognitive deficit, rather than, e.g., social or interactionist constructions where cultures bear responsibility for disabling those individuals through failure to support diverse paths of development.

This relationship between representation types and assumptions about 'ability' is cyclic: normative hierarchical assumptions regarding age-appropriate arithmetical understanding, calculation methods, and representational strategies not only derive from but feed back into perceptions of learner attributes such as attainment, educational dis/ability, individual capacities for 'abstract' or 'symbolic' reasoning, and more. Thus, observation of learners already judged as 'high/low ability' or 'normal/disabled' (etc.) leads to views of what types of representational strategy are desirable, while hierarchies of representational strategy lead to the judging of those who use them as having 'ability', being 'normal', etc.—or not. There have been attempts to use representations to categorize learners in ways which are intended to convey neutral differences rather than positive/negative judgements (e.g., as 'visualizers' or 'verbalizers'), but these still result in simplistic assumptions about individuals and their capacities (e.g., for 'abstraction') which may narrow future learning experiences (Cox, 1999).

While consideration of the ethical implications of diversity within student populations may be found elsewhere in educational research (e.g., studies of academic selection and grouping in schools), it is not as yet sufficiently acknowledged within the subset dealing with mathematical representation.

2.3 Categorizing and describing representations

As well as the ethical issues of classifying learners discussed above, there are also methodological issues relating to the actual assignment of representations to unilateral categories. There have been many taxonomies employed, and the aim here is not to critique individual systems, but to address the ongoing cumulative issues of this theoretical practice overall in (mis-)shaping how we understand learners' mathematics.

There are two particularly influential forms of empirical research in arithmetical representation which historically contributed to oversimplified analytical frameworks and overgeneralized conclusions. The first is teaching experiments where participants were instructed to perform tasks either with or without specific researcher-provided manipulatives (see Carbonneau et al. (2013) for examples). The second is analyses of graphic imagery consisting of a set of preordained theoretically-generated types—either involving images chosen by researchers and presented to participants, or with participants being directly instructed to draw a 'diagram'/'picture'/etc. for tasks. Some examples of popular categorizations are *decorative*, *representational*, and *organizational* (originally from Levin (1981), referenced in Carney and Levin (2002); Elia et al. (2007); and Elia and Philippou (2004)); *concrete*, *pattern*, *kinaesthetic*, and *dynamic* imagery (Presmeg, 1986); or *dramatic*, *physical*, *pictorial*, *verbal,* and *symbolic* representations (Verschaffel & De Corte, 1996). One issue pertains to how the taxonomies cited above have been recruited by later researchers in different pedagogical and research contexts—for example, using them to assign a single assumed function or role to participant-produced representations, with no additional data for triangulation (e.g., their accompanying verbalizations, gestures or explanations).

A related issue is that many participant-produced representations do not fall neatly into categories, even within quite narrow methodological limitations. For example, learners who are not forced into either concrete modelling of tasks or drawing may combine both (see Fig. 2), perhaps along with symbols. Representations may be neither wholly 'decorative'/'pictorial'/etc. nor 'organizational'/'schematic'/etc. but combine elements of both (see Fig. 4). This has not stopped some researchers choosing still to unilaterally divide all participant-produced representations into only two categories—e.g., deciding that they must all be either 'schematic' or 'pictorial' (Hegarty & Kozhevnikov, 1999). While some research explicitly acknowledges the non-dichotomous nature of their data (e.g., Ainsworth et al. (2002), who included 'mixed (pictorial and mathematical)' as a category), this still frames representations that fit neatly into categories as the norm, and those that do not as atypical. I suggest that diverse 'mixed' arithmetical-representational strategies should be both expected and valued.

Of the studies involving collecting and analysing the images and models produced by students during mathematical problem-solving, some choose descriptive rather than taxonomic approaches, which allows greater nuance and detail. For example, Saundry and Nicol (2006) used thick description of how students produced and interacted with their images. Such approaches have been effective in identifying the misconceptions and partial understandings of learners struggling with basic arithmetical reasoning (Karsenty et al., 2007), or different ways children use imagery in their numerical processing (Gray et al., 2000). Regarding the ethical concern above, studies using more descriptive qualitative approaches are less likely to contrast groups of learners pre-judged as 'high' or 'low', but have rather indicated significant intra-group variation in arithmetical problem-solving (Brown et al., 2008) and associated representational strategies (Deliyianni et al., 2009). However, there are limitations in analysis based predominantly on rich description: it does not lend itself well to structured analysis of inter- or intra-participant patterns.

The rise in qualitative microgenetic methodology (see Siegler and Crowley (1991) for an introduction) has been particularly helpful in shedding light on individual developmental trajectories (Meira, 1995; Voutsina, 2012) and understanding differences as well as difficulties (Fletcher et al., 1998). Meanwhile, dynamic testing and assessment approaches have been positively influential in developing targeted educational interventions, not to mention critiquing the classification of children (Elliott et al., 2018). Wide availability and ease of recording and digitization of data have provided increased opportunities for observing microprogressions, cooperative work, and more. When these methodologies are applied to arithmetical representation, more flexible analytical frameworks are required to unlock the potential of the data.

The analytical framework described below is intended to illustrate a way of combining the structured nature of categorical models with the detailed nuance of descriptive methods and so address some of the main issues that can constrain the study and appreciation of diverse representational strategies in mathematics. An effective analysis of visuospatial representations should be able to address issues of both form and function while acknowledging their interrelation, and the ways an inclusive spectrum of participants actually interact with and employ the various representational elements. Thus I propose conducting analysis of arithmetical representations not in terms of any singular category choice, but along *multiple* simultaneous aspects of representational strategy, as outlined below.

3 Research context

Although the main concern of this paper is theoretical principles and methodology for qualitative analysis, not the reporting of empirical results, an explanation of the context in which the framework was developed is needed, as some research data are included as illustrative examples. The overarching project (Finesilver, 2014) used microgenetic methods to study emerging and developing multiplicative structure in students' visuospatial representations.

3.1 Participants

Appropriate representational experiences are of importance for not only younger learners encountering new concepts (Goldin & Shteingold, 2001) but older ones who are still struggling (Jitendra et al., 2016). With an aim to exploring the nature and extent of representational diversity in this latter group, I collected/recorded representations produced by or with 11–15-year-old students during problem-solving activities in a specific area of arithmetic (multiplication and division) with which they were not as secure as might be assumed. The participants attended two typical inner London schools and were selected through identification by teachers as struggling the most in mathematics compared to their peers. Unsurprisingly, this group turned out to be diverse across multiple axes of identity, including neurodiversity. While neurological diversity is 'an inherent and valuable part of the range of human variation' (Dyck & Russell, 2020), it is invisible in much research literature, and thus the opportunity for a methodologically inclusive approach was welcome.

3.2 Data

The data derived from a series of six interactive problem-solving interviews methodologically situated at a point between clinical interview and naturalistic pedagogic activity, with the dynamic assessment principle of first gauging unassisted capability on a task, then calibrating support (if any) in situ to the needs of the individual (Elliott et al. 2018). The guiding principles regarding representation were (1) strong encouragement of students' freedom to follow their own preferences, ideas, and strategies and (2) absence of time pressure on tasks (see Finesilver (2017b) for further discussion of this point).

The two main task scenarios employed were 'Biscuits' (partitive division, where a number of biscuits is to be shared between a given number of children) and 'Passengers' (quotitive division, where one calculates the number of vehicles required to transport a given number of passengers). A full list of tasks and individual rationales may be found in Finesilver (2014), and detailed analysis of two particular tasks in Finesilver (2009, 2017a). The representational media available were multilink cubes, coloured pens, and paper.

The main dataset consisted of over 200 unique visuospatial representations collected from task-based activity (an exact figure is not possible, as participants sometimes re-appropriated whole or parts of their prior representations for subsequent tasks), many of which contained multiple mixed concrete, drawn, symbolic, and/or textual elements that defied any categorization to be found in prior literature. These were accompanied by audio recordings and contemporaneous field notes for indications of how the visuospatial representational elements related to calculation.

3.3 Analysis

Using a bottom-up research methodology influenced by grounded approaches, I collected the data first and then identified patterns through a process beginning with thick description of the arithmetical-representational activity, followed by repeated coding and sifting of the full set of collected representations, comparing representational elements and the relations between them, and their use over time, participants, and task types (see Finesilver (2014)). Constructing the analytical framework involved returning to the data repeatedly to check the constructs and sense-making processes, both reflexively and with peer debrief (Creswell & Miller, 2000). This resulted in the final set of thirteen analytical aspects in Table 1, which were tested qualitatively for their power in pinpointing changes, discriminating between superficially similar but differently functioning representations, and teasing apart the interconnected and independent variations in arithmetical-representational strategies.

4 The analytical framework

The complete list of analytical aspects is presented here, organized for ease of application into three groups. This is followed by further explanatory detail. Note that while certain aspects are specific to multiplicative relationships with natural numbers, some could also be applied to other arithmetic (e.g., additive structures, rational number work) or problem-solving more broadly.

Table 1 General framework of aspects for qualitative analysis of visuospatial arithmetical-representational strategies

Aspects that may be determined from the finished representation	
Media	means of production, e.g., cubes, pen/paper, fingers, pixels on screen
Mode(s)	means of meaning-making, e.g., modelling, drawing, words, symbols
Resemblance	visual resemblance of the drawing/model to the task scenario
Spatial structuring	visuospatial organization of representational elements (e.g., groups of units) through separation in space, use of containers, alignment in one or more dimensions, etc
Unitcountability	if unitcountable, each represented unit (cube, tally mark, etc.) = 1, and enumeration could be achieved by direct unitary counting (i.e., counting them *in ones*); if non-unitcountable, the 'ones' are not individually represented
Completeness	whether the student produces a complete set of observable externally represented elements (a special form of *consistency*)
Strategic soundness	whether the strategy would produce a correct solution if no errors are made in its execution
Aspects of arithmetical-representational strategy to observe in action	
Motion	whether the representation is static once created, or involves ongoing movement of elements while in use for calculation
Enumeration	how quantities are derived, e.g., unit-counting, step-counting, number fact retrieval
Consistency	whether a single coherent strategy is used from start to finish, or changes occur
Execution errors	e.g., in number fact retrieval, verbal count sequence, finger pointing movement, notation
Aspects of interactive co-production	
Verbal	e.g., teacher's (or other person's) spoken prompts, suggestions for calculation
Visuospatial	e.g., teacher's (or other person's) gestures, participation in modelling/drawing

4.1 Aspects that may be determined from the finished representation

First are the immediately observable characteristics of a representation, which do not necessarily relate to the precise ways in which the elements actually do function in calculation (or in terms of focus, affect, etc.).

Traditionally, representations for scenario-based tasks have been compared based on how much they visually resemble the stated scenario or its underlying structure. However, as discussed, these have often been in constructed binaries, for example, 'abstract' versus 'concrete' (e.g., Gray et al., 2000) or 'schematic' versus 'pictorial' (e.g., Hegarty & Kozhevnikov, 1999). These tend to conflate three aspects which may actually vary somewhat independently. *Media* and *mode* are common aspects of representational analysis deriving from social semiotics (see, e.g., Bezemer and Kress (2008) for further details); mathematical problem-solving representations are often multimodal and maybe multimedia. Furthermore, conventional media and mode choices are not intrinsically more 'abstract' or mathematically advanced than others. For the purposes of more detailed qualitative analysis, media, mode, and resemblance should be considered separately, with *resemblance* better considered not categorically but on a spectrum between literally enacting the scenario with the actual objects stated (e.g., sharing out some actual biscuits between actual people) and calculating using a standard configuration of symbols which could represent any isomorphic calculation. Two or more representations (whether by different individuals, or the same individual at different points in time) might be compared as having higher or lower resemblance through noting the presence/absence and depiction of different elements.

The *spatial structuring* of representational elements and the methods of *enumeration* for which they are employed are also sometimes unhelpfully conflated, as is the *unitcountability*—often the result of attempts to focus on only one of these aspects, but then not wishing to entirely ignore the others. The study of calculation aspects of arithmetic (such as whether and how a participant counts, calculates, or retrieves an answer from memory), while receiving a great deal of attention, has historically taken place separately from the analysis of visuospatial representations. The organization of representational elements within the workspace (where present) is highly relevant to the calculation process—but again, they can vary independently. This is particularly the case in unitcountable representations (e.g., arrays of dots and squares in representing multiplicative structure, as compared by Izsák (2005), or container and array representations in scenario tasks by Van Dooren et al., (2013). Individuals may represent the same calculation in different ways and, conversely, calculate differently with the same representation; furthermore, equivalent spatial structurings may be found in different modes/media (see Example 2 below). While unitcountability and a great deal of the spatial structuring may be determined by looking at an inscription or model alone, they must be observed in use to see whether the user actually counted (in ones or otherwise), or the order in which structures were constructed; it is necessary to separate how the representation appears from how it is employed. Note that unitcountability was highly salient in this particular project, as one-to-many correspondence is considered by some to be the origin of multiplicative reasoning (Nunes & Bryant, 1996), and a learner moving from one-to-one correspondence to using a single element (whether symbol, icon, or object) to stand for a larger number, is clearly a very important indicator of arithmetical development; it may be less so elsewhere.

In common usage, *completeness* has positive associations and incompleteness negative; this is not the case here. Like the move from unitcountable to non-unitcountable representation, an incomplete representation means that not all units are individually represented, which is particularly telling when combined with a *sound strategy* and/or correct answer. However, it is noted that these are not always separated out in quantitative research paradigms, although they provide different information about the participant's arithmetic.

4.2 Aspects of arithmetical-representational strategy to observe in action

Motion has for some time been considered a relevant aspect of visuospatial representation, whether it be embodied cognition of gesture in expressing mathematical concepts (Broaders et al., 2007), the kinetic deployment of fingers as countable media (Anghileri, 1995), rearranging concrete units into equal group configurations (Squire & Bryant, 2002), or dynamic virtual manipulatives (McLeod et al., 2012). Motion has been analysed qualitatively as well as quantitatively and is not confined to particular media, e.g., Saundry and Nicol (2006) described patterns of movement in students manipulating pictures on the page, moving, eliminating, sharing, and distributing. For an example of how motion relates to both spatial structuring and enumeration, consider a student counting while drawing, pointing to, or moving units (see Example 2 and Finesilver (2017a)).

Enumeration, as mentioned above, should ideally be determined from observation. While there may be indications that remain—for example, a symbolic notation of calculation—these can be misleading, e.g., a learner writing what they believe is culturally expected, having in fact derived their answer a different way. Similarly, observing how an arithmetical-representational strategy was carried out may provide salient information on what *execution errors* caused an incorrect answer from a *sound strategy*. *Consistency*, like completeness, is a neutral descriptor, and inconsistency can be a positive sign; for example, while changes in spatial structuring and unitcountability may be seen from the finished representation, the researcher may observe an individual begin their enumeration by unitary counting and then change to step-counting partway through.

4.3 Aspects of interactive co-production

These aspects refer to the potential involvement of a third party in the representational activity. It is increasingly the case that teacher-researchers, particularly those of a more Vygotskian bent, interact with participants in 'co-construction' (Carruthers

& Worthington, 2011), becoming involved via verbal or visuospatial interactions in a joint representational/semiotic process. The distinction between reasoning with self-produced representations versus those presented by others is vital (Papert, 1993), and it is not coincidental that the need for this framework became apparent during a study where learners were explicitly encouraged to be creative in their representational strategies.

5 Applying the framework: further examples

Clearly the tiny subset of representations reproduced in this paper cannot give a fair impression of the variety of representational strategies collected in the project, let alone of all those that the reader might have observed (or can imagine) diverse and creative problem-solvers using; neither are the brief commentaries included here with each intended to substitute for thorough analysis. They are selected for the purpose of providing a varied set of 'snapshots' into some emerging multiplicative thinking, and how a multidimensional qualitative framework may be applied. In Example 1 (above), I focused on *inconsistency* in the *spatial structuring* of two *unitcountable* representations, indicating decreased need for container elements and more reliance on layout of units. Examples 2 and 3 focus on changes in *media/mode* and *unitcountability*; Example 2 is chosen as a more strongly researcher-led interaction on a single task strategy, while Example 3 shows more independent open-ended work on multiplicative relationships. Example 4 is used to further explore the role of *resemblance* when working with larger quantities.

5.1 Example 2: A 'sharing biscuits' scenario represented in physical modelling and drawing

Paula, a 14-year-old student with severe quantification difficulties (potentially dyscalculia) and limited grasp of symbolic notation, was working on partitive division ('Biscuits') tasks, which required a high level of supportive interaction. She initially failed even to share a set of physical objects into a requested number of equal groups, so the researcher employed a *visuospatial prompt* of drawing circles ('plates' for the 'biscuits'), into which she moved cubes (i.e., *spatial structuring* of *unitcountable* units). After some practice 'dealing' (see

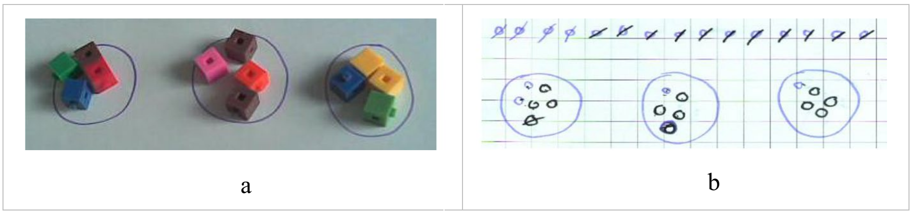

Fig. 2 **a–b** Two partitive division representations (Paula with researcher). Changing: media, mode. Constant: spatial structuring, motion (also resemblance, unitcountability, completeness, consistency)

Finesilver (2019) for detailed discussion), she was able to reliably create equal groups (Fig. 2a). Following this, she was encouraged to try moving from an enactive mode using physical media to an iconic mode using only drawing (Fig. 2b). Initially very confusing to her, this was achieved through maintaining identical *spatial structuring*, an equivalent level of *resemblance* to the task scenario and, especially, equivalent *motion*: her hand traced a similar sequential pattern in the workspace when picking up cubes from an initial collection above the 'plates' containers to distribute in sequence, and when deleting dots from an initial collection above to redraw them inside the containers. By maintaining these strong links with a secure strategy, she achieved the transition more smoothly than might be expected. While, as mentioned above, there is no theoretical difference in mathematical sophistication between these two representations, having the capability to solve arithmetical tasks with a pen and paper rather than having to find a collection of suitable objects to count out is of practical benefit for ease of participation in lessons and formal assessments, and for self-esteem amongst peers who work predominantly with pen and paper.

5.2 Example 3: Multiplicative relationships with 'unit containers' and 'number containers'

Tasha, a 12-year-old student with ADHD and a history of low performance in school mathematics, was initially comfortable with exploring multiplication and division only through *unitcountable* representations such as Fig. 3a, enumerating by grouped unitary counting. It is relevant that she used identical container forms to *spatially structure* equal groups of units for both multiplication and division tasks: this reinforces the relationship between the operations. In Fig. 3b, she was finding the number of 5-seater taxis needed to transport 30 people and decided to record the number in each group with symbols as well as unit marks. This choice to duplicate information, which may seem redundant to the casual viewer, provided her a link between her previous secure strategy and the new one shown in Fig. 3c (where she was exploring the different possible divisions of 30 into equal groups). Note that, in contrast to Example 1, Tasha retains the familiar containers, even after replacing the groups of units with number symbols. Again, these elements may seem 'unnecessary', but this nonstandard form, 'number containers', likely provided the link supporting the major cognitive leap from *unitcountable* to *non-unitcountable* representation (i.e., marks representing ones, to marks representing higher numerosities).

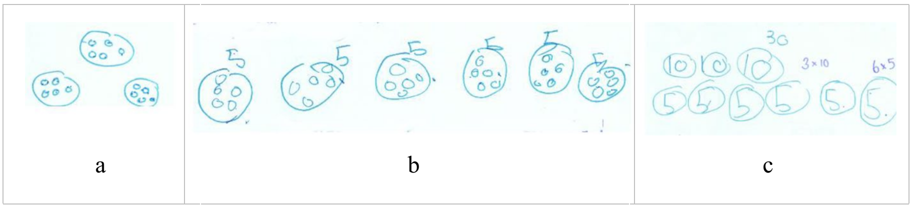

Fig. 3 a–c A selection of one student's container-based representations of various equal group calculations (Tasha). Changing: mode, unitcountability. Constant: spatial structuring (also media, completeness)

5.3 Example 4: Number containers, 'decoration', and text in tasks with larger quantities

Some participants worked on tasks involving larger quantities (e.g., calculating the number of coaches or planes required for > 100 passengers). Here, two students independently and successfully employ *non-unitcountable* representations with *spatial structuring* by *number containers*, as seen in Example 3, but with some non-mathematically-functional elements that deserve particular attention.

Figure 4a–b are later productions by Wendy (of the disappearing van in Fig. 1a). In that earlier unitcountable representation that was enumerated by rhythmic group-counting, she had discarded both containers and decoration. Here, she has gained confidence in using number symbols (i.e., *non-unitcountable* representations) and enumerates by repeated addition. The point of note is that numbers are spatially structured with containers, and *resemblance* is increased through pictorial elements (wheels on the buses, aeroplane-shaped containers with windows). Wendy was a highly motivated, engaged student who never doodled or wasted time, so the significant time she chose to spend drawing these elements indicates that while they do not contribute directly to enumeration, there was some other reason she chose to include them. She had made considerable progress in sessions even by 'typical' classroom standards and was tackling increasingly challenging division calculations. This was achieved through highly visual metaphoric reasoning (imagining quotitive division as an initial quantity of people filling up equally-sized vehicles), which allowed her to manipulate numbers that were still very intimidating when presented in bare form; I speculate that pictorial elements here acted affectively to reinforce the reassuring and helpful metaphor, avoiding anxiety.

In Fig. 4c, Sidney also uses number containers to represent a set of coaches each carrying 21 people but prefers to label them with text than make the containers visually resemble coaches. This is an unusual choice (particularly the repetition of the text label) but equally valid. Perhaps even more surprising is Fig. 4d, where a number of taxis are drawn and labelled to successfully represent the quotient, but neither the dividend, divisor, nor calculation were visually represented. (While drawing, he step-counted aloud in fives up to the required total.) This representational choice may have been helpful to him in some way and, while unexpected, does no harm.

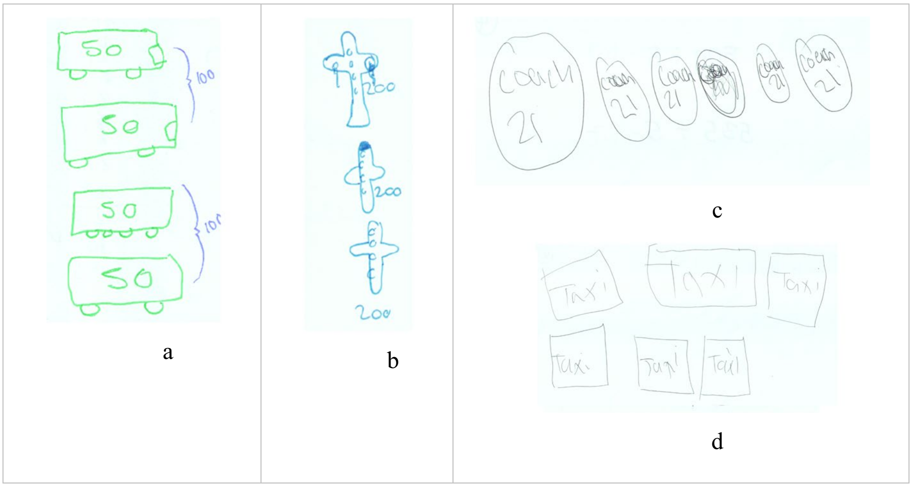

Fig. 4 **a–d** Representations that include non-mathematically functional elements, used for four of the later 'Passengers' tasks (**a–b** Wendy, **c–d** Sidney)

6 Discussion

6.1 Diversity of arithmetical-representational strategies

I set out to investigate the diversity to be found in student-produced arithmetical-representational strategies. Despite employing limited media and a basic task set with only natural numbers, by allowing an inclusive group of students the time, space, and encouragement to work with arithmetical relationships in the ways that made most sense to them, they produced a diverse set of multimodal data. The full set of 200+ may be viewed in Finesilver (2014), but the few examples shown here give a flavour. If one were to study a larger and more diverse set of participants, or expand the tasks set, even greater diversity might be expected. If we wish to understand the nuances of developing mathematical understanding, looking in depth and detail at learners' representations is important in taking account of the 'messiness' of variation and development, as opposed to forcing them into artificially delineated categories or expected stages. Frameworks are required with the capacity to address multiple aspects of arithmetical-representational strategies which may vary (semi-) independently and non-linearly.

Some structural and functional similarities, differences, and changes may be studied through the dissection of completed inscriptions. Ideally, though, arithmetical-representational strategies should be studied through observation of their production in context, and dynamic usage. For example, what directional relations and interplay are observed between *spatial structuring, enumeration,* and *error types*? (I have used a subset of this framework to focus on these aspects in a cuboid array task (Finesilver, 2017a).) While observation is not always practicable, researchers cannot presume to know all the thinking involved in a representation from only seeing the end product.

6.2 Non-mathematically functional elements

Research on arithmetical representations has, understandably, focused on their direct usage in calculation. However, there are other factors affecting representational choices, perhaps particularly those of struggling learners. All elements of a representation are created for some reason and may indirectly be supporting problem-solving activity, rather than directly denoting quantities or organizational structures. The 'decorative' elements in Examples 1 and 4 are not there by chance, nor the back-and-forth motion in Example 2, or the container circles in Example 3. It could be entirely appropriate for learners to make choices that help them maintain focus, reduce stress, etc. rather than those associated with the perceived efficiency, 'abstraction', or conformity that teachers generally prize.

An anxious affective state particularly impairs the reasoning required for mathematical problem-solving (Trezise & Reeve, 2017) and can also cause students to feel they are in a 'race' (Dowker, 2019). Thus, students stressed by their mathematical environment may not choose arithmetical-representational strategies that take longer, even when they could help. That my research data contradicted this from the start (in a Cartesian product task (Finesilver, 2009), for which a partial early version of this framework was used) may almost certainly be attributed to absence of (a) time pressure and/or (b) teachers or peers overlooking their work. It may be that struggling learners benefit from being allowed/encouraged to include familiar elements or aspects which might not seem necessary to the observer, to maintain emotional security when working on challenging and potentially anxiety-provoking tasks. This deserves further study under de-pressured and exploration-encouraging conditions.

6.3 Change and capabilities

Where particular symbolic notations are an idealized ‘end product’ of school mathematics, many potentially limiting assumptions are made about levels of understanding and capability of learners employing nonstandard representational strategies, whether they have been labelled in some way, e.g., as disabled or neurodivergent, or are seen through a more general lens of (prior) low attainment. Broad categorizations and hegemonic perceptions of development such as those referenced earlier in this paper implicitly feed into this tendency, reading learners’ attainment in fixed ways, and contributing to stigmatizing rather than supporting. However, by naming and separating out arithmetical-representational aspects, one can capture exactly what is constant and what changes, ‘regressions’ that may take place alongside ‘progressions’, and the many other tiny changes (or ‘microprogressions’) which can be particularly relevant for struggling learners. (I have used subsets of aspects of this framework in this way, in microgenetic case study approaches in atypical individuals’ problem-solving tuition (Finesilver, 2014, 2019).)

I suggest that using flexible multidimensional frameworks, such as the one proposed in this paper, could be a factor in challenging perceptions of individual deficits and designing more inclusive methodologies for qualitative research, to form more nuanced assessments of learners, and find mathematical capabilities, ingenuities, and possibilities which go unseen under crude taxonomies.

7 Concluding comments

Learners’ explorations of nonstandard visuospatial representations are a valuable way for them to explore and understand arithmetical relationships and a valuable source of data for researchers and teachers to understand and value diverse learning trajectories. They have been undervalued and oversimplified partly because of a history of analytical approaches based on over-broad taxonomic categories, and also resulting from overly-hegemonic representational hierarchies. Both of these factors can result in unduly limiting judgments being made on children’s capabilities and potential, which should be an ethical concern. I have argued the need for a more flexible, multidimensional approach, which avoids unnecessary binaries and hierarchies and allows for more fine-grained, in-depth, and dynamic analysis of arithmetical-representational strategies and the mathematical thinking of those who use them. Rather than just choosing a different set of categories in which to slot representations, I have proposed the alternative approach of employing a set of independently analytical aspects.

To illustrate this principle, I have presented a framework and showed how aspects can be used to compare and contrast various elements of visuospatial representations for learners engaged in arithmetical problem-solving, highlighting structural and functional similarities, differences, and changes. The characteristics that make this framework unique (to my knowledge) and particularly effective are as follows: (1) the level of detail provided by treating the different aspects of representation as varying independently, combined with a systematic organizational structure for describing representational activity and (2) that it can be applied to drawing, modelling, enactive, and other visuospatial strategies, plus any mixture of modes and media. Although not unique in this, further advantages are that (3) it highlights interactive relationships between representation and calculation and (4) it does not impose a single hierarchical structure on representations or the learners who use them.

While the genesis of this framework was within the specific context of natural number multiplication and division, I suggest the principles may have wider utility in various research, diagnostic, and practitioner situations, as there are many occasions when it may be useful to specify in detail similarities and differences of representational strategy. For example, one might need to make detailed comparisons within a static dataset (e.g., multiple students' representations of a mathematical task on a single occasion) or a dynamic one (e.g., an individual student's representations changing longitudinally). It is appropriate for both research and practice, where arithmetical representations with visuospatial elements are involved, and may be helpful in encouraging closer study of the diverse mathematical behaviours and competencies of more inclusive groups of learners. I welcome broader growth in the use of this kind of approach, with the developing of similar qualitative analytical frameworks of aspects for other areas of mathematics or related subjects.

Availability of data and material Not applicable.

Code availability Not applicable.

Declarations

Competing interests The author declares no competing interests.

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