Educational Studies in Mathematics (2021) 108:473–491
https://doi.org/10.1007/s10649-021-10044-1

Algebraic and fractional thinking in collective mathematical reasoning

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Accepted: 21 February 2021 / Published online: 7 June 2021

Abstract
This study examines the collective mathematical reasoning when students and teachers in grades 3, 4, and 5 explore fractions derived from length comparisons, in a task inspired by the El´konin and Davydov curriculum. The analysis showed that the mathematical reasoning was mainly anchored in mathematical properties related to fractional or algebraic thinking. Further analysis showed that these arguments were characterised by interplay between fractional and algebraic thinking except in the conclusion stage. In the conclusion and the evaluative arguments, these two types of thinking appeared to be intertwined. Another result is the discovery of a new type of argument, identifying arguments, which deals with the first step in task solving. Here, the different types of arguments, including the identifying arguments, were not initiated only by the teachers but also by the students. This in a multilingual classroom with a large proportion of students newly arrived. Compared to earlier research, this study offers a more detailed analysis of algebraic and fractional thinking including possible patterns within the collective mathematical reasoning. An implication of this is that algebraic and fractional thinking appear to be more intertwined than previous suggested.

Keywords Algebra · Davydov curriculum · Fractions · Mathematical reasoning

1 Introduction

Several studies conclude that algebraic thinking rests on an understanding of the concept of fractions and the ability to manipulate common fractions (e.g., Lee & Hackenberg, 2013; Norton & Hackenberg, 2010; Reeder, 2017). For instance, in algebra quotients are almost always represented as fractions (Peck & Matassa, 2016), which means that knowing fractions

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is essential if one is to learn algebra. Some researchers go even further and claim that students should not be taught algebra before they can undertake general reasoning with fractions (DeWolf et al., 2015), or that algebra cannot be learnt until students understand structures in rational numbers, particularly the structures within fractions (Lee & Hackenberg, 2013; Norton & Hackenberg, 2010). In many school systems, this implies a late start of algebra as an area of mathematical study.

At the same time, a growing body of research concerning younger children's mathematical reasoning shows that children are capable of algebraic thinking, such as analysing relations between quantities, performing generalisations, solving mathematical problems, and justifying, without formal education in fractions or rational numbers (Cai & Knuth, 2011; Carraher et al., 2006; Chimoni et al., 2018; Kieran, 2018). Some stress that algebra should be used to develop proficiency, with an emphasis on arithmetic, instead of being treated as a distinct content area apart from arithmetic and, in particular, that algebra should be used in measurements instead of just counting and operations to enhance number sense (Davydov, 2008; Izsák & Beckman, 2019; Schmittau, 2011; Simon et al., 2018; Venenciano & Heck, 2016). This would require a different treatment of algebra and fractions.

When looking at these two stances, it is unclear how algebraic thinking and fractional thinking are linked independent of the stance. Also, the question if and how algebraic and fractional thinking interplay in mathematical reasoning, in the sense how different arguments interact with each other, has not gained much attention. In addition, researchers espousing both stances have identified a need for further research into algebra, especially regarding younger students (e.g., Carraher et al., 2006; Chimoni et al., 2018; Lee & Hackenberg, 2013; Norton & Hackenberg, 2010; Nunes et al., 2009). Therefore, there is a gap in the research body, and the present study aims to address this gap by investigating collective mathematical reasoning in relation to algebra and fractions. The research questions are as follows: (1) What mathematical properties are arguments anchored in when students are analysing the fractional part of a mixed number? (2) What aspects of algebraic thinking and fractional thinking are these mathematical properties indications of? (3) In what way do algebraic and fractional thinking interplay in collective mathematical reasoning?

2 Background

Three areas will be used as background to this study, namely, mathematical reasoning, fractional thinking, and algebraic thinking, with the last two having been treated in research for many years. Selected aspects of each research area that are central to the aim of this study will be covered.

2.1 Mathematical reasoning

Mathematical reasoning can be viewed in different ways (Jeannotte & Kieran, 2017; Sumpter, 2013). Here, the starting point is to view mathematical reasoning as "[…] the line of thought adopted to produce assertions and reach conclusions in task solving" (Lithner, 2008, p. 257), where we see the line of thought as a result of collective joint activity aiming at meaning making (Sumpter & Hedefalk, 2015, 2018). Compared to collaborative reasoning (e.g., Granberg & Olsson, 2015), the emphasis is not on a specific individual or the characteristics of reasoning such as different types of imitative reasoning. Here, the decisions and arguments are created by a group of people where there is an emphasis on the collaboration (Sumpter, 2016), and learning and teaching are seen as ongoing changes in human behaviour (Sumpter & Hedefalk, 2018). This allows us to focus on different arguments that are given for different choices made during the reasoning process, instead of, for instance, different processes within reasoning or kinds of reasoning generated by different tasks (e.g., Jeannotte & Kieran, 2017; Lithner, 2008, 2017). However, compared to studies focusing on collective argumentation (e.g., Knipping, 2008), in order to have a more detailed analysis of arguments highlighting the different mathematical properties, we follow Lithner's (2008) notion of anchoring including the structure of reasoning.

Hence, mathematical reasoning can be organised as a process that includes the following four steps (Lithner, 2008, 2017): (1) a (sub)task is explored (TS); (2) a strategy choice is made (SC); (3) the strategy is implemented (SI); and (4) a conclusion is suggested (C). For each of the last three steps, arguments can be connected to the choices made (Hedefalk & Sumpter, 2017; Lithner, 2008). Analysing arguments and their mathematical properties is one way of studying mathematical reasoning (Lithner, 2008). Research has identified three types of arguments (Hedefalk & Sumpter, 2017; Lithner, 2008; Sumpter & Hedefalk, 2018). Predictive arguments are linked to the strategy choice and are intended to answer the question "Why will the strategy solve the task?", while the strategy implementation can be supported by verifying arguments asking "Why did the strategy solve the task?" (Lithner, 2008). A third type of argument was discovered when analysing preschool children and teachers' collective mathematical reasoning (Hedefalk & Sumpter, 2017; Sumpter & Hedefalk, 2018). It was then noted that an argument can also be linked to the conclusion answering the question "How does the conclusion answer the question for the (sub)task explored?" These arguments are called evaluative arguments. Arguments can be anchored in mathematical properties, which are described as objects, transformations, and concepts (Lithner, 2008). Here, objects in relation to properties can be defined as numbers, variables, and functions. Transformations are what is done to these objects, such as adding a fractional part to an integer. A concept is based on a set of objects, transformations, and their properties—that is, actions in relation to an object. In this study, we are interested in properties related to fractional thinking and algebraic thinking.

2.2 Fractional thinking

There are several ways of describing fractional thinking (Lamon, 2012; Nunes et al., 2009). In this paper, we are interested in the mathematical content included in arguments, and the choice is to use the fraction scheme framework presented by Steffe and Olive (2010). It has been employed in studies describing students' understandings of fractions in different mathematical situations, such as written tests (e.g., Norton & Wilkins, 2009) and verbal arguments (e.g., Boyce & Norton, 2016). Likewise, this framework uses a hierarchy and is often utilised to describe a common progression in the development of understanding of fractions among students (Boyce & Norton, 2016; Steffe & Olive, 2010). However, as these schemes also describe students' actions in relation to rational numbers (e.g., Norton & Wilkins, 2009), we chose not to concentrate on the progression including the hierarchy, but on the different descriptions of fractional thinking schemes presented in Table 1.

The first fraction scheme is the part–whole scheme, which provides a way of producing and conceptualising any proper fraction, but not necessarily improper fractions (i.e., fractions greater than one). The second fraction scheme is described as the partitive unit fraction

Table 1 Fractional thinking schemes according to Steffe and Olive (2010)

Scheme	Actions
(1) a part–whole fraction scheme	Unitising proper fractions (smaller than one, but not necessarily improper fractions)
(2) a partitive unit fraction scheme	Generating fraction language Unitising an undivided whole as a proper fraction
(3) a partitive fraction scheme	Conceptualising, for example, three-fourths as three one-fourths
(4) a reversible partitive fraction scheme	Splitting an unpartitioned piece of a larger whole to recreate the whole
(5) an iterative rational scheme	Splitting an unpartitioned piece of a smaller whole to recreate the whole

scheme, in which students also generate fraction language. Also, compared with the part–whole scheme, the second scheme includes the division of a whole, even if this whole was initially undivided. This could be seen as a set of splitting actions (Confrey, 1994). Some of the actions included in this scheme are sharing, folding, dividing symmetrically, and magnifying.

The third scheme is the partitive fraction scheme described as generalisation. This is when students use a more general scheme to conceive of a proper fraction, for example, three-fourths understood as three one-fourths of the whole. This scheme facilitates the development of splitting actions and of higher fraction schemes (Norton & Wilkins, 2009). The fourth scheme is the reversible partitive fraction scheme, which is the first scheme to rely on splitting operations. For example, this scheme produces an implicit whole from a proper fractional part of the whole, such as the task of deciding how much of a given bar is 4/5 (Norton & Wilkins, 2009). The fifth and final scheme is the iterative and reversible fraction scheme. In this scheme, students can produce an implicit whole from any fraction, including improper fractions (i.e., m/n where $m > n$). An example of this scheme is the task of drawing a bar that is 4/3 of a given bar (Norton & Wilkins, 2009).

2.3 Algebraic thinking

Algebraic thinking can be broadly defined as a process by which students are analysing relationships between quantities, noticing structures, studying changes, generalising, solving problems, modelling, justifying, proving, and predicting (Kieran, 2004). Similar descriptions treat algebraic thinking as working with unknown numbers when analysing relations between and structures within numbers when these unknowns can be named or symbolised, even in a non-symbolic way (Radford, 2013), or as an arithmetic–algebraic work space (Hitt et al., 2016). Another suggested definition of algebraic thinking entails identifying four component abilities (Blanton & Kaput, 2005): understanding patterns, relations, and functions; representing and analysing mathematical situations and structures using algebraic symbols; using mathematical models to represent and understand quantitative relationships; and analysing changes in various contexts. What these descriptions share is a focus on actions more than mathematical properties, and although they could function as starting points, when analysing arguments in mathematical reasoning, further specification is needed.

Therefore, the choice here is to use a description of algebraic thinking that also includes mathematical content, which allows us to explore and contrast different arguments rather than just describe a process. We follow Kaput's (2008) description of algebraic reasoning, which defines two core aspects that cover a general description of algebraic thinking as "systematically symbolizing generalisations of regularities and constraints" (p.11) and "syntactically guided reasoning and actions on generalisations expressed in conventional symbol systems" (p. 11). These two core aspects are then expressed in three strands. The first strand is called "the study of structures and systems abstracted from computations and relations, including those arising in arithmetic (algebra as generalised arithmetic) and quantitative reasoning" (Kaput, 2008, p. 11). This includes building generalisations of structures in arithmetic, which could be considered the primary route into algebra, for instance, building generalisations about particular number properties or relationships (e.g., Bourbaki, 1974; Davydov, 2008; Radford, 2013). It should be stressed that, in relation to emergent algebraic thinking, generalisations are not necessarily explicit (Zazkis & Liljedahl, 2002). This strand includes computation strategies, conventional as well as student invited, and critical steps such as seeing the "=" sign as equivalence instead of an operation in itself or as a separator of operations.

The second strand concerns "the study of functions, relations, and joint variation" (Kaput, 2008, p.11). This includes analytical aspects of the idea of functions but also includes generalisations of patterns, both number patterns and patterns of figures. One example of such elementary patterning activities is the comparison of different expressions of a pattern to determine whether they are equivalent. The patterns are thought to be necessary precursors to other forms of mathematical generalisations and different mathematical concepts with the aim of developing symbol sense (Blanton & Kaput, 2005). Here, we have objects, transformations, and concepts that could be viewed as more algebra specific, such as understanding in what ways two (physical) quantities are proportional to each other, as in measurement tasks from the Davydov curriculum (Davydov & Tsvetkovich, 1991). The second strand also includes concepts from other mathematical areas, such as elements in a series (as in the above example), and variables, including transformations such as finding equivalent patterns and analysing functions, for instance, the use of "function machines" (Kaput, 2008).

The third strand is about modelling (Kaput, 2008), of which there are three types. The first is number- or quantity-specific modelling, in which a variable is regarded as unknown rather than as representing a situation. This means that we work on a specific case more than striving to understand the representation of a class of situations. The second type of modelling is related to the first core aspect, meaning that the domain of generalisation of what is being modelled is emphasised and that variables are usually used. This could, for instance, be a pattern expressed as a basic series or as a function. This type of modelling is also situation-specific. The third type of modelling involves further generalisations, including comparisons with other models and situations; the expressions use variables, usually in the form of parameters.

Here we use these three frameworks to explore the collective mathematical reasoning when students and a teacher are working on measurement tasks. The framework of collective mathematical reasoning builds on mathematical arguments anchored in mathematical properties in objects, concepts, and transformations, and the two chosen frameworks regarding algebraic and fraction thinking illuminate such mathematical properties.

3 Methods

Given our aim to investigate collective mathematical reasoning in relation to algebra and fractions, the task design becomes important. Since the El´konin and Davydov mathematical curriculum was created to use algebra as a tool in joint activities focusing on students' agency to develop knowledge about numbers (Davydov, 2008), we decided to use it as a stimulus to generate arguments about fractions and algebra. These lessons happened once a week. The empirical data for the present study came from three lessons in two collaborative projects, both with the purpose of exploring this particular curriculum (H. Eriksson, 2015; Eriksson & Eriksson, 2016). The lessons were designed by the participating teachers ($n = 7$) and the first author in an iterative process following a learning study set up, of which the lessons in focus are number five in the process (Eriksson, 2015). The lessons were taught by the ordinary class teachers and video recorded by the first author.

3.1 Task design

The tasks were inspired by Davydov and Tsvetkovich (1991) and build on the idea of measurement in which fractions are viewed as the results of comparisons that do not represent equality (Davydov, 2008). All tasks were formulated as "measure one length with another length," with the aim to generate fractional thinking using algebra as a tool. Cuisenaire rods were used in comparing lengths in these lessons, which represented different degrees of difficulty. The specific tasks were chosen based on previous lessons in which the students were constructing measurement on their own choosing which rods to measure and choosing the different units of measures. Therefore, depending on what measurements the students found problematic, the following tasks were chosen: the measurement in grade 3 was to compare the black rod with green rods, grade 4 to compare the black rod with red rods, and grade 5 to compare the blue rod with yellow rods (see Fig. 1).

The core of each task was to identify the fractional part of the unit of measure. In the lesson design, the students first had to identify a length to be measured (the black and blue rods), the units of measure (the green, red, and yellow rods), and the fractional part of the unit of measure (i.e., the smaller unit of measure, here represented by the white rods). Given that the black and blue rods are both variables (i.e., lengths) treated as unknown, the task itself is within the third strand of algebraic thinking, i.e., modelling (e.g., Kaput, 2008). In the series of lessons, the teachers followed a design suggested by Davydov (2008) based on analysing relationships among quantities grounded in measurements. Therefore, in all three tasks, the first step was to find inequalities. The second step is to construct equality using a remainder, and the third step is to explore the remainder. Here, W stands for the whole (i.e., the unit of measure, hence a length that varies) a notation chosen by the teachers, and r stands for the remainder, which depends on W (Fig. 2). In the remainder, m stands for the numerator and n for the denominator,

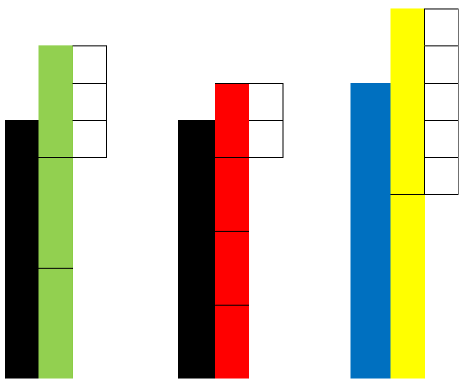
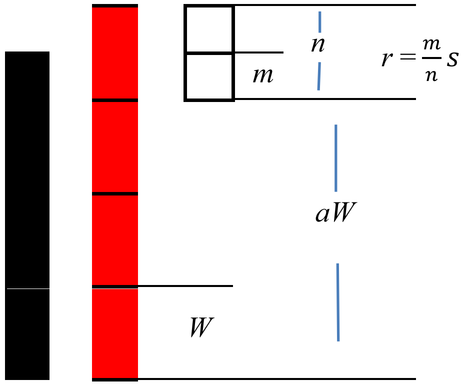


Fig. 1 The three sets of rods used, left to right: black and green, black and red, and blue and yellow. White rods are the smaller unit of measurement

Fig. 2 The notations used in the analyses for this paper



where m and n are lengths in s, the smaller unit of measure. In this sense, the task is a system of equations. The following notation is used for the equations:

$$\begin{cases} aW < Black \text{ or } Blue < (a+1)W & (1) \\ Black \text{ or } Blue = aW + rW & (2) \\ r = \frac{m}{\mathrm{n}}s, \text{ where } W = ns; a, m, n \in \mathbb{Z}^{+} & (3) \end{cases}$$

In the previous lesson, the students had worked with measurements, in which the object to be measured was a whole number of the unit of measure; it also included problems in which the lengths to be compared were not equal, meaning that the task and some of the notations, such as W, were not completely new.

3.2 Data collection

The data consist of transcribed videotapes capturing collective mathematical reasoning from three lessons, each lasting an entire class session, one from each of grades 3, 4, and 5. The primary school had a high proportion of newly arrived (to Sweden) students; approximately twenty to twenty-five native languages are spoken at the school. This meant considerable variation in the ages of the students: the grade 3 lesson involved 22 students of 8–10 years old, the grade 4 lesson involved 20 students of 9–11 years old, and the grade 5 lesson involved 25 students of 9–13 years old. All the students and their parents signed letters of consent allowing the students to participate in the project. The agreement covered the video documentation of the research lessons and permission to use the video recordings and other material, such as the students' worksheets, in research. This letter was translated orally for parents who did not read Swedish. Other ethical considerations stipulated by the Swedish Research Council (Vetenskapsrådet, 2017) were followed, including keeping the informants' identities anonymous.

3.3 Method of analysis

The data were transcribed, and the students were assigned fictive names in the transcripts. A preliminary analysis found no differences between the three lessons in terms of the mathematical reasoning, here referring to the main line of thought regarding strategy choice and implementation. The three lessons are therefore treated in the same way in the analysis. In total, 160 min of data were generated from these three lessons. The transcripts reported the students' and teachers' actions, including gestures and oral and written communication. Given that so many students had different first languages and that many of them were newly arrived, the teachers used repetition to support language development. These repeated passages, and others judged irrelevant to the purpose of this paper, such as talk about discipline, were marked as "[…]" and not included in the analysis.

The data were analysed in five steps. The transcripts were organised using Lithner's (2008) reasoning structure, and the first step was to detect episodes in each lesson that addressed the problem in focus. The second step was to identify the task situations (TS), strategy choices (SC), strategy implementations (SI), and conclusions (C) that followed in these episodes. Note that it is possible to expand or reduce the task situations to include more or less information, so an appropriate granularity was chosen for the analysis of each episode. As a third step of the analysis, we identified predictive or verifying arguments connected to the SC and SI (Lithner, 2008), or evaluative arguments found in the conclusion (Hedefalk & Sumpter, 2017).

In the fourth step, the mathematical properties of the components anchoring the different arguments were identified and analysed. Here, the focus was on fractional thinking and algebraic thinking and the possible interplay between the two. Regarding fractional thinking, we used the fraction scheme framework (Norton & Wilkins, 2009; Steffe & Olive, 2010). Given the nature of the tasks, only the first two schemes are relevant to this study, the part–whole fraction scheme and the partitive unit fraction scheme, both connected to the mathematical idea of splitting the whole (Confrey, 1994). The part–whole scheme can be illustrated by arguments made when the whole is already divided into parts, while the partitive unit fraction scheme can be illustrated when the whole is being split by the children. Some examples of mathematical properties considered indications of fractional thinking were objects such as remainders, concepts such as rational numbers, and transformations such as division.

In categorising algebraic thinking, the three strands of the two core aspects of algebra considered algebraic reasoning by Kaput (2008) gave us tools both for highlighting overarching processes such as generalisation and for talking about, for instance, different aspects of modelling. Note that we apply the idea of emergent algebraic thinking (e.g., Radford, 2013; Zazkis & Liljedahl, 2002), meaning that talk and actions that could be seen as first steps to a generalised idea will be interpreted as such. Some of the mathematical properties considered algebraic thinking concerned structures such as mathematical objects and transformations, which are included in mixed numbers, and relations such as mathematical concepts and transformations, which are involved in the development of equalities. As the last step, to explore whether algebraic and fractional thinking might overlap, we compared the different steps of the reasoning (i.e., TS, SC, SI, and C). In these comparisons, we identified arguments that include properties categorised as both fractional thinking using fraction schemes and algebraic thinking seen as algebraic reasoning.

Table 2 Analytical structures and the unit of analysis

Reasoning steps	Arguments	Mathematical properties	Fraction schemes	Algebraic reasoning
TS				
SC	Predicting			
SI	Verifying			
C	Evaluating			

In this way, Lithner's (2008) framework provided a structure for ordering the data and, together with the theoretical tool of Hedefalk and Sumpter (2017), for identifying different types of arguments. The unit of analysis is the content of these arguments, the different mathematical properties. These are compared with the fraction schemes (Steffe & Olive, 2010) and the core aspects and strands presented in the algebraic reasoning framework (Kaput, 2008); see Table 2.

4 Results

The task situations chosen to illustrate the results come from grade 4, thus the measurement of the black rod with the red rod as a unit of measure (Fig. 1). Similar reasoning, here referring to the particular mathematical properties in focus, was evident in grades 3 and 5 as well. The main reasoning evident in the data is algebraic thinking, such as modelling, and fractional thinking, such as dividing a whole, and the focus is on specific arguments. In each task situation, W is a fixed length and is therefore not denoted as a variable, whereas r is treated as a variable depending on the students' choice of a smaller unit of measure. The teacher is noted as "T". Tables including the Swedish language of the students is presented in Tables 7, 8, and 9.

4.1 The task situation 1 and 2

4.1.1 The first task situation

The first task situation was about identifying inequality (equation 1). When this situation starts, the students and the teacher have just begun to measure the black rod using the red rods (see Table 3). The task was chosen since some of students had worked with it the day before, but had struggled to construct a solution. In this first section, the teacher reformulates the problem to a new task situation, presenting a new model.

In the first task situation, the focus is on trying to understand the inequality where the red rods are longer than the black rod, which is a study of structure. The reasoning ends at 01:55 when the teacher redirects the students to a new task situation and asks for arguments about this new problem, to identify the central mathematical properties of the task. One can see this question as a stimulus for a new type of argument not explored before, namely, identifying arguments. Chaid stresses the remainder in the inequality, which is a part of the study of structure. After some guessing out loud, where the students are using fraction language (second fraction scheme) and the teacher is quiet, Bayar takes command of the reasoning by confirming an earlier guess of 3½. Two inequalities are constructed through the division of the whole into parts that could be interpreted as the second fraction scheme. There is also a suggestion of how to construct equality using r, "a little bit more". As a study of inequality striving for equality, this could be seen both as a study of structures and as emergent modelling.

Table 3 Grade 4: identifying inequality, equation 1

Timecode	Data	Analysis of Arguments	Identification of mathematical properties
	[The students are entering the classroom and the teacher is drawing a line segment on the board. On the board is the measure seen here.]		
00.14.04	T: Some of you constructed this measure yesterday. You decided to measure a black rod with red rods only.	TS: Estabishing TS. SC to measure.	TS is study of structure. SC is dividing a whole in parts.
00.59.01	Students together: It didn´t work.	Confirming TS	
01.01.26	T: But…was it just to measure? How did it work? Does anyone remember the difficulty? What is our problem? [The teacher is pointing at the measurement on the board.]	T asking for verifying arguments about the SI.	
01:30.14	Dana: It is too much. It is longer.	C: evaluative arguments provided.	Inequality is noted Black rod < $(a+1)W$
01:35.02	T: What's the problem? What did she say?	T asks for identifying arguments.	
01:39.13	Adam: She said we could measure the red rods with the black one.	SC: suggestion to measure red rods with black rod, SC not accepted.	
01:45.00	Mathew: Take other colours.	SC: suggestions to change rods.	
01:55.00	T: That's another task. Interesting. We can do that another time. What's our problem right now?	T direct students back to TS. Asks for identifying arguments.	
01:57.17	[The teacher takes the last unit, the last red rod away.] 	T constructs new TS.	New model: $B > aW$
02:13.05	Chaid: A very small piece of the red is missing	Provides identifying arguments: not an equality since the units of measure (aW) are less than the object to measure (B).	$B = aW + rW$. Therefore, inequality $B > aW$.
02:52.02	[The students are guessing answers like three and a half, four and a half etc. T is quiet through this process.]	SC: guessing a and r.	Using fraction language.
03:48.17	Bayar: It is three and a half. It is three of these, and then it was one red that was longer, and if you add one more it would be more than the black rod. But if you just add a little bit they would be equal. If you add one half.	C: 3 ½, provide evaluative arguments. R is suggested ½ W, add "a little bit more". No further arguments provided to why $r = ½W$.	Constructs inequalities by dividing the whole into parts: a=3, if a= 4, then $B<4W$. $aW<B<(a+1)W$ Equality is constructed $aW+rW=B$

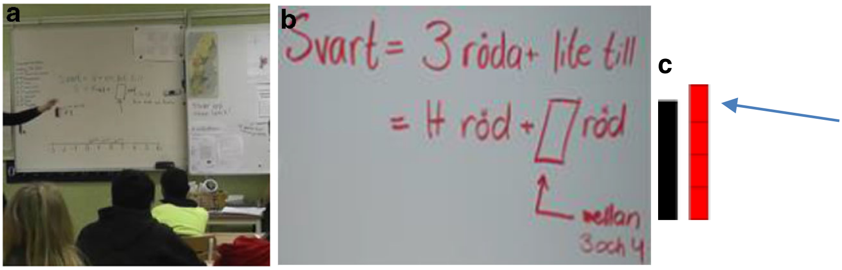
4.1.2 The second task situation

The next step was to construct equality. When the students discussed this construction, they were investigating the relations between three different units. In the process of constructing equality, they found a but were now exploring r (equation 2). Here, the task situation has been

Table 4 Grade 4: study of relations, equation 2

Timecode	Data	Analysis of arguments	Identification of mathematical properties
05:02.45	Leart: Maybe we can measure? We do need to measure the red rod, don't we? And the little bit to the left?	Sub-TS identified, asked to be confirmed: to measure W and r. First suggestion of SC.	The whole length is divided into aW and rW; smaller unit of measure not chosen
05:03.18	T: Now we have to be sure [The teacher is writing W next to the red rods].	Confirms, suggesting the symbol W	The red rod is denoted W
05:15:30	T: This is the whole, this is W. And then you know it is the piece marked on the number line.	SC: Confirming the initiation of SC, to start with W	The whole unit of measure
05:24.15	Mehmet: Actually, there is a little bit more …	SI: Verifies SC with added focus on the missing bit, the remainder. Re-establish the TS	Stress r in the equation Black = $3W + rW$
05:46.25	T: We do need to go back … what is the problem? What are we going to solve? What are we doing?	TS: Confirming. Stress the TS. T asks for identifying arguments about the TS.	
06:00.23	Dana: We are going to measure the black rod.	SC: Responding and establishing TS.	
	[The teacher is writing "Black = 3 red + a little bit more".]	SI: Verifies the strategy	Study of structure: Black = $3W + rW$

[The students and the teacher continue with various suggestions, during which the following was established and written down on the board: *Black* = *W* Red +  Red* (Image a). The arrow pointing at the box in the reconstructed figure is clarified "between 3 and 4" (Image b). T points at the four red rods, the fourth rod (Image c).]



a) A photo from the classroom. b) a reconstruction of what is on the white board. c) A reconstruction of the measure in which the teacher is pointing

Timecode	Data	Analysis of arguments	Identification of mathematical properties
11:54.23	Dana: It is the red rod above that we need to measure [looking at the fourth red and the distance between 4W and the black rod].	C: In order to measure the black rod, we also need to know the length of W.	$W = ns$, s is the smaller unit of measure
12:10.40	T: Why do we have to measure the red rod above?	Tasks for evaluative arguments for the C above.	
12:20.14	Bayar: Because it is bigger [reference to 4 red rods].	Implicit argument: four red rods are too long.	$4W > B$
12:25.06	Mehmet: How long is the red rod in order to become [as long as the] the black rod? So they become equal?	Stress the remainder, r, in order to create equality	$rW + 3W = B$
12:33.15	Dana: Then we need something that is smaller to measure with.	C: An evaluation is made. In order to answer TS, a smaller unit of measure is needed.	System of equations: $r = ms/ns$ and $W = ns$

*Represents an empty box

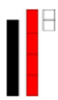
divided into two sub-tasks (see Tables 4 and 5). The sequence starts when the teacher has decided that the symbol W represents "the whole", a red rod.

This sequence started with Leart asking for confirmation of what mathematical properties the TS was about, including a suggestion of SC. Implicitly, it was a part–whole fraction scheme in which the whole length was divided into three red rods and a remainder. This is a question that could generate identifying arguments. This new type of argument was again evident at 05:46.25 when the teacher asked the class to identify the central mathematical properties of the task. The teacher posed this question just after Mehmet stressed that the difficult part of the task was to find a way of measuring r, which could be seen as an emergent step towards studying relations. Dana responded by rephrasing the sub-TS, to find the length of the black rod, and the strategy choice, to measure. The teacher then asked for arguments concerning the TS, generating identifying arguments that can be interpreted as the study of structure: $B = 3W + rW$. Dana proposed the conclusion that what remains to be understood is the point that Mehmet raised earlier, namely, how to measure r. In the second phase, the reasoning was driven by the students, Bayar and Mehmet, who offered evaluative arguments about the study of inequality and equality that end with the preliminary conclusion that, to answer the teacher, a smaller unit of measure is needed. The final evaluation of the conclusion was stated by a student stressing the system of equations: that W and r are related through the smaller unit of measure. This could also be seen as a first step towards the second fraction scheme, to see the remainder as a proper fraction; at the same time, it is also a study of relations.

The second sub-task deals with the solution of equation 3 (see Table 5).

In Table 5, we can see that the reasoning was driven mainly by the teacher, although each step was suggested by the class. The conclusion was presented by Evin, with the equation ms/ns being solved, resulting in ½W. This is both algebraic thinking as the study of relations and a sign of working with fractions as in the second fraction scheme. The final conclusion that $Black = 3W + \frac{1}{2}W$ was confirmed by the whole class.

Table 5 Grade 4: solving the system of equations, equation 3

Timecode	Data	Analysis of arguments	Identification of mathematical properties
14:24.22	[The teacher puts white rods beside the red rod.] T: How many of the red rods do we need? How many white rods fit the little bit more? How many of the white rods are needed to construct the black rod?	T formulates TS: What is m? SC: Use white rods as a measure and count. 	
14:38.27	Students together: One!	SI: Straightforward, C: 1	$m = 1s$
15:55.06	T: How many white rods do we need to construct the red rod? One [referring to m] of how many [referring to n]?	TS: What is n? SC: put white rods next to the red rods and measure.	
16:02.04	Students together: Two! [The teacher points to the empty box written on the whiteboard.]	SI: straightforward. C: 2	$n = 2s$
16:12.17	T: How should we write this then?	TS: What is the final answer?	
16:45.05	[Evin goes to the board and writes: 1 white rod/2 white rods.] [While Evin writes, the teacher encourages the class to comment to confirm the conclusion.]	C: Evin fills in the gap: Black = $3W + (1s/2s)W$	$B = 3W + ½W$

4.2 Algebraic thinking and fractional thinking

The last step of analysis focuses on how algebraic and fractional thinking interplay in collective mathematical reasoning. The arguments displaying algebraic reasoning and the fraction schemes were identified. Here, as an illustration, the arguments and the mathematical properties were collected from Tables 3 and 4. The pattern that emerged is presented in Table 6.

As Table 6 illustrates, algebraic and fractional thinking appear to interplay, here seen as interaction between different arguments. The arguments in focus are the identifying, predicting, and verifying arguments when the students or teachers suggest or implement strategies or talking about some of the mathematical properties that are part of the task situation. When looking at the conclusions, another interaction appears. In most of the conclusions, algebraic thinking and fractional thinking appear to be intertwined, meaning that the evaluative argument contains concepts, objects, or transformations that have mathematical properties from both mathematical areas, for example, Bayar's argument in 03:48.17 in Table 3. In evaluative arguments, the core of the (sub)task is more explicit regarding intrinsic mathematical properties (i.e., in what way is this an answer to the (sub)task); in comparison, in predictive and verifying arguments, the focus is on the strategy and its implementation.

Table 6 Grade 4: arguments displaying algebraic reasoning and the fraction scheme in the first task and second situations

Timecodes	Task situation	Arguments	Properties	Algebraic reasoning	Fraction scheme
00:14.04	TS	(Identifying)	Inequality	Study of structure	
	SC	Predicting	Starting with B		Dividing a whole in parts
01:30.14	C	Evaluating	$B < (a+1)W$	Study of structure	
01:39.13	SC	Predicting	Other measures		Evading fractions
01:55.00	TS	(Identifying)	New inequality	Study of structure	
02:13.05	C	Evaluating	$B = \mathrm{a}W + \mathrm{r}W$	Study of structure	Dividing a whole unit of measure
	SC	Suggestions	3 and a half		Fraction language
03:48.17	C	Evaluating	$B = 3W + rW$ $B = aW+$ *a little bit*	Study of structure	Dividing the undivided unit
05:02.45	TS	(Identifying)	$B = aW + rW$	Study of structure	
05:15.30	SC	Predicting	Starting with W	Symbolising using letters	
05:24.15	SI	Implementing	Stress r in the equation $\mathrm{B} = 3W + rW$	Study of structure	
05:46.25	TS		Back to inequality		
06:00.23	SC	Predicting	To measure		
	SI	Verifying	$B = 3W + rW$	Study of structure	
11:54.23	C	Evaluation	$W = ns$	(Modelling) Study of structure	Dividing the undivided unit
	TS	Asking for evaluation		Study of structure	
12:33.15	C	Evaluation	$B < 4W$ $r = ms/ns$ and $W = ns$	Study of structure	Dividing the undivided unit

5 Discussion

The aim of this article was to investigate the collective mathematical reasoning concerning algebraic thinking and fraction schemes evident in three lessons. Starting with the first two research questions, we see that the arguments that were part of collective reasoning were anchored in mathematical properties related to algebraic thinking and fractional thinking as described by Kaput (2008) and Steffe and Olive (2010), respectively. These results are in line with the growing body of research suggesting that young children are capable of different types of algebraic thinking (e.g., Cai & Knuth, 2011; Carraher et al., 2006; Chimoni et al., 2018; Kieran, 2018). The task provided by the Davydov (2008) curriculum features algebraic modelling using measurements requiring the splitting of a whole (e.g., Confrey, 1994). In this way, the children arrive at the concept of the fraction using various arguments. However, instead of focusing on whether mathematics education should start with fractions or algebra, we would like to conclude that a task design allowing interplay between the two mathematical areas would be fruitful. Such a conclusion is supported by previous studies (e.g., Davydov, 2008; Izsák & Beckman, 2019; Schmittau, 2011; Simon et al., 2018; Venenciano & Heck, 2016). What we can offer, compared with previous research, is further description and more detailed analysis of the arguments, allowing us to highlight not only the types and aspects of algebraic and fractional thinking that are in focus but also patterns evident within the mathematical reasoning. This is a contribution to the field.

The third research question is about such patterns: whether algebraic and fractional thinking interplayed in collective mathematical reasoning. Our results indicate that the predicting arguments concerning the strategy choice and the verifying arguments about the implementation of the strategy either are based in fractional thinking (e.g., splitting a whole into equal parts with a reminder) or algebraic thinking (e.g., the study of inequalities). The exception concerns evaluative arguments connected to the conclusions in which we interpret the different ways of thinking as intertwined. One possible explanation is that this is due to the nature of the arguments: predictive and verifying arguments are about the strategy (e.g., Lithner, 2008), whereas evaluative arguments focus on and in what way the conclusion addresses the task situation (Sumpter & Hedefalk, 2018). Such a focus allows a broader range of mathematical properties to be seen as intrinsic and thereby relevant, since it is not only about a specific strategy choice.

The focus on the intrinsic mathematical properties of the task is what characterises the identified argument type that is an unexpected result of this study, the identifying arguments. This type of argument has not been visible in previous studies of collective mathematical reasoning and teachers' roles (e.g., Sumpter & Hedefalk, 2018). Here, it is illustrated not only by the teachers, but also by the students who ask the question "What is the task really about?" Identifying arguments are defined as arguments that aim to answer such a question and deal with mathematical properties that are intrinsic to the task, at the core of the task one is trying to solve. This may be different for various strategies, of which predicting and verifying arguments are dealt with, or the conclusion and evaluating argument. One implication of this finding is that a better understanding of the types of questions that could stimulate creative mathematical reasoning (e.g., Hedefalk & Sumpter, 2017; Lithner, 2008; Sumpter & Hedefalk, 2018) would more clearly guide teacher educators and teachers when discussing collective mathematical reasoning. Our results indicate that it is possible to construct classrooms where it is not just teachers who initiate questions and arguments but also students, here in multilingual classrooms with many newly arrived students. This is an important result.

These results raise some questions. We need further research into the relationship between key questions and mathematical reasoning to understand why certain questions stimulate certain arguments (e.g., Sumpter & Hedefalk, 2018). The other area of interest is how teachers get students to generate different types of mathematical arguments and to be actors with power in the creative process. This includes how these students perceive themselves as actors in such classrooms. Such studies would support the idea of collective reasoning as a tool for mathematics teaching and learning as suggested by Sumpter (2016).

Appendix

Table 7 Grade 4: Identifying inequality, equation 1. The Swedish language included in the excerpt

Timecode	Data	Analysis of Arguments	Identification of mathematical properties
	[The students are entering the classroom and the teacher is drawing a line segment on the board. On the board is the measure seen here.]		
00.14.04	L: Det var någon av er som mätte en sådan här svart stav igår. Då hade man bestämt att man skulle mäta en svart stav med bara röda.	TS: Estabishing TS. SC to measure.	TS is study of structure. SC is dividing a whole in parts.
00.59.01	Flera elever. Men det gick ju inte.	Confirming TS	
01.01.26	L: Men…var det bara att mäta? Hur fungerade det? Är det någon som kommer ihåg bekymret? Vad är problemet?	T asking for verifying arguments about the SI.	
01:30.14	Dana: Det är lite för mycket. Det sticker upp.	C: evaluative arguments provided.	Inequality is noted Black rod < $(a+1)W$
01:35.02	L: Vad var problemet? Vad sa hon?	T asks for identifying arguments.	
01:39.13	Adam: Hon sa att vi kan mäta de röda stavarna med den svar.	SC: suggestion to measure red rods with black rod, SC not accepted.	
01:45.00	Mathew: Tag andra färger.	SC: suggestions to change rods.	
01:55.00	L: Det blir en annan uppgift. Spännande. Den kan vi ta en annan gång. Vad var problemet här?	T direct students back to TS. Asks for identifying arguments.	
01:57.17	[The teacher takes the last unit, the last red rod away.]	T constructs new TS. 	New model: $B > aW$

Table 7 (continued)

Timecode	Data	Analysis of Arguments	Identification of mathematical properties
02:13.05	Chaid: Det fattas jätte-lite till av den röda.	Provides identifying arguments: not an equality since the units of measure (aW) are less than the object to measure (B).	$B = aW + rW$. Therefore, inequality $B > aW$.
02:52.02	[The students are guessing answers like three and a half, four and a half etc. T is quiet through this process.]	SC: guessing a and r.	Using fraction language.
03:48.17	Bayar: Det är tre och en halva. Det är tre stycken såna där, och sen var det en röd som var längre och sen om du skulle lägga till en så skulle det vara längre än en svart. Men om man la lite till skulle man vara lika. Om man la till halva.	C: 3 ½, provide evaluative arguments. R is suggested ½ W, add "a little bit more". No further arguments provided to why $r = ½W$.	Constructs inequalities by dividing the whole into parts: a=3, if a= 4, then $B<4W$. $aW<B<(a+1)W$ Equality is constructed $aW+rW=B$

Table 8 Grade 4: Study of relations, equation 2; * represents an empty box. The Swedish language included in the excerpt

Timecode	Data	Analysis of arguments	Identification of mathematical properties
05:02.45	Leart: Vi kan kanske mäta? Vi måste väl mäta den röda, och den lilla biten som är kvar?	Sub-TS identified, asked to be confirmed: to measure W and r. First suggestion of SC.	The whole length is divided into aW and rW; smaller unit of measure not chosen.
05:03.18	L: Nu måste vi hålla ordning här	Confirms, suggesting the symbol W	The red rod is denoted W
[The teacher is writing W next to the red rods].			
05:15:30	L: Det här är de hela, det här är h. Och då känner ni igen att det är de streckade på tallinjen.	SC: Confirming the initiation of SC, to start with W	The whole unit of measure
05:24.15	Mehmet: Och sen är det ju faktiskt en liten bit till som vi….	SI: Verifies SC with added focus on the missing bit, the remainder. Re-establish the TS	Stress r in the equation Black = 3W + rW
05:46.25	L: Vi kan kanske behöva backa...vad är vårat problem? Vad är det vi ska lösa? Vad är det vi håller på med?	TS: Confirming. Stress the TS. T asks for identifying arguments about the TS.	
06:00.23	Dana: Vi ska få till det svarta.	SC: Responding and establishing TS.	
	[The teacher is writing "Svart = 3 röda + lite till".]	SI: Verifies the strategy	Study of structure: Black = 3W + rW
[The students and the teacher continue with various suggestions, during which the following was established and written down on the board: Svart = H röd + röd* (Image a). The arrow pointing at the box in the reconstructed figure is clarified "mellan 3 och 4" (Image b). T points at the four red rods, the fourth rod (Image c).]			

Table 8 (continued)

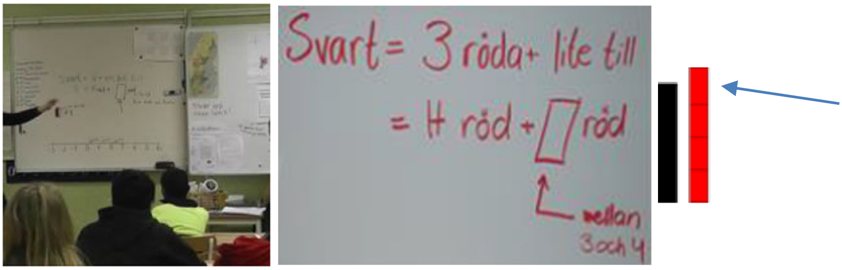
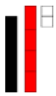
Timecode	Data	Analysis of arguments	Identification of mathematical properties
			
a) A photo of the white board in the lesson. b) A reconstruction of what is on the white board. c) A reconstruction of the measurement in which the teacher is pointing			
11:54.23	Dana: Det är den sista röda vi behöver mäta. [looking at the fourth red and the distance between 4W and the black rod].	C: In order to measure the black rod, we also need to know the length of W.	W = ns, s is the smaller unit of measure
12:10.40	L: Varför behöver vi mäta den röda?	Tasks for evaluative arguments for the C above.	
12:20.14	Bayar: För att den är längre. [reference to 4 red rods].	Implicit argument: four red rods are too long.	$4W > B$
12:25.06	Mehmet: Hur lång är den röda då, så att de blir lika?	Stress the remainder, r, in order to create equality	$rW + 3W = B$
12:33.15	Dana: Då behöver vi mäta med något som är mindre.	C: An evaluation is made. In order to answer TS, a smaller unit of measure is needed.	System of equations: $r = ms/ns$ and $W = ns$

Table 9 Grade 4: Solving the system of equations, equation 3. The Swedish language included in the excerpts

Timecode	Data	Analysis of arguments	Identification of mathematical properties
14:24.22	[The teacher puts white rods beside the red rod.] L: Hur manga röda enheter behöver vi? Hur många vita behöver vi? Hur många vita behöver vi för att mäta den svarta?	T formulates TS: What is m? SC: Use white rods as a measure and count. 	
14:38.27	Många elever: En!	SI: Straightforward, C: 1	$m = 1s$
15:55.06	L: Hur många vita behöver vi för att mäta den röda vi behöver? En [refererar till m] av hur många [refererar till n]?	TS: What is n? SC: put white rods next to the red rods and measure.	
16:02.04	Många elever: Två! [The teacher points to the empty box written on the whiteboard.]	SI: straightforward. C: 2	$n = 2s$
16:12.17	L: Hur skriver vi det?	TS: What is the final answer?	
16:45.05	[Evin goes to the board and writes: 1 white rod/2 white rods.] [While Evin writes, the teacher encourages the class to comment to confirm the conclusion.]	C: Evin fills in the gap: Black = $3W + (1s/2s)W$	$B = 3W + ½W$

Funding Open access funding provided by Stockholm University.

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