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Minding mathematicians' discourses in investigations of their feedback on students' proofs: a case study

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Abstract
This article presents a research apparatus for investigating and making sense of stories that emerge from feedback that mathematicians provide on students' proofs. Using the commognitive framework, the notion of *didactical discourse on proof* is developed as a lens for conceptualizing mathematicians' practice of feedback provision. The notion is accompanied by a tentative organizational frame, within which didacticians and mathematicians can operate as partners. The methodological affordances of this apparatus are illustrated with a case study of a research topologist, who taught a small, graduate course in topology. The emerged characteristics of her feedback and discourse are situated in the literature and used to sketch future research avenues.

Keywords Commognition · Mathematicians' teaching · Proof and proving · University mathematics education · Feedback

1 Introduction

In university mathematics, assigning mathematical statements for students to prove and providing written feedback on their submissions appears as a universal practice. Miller, Infante, and Weber (2018) suggest that, through this practice, mathematicians teach their students what a written proof should look like, and students learn about mathematicians' values in proof writing. Pinto and Cooper (2019) explore a novel assessment scheme, where students engage in multiple rounds of proof revision and resubmission based on their lecturer's feedback. Moore (2016) and Spiro, Hanusch, Miller, Moore, and Fukawa-Connelly (2019) show that mathematicians can invest considerable time in commenting on students' proofs, even if these are not expected subsequently to be resubmitted. I am not familiar with additional studies on the topic. These four articles, though, convincingly argue for the instrumentality of mathematicians' feedback for students' comprehension, their capability to produce written

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proofs, and their development of the notion of proof itself. This argument is in tune with higher education research that discusses teachers' feedback as an important component in teaching and learning (e.g., Hattie & Timperley, 2007; Mullner & Tucker, 2015).

My interest in this article is on the discursive characteristics and didactical reasoning that underpin mathematicians' feedback. With all the effort and time that mathematicians invest, they must assume that their feedback is more than an arbitrary set of annotations. Indeed, the argument about feedback contributing to students' learning is inseparable from the assumption that the stories that mathematicians tell by means of their feedback are not only graspable but even informative for students' future proving (see Moore, Byrne, Hanusch, & Fukawa-Connelly, 2018 for a student perspective on this matter). In this way, this article hopes to contribute to the theoretical, analytical, and methodological apparatus available for mathematics education research to investigate and make sense of stories that emerge from mathematicians' feedback.

The aim of this article is twofold. First, it is to argue that the commognitive framework (Sfard, 2008) can offer insights to investigate mathematicians' feedback on students' proofs and the practice of feedback provision. To make this argument, I introduce the notion of *didactical discourse on proof* in Section 3.1. Drawing on the literature review in Section 2, I suggest that investigations of this sort do not need to be exclusively research-oriented, and in Section 3.2, I sketch a tentative organizational frame, within which didacticians and mathematicians[1] can operate as partners (Wagner, 1997). The second aim is to illustrate how the notion and the frame can be operationalized and, specifically, to demonstrate their explanatory affordances in data analysis. To this end, in Sections 4 and 5, I take a close look at the case of a research topologist, who taught a "small" graduate course in topology. The contribution of this case study to the existing literature and further research is discussed in Section 6.

2 Mathematicians' didactical practices of proof

Before reviewing the four studies mentioned in Section 1, I note their difference in the terminology to refer to the mathematicians' practice under scrutiny, namely, "evaluation" in Moore (2016), "assessment" in Miller et al. (2018), "feedback" in Pinto and Cooper (2019), and Spiro et al. (2019). I use "feedback" or "feedback provision" due to the similarity of this student–mathematician interaction with the renowned *I-R-F* (initiation, response, feedback) pattern (e.g., Sinclair & Coulthard, 1975). In this case, *I* can be associated with mathematicians' assigning statements for students to prove, *R* pertains to a proof that a student submits, which is followed by *F* consisting of the mathematician's comments and points. The original pattern was identified in the context of school classroom talk that occurs in real time. The

[1] It is never easy to choose appropriate terms to refer to humans. This is especially hard when one needs to demarcate between two academic cohorts with advanced mathematical knowledge and teaching experience in university-level mathematics, and who pursue research interests in intimately related disciplines. Throughout this article, I use *mathematicians* in the sense of people whose research expertise predominantly lies within mathematical domains. With *didacticians*, I refer to those whose research focuses on "the art and science of teaching and learning" mathematics (Blum, Artigue, Mariotti, Sträßer, & van den Heuvel-Panhuizen, 2019, p. 2). Note that there are regional variations in the usage of both terms.

context of interest can be viewed as a written equivalent of this pattern, when a far more significant time separates among *I*, *R*, and *F*.[2]

In Moore (2016), four mathematicians first evaluated students' elementary proofs in discrete mathematics and geometry, and then re-evaluated some of them after reading the grades and comments of their colleagues. Miller et al. (2018) asked nine mathematicians to score proofs in the context of a transition-to-proof course. Both studies show that proof evaluation requires sophisticated judgments about the seriousness of the identified flaws and interpretation of student cognition. Spiro et al. (2019) and Pinto and Cooper (2019) focus on mathematicians' comments. Drawing on the data collected in abstract algebra and real analysis courses, Spiro et al. (2019) introduce a scheme to code mathematicians' comments. The researchers distinguish between comments that refer to a proof as a whole and those directed at a specific piece of a student's text. The vast majority of the explored comments were of the latter type, and these comments introduced direct edits, demonstrated the correct use of notation, and pointed at changes that needed to be made. Similar dichotomization was made in Pinto and Cooper (2019). Their case study shows that when a proof–feedback exchange contains multiple cycles, a mathematician can prioritize giving feedback on holistic issues, while leaving commenting on more local matters for later rounds of response.

The foregoing studies can be associated with a broader research area interested in mathematicians' teaching of proof. This, in turn, has paid close attention to mathematicians' views and perspectives on proof teaching and learning (Alcock, 2010; Harel & Sowder, 2009; Hemmi, 2010; Lai & Weber, 2014; Weber, 2012). Access to these views and perspectives has mostly been gained through interviews (e.g., the first three cited above), while some studies (e.g., Weber, 2012) also engage their participants in artfully designed tasks that are faithful to proof teaching. Overall, only a handful of studies draw on authentic proof-teaching practices (e.g., Pinto, 2018; Weber, 2004).

Some recent studies suggest that mathematicians' didactical practices are influenced by their research (e.g., Biza, Giraldo, Hochmuth, Khakbaz, & Rasmussen, 2016). For instance, the mathematicians discussed in Madsen and Winsløw (2009) indicated that, without their research, they could not teach the way they do. Yet, some admitted that the more theoretical their research is, the more difficult it is to integrate it in their instruction. Differences between mathematicians working in different disciplinary areas have also been documented. Inglis, Mejia-Ramos, Weber, and Alcock (2013) identified substantial disagreements among pure and applied mathematicians on whether a particular calculus argument should be sanctioned as a proof. Weber and Czocher (2019) report on a similar finding and show that the disagreements can emerge around visual and computer-assisted arguments.

Another noteworthy issue concerns mathematicians' opportunities to develop their didactical practices (Biza et al., 2016). One of the key themes already recognized within this research pertains to mathematicians gaining relevant insight into teaching (e.g., Jaworski, Mali, & Petropoulou, 2015; Nardi, 2016). This theme raises a question about opportunities that mathematics education researchers provide for their mathematician participants to grow as university teachers.

In the school context, Wagner (1997) discerns three forms of partnerships that didacticians and teachers can establish as part of a research investigation. The first form is a data extraction agreement, in which didacticians have full agency over the process and the teacher agrees to

[2] Mehan (1979) changed the pattern to I-R-E, where E stands for evaluation. However, the analysis in Section 5 showcases that mathematicians' feedback is not always evaluative.

exhibit the practice for didacticians' scrutiny. The didacticians' learning from the investigation is reported in a way that is targeted at other didacticians and a mediating effort is needed to "translate" the insights for other teachers. Clinical partnerships are the second form of partnership. Here, didacticians still lead the investigation but teachers have the space to articulate their expertise and contribute to the research endeavor, when the investigation itself is oriented towards a joint work. In the third form of colearning agreements, teachers and didacticians aid one another to learn something new and valuable about their practice and themselves.

On the face of it, most research on mathematicians' proof teaching is conducted with some sort of didactician–mathematician agreement. Yet, the reports are typically silent about what the mathematicians learned from participating in a study, as well as their roles beyond data generation. I argue that if appropriate organizational frames are put in place, investigations into mathematicians' didactical practices can provide systematic opportunities through which mathematicians can develop as mathematics teachers, for instance, through raising their awareness of their implicit and habitual actions.

3 Research apparatus for investigating mathematicians' feedback on students' proofs

In a university course setting, the phenomenon of mathematicians providing feedback on their students' proofs can be construed as a form of communication that unfolds as part of mathematics teaching and learning. It seems reasonable, therefore, to consider this phenomenon through some discursive lens (e.g., Sfard, Forman, & Kieran, 2001). My choice rests with the commognitive framework (Sfard, 2008). Being interested in mathematicians, mathematics teaching, and proving, it seems justifiable to choose a framework that "has been developed within the field of mathematics education and is designed to address the problems arising in this field" (Morgan, 2020, p. 226). Furthermore, commognition has been acknowledged for its capability to account for the complexity of university mathematics education, while offering analytical tools for capturing its fine-grained intricacies (e.g., Nardi, Ryve, Stadler, & Viirman, 2014). These affordances have been used before to explore the teaching and learning of proof (e.g., Pinto, 2018). Lastly, as it will be argued shortly in Section 3.2, a mathematician can be a contributing partner for a didactician in investigating the former's didactical practice (cf. Wagner, 1997). This turns the communication between mathematicians and didacticians into another object of research interest. From the commognitive standpoint, both types of communication (i.e., (a) between a mathematician and the students, and (b) between the mathematician and the didactician about (a)) constitute ontologically compatible entities for analysis.

In the first part of this section, I use the commognitive framework to introduce the notion of didactical discourse on proof. In the second part, I sketch a tentative organizational frame, within which a mathematician and a didactician can engage in this discourse as part of a research investigation.

3.1 Didactical discourse on proof

To access the commognitive apparatus, one needs to delineate a *discourse* within which the practice of feedback provision on students' proofs resides. Similar to many socio-cultural schools of thought that draw on the works of Vygotsky, commognition acknowledges the social nature of human activity. Accordingly, I start with the social to arrive at the individual.

Commognition construes discourses as "different types of communication, set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard, 2008, p. 93).[3] Object-wise, the discourse of our interest can be conceived as didactics of mathematical proof (hence, didactical discourse on proof, or *DDP*, for short). To better position it, I consider DDP through the lens of related discourses. With proof at its heart, DDP can be viewed as a part of a pedagogical discourse, which Heyd-Metzuyanim and Shabtay (2019) describe as the one that orients teachers "towards *what* to teach students, *how* to teach them, *why* certain teaching actions are more effective than others and, often not talked about but still very important, *who* can learn (or not learn)" (p. 543, italics in original). From a complementary perspective, DDP can be viewed as consisting both of mathematical and of pedagogical components: the mathematical component concerns contemporary norms that are accepted within a particular community regarding what constitutes a mathematical proof and how it can come about; the pedagogical component focuses on initiating newcomers into proving (see Cooper, 2014, for revisiting the Mathematical Knowledge for Teaching framework from the commognitive standpoint).

Discourse can also be charted through its settings and communicating actors. Regarding DDP, a few of its residences come to mind: *Educational Studies in Mathematics*' special issue on "Research-based interventions in the area of proof," a thematic working group on "Argumentation and proof" in the Congress of European Society for Research in Mathematics Education, the blog "On teaching and learning mathematics" on the website of the American Mathematical Society, proof-related professional developments, corridor conversations, and interviews settings where teachers share their approaches to teaching proof. Thus, mathematicians are not the only community to participate in DDP. School mathematics teachers, didacticians, teacher educators, textbook writers, and curriculum designers can also find themselves engaging in this discourse, although the communicational practices may vary significantly from one DDP community to another.

The existence of various DDP communities offers a pragmatic response to the question of what makes DDP deserving to be discerned from a general discourse on mathematics education—there are many people that find it meriting such a discernment. And they have good reasons for it. Some of these reasons have been provided by mathematics education research repeatedly pointing at proof as a universal challenge for newcomers to a mathematical discourse (e.g., Stylianides, Stylianides, & Weber, 2017). An alternative justification for the discernment stems from the historical evolution of proof in the mathematics community and its far-reaching entailment to mathematics (e.g., Hanna & Barbeau, 2002; Kleiner, 1991). This evolution sits at a plural crossroads of epistemology, ontology, philosophy, and logic, pertaining to a broader polemic on the nature of mathematics and its practice (e.g., de Freitas, Sinclair, & Coles, 2017).

Having addressed DDP as a collective endeavor, I now shift to its individual versions. Evidently, individual discursive actions are shaped by context and circumstances. Nevertheless, Sfard (2008) maintains that "some patterns of the person's discursive actions are likely to remain relatively stable across her interaction with different interlocutors" (p. 128). These personal and publicly accessible patterns are construed as personal discourse.

[3] Several schools of thought have associated the notion of discourse with the communicational activity of a specific community (e.g., Bakhtin, 1986; Foucault, 1972; Gee, 1997; Halliday, 1978).

Accordingly, one's DDP can be associated with any discursive activity that has to do with one's didactics of mathematical proof. This includes the activity of proof teaching and its communicational congener of talking about proof didactique. Indeed, while the participants of DDP often undertake these activities in different settings, with different motives, and with different interlocutors, both endeavors are social, patterned, and involving a repertoire of permissible actions and re-actions (cf. Sfard, 2008). Thus, positioning one's proof teaching and talking about this practice in the same ontological category warrants their investigation with the same analytical machinery. The commognitive machinery is attentive to one's keywords, visual mediators, narratives, and routines (see Sfard, 2008 for the original framework and Tabach & Nachlieli, 2016 and Nardi et al., 2014 for summaries).

3.2 A tentative organizational frame

The proposed frame is built around a didactician's analysis of comments and points that a mathematician gives on their students' proofs.[4] The distinctive element of the frame pertains to *reflection sessions*, where the didactician shares with the mathematician (possibly preliminary) narratives emerging from the feedback analysis. In a typical interview, a didactician can either observe the mathematician providing feedback "here and now," or a mathematician can be invited "to relive an original situation [i.e., feedback provision] with vividness and accuracy" (Bloom, 1953, p. 161). In turn, reflection sessions are designed as a communicational platform for the mathematician to react to the shared narratives and for both partners to engage in DDP.

What can the partners learn from their communication in reflection sessions? Much has been written about the value of reflection in teaching, with particular attention to obstacles that impede many teachers from "simply" reflecting on their practice (e.g., Artzt, Armour-Thomas, Curcio, & Gurl, 2015; Goodell, 2006). Reflection sessions can potentially provide the mathematician with a space for such communication with a genuinely interested didactician who offers systematically derived narratives that the mathematician might be unlikely to discover alone. Such narratives can raise the mathematician's awareness of their own didactical doings and provide opportunities to reconsider habitual (often implicit) practices. From the didactician's side of things, the mathematician's reflection may help to situate the analytical narratives in the DDP that goes beyond feedback provision and additional discourses in which the mathematician participates. A reflection may offer reasons and justifications for the identified discursive regularities, and it can illuminate links to other aspects of the mathematician's teaching that may have been glossed over by the didactician in the initial feedback analysis. In this way, the mathematician's reflection not only enriches the data corpus, but may also present promising angles for new analytical rounds. Last, but not least, alternating between the perspectives of an insider and outsider to the discourse of interest is a key principle in commognitive analysis (Sfard, 2008). Communication with the mathematician can assist the didactician to grow as an insider in the mathematician's DDP and potentially lead to more informed interpretations.

Before illustrating what collaboration within the sketched frame may look like, let me offer two comments. The first concerns the status of the mathematician's reflections on the

[4] The communication described herein requires a professional vulnerability, openness, and trust from the interlocutors. In my personal, albeit limited, experience with several mathematicians, communication of this sort can be established in pairs (for another example, see Kontorovich & Bartlett, accepted pending revisions). Hence, I present the frame as a partnership between one didactician and one mathematician, while not discarding the possibility of using the frame in larger groups.

didactician's analytical narratives. From the commognitive standpoint, analytical narratives are stories that a didactician tells about the mathematician's proof teaching. Hence, an accurate application of the framework should shape the didactician's persuasion in the validity of the devised narratives. The mathematician's reflection, in turn, constitutes a story that a mathematician tells about the didactician's story. Thus, reflections and analytical insights differ in their rules for narrative building and endorsement. To put it bluntly, reflection sessions are not intended for a member check technique (e.g., Lincoln & Guba, 1985), and so, there is no reason to celebrate if a mathematician affirms an analytical narrative, nor it is sensible to be concerned if it is rejected.

The second comment pertains to the accessibility of the analytical narratives to the mathematician. Developed within the commognitive framework, an analytical narrative can lie beyond the mathematician's reach. This communicational divide cannot be underestimated, and different partnerships may well bridge it in different ways. Some may decide to broker the relevant commognitive constructs and add an interesting angle to their communication. Other partnerships may prefer to let the theory impact their communication from "behind the scenes" without becoming an explicit object of discussion. In the latter case, the narratives that the didactician shares with the mathematician can revolve around identified discursive regularities, before hanging them in a "theoretical coat closet" (Maxwell, 2013). Such pattern-oriented narratives are often quicker to generate, which might be useful if partners are to maintain regular reflection sessions.

4 The case study

To demonstrate the DDP and the organizational frame from Section 3 in action, I present the case of a research topologist, Dr. Brownstone. Initially, the case was chosen due to a critical mass of felicitous affordances, which are described in Section 4.1. However, as it will be argued in Section 6, the case findings offer insights to the existing literature and further research. The purpose of the analysis is to showcase how a mathematician's practice of providing feedback on her students' proofs can be illuminated through the lens of her DDP. The two questions underpinning the analysis are:

(1) What are the discursive characteristics of a mathematician's comments and points on student-submitted proofs?
(2) How can the statements about proof generated in reflection sessions be utilized to explore and make sense of a mathematician's feedback on student proofs?

The methods that were used to address the questions are presented in Section 4.2.

4.1 Topology, Dr. Brownstone, and the research setting

In the research mathematics community, topology enjoys a special standing of a vibrant field enrooted in the works of highly regarded mathematicians (e.g., Euler, Cauchy, Riemann, Poincaré). This is also a field that has become fundamental to many mathematical disciplines (for elaborated reviews, see Franklin, 1935; Richeson, 2008). When speaking colloquially, some topologists describe their discipline as "rubber-sheet geometry" that studies non-rigid shapes (e.g., Franklin, 1935). These shapes often require a re-defining of the key concepts of geometry and additional fields and raising them to higher levels of abstraction (e.g., Richeson, 2008). Another characteristic of topology is its at best modest interest in numerical measurements. As a distinguished topologist Burt Totaro (2008) phrased it, "[t]opology allows the possibility of making qualitative predictions when quantitative ones are impossible" (p. 383). These characteristics turn proving theorems about topological objects into a key endeavor in a topological research discourse. This allows wondering whether this special status of proof may permeate topologists' DDP, especially the ones that they lead in their topology classrooms.

Educational research on topology is in its infancy (for exceptions, see de Freitas & McCarthy, 2014; Gallagher & Engelke Infante, 2019; Stewart, Thompson, & Brady, 2017), and it does not allow determining whether the abovementioned wonder is generally true. Yet, it was true in the case of Dr. Brownstone, who allocated a substantial space in her course for proving. For instance, almost all the course tasks assigned as part of the homework, a mid-term test, and in a final exam asked students to "prove that …."[5]

Dr. Brownstone is a well-regarded topologist with dozens of publications in professional research journals. She has more than a decade of mathematics teaching experience in school and university, and a genuine interest in improving her students' learning. The focal course lasted over a single semester in a large, English-speaking university in the Southern Hemisphere,[6] and it was concerned with standard topics in point-set topology (e.g., continuity, convergence, compactness) and in algebraic topology (e.g., covering spaces, fundamental groups, homology theory). The course cohort consisted of six students at the beginning of their professional mathematical journeys. Four of them were studying towards post-graduate degrees in mathematics, and two students were at the end of their undergraduate mathematics major taking the course as enrichment.

A few words seem in order regarding how our partnership came about. Dr. Brownstone and I have been acquainted for several years. At some point, she invited me to visit her topology course and discuss possible ways "to increase the students' engagement" (her phrasing). Dr. Brownstone explained that the course requires "heaps of proving," and in her experience, this is a common point of weakness for students. I did not take Dr. Brownstone's reaching out for granted. In terms of Section 3.1, it marked her active participation in DDP, at least on the level of noticing and reflecting on her own teaching. The invitation also signaled her potential readiness to make changes in aspects of her teaching in light of my feedback.

4.2 Data corpus and methods

Towards the middle and at the end of a 12-week long semester, Dr. Brownstone prepared two homework assignments that were intended for individual solution by the course students. The assignments comprised four and five problems respectively (see Fig. 1 for a representative example). Some of the problems contained subproblems. The students were given nearly 2 weeks to complete the assignments, each of which contributed 7.5% to their final grade.

After the students' assignments were collected and before providing her feedback, Dr. Brownstone and I had a 1-h-long session where she shared her general approach to assessing assignments and situated this practice in her pedagogical discourse. The session ended with Dr. Brownstone asking, "So how do you want me to go around [assessing] the assignments this time?" and me

[5] Another topologist instantiating this permeation is Robert Lee Moore and his renowned method of teaching (see Coppin, Mahavier, May, & Parker, 2009 for details).

[6] This general description is aimed at protecting the confidentiality of Dr. Brownstone and her students.

Cantor's ternary set, C, is defined to be the intersection of the sequence $C_0, C_1, \ldots$, where $C_0 = [0, 1]$ and for $i > 0$, C_i is the union of 2^i closed intervals obtained from the 2^{i-1} closed intervals of C_{i-1} by removing from each of these intervals the open middle third of the interval. Thus

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right], C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right], \text{ etc.}$$

Prove that C is homeomorphic to $\mathbf{2}^{\mathbb{N}}$ where $\mathbf{2} = \{0, 1\}$ and has the discrete topology, and $\mathbf{2}^{\mathbb{N}} = \prod_{\alpha \in \mathbb{N}} X_\alpha$ where $X_\alpha = \mathbf{2}$ for each $\alpha \in \mathbb{N}$.

Fig. 1 A representative problem from the course homework assignment

asking her to continue providing feedback to students as she usually does. My response was driven by the acknowledgment of her professionalism and an interest in her typical practice of feedback provision. When asked in a later reflection session about how this feedback was different from usual, she replied, "I assessed in a way that I would usually do. Since I knew that we were going to talk about it, maybe I was a bit more detailed in my comments to the students."

In accord with Section 3.2, our interaction cycled between my analyses and our reflection sessions. To address the first question about discursive characteristics of the feedback, I distinguished among local comments (cf. Spiro et al., 2019) that she provided on particular words, symbols, and sentences of a student (*line-level comments*), global comments about a submitted text as a whole (*text-level comments*), and numerical points that she allocated. Table 1 overviews the distribution of her comments in the data corpus.

My analysis of Dr. Brownstone's feedback was informed by the principles of commognitive research and the conceptual–analytical apparatus that the framework offers (Chan & Sfard, 2020; Sfard, 2008). From the commognitive standpoint, one can participate in mathematical discourse by *mathematizing*, i.e., producing narratives about mathematical objects, and by *subjectifying*, i.e., narrating about discourse participants. Accordingly, Dr. Brownstone's comments were first broken down into sentences (and sometimes parts of a sentence), each of which was identified as an instance either of mathematizing or of subjectifying (see Table 2 for examples). To grasp the diversity of line-level comments, they were further categorized into *object-level mathematizing*—producing or contributing to the production of mathematical objects, and *meta-level mathematizing*—addressing students' mathematical actions, both taken and omitted at some higher level (for a similar analytical approach, see Chan & Sfard, 2020). Table 3 illustrates this distinction and introduces additional subcategories that emerged from iterative analytical rounds. The subcategories bear

Table 1 Overview of the analyzed comments

	Assignment I		Assignment II		Total	
Students	Text-level	Line-level	Text-level	Line-level	Text-level	Line-level
I	6	44	6	12	12	56
II	8	40	18	21	26	61
III	8	12	6	9	14	21
IV	6	10	6	15	12	25
V	6	14	5	17	11	31
VI	8	14	6	15	14	29
Total	42	134	47	89	89	223

Table 2 Examples of mathematizing and subjectifying instances

	Comment level	*Examples*	*What is the instance about (the theme)*
Mathematizing	Text-level	"A very detailed proof"	Proof
	Line-level	Proof: B is closed in A so then A \ B is open in A. *[red:]* so A \ B is open in X ? X is open in X. So (A \ B) ∩ X is open in X	$A \backslash B$
Subjectifying	Text-level	"You articulate your arguments very well"	Student's discursive action of argument articulation
	Line-level	Let T_C be the subspace topology on C induced by the usual topology on ℝ. Then $T_C = \{C \cap U \mid U$ is open in ℝ$\}$. Hence a basis of T_C is $\mathcal{B}_C = \{U \subset C \mid \forall p<q \in C\,[p<r<q$ and $r \in C] \Rightarrow r \in U\}$, i.e. a basic open set in C must be "connected" → easy to check $\mathcal{B}_C$ is a basis of C. *[red:]* What do you mean?	Student's discursive action of meaning with the word "connected"

Table 3 Categorization of line-level comments

Mathematizing	*Types*	*Examples*
Object-level	(1) Amending words or symbols that a student used to signify mathematical objects; or (2) expanding mathematical narratives about these objects	To show f is onto: Pick $y_1 \in 2^{\mathbb{N}}$ $y_1 = \left(\frac{a_n}{2}\right)_{n\in\mathbb{N}}$ *[red:]* for some $a_1, a_2, \ldots \in 2$
	Making an unaddressed inference that is "symmetrical" to the one appearing in the text	Without loss of generality, ~~let~~ since X is T_0, there is open subset of X, denoted U, such that x∈U and y∉U. *[red:]* or y∈U and x∉U
Meta-level	Seeking definitions, explanations, or substantiations from a student regarding notation, words, or statements	(c) Suppose X is a regular T_0 topological space, and let $x, y \in X$ such that $x \neq y$. Since X is T_0 there is an open set $U \subseteq X$ containing exactly one of x and y. If $x \in U$ we are done so suppose instead that $y \in U$. Since X is regular, and $C := X \setminus U$ is closed with $y \notin C$, there are open sets $V, W \subseteq X$ with $y \in V, C \subseteq W$, and $V \cap W = \emptyset$. Since $C \subseteq W$, and $x \in C$, we have $x \in W$. Moreover, $W \cap V = \emptyset$ so since $y \in V$ we have $y \notin W$. Therefore X is T_1. *[red:]* Why?
	Indicating an issue without unpacking it	So f maps basic open sets to basic open sets So f^{-1} is continuous at c c is arbitrary, so f^{-1} is continuous. f is bijective So f is continuous So f is homeomorphic *[red:]* Something is missing.
	Contradicting a statement	Let $x, y \in X/A$ with $x \neq y$. Suppose x is the element of X/A corresponding to the set A in X, and y is an element of X with $y \notin A$. X is regular, so there are disjoint open sets $U, V \subseteq X$ with $y \in U$ and $A \subseteq V$. In X/A, the set V corresponds to $(V \backslash A) \cup \{x\}$, and U stays the same. *[red:]* Techically this is not the case Members of X/A are equiv. classes. Therefore $y \in U$ and $x \in V$ for open $U, V \subseteq X/A$ with $U \cap V = \emptyset$. Now suppose neither x nor y correspond to the set A in X. $x, y \in X$ and X is T_2, so there are disjoint open sets $U, V \subseteq X$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$. In X/A, U and V correspond to disjoint open sets U' and V' with $x \in U'$ and $y \in V'$. Therefore X/A is Hausdorff. *[red:]* what if $U \cap A \neq \emptyset \neq V \cap A$? * The two comments exemplify the categories correspondingly.
	Pointing at an alternative inference or a case substantially different from those considered	

resemblance to feedback classifications familiar from the literature (e.g., Hattie & Timperley, 2007). Overall, the categories of subjectifying, object-level and meta-level mathematizing were used to capture issues in students' proofs that Dr. Brownstone articulated on the line- and text-levels and to discern those issues that entailed point reduction.

In reflection sessions, I shared preliminary characteristics emerging from the feedback analysis with Dr. Brownstone according to the principles sketched in Section 3.2. Specifically, I elaborated on the analytical narratives, provided illustrations, and generally described how my analytical narratives came about. As will be visible shortly in the next section, Dr. Brownstone responded with statements many of which revolved around proof. These statements often brought new aspects to the fore and sometimes even provided terms that I used in the analysis rounds that followed. Overall, we conducted three reflection sessions lasting for 35, 20, and 25 min, each of which spurred a new round of analysis.

5 The story of Dr. Brownstone's feedback and practices of feedback provision

This section consists of three parts. The first reports on the emergence of a narrative that turned out to be especially useful for making sense of Dr. Brownstone's feedback. The second part concerns regularities identified among points, line- and text-level comments that she provided. Two types of written "talks" that Dr. Brownstone had with her students through her feedback are in focus in the third part. Each part is structured to highlight the usage of the proof statements that she made in our reflection sessions to advance the feedback analysis (see the second question in Section 4).

5.1 "In the homework problems, students need to acquire the proof idea and represent it accurately in writing for Dr Brownstone"

The analysis of the subjectifying sentences that Dr. Brownstone wrote as part of her feedback revealed two sets of verbs that she tended to use to refer to students and their proof-related actions. Table 4 illustrates that some of the verbs related to students *possessing* some sort of a mathematical commodity (e.g., "terms," "the right idea," "concepts," "understanding"), while the right column highlights that the students *represented* these commodities in their texts. Hence, an analytical narrative that I sketched based on this observation stated, "Proving of assigned statements required students to possess and represent some mathematical commodity." I shared this sketch with Dr. Brownstone with an expectation for her reflection.

Table 4 Instances of subjectifying comments

Possession	Representation
"You *have* a very good grasp on how to accurately express terms"	"You *articulate* your arguments very well"
"You *have* the right idea"	"You have *demonstrated* that you understand what you write"
"I think you *have* the right idea"	"You *show* a good understanding of the concepts"
"You *understood* the main concepts involved in this proof"	"You *show* a good understanding of quotients"

One part of Dr. Brownstone's reflection can be associated with the "possession" component of the narrative. She noted that she was convinced that there were cases in which the students were using the internet to solve the assigned problems. Moreover, she added that sometimes she intentionally name-dropped mathematical notions in problem formulations for an internet search engine to take the students to a specific web resource. For example, regarding a product topology and the Cantor set in the problem presented in Fig. 1 she explained:

> I am absolutely sure they used external resources through the problem, and this particular problem I'm happy with. Because my goal for them is to understand what is going on with product topology and also get the benefit of finding out a bit more about a Cantor set.

In this way, Dr. Brownstone's reflection specified my analytical narrative, clarifying that she did not expect students to "possess the mathematical commodity" a priori (for instance, from a classroom), but that they would emerge from working on the assigned problems. In some cases, she expected students to take physical actions, like searching the internet, to "acquire the mathematical goods." Overall, Dr. Brownstone talk suggested that she adhered to the acquisitionist view on learning (see Sfard, 1998 for the metaphor of learning as acquisition and Heyd-Metzuyanim & Shabtay, 2019 for acquisition pedagogical discourse).

In another part of her reflection, Dr. Brownstone drew parallels between some of the practices that she institutionalizes in her course and mathematics research.

> In a sense, I think of assignments as reflecting what you would do normally in a research publication. […] Because there are two elements to a proof, loosely speaking: one is this spark of insight as to how the proof works – what makes it go through. And the other is writing it down in a logical manner, which supports the idea and makes it true mathematically for the editor and reviewers. So they [the course students] have to obviously have the idea of how the proof works, how you get from here to here. And then, they have to write it down in the way that conveys to me that it is logical.

In this reflection, Dr. Brownstone stressed the similarities between "proofs" in homework assignments and in a research discourse. There, she also distinguished between "the spark of insight as the how the proof works" and "writing it down […] for the editor and reviewers." In this way, her reflection can be interpreted as a substantiation for the abovementioned analytical narrative since "this is the process that students need to go through because this is what you would normally do in research." Yet, Dr. Brownstone emphasized the importance of written representation and its intended audience. In this way, this analysis–reflection cycle resulted in a refined version of the narrative, suggesting that "in the homework problems, students need to acquire the proof idea and represent it accurately in writing for Dr Brownstone."

I acknowledge that the foregoing narrative makes considerable room for interpretations, such as what "the idea of proof" stands for and what is meant with "representation for Dr Brownstone." Yet, I was content with the narrative as it was faithful to the analytically identified pattern and accounted for Dr. Brownstone's subsequent reflection. The two components of the narrative also resonated with how proof is sometimes discussed in the mathematics education literature. The notion of "the proof idea" echoes with what Hanna and Mason (2014) call "key ideas," when Dreyfus and Hadas (1996) view a proof as an argument that explains why a statement is true ("what makes it go through"). In turn, the representation component resembles Harel and Sowder's (1998) approach, arguing that a proof needs to provide certainty to a mathematician.

5.2 Line-level comments, text-level comments, and point reduction

Tables 1, 2, and 3 show that the scope of line-level comments that Dr. Brownstone provided was rich and diverse, when only some of them were also addressed on the text-level and entailed point reduction. As an example, let us consider Fig. 2 showing a text that one of the students, Diego, submitted as a response to the problem presented in Fig. 1. In her meta-mathematizing comments, Dr. Brownstone draws on the fact that the set 2 = {0,1} is the one having a discrete topology, while $2^{\mathbb{N}}$ is an infinite product with a conventional product topology. In this way, the deductive chain that Diego developed on his way to the claimed homeomorphism *F* contained a faulty assumption. This is probably what instigated Dr. Brownstone to suggest on the text-level for Diego to check his understanding of the $2^{\mathbb{N}}$

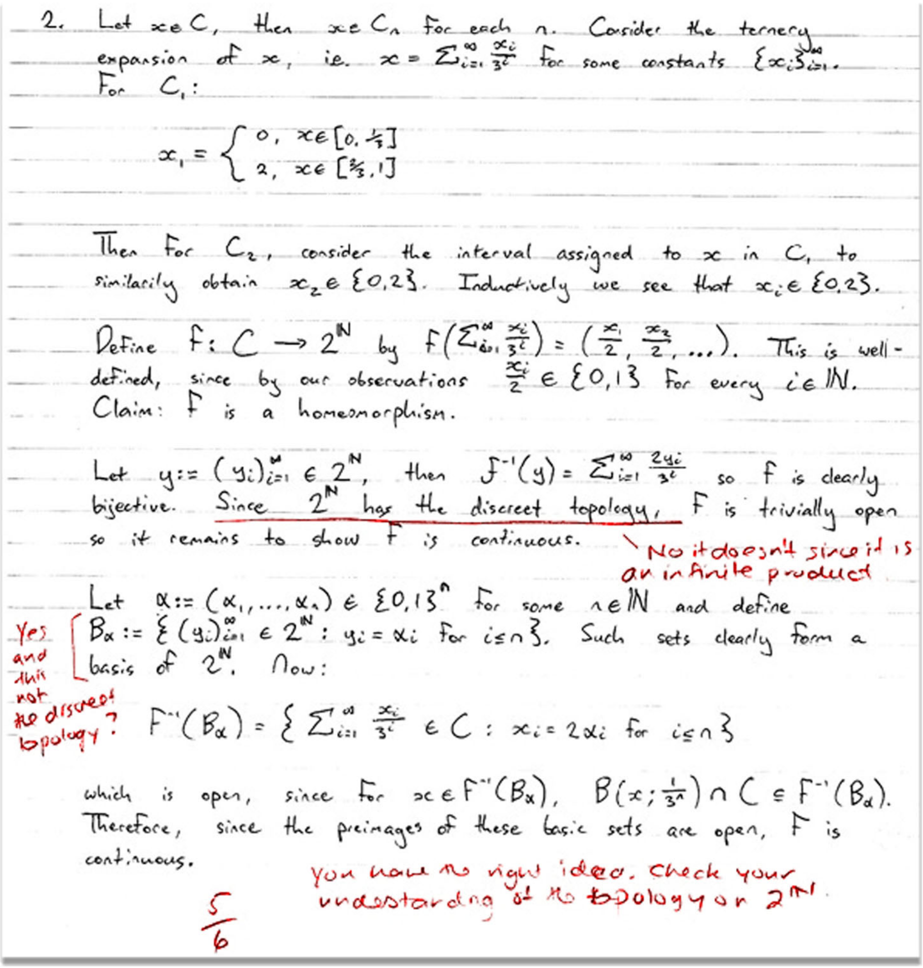

Fig. 2 Diego's solution to the problem in Fig. 1

topology. Notwithstanding, he received five points out of six. This example illustrates that the relation between the comment and the reduced grade was not always straightforward.

In the reflection session, Dr. Brownstone referred to Diego's text as follows:

> Dr Brownstone: The components of what he is doing are on the right track. Because he misinterpreted the product topology he is going to fail, he is not about to prove it [the homomorphism] because he is not proving the right thing […]. But the basic idea, the format of his proof is correct, he just misinterpreted the topology on $2^{\mathbb{N}}$, which was one of the things that I was hoping he would take out of this question. […] I must've thought that he did everything else very well because I've taken only one mark off.
> Igor': What is 'else' in this case?
> Dr Brownstone: Well, the components of the proof are to describe the Cantor set in a way that you can talk about a bijection with the $2^{\mathbb{N}}$, and then you need to show that it is bijection, and then you should show that it is open and continuous. But setting up this bijection is key. […] So he has understood all of that, but he has misinterpreted the topology on $2^{\mathbb{N}}$.

Note that in her reference to Diego's text, Dr. Brownstone stresses "the basic idea," "the components," and "the format"—keywords aligning with her notion of the proof idea from Section 5.1. When describing her expectations, she mentions three "components of the proof": capturing the Cantor set in a way that allows manipulation, creating a bijection with $2^{\mathbb{N}}$ and showing that the bijection is continuous and open. All three are evident in Diego's text: he represented a general point x in the Cantor set as $\sum_{i=1}^{\infty}\frac{x_i}{3^i}$ ($x_i\in\{0, 2\}$), and he introduced a bijective function F that matches the set to the infinite binary sequence $\left(\frac{x_1}{2}, \frac{x_2}{2}, \ldots\right)$. The assumed discreteness yielded that F is trivially open, and then Diego proceeded to show its continuity. The issue with the discrete topology emerged in a single "component," hence a reduction of a single point. Note that representation aspects were never addressed in this case.

Echoing patterns emerged from the analysis of Dr. Brownstone's comments. Most students' texts received at least one line-level comment about an issue with their mathematizing. Yet, only just over 20% of the text-level comments mentioned these issues again, meaning that many local issues got lost between the levels. Specifically, nearly 90% of the mathematizing comments Dr. Brownstone made on the line-level were not mentioned again on the text-level. In terms of meta-mathematizing, seeking further explanations and substantiations were re-raised in about 35%, when contradicting, pointing at issues, and providing alternative conclusions were re-addressed in more than 70% of the cases. In this way, an issue with the proof idea had higher chances of sprouting to the text-level than the issue being about representation. Table 5 presents the most common issues raised on the text-level. The table shows that the

Table 5 Common issues at the text-level and accompanying points

Type of issues	Proof idea vs representation	Point reduction
Proofs with missing components (e.g., "Your proof is correct (as far as it goes)")	Proof idea	All cases
Misunderstanding of a mathematical object or its properties (e.g., Fig. 2)		
Denoting different mathematical objects with the same notation (e.g., "Just be careful that if you fix a point x, then don't use x as a variable")	Representation	Never
Insufficient explanations (e.g., "a few extra details are required in places")		
Undefined mathematical objects (e.g., "Just be careful to define any sets you introduce")		

issues concerning the proof idea came hand in hand with a lower grade, which was not the case for their representation congeners. To sum, in Dr. Brownstone's feedback, issues with the proof idea appear as something serious enough to be communicated to students several times and via different media, when many issues with representation are not worth mentioning twice.

5.3 Talking to "a fellow mathematician" and to "a dear student"

Most text-level comments described students' texts, their mathematizing, or students themselves in distinctively positive terms (see Table 2 for examples). Yet, the analysis revealed a peculiar connection between the allocated scores and the object of Dr. Brownstone's comment. Table 6 shows that comments on fully scored texts mostly revolved around mathematical objects (usually, "proofs"), while a reduced grade was accompanied by a comment with some subjectifying component.

> I shared the foregoing observation with Dr. Brownstone, and she responded:
> That's interesting, I haven't thought about it at all… I don't know, I think if the proof does stand alone then I talk about the proof. If it is not working, then I feel like I'm talking to a person that I need to explain what needs to be done to make it right. I suppose in a way, if the proof goes well, then I am talking to a fellow mathematician and we talk about the math. If the proof does not go well, then I put my teacher's hat and say 'OK dear student, this is what you need to think about. This is me teaching you to do something in this way'. Maybe, I don't know…

While acknowledging her unawareness about the identified discursive regularity, Dr. Brownstone embraced and rationalized it by coining the terms a (written) *talk to a fellow mathematician* and a *talk to a dear student*. Two notes can be made on her reflection. On the one hand, she distinguished between the talks in terms of "the proof [that] does stand alone," is "working", and "goes well"—formulations that resonate with the proof idea more than with the representation element in the narrative in Section 5.1. This is also in line with Section 5.2, showing that many issues with representation did not sprout text-level comments, and then Dr. Brownstone did "talk about [the quality of] the proof." On the other hand, putting her "teacher's hat" on and teaching a "dear student […] to do something in this way" were not always as explicit as it sounds. Only in less than 10% of the cases, did Dr. Brownstone explicitly lead the students towards a specific mathematical action, as she did in her comments to Diego (see Fig. 2).

Endorsing Dr. Brownstone's terms, I identified a talk to "a colleague mathematician" with text-level comments that she provided when allocating a full score on a student's text; the remaining comments were associated with a talk "to a dear student." The analysis revealed that the two talks differed in two sets of keywords that they used. Table 7 shows that the words "proof" and "argument" were mostly used in the talk to a colleague mathematician, when in

Table 6 Relation between text-level comments and score

	Full score	Reduced score
Comment without subjectifying components	91%	13%
Comments with a subjectifying component	9%	87%

Table 7 References to students' texts

Signifying words	"Proof" and "argument"	"Work", "this," and "well done"
Talk to a colleague mathematician	84%	24%
Talk to a dear student	16%	76%

the talk to "a dear student," the texts were often encapsulated as "work," "this," and "well done." Manin (1977) writes that "a proof becomes a proof after the social act of 'accepting it as a proof'" (p. 48). In this case, such an acceptance materialized in Dr. Brownstone sanctioning students' texts with words that she used to refer to them.

The second set of keywords pertains to the possessive pronouns "your" and "yours," through which Dr. Brownstone attributed mathematical objects (mostly "proof," "argument," "work") to the students (e.g., "Your proof is correct (as far as it goes)" compared with "Good clear proof"). The analysis showed that when talking to "a dear student," the mathematical objects were attributed to them in just under 85% of the cases, when the ownership of objects of "colleague mathematicians" was recognized in just over 15% of the instances. Research on personal pronouns (e.g., Herbel-Eisenmann & Wagner, 2010; Pimm, 1987) suggests that they can be used to identify one's inclusion or exclusion in some community, in the community of mathematicians for instance. Accordingly, the discerned discursive patterns can be explained from two perspectives. By sanctioning a submitted text, Dr. Brownstone transformed its status from an exclusively student's belonging to a cooperative object. Accordingly, not using a possessive pronoun can mark a shared ownership of a "proof." A complementary interpretation is also possible. To recall from Section 5.1, Dr. Brownstone expected students to "acquire the proof ideas," while being aware that these can be found "out there." Hence, the absence of the pronoun may be interpreted as an acknowledgment that an intended acquirement did occur. Note that both interpretations lead to a similar explanation regarding not "proofs." Such texts contained unintended issues, and, accordingly, texts of this sort were attributed to those who had submitted them.

6 Summary and discussion

Assigning statements for students to prove over time and providing written feedback on the submissions constitute typical didactical practices in many university courses. Recently, the practice of mathematicians' feedback provision has started to attract research attention in the mathematics education community (Moore et al., 2018; Moore, 2016; Pinto & Cooper, 2019; Spiro et al., 2019). Construing proof–feedback exchanges as communicational activity by which students and a mathematician participate in a mathematical discourse, I introduced didactical discourse on proof (DDP) as a potentially useful lens to investigate mathematicians' contribution to such exchanges.

DDP might be of interest for future research in the area of mathematicians' proof teaching not only because of the analytical potential that it inherits from the commognitive framework (e.g., Morgan, 2020; Nardi et al., 2014; Tabach & Nachlieli, 2016). From the commognitive standpoint, mathematicians do not hold the explanatory keys to their discourses as they are too preoccupied to participate in them. Then, didacticians' sense-making of mathematicians' discourses, as part of their research endeavor, comes hand in hand with impacting the objects of their investigations (cf. Wagner, 1997). In Sfard's (2008) words, "The only dimension to play with in commognitivist data gathering is the degree of a researcher's proactivity" (p. 278). This proactivity can be enacted in different ways, for instance, by sharing emerging findings with mathematician participants as part of the investigation. As one mathematician sagaciously remarked in his blog entitled "The things in proof are weird: a thought on student difficulties,"

> For the enculturated capacities, the prima facie teaching challenge is to inculcate them, i.e., to cause the capacities to be developed in the first place. But there is also a prior, less obvious challenge: we have to know they're there. Since as instructors we tend to be already very well-enculturated, our culture is not always fully visible to us. If we can't see what we're doing, it's harder to show students how to do it. (Blum-Smith, 2020)

Assisting mathematicians "to see what they are doing" seems especially appropriate given typical institutional opportunities for mathematicians to grow as teachers (e.g., Biza et al., 2016).

In tune with the above, I sketched a tentative organizational frame, in which making sense of the mathematician's feedback provision is structured as a partnership between a didactician and a mathematician (cf. Wagner, 1997). The presented case of Dr. Brownstone can be viewed as a "proof of concept," showing that the narratives that are generated within such a frame can be of a mutual benefit to both partners. Let me acknowledge that such cooperation is based on special professional and personal relations that involve mutual respect, trust, openness, and interest in each other's scholarship. Unpacking the building blocks, processes, and organizational arrangements that can support the establishment of mutually contributing partnerships of this sort goes beyond the scope of this article. However, the literature on productive collaborations between and rapprochement of mathematicians and didacticians makes me believe that such partnerships are achievable also elsewhere (e.g., Jaworski et al., 2015; Nardi, 2016).

I now turn to the case study of Dr. Brownstone. The first question underlaying this investigation concerned discursive characteristics of her feedback. The findings presented in Section 5 can be brought together in the following story. Being generally concerned with proof teaching and learning, Dr. Brownstone allocated a substantial space for students to prove in her course. She drew parallels between students submitting written proofs in her classroom and submitting publications for review as part of mathematical research. In her view, in both settings, a prover needed to possess the proof idea and represent it in writing to some assessor. Dr. Brownstone was aware that students are likely to use external resources when working on the assigned problems, and most of her comments referred to proof ideas in terms of something to be acquired rather than to be developed or constructed (see Section 5.1). Overall, the importance of having the proof idea and representing it, something that Dr. Brownstone stressed in the reflection sessions, was evident in the line-level comments that she provided. Yet, the issues with proof ideas were rearticulated at the text-level more often, and they came hand in hand with reduced scores. Issues with representation, though, sprouted the text-level comments more rarely and almost never entailed a score reduction (see Section 5.2). Notable, many statements that Dr. Brownstone made in the reflection sessions shed a nuanced light on the narratives emerging from the feedback analysis, provided new angles for further analytical rounds, and even offered terminology for capturing aspects of feedback. This addresses the second question about the potential of proof-related statements generated in reflective sessions to enrich the didactician's data analysis.

Next, I present six ways in which the case of Dr. Brownstone links to the literature, wherein some of these ways may be of interest for future research. First, albeit in a case of a single mathematician, the analysis demonstrates that providing comments and allocating scores invoke repetitive and patterned practices, the intricacies and the “inner logic” of which are discernable with appropriate conceptual and analytical tools. While employing different research instruments, Moore (2016) and Miller et al. (2018) arrive at compatible conclusions with respect to grading proofs. The contribution of the analyzed case is in showing that the two activities can be related albeit not straightforwardly.

Second, the analysis illustrates how sensitive, nuanced, and complex mathematicians' feedback on students' proofs can be. For instance, Dr. Brownstone distinguished among (i) texts that she considered as not proofs; (ii) instances where the proof idea was in place but where there were issues with its representation; and (iii) proofs that were in tune with what she expected (see Section 5.3). In the first two cases, her mindfulness about the issues with proof ideas and a more accommodating stance towards issues with representation can be reinterpreted using Moore's (2016) notion of seriousness of a flaw. The identified flaws were considered serious enough for score reduction in case (i); in the latter two cases, a maximum grade was awarded. This grading policy is not obvious as some of the mathematicians in Miller et al. (2018) did not give full credit to proofs that they considered correct. The distinction between (i), (ii), and (iii) builds on the words that Dr. Brownstone used to refer to a student's text (e.g., “proof” vs “work”), its attribution to the student (i.e., “your proof” vs “proof”), and the combination of line- and text-level comments. From the methodological viewpoint, this finding may spur future research to attend to discursive features of mathematicians' DDP as these might reflect different appreciations of the proof under consideration.

Third, Moore (2016) reports that when judging the seriousness of a flaw, his mathematicians assessed student cognition. Dr. Brownstone reflected that when the proof “went well,” she talked to “a colleague mathematician” about the quality of the proof; however, when this was not the case, she “put her teacher's hat” on and taught the student “to do something” in a particular way (see Sections 5.2 and 5.3). These echoing findings can instigate further research to explore how common it is for mathematicians to consider students' mathematical thinking when engaging with their proofs, and whether such considerations are inherent to the practice of feedback provision or whether they are triggered by the identification of serious flaws.

Before embarking on the abovementioned research paths, a methodological observation seems in order. One of Moore's (2016) findings suggests that, “mathematics professors focus on evaluating the student as much as the proof itself” (p. 269). Similar conclusions emerge from Miller et al. (2018), and the case study reported herein contributes additional evidence from a mathematician teaching and evaluating proofs in the context of graduate-level topology. Accordingly, future research on mathematicians' feedback might choose to be mindful of the relationships between the participating mathematicians and proofs on which they give feedback with respect to the image of students who produced the proofs (e.g., do mathematicians provide feedback on proofs that *they* assigned in *their* course to *their* students, or is it a feedback exercise designed for the sake of research?). Overall, several studies reported that context matters when mathematicians engage with someone's proof (e.g., Inglis et al., 2013; Lew & Mejía-Ramos, 2020). Thus, accounting for the foregoing relationships seems important for interpreting research findings and drawing conclusions. My view on this methodological dilemma is that the motivation for exploring mathematicians' feedback is practice-driven and, as such, research may be more informative if it concerns authentic practices of feedback provision. Such explorations would also be a welcome contribution to research on mathematicians' proof teaching, which currently contains only a handful of studies that observed the endeavor in action (e.g., Pinto, 2018; Weber, 2004).

The remaining three comments concern the argument on mathematicians teaching their students to prove better through providing feedback on their proofs (Moore et al., 2018; Miller et al., 2018; Moore, 2016; Pinto & Cooper, 2019; Spiro et al., 2019). Fourth, it is well known that proofs have somewhat different statuses in different mathematical disciplines, and research has shown that these disciplinary differences can also materialize into mathematicians reacting differently to the same even elementary proofs (e.g., Inglis et al., 2013; Weber & Czocher, 2019). The presented case produced some parallels between Dr. Brownstone's research discipline and the way she taught and talked about proof teaching (cf. Madsen & Winsløw, 2009). Drawing such parallels makes sense from a discursive perspective, in which mathematicians stand at the nexus of two discourse communities—one of disciplinary research and one that they lead in their classrooms. After all, discourses are not distinct and the distance between a research discourse and DDP often measures to the length of the corridor that a mathematician needs to cross from their office to a classroom. Then, it may be interesting to further explore the relations between mathematicians' disciplinary discourses and the way they teach proof to their students.

Fifth, the complexity of Dr. Brownstone's feedback and what appeared sometimes as not fully coherent messages to her students (see Section 5.2) make me wonder whether feedback provision is a purely teaching activity, in the sense of mathematicians having an exclusive motive with it "to bring the learners' discourse closer to a canonic discourse" (Tabach & Nachlieli, 2016, p. 303). Indeed, when I asked Dr. Brownstone about her tendency to write praising comments even when she identified serious flaws, she responded:

> So the idea being is that they will take away a lesson that will influence the way they write in the future? Goodness gracious, the responsibility of marking! [Laughs]. Hmm… [Laughs]. Sorry, I was just trying to give them positive feedback on what they have done.

Moreover, many of Dr. Brownstone's comments communicated her emotions about the students' mathematizing, offered stylistic improvements, praised the proofs, and sought further explanations. These communicational actions seem less in tune with those of a pedagogue—in Greek etymology "a person who leads a child"—and they appear more characteristic to a colleague prover or an interested reader (cf. Morgan, 2002). Accordingly, it might be of value to separate between the argument of mathematicians' feedback being instrumental to their students' proof learning and of mathematicians' in-the-moment motives when writing a comment on a student's proof.

Lastly, the presented case study shows that in their feedback, mathematicians can not only write about mathematical objects but also subjectify by making explicit references to student understanding. Thus, as an instance of teaching, feedback can potentially have an impact not only on students' mathematical discourses but also on their mathematical identity. Investigation into the impact of mathematicians' feedback on students' identity building is another interesting path in a barely charted territory.

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