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Strategic competence for multistep fraction word problems: an overlooked aspect of mathematical knowledge for teaching

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Abstract
Prior work on teachers' mathematical knowledge has contributed to our understanding of the important role of teachers' knowledge in teaching and learning. However, one aspect of teachers' mathematical knowledge has received little attention: *strategic competence for word problems*. Adapting from one of the most comprehensive characterizations of mathematics learning (NRC, 2001), we argue that teachers' mathematical knowledge also includes strategic competence, which consists of *devising a valid solution strategy*, *mathematizing the problem* (i.e., choosing particular strategies and presentations to translate the word problem into mathematical expressions), and *arriving at a correct answer* (executing a solution) for a word problem. By examining the responses of 350 fourth- and fifth-grade teachers in the USA to four multistep fraction word problems, we were able to explore manifestations of teachers' strategic competence for word problems. Findings indicate that teachers' strategic competence was closely related to whether they devised a valid strategy. Further, how teachers dealt with known and unknown quantities in their mathematization of word problems was an important indicator of their strategic competence. Teachers with strong strategic competence used algebraic notations or pictorial representations and dealt with unknown quantities more frequently in their solution methods than did teachers with weak strategic competence. The results of this study provide evidence for the critical nature of strategic competence as another dimension needed to understand and describe teachers' mathematical knowledge.

Keywords Fractions · In-service teachers · Word problems · Mathematical knowledge for teaching · Content knowledge · Strategic competence

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A substantial body of research has highlighted the importance of teachers' understanding of the subject matter in teaching and student learning (e.g., Copur-Gencturk, 2015; Baumert et al., 2010; Borko et al., 1992; Campbell et al., 2014; Hill et al., 2008; Hill, Rowan, & Ball, 2005; Tchoshanov, 2011). When looking across these studies, we noticed that some aspects of teachers' content knowledge have been overlooked. In fact, the majority of prior work has focused on teachers' conceptual knowledge (their understanding of concepts, rules, and the relations among them) or specialized content knowledge (their knowledge of the mathematics used in teaching, such as evaluating alternative solutions, e.g., Lee, 2017; Lo & Luo, 2012; Olanoff, Lo, & Tobias, 2014; Van Steenbrugge, Lesage, Valcke, & Desoete, 2013). In this study, we focused on *strategic competence for word problems* as another manifestation of teachers' mathematical knowledge that deserves investigation. We grounded our rationale in the National Research Council's conceptualization of learners' mathematical proficiency (NRC, 2001), which is one of the most comprehensive reports conceptualizing mathematical understanding in mathematics education to date. The NRC's conceptualization of mathematical understanding includes the types of knowledge and understanding that many current conceptualizations of teachers' content knowledge include, such as procedural knowledge and conceptual understanding (e.g., Krauss et al., 2008; Saderholm, Ronau, Brown, & Collins, 2010; Tchoshanov, 2011). However, it also includes *strategic competence* as an added component because "having a deep understanding [of mathematics] requires that learners connect pieces of knowledge, and that connection in turn is a key factor in whether they can use what they know productively in solving problems" (NRC, 2001, p. 118). Specifically, our definition of teachers' *strategic competence for word problems* is adapted from the NRC's characterization and includes three key indicators:

(1) *devising a solution strategy*, which refers to whether teachers can apply the appropriate concepts and procedures to come up with a valid solution method;
(2) *mathematizing the problem*, which refers to choosing particular strategies and presentations to translate the word problem into mathematical expressions; and
(3) *arriving at the correct answer*, which refers to executing a solution method to arrive at the final correct answer.

We acknowledge that our conceptualization is narrower than the original, in that we focus only on teachers' strategic competence for word problems rather than nonroutine mathematical problems that could occur in real-life situations. However, as the NRC (2001) has pointed out, one's performance on multistep word problems is an indicator of one's strategic competence. The NRC (2001) also called attention to US students' performance on multistep word problems to provide evidence of US students' strategic competence (NRC, 2001). We have applied the same theoretical and empirical justification to study teachers' strategic competence for multistep word problems, with the understanding that teachers' performance on multistep word problems could reveal their strategic competence and inform future work that looks to link teachers' strategic competence of concepts with how they teach these concepts to students. Our decision to hone in on word problems is also supported by evidence from a study by Li and Kulm (2008), in which major discrepancies in teachers' performance were observed when they were asked to solve one-step problems versus multistep problems.

We also focused on teachers' strategic competence for fraction word problems. We grounded our rationale in the view that strategic competence may depend on specific mathematical concepts. For instance, although an individual can have a strong content knowledge of whole numbers, the same individual may not have a similar level of understanding of fractions. The same argument can be applied *mutatis mutandis* to teachers' strategic competence for word problems. Investigating teachers' strategic competence for fraction word problems also allows us to ascertain how they manifest strategic competence in addition to other components of their knowledge of fractions. Given the substantial amount of prior work on teachers' knowledge of fractions (e.g., Armstrong & Bezuk, 1995; Borko et al., 1992; Hohensee & Jansen, 2017; Izsák, 2008; Izsák, Jacobson, & Bradshaw, 2019; Lee, 2017; Ma, 1999; Newton, 2008; Zhou, Peverly, & Xin, 2006), teachers' conceptual understanding of fractions is well known. Thus, investigating teachers' strategic competence for fraction word problems could contribute to our understanding of the distinct role strategic competence plays in solving fraction word problems.

Along with the theoretical implications of the conceptualization of teachers' mathematical knowledge, studying teachers' strategic competence for fraction word problems has important implications for research on teaching and student learning. Word problems involving fractions are particularly difficult for students (e.g., Aksu, 1997; Brown & Quinn, 2007), and according to the Trends in International Mathematics and Science Study (TIMSS) 2015, only about 20% of the students in Grade 8 correctly answered a multistep fraction word problem (Mullis, Martin, Foy, & Hooper, 2016). Given that learners' competence in solving word problems is tied to the kind of instruction they receive on this topic (Verschaffel, Schukajlow, Star, & Van Dooren, 2020), studying teachers' performance on multistep fraction word problems could provide insights into why students have difficulties with word problems. Teachers also report teaching students the strategies they themselves use to solve problems (e.g., Fisher, 1988), which provides further justification for investigating teachers' strategic competence in solving word problems to better understand the learning environment they may create for their students.

1 Conceptual framework

As mentioned previously, we define *strategic competence for word problems* through three key indicators: (1) devising a valid solution strategy, (2) mathematizing the word problem, and (3) executing a strategy to solve it correctly. However, before we discuss strategic competence in detail, let us first define *word problems*, which are:

> "verbal descriptions of problem situations, presented within a scholastic setting, wherein one or more questions are raised, the answer to which can be obtained by the application of mathematical operations to numerical data available in the problem statement or on numerical data derived from them" (Verschaffel et al., 2020, p. 1).

As indicated in the definition, word problems are verbal definitions of situations that solvers (in this case teachers) must translate into mathematical expressions to find the correct answer. Three aspects of strategic competence play a significant role in the process of solving a word problem. The first one, *devising a valid solution strategy*, captures whether a solver applies the appropriate concepts and procedures to come up with a valid approach to solve the problem. For instance, a one-step multiplication problem could be represented as repeated addition or it could be solved by using a multiplication algorithm. Regardless of which solution strategy is being used, devising a valid solution method is a manifestation of strategic competence (NRC, 2001). Similarly, the failure to devise an appropriate strategy could reveal insights into solvers' strategic competence. Indeed, prior work on teachers' solutions to word problems has indicated that some teachers have difficulties in coming up with a valid strategy to solve a given problem (e.g., Taplin, 1998).

The second indicator of strategic competence, *mathematizing the word problem*, refers to solvers constructing a "model of the variables and relationships described in the problem" (NRC, 2001, p. 124). Recall that a word problem involves a verbal description of a mathematical situation; thus, *mathematizing the word problem* includes choosing specific strategies and using representations to characterize the mathematical relationships between the quantities in a given problem. In addition to a specific solution strategy, how the known and unknown quantities are represented in a solution strategy (e.g., algebraically, visually) is part of mathematizing through mathematical representations. The first indicator of strategic competence tells us whether teachers devise a valid strategy to solve a problem, whereas mathematizing the word problem tells us which specific *strategies* they used in their solution as well as how they dealt with known and unknown quantities in their *representations* of the word problem. Prior work on teachers' correct and incorrect strategies has provided evidence that specific strategies and presentations used to solve a problem may reveal one's mathematical knowledge (Lee, 2017; Newton, 2008). For example, solving a multiplication problem by using repeated addition may indicate teachers' content knowledge of the relationships between addition and multiplication.

Incorrect strategies can provide further insights into teachers' strategic competence and mathematical knowledge (Lee, 2017; Newton, 2008). This is mainly because "there are mutually supportive relations between strategic competence and both conceptual understanding and procedural [knowledge]" (NRC, 2001, p. 127). Thus, errors in teachers' mathematization of fraction word problems may be related to their conceptual understanding of fraction concepts (i.e., their comprehension of concepts, algorithms, and the relationships among concepts). For instance, teachers who have a partial understanding of referent units may make errors in their solution strategies on problems involving fractions with different wholes by using the same referent whole for every fraction.

The final indicator of strategic competence is teachers' ability to execute their solution strategy well (i.e., to solve the problem correctly). Teachers may devise a valid strategy, but their success in solving the problem also depends on executing the strategy accurately to arrive at the final correct answer, regardless of which strategy is used. For instance, during the execution process, teachers may make arithmetic errors that prevent them from finding the correct answer.

The different kinds of errors that teachers make in the process of solving a word problem can provide insights into various aspects of strategic competence. Specifically, teachers may make a translation error, which can be defined as not being able to come up with a valid solution strategy. This type of error provides evidence of the first indicator of strategic competence. Or teachers may make errors in their mathematization of the word problem. These errors may be related to teachers' understanding of mathematical concepts (i.e., conceptual errors) or to their understanding of the information given in a problem (i.e., an interpretation error). Finally, teachers may make arithmetic errors (i.e., computation errors) as they are executing a valid solution strategy to find an answer. This type of error can offer insights into the third indicator of strategic competence.

2 Current study

In this study, we focused on teachers' strategic competence for fraction word problems, which included teachers' capacity to devise a valid solution method (the first indicator of strategic competence), the specific strategies and representations used to characterize the word problem mathematically (the second indicator of strategic competence), and the ability to carry out a strategy to solve the problem correctly (the third indicator of strategic competence). We investigated teachers' strategic competence by using fraction word problems with two or more steps, because increasing the number of steps involved in solving a word problem also increases the number of solution strategies, concepts, and representations teachers might use to solve the problem. Subsequently, using such problems can provide a more nuanced understanding of teachers' strategic competence with word problems. In fact, prior work on teachers' performance on word problems supports this decision, given that teachers' performance on one-step word problems can lead to overestimating teachers' strategic competence on multistep word problems and that teachers who can formulate a one-step word problem may not do the same when the problem requires two or more steps (Li & Kulm, 2008).

We investigated teachers' strategic competence by analyzing their solutions to a set of multistep fraction word problems. During the analysis, we explored the percentages of teachers who devised a valid solution strategy or who arrived at a correct answer for each problem (i.e., the first and third indicators of strategic competence) and focused on the mathematization of word problems (i.e., the specific strategies and representations teachers used in their solution methods (e.g., algebraic, pictorial)). We also analyzed the kinds of errors that teachers made for each problem to provide further insights into their strategic competence. Finally, we compared the strategies and representations used by teachers who showed more strategic competence with those used by teachers with less strategic competence to better understand how teachers with strong strategic competence approached the same problem compared with those with less strategic competence.

3 Relevant literature

Prior work on teachers' strategic competence is relatively sparse. Most research to date has been conducted with preservice teachers (Adu-Gyamfi, Schwartz, Sinicrope, & Bossé, 2019; Baek et al., 2017; Cullen et al., 2017; Lee, 2017; Nillas, 2003; Olanoff et al., 2014; Taplin, 1998) and suffered from several methodological limitations, such as using a single problem to investigate teachers' strategic competence, requiring participants to use certain strategies or representations, and collecting data from a group of preservice teachers who were taking the same methods class (e.g., Adu-Gyamfi et al., 2019; Baek et al., 2017; Cullen et al., 2017; Li & Kulm, 2008; Lo & Luo, 2012; Son & Lee, 2016). Partly because of these limitations, evidence is mixed regarding preservice teachers' strategic competence (cf. Adu-Gyamfi et al., 2019; Baek et al., 2017; Cullen et al., 2017; Lee, 2017; Nillas, 2003). For instance, Lee (2017) analyzed 111 preservice teachers' written solutions to a one-step fraction division word problem that required them to use a drawn length model. Lee found that 52% of preservice teachers devised a valid solution strategy and arrived at a correct answer. In contrast, Baek et al. (2017) reported in their analysis of 93 solutions provided by 77 preservice teachers to a multistep fraction word problem requiring them to use pictorial representations, that 81% of the solutions devised a valid strategy and arrived at a correct answer. Such variation across studies in teachers' arriving at a correct answer could also be related to the fact that most of the prior work has documented teachers' strategic competence on a single problem (e.g., Adu-Gyamfi et al., 2019; Lee, 2017). Single word problems are limited in terms of their ability to capture teachers' strategic competence adequately because a single item can capture only a limited amount of information. Thus, studies involving multiple problems could provide a more nuanced understanding of teachers' strategic competence.

Another methodological limitation of prior work is that teachers have been asked to use prescribed representations or strategies when solving word problems, which appears to affect how these teachers may be mathematizing the word problems (e.g., Baek et al., 2017; Cullen et al., 2017; Lee, 2017). Restricting teachers to using certain strategies may have an impact on their strategic competence, because a teacher who may not be comfortable with using a certain representation may otherwise successfully solve the problem. Our concern regarding the potential of requiring teachers to use prescribed strategies to characterize teachers' strategic competence inaccurately is supported by the results of two studies conducted with the same elementary and middle school preservice teachers who were asked to solve the same problem (Baek et al., 2017; Cullen et al., 2017). Specifically, the way in which preservice teachers mathematized the word problem and arrived at a correct answer changed when they were asked to use pictorial or algebraic representations (cf. Baek et al., 2017; Cullen et al., 2017). Although asking teachers to use a certain strategy could capture what they know about that specific strategy, it could also limit researchers' ability to capture their strategic competence accurately as it reveals in classrooms. Given that teachers have reported teaching the strategies they naturally use (Fisher, 1988), it is important to investigate how teachers solve fraction word problems naturally to better understand what strategies they use with their students. Bringing these considerations together, we asked teachers to solve such problems naturally (i.e., without suggesting what representations or strategies they should use) because giving them the flexibility to use any method they were comfortable with could reveal their strategic competence for word problems more accurately.

As reflected in existing work on teachers' performance on word problems, it also underscores the importance of using multistep word problems to depict teachers' strategic competence more accurately. Although the majority of existing studies have tended to use one-step word problems (Lee, 2017), the use of such problems can potentially overestimate teachers' strategic competence (Li & Kulm, 2008). For example, Li and Kulm (2008) found that 52% of 46 preservice teachers correctly answered the question, "How many ½'s are in 1/3?" However, only 39% of these preservice teachers solved a multistep fraction word problem correctly. Such a discrepancy in teachers' performance on one-step and multistep word problems indicates teachers' strategic competence might be more fragile than has been found in studies that focused solely on one-step word problems. Thus, investigating teachers' strategic competence by using multistep word problems is vital to understanding their strategic competence accurately.

Studies involving fraction word problems have indicated that preservice teachers use formal strategies, such as division algorithms, in their solutions to one-step fraction word problems (e.g., Lee, 2017). These studies have also documented that teachers' conceptual understanding is related to the strategies they use in their solutions (Lee, 2017; Newton, 2008). For instance, Lee (2017) reported that preservice teachers who solved a one-step fraction division word problem incorrectly revealed their partial conceptual understanding of fractions by using a multiplication operation rather than a division operation or by making procedural errors as they executed their solution method.

Although these studies provide a glimpse into teachers' strategic competence before they enter teaching, the results may look different for in-service teachers. In fact, scholars argue that teachers can enhance their mathematical knowledge through teaching (e.g., Hiebert, Morris, Berk, & Jansen, 2007); thus, teachers' strategic competence may look different after they gain professional knowledge. In-service teachers can enhance their mathematical knowledge through their interactions with students and the curriculum materials or through their completion of a professional development program (e.g., Copur-Gencturk, Plowman, & Bai, 2019; Copur-Gencturk & Thacker, 2021). In contrast, preservice teachers have relatively less experience in teaching and learning from teaching, which may have an impact on how they approach solving word problems. Therefore, although studies conducted with preservice teachers have noted that preservice teachers use more formal strategies, in-service teachers my use more informal strategies. Relying on research on preservice teachers' strategic competence to infer in-service teachers' strategic competence may hinder our understanding of in-service teachers' knowledge of mathematics. Indeed, scholars who have conducted studies with preservice teachers have recognized the importance of engaging in the same research issues with in-service teachers (Son & Crespo, 2009).

In addition to the problem of generalizability of the research findings of preservice teachers to the in-service teacher population, prior studies with preservice teachers have another methodological limitation that may affect the observed results. In prior work conducted with preservice teachers, participants were from the same institution or were completing the same methods course (e.g., Baek et al., 2017; Cullen et al., 2017; Lee, 2017; Lo & Luo, 2012; Newton, 2008); therefore, how the participants solved the problems may have been affected by the content and instruction in the course. For instance, if the teachers learned a solution strategy to solve a word problem, they may have used this strategy as a result of their specific learning experience. Thus, data collected from teachers with similar educational backgrounds could affect the observed results, which has important implications for the theorization of teachers' knowledge and research. Although such studies provide insights into teachers' understanding, they make it harder to deduce whether what is found could be patterns observed in teachers from different universities or in those who have taken different methods courses. Thus, studying a sample of teachers from a wide range of backgrounds allows a better depiction of how teachers deal with multistep fraction word problems.

The current study aims to contribute to an understanding of teachers' strategic competence for fraction word problems by collecting data from 350 elementary school mathematics teachers from across 48 different states in the USA. This design has the potential to address the limitations of prior work in which data were collected from a sample of teachers from a single institution or course. Additionally, teachers in the current study were allowed to use any strategies and representations they chose to solve word problems in an effort to capture each indicator of strategic competence more accurately. To contribute to the field's conceptualization of teachers' mathematical knowledge, manifestations of teachers' strategic competence were investigated for a set of fraction word problems. Specifically, we aimed to answer the following research questions:

- What can be said about teachers' strategic competence for fraction word problems in terms of teachers' ability to devise a strategy and arrive at a correct solution?
- How are teachers mathematizing the word problems (i.e., what strategies and representations are they using to answer these word problems or what kinds of errors are they making in their solutions)?
- What is the difference in teachers' mathematization of word problems by those with strong strategic competence compared with those with weak strategic competence?

4 Methods

4.1 Context of the study and participants

We collected data from elementary school teachers who teach fraction concepts across 48 states in the USA. To do so, we used the Qualtrics Panel, which recruits teachers from across

Table 1 Background characteristics of teachers in the present sample compared with a nationwide sample

Background characteristic	Study sample	Nationwide sample of US elementary school teachers [a] (2015–2016)
Female	88%	89%
White	83%	80%
Regular certification	94%	91%
Years of teaching experience		
Less than 3	10.3	10.1
3 to 9	36.9	28.3
10 to 20	38.6	39.3
More than 20	14.3	22.3

[a] Source: Digest of Education Statistics, https://nces.ed.gov/programs/digest/d17/tables/dt17_209.22.asp?current=yes

the USA, along with teachers we personally recruited through partner professional development organizations[1]. Teachers who were eligible to participate in the study were identified by a screening survey we created. Eligibility criteria were the following: (1) being a fourth- or fifth-grade teacher in the USA and (2) teaching mathematics in the same year the data were collected. All eligible participants were compensated for their participation in the study with an online gift card. We restricted our analyses to fourth- and fifth-grade teachers because fraction operations are mainly taught at these grade levels according to the Common Core Standards, which most US states have adopted. Our analytical sample was restricted to the 350 teachers who submitted a solution for at least one of the four-word problems included in the survey.

As shown in Table 1, the background characteristics of the study sample were similar to those of the US elementary school teacher population in terms of gender, race, and holding a regular teaching certificate. The study sample included more teachers with teaching experience in the 3- to 9-year category but fewer teachers in the more than 20-year category. Sixteen percent of the sample had the credentials to teach mathematics in middle school and high school, whereas 67% of the sample taught multiple subjects in Grades K-6. The teachers in the study sample had on average 10 years of mathematics teaching experience, with a standard deviation of 7.7 years.

4.2 Materials and procedure

Participating teachers were asked to solve the four fraction word problems shown in Fig. 1. These word problems were presented in a randomized order to reduce potential question order bias. We provided instruction regarding how teachers could upload their solutions to the online survey, that even if they could not solve the word problem, they were required to submit their work to continue taking the survey. Our rationale for this decision was to capture not only the responses that teachers felt were correct but also solutions they felt were incorrect so that we could better capture their strategic competence.

We purposefully adapted these four-word problems from published materials and existing resources (e.g., Schoenfeld & Kilpatrick, 2008; Schuster & Anderson, 2005) so that the problems would be similar to ones the teachers would encounter in teaching mathematics.

[1] Although our study sample is a national sample of US fourth- and fifth-grade teachers, it is important to highlight that we do not claim that the sample is nationally *representative* of US elementary school teachers.

Sally began reading a book yesterday. Today, she read $\frac{3}{8}$ of the book, and now has just $\frac{7}{12}$ left to finish the book. What portion of the book did she read yesterday? Please <u>show all work on a piece of paper</u>, and then upload an image of your work below.
During one evening, Dejon devoted his study time to biology, math, and Spanish. He spent $\frac{2}{5}$ of his study time on biology and $\frac{1}{3}$ of the study time on math. The remaining time left for Spanish was 32 minutes. How much time did he spend studying biology? Please <u>show all work on a piece of paper</u>, and then upload an image of your work below.
Suppose that Max, the Wonder Dog, ate $\frac{1}{4}$ of a bag of treats on Sunday. Each night after that, Max ate $\frac{3}{16}$ of the bag. How many nights (including Sunday) would it take to finish the whole bag? Please <u>show all work on a piece of paper</u>, and then upload an image of your response below.
Pat has taken the train to go to work. After traveling $\frac{1}{4}$ of the trip, she fell asleep. When she wakes up, she realizes that she has only 20 miles to arrive at her destination. If the distance she needs to travel after she wakes up is $\frac{2}{7}$ of the distance she traveled while sleeping, what is the total distance of the trip? Please <u>show all work on a piece of paper</u>, and then upload an image of your response below.

Fig. 1 The four-word problems and prompts provided on the survey

Moreover, these word problems were selected to obtain thorough information on the three indicators of teachers' strategic competence. Specifically, the use of problems with a combination of different fraction concepts or operations would provide insights into the extent to which teachers would devise a valid solution strategy regardless of the content and operation involved. Furthermore, using these problems allowed us to capture how teachers mathematized word problems because "different problem types elicit different solution strategies" (Langrall & Swafford, 2000, p. 255). Finally, using problems with different combinations of operations and fraction concepts also allowed us to see the extent to which teachers would execute solution strategies to solve them correctly.

The first author and another scholar, who had expertise in children's acquisition of fraction concepts, created a first draft of the word problems by using the existing literature and resources. This step was followed by a process of receiving feedback from scholars who research fractions or mathematics education ($N = 10$). We then revised these word problems based on the feedback and began conducting interviews with 20 in-service teachers in Grades 4 and 5 to ensure that the word problems were capturing teachers' strategic competence. We further revised these word problems and made some minor corrections to ensure that wording of the problem would be clear to teachers based on what we learned from the interviews.

4.3 Coding schemes

We coded each response to capture all three indicators of strategic competence: devising a valid strategy, mathematizing the word problem, and arriving at a correct answer. As part of capturing the teachers' strategic competence, when they made an error (e.g., conceptual error or computational error), we also coded the kinds of errors they made.

Coding for devising a valid strategy and arriving at a correct answer We used three categories to group teachers' solutions based on their ability to devise a valid strategy and arrive at a correct answer. Specifically, a score of 1 indicated that the solution did not include a valid strategy and the final answer was incorrect. A score of 2 indicated that teachers devised a valid strategy to solve the word problem but did not arrive at a correct answer, and a score of 3 indicated that teachers devised a valid strategy and executed that strategy correctly.

Coding for mathematization We also created a coding system to characterize how teachers mathematized these problems according to the strategies and representations they used. Because our study focused on fraction word problems in addition to fraction operations, teachers' choice of strategies could depend on the fraction concepts; thus, our coding strategies included fraction operations, fraction concepts (unit fractions, ratios, and proportional reasoning), and representations such as drawings and algebraic notation. We examined prior literature on strategies used by teachers in solving word problems (both in-service and preservice; see Table 2 for some of the strategies found in earlier literature). We created our coding scheme based on the definitions and sample solutions provided in earlier work, and we used these categories in our initial training. When the strategies identified in earlier work failed to capture a strategy used in our sample, we created a new category to define that strategy. For instance, we noticed that for problems 1 and 2, the way teachers mathematize word problems is dependent on how they dealt with given and unknown quantities (see Table 2); thus, we added four categories to describe teachers' representation of these problems mathematically. This iterative process continued until the list of strategies had thoroughly captured all the common strategies. We retained an "other" category for instances in which a strategy did not fit any of the other categories.

Table 2 includes all the strategies and representations we identified in our data, along with a description of each strategy. As shown in the table, teachers' mathematization fell into five categories: fraction operations, fraction concepts, pictorial representations, algebraic notation, and other. As presented in Table 2, how teachers used addition and subtraction operations depended on whether they were dealing with known quantities or the unknown quantity in their solution strategy; thus, we captured such nuances in our coding. We allowed a solution to be assigned several codes.

Coding for errors Recall that the types of errors teachers made reveal their strategic competence. In generating our coding scheme, we drew on previous literature and on the kinds of errors we noticed during the interviews we conducted with the 20 teachers. We first sought to identify the error categories relevant to the context of solving fraction word problems through a careful analysis of the relevant literature. We then investigated the interviews we conducted with teachers during validation of the items to ensure that our categories were capturing the errors teachers made. Specifically, what counted as conceptual errors was based on an analysis of the interview data. We tried to determine whether this initial coding scheme was comprehensive enough to capture the range of teachers' errors, and we continued to revise our categories until we had thoroughly captured their errors. The final set of error codes illustrated in Table 3 captured the majority of errors we noticed in our data.

We coded the errors as falling into one of the six categories shown in Table 3. The *computation error* category indicated that teachers made arithmetic errors. These errors occurred as teachers were executing a valid solution strategy. The *translation error* category captured whether teachers devised a valid strategy to solve the word problem. For instance, teachers in this category tried out every operation with the given quantities to figure out a solution. *Conceptual errors* were related to teachers' knowledge of fraction concepts and affected teachers' ability to devise a valid solution strategy. We determined whether a

Table 2 Overview of the coding categories for mathematization of the word problems along with their descriptions

Code	Description
Fraction operations	
Addition/subtraction combinations	
Unknown quantity = whole − known quantity 1 − known quantity 2	A solution that included representing an unknown quantity on one side of the equation and known quantities along with the whole on the other side of the equation received this code.
Known quantity 1 + unknown quantity = whole − known quantity 2	A solution that included representing the unknown quantity and one of the known quantities on one side of the equation and the whole and the other known quantity on the other side of the equation received this code.
Whole = known quantity 1 + known quantity 2 + unknown quantity	A solution that included representing the whole on one side of the equation and the unknown quantity and known quantities on the other side of the equation received this code.
Unknown quantity = whole − (known quantity 1 + known quantity 2)	A solution that included representing the unknown quantity on one side of the equation and the whole on the other side of the equation received this code. Some solutions that included two steps (first adding two known quantities and then subtracting from the whole to find the unknown quantity) were also coded in this category.
Repeated addition/subtraction	A solution that included successive addition of the same amount received this code (Kent, Empson, & Nielsen, 2015), or a solution that involved successive removal of the same amount from the dividend received this code (Son & Crespo, 2009).
Fraction multiplication	A solution that involved conceptualizing the part–whole relationship between a given and an unknown quantity received this code (Lo & Luo, 2012), or a solution that involved solving the problem as a fraction multiplication problem received this code.
Fraction division	A solution that involved solving the problem as a fraction division problem received this code (Lo & Luo, 2012).
Fraction concepts	
Unit fraction	A solution that included identifying the "unit fraction" for a given quantity to find an unknown quantity received this code (Lo & Luo, 2012).
Proportional reasoning	A solution that indicated multiplicative reasoning other than setting up a proportion received this code (Fisher, 1988).
Setting up a proportion	A solution that included setting up a proportion to solve for an unknown quantity was coded as falling into this category (Arican, 2016; Cullen et al., 2017; Lo & Luo, 2012).
Pictorial/direct modeling	A solution that involved drawing a picture representing the problem, using a picture as a reference, or solving the problem by using direct modeling was coded as pictorial/direct modeling (Cullen et al., 2017; Kent et al., 2015; Lo & Luo, 2012).
Algebraic	A solution that included using an algebraic expression, a symbol for an unknown quantity, or a formula to denote the relationship received this code (Arican, 2016; Fisher, 1988). A solution that used an algebraic expression to set up a ratio was not included in this category if the expression was not used during solving problems.
Other	A strategy that could not be placed into any other correct category was coded in this category (Fisher, 1988)

conceptual error had been made based on the interview data. Specifically, we noticed that across the different word problems, teachers who answered these problems incorrectly used the

Table 3 Overview of the error categories along with their descriptions

Code	Description
Computation error (arithmetic error)	Arithmetic error made during the execution of a valid solution strategy (Taplin, 1998).
Translation error	A valid solution strategy is not devised (Taplin, 1998).
Conceptual error	An error made in the solution strategy related to fractional concepts, such as using incorrect referent wholes (Baek et al., 2017; Lee, 2017).
Interpretation error	An error made in a correct solution strategy by missing some information or interpreting it incorrectly (Taplin, 1998).
No attempt	The teacher did not provide a solution to a given problem (Fisher, 1988; McAllister & Beaver, 2012).
Other error	An error that could not be placed into any other error category (Fisher, 1988).

incorrect whole or referent unit for some fractions or reported not knowing the referent whole. For instance, one teacher who had difficulty in devising a valid strategy for the second word problem said: "I was like, 'Well, what was the length of his study time?' For some reason, . . . I couldn't see it. I couldn't get my head around the whole." Thus, those who were reporting similar issues when solving problem 2 were coded as making a conceptual error in addition to a translation error. An interpretation error included responses that overlooked some of the information provided in the problem even if the solution strategy was correct. For instance, some teachers' mathematization did not include that Max the wonder dog had already finished ¼ of the bag, and solved the problem for the full bag ($3/16 \times x = 1$ rather than $3/16 \times x = 3/4$). Several teachers noted that they had difficulties in solving the given word problem; thus, we coded these as "no attempt". Finally, if the error teachers made did not fall into any of these categories, we coded it as "other."

4.4 Coding the data

For all three coding areas, we followed the same approach, in which we coded the data in three stages. In the first stage, we independently coded a subsample of solutions according to the definitions and examples in the coding scheme, and then we discussed our rationale for coding, resolved our disagreements, and refined the initial codes until we arrived at consistency in our coding. Next, we used the refined codes as a basis for further coding. In the second stage, we independently coded a subsample of solutions to establish interrater reliability. Our agreement for devising a valid strategy and arriving at a correct answer across the four-word problems was over 95%, the interrater agreement for mathematization for each word problem was above 96%, and the interrater agreement for errors for each word problem was over 97%. In the final stage, we coded the remaining solutions independently and resolved any differences and disagreements through discussion; thus, both authors coded all the solution strategies.

4.5 Data analysis

The analyses were based on a qualitative examination of teachers' written solutions to the four fraction-related problems. More specifically, we examined teachers' written solutions to analyze three manifestations of strategic competence. The solutions for each of the four problems were coded with reference to the coding schemes. After the solutions were coded, descriptive statistics were calculated; these findings are presented below.

To answer the first research question (i.e., teachers' strategic competence in terms of devising a valid strategy and arriving at a correct answer), we reported the percentage of teachers who devised a valid strategy (or not) and arrived at a correct answer (or not). To investigate how teachers mathematized the word problems (i.e., the second indicator of strategic competence), we reported the percentage of teachers who used certain strategies and representations to mathematize word problems and who made certain errors. We also reported the common ways teachers mathematized the word problems, paying particular attention to how they dealt with known and unknown quantities in their solutions, given that such information also revealed their strategic competence (NRC, 2001). To further understand the characteristics of teachers' strategic competence, we investigated differences in the mathematization of word problems by teachers with strong strategic competence and those with weak strategic competence. We defined teachers with strong strategic competence as those who answered all the word problems correctly (i.e., those who received a score of 3 for all the problems; $N = 76$). We defined teachers with weak strategic competence as those who answered at least one of the word problems correctly (i.e., those who received a score of 0 on at least one of the problems; $N = 207$). We then aggregated the strategies and representations used across word problems at teacher level and reported the average percentage of teachers within each group who used those strategies[2]. We also examined whether there was a relationship between the strategies and representations used by the two groups by using a chi-squared test. For the 12 analyses conducted for each strategy category, the p-value was adjusted to 0.004.

5 Results

5.1 Devising a valid solution strategy and arriving at a correct solution

The percentage of teachers who devised a valid strategy and arrived at a correct answer when solving fraction word problems ranged from 36 to 91% for the four problems. As shown in Fig. 2, when teachers devised a valid strategy, they tended to solve the problem correctly (see the low percentages for valid strategy and incorrect answer). Additionally, the most common type of error preventing teachers from arriving at a correct answer was the translation error (i.e., not devising a valid strategy; see Fig. 3). Taken together, these findings suggest that the extent to which teachers devised a valid strategy appeared to be associated with their arriving at a correct answer.

A closer look at the kinds of errors teachers made also indicated that teachers' knowledge of fraction concepts, especially their knowledge of the referent unit, seemed to be associated with their strategic competence. In fact, for all the word problems in which teachers made conceptual errors, more than 95% of the teachers who made a conceptual error also did not devise a valid strategy to solve the problem. However, the opposite was not true: 66% of

[2] Because in this question we were more interested in whether or not certain strategies were used by these two groups of teachers, and we did not focus the frequency of strategies used by teachers. However, we also ran an analysis by taking into account of the frequency of the use of the strategies. The results were similar, supporting our confidence in our findings.

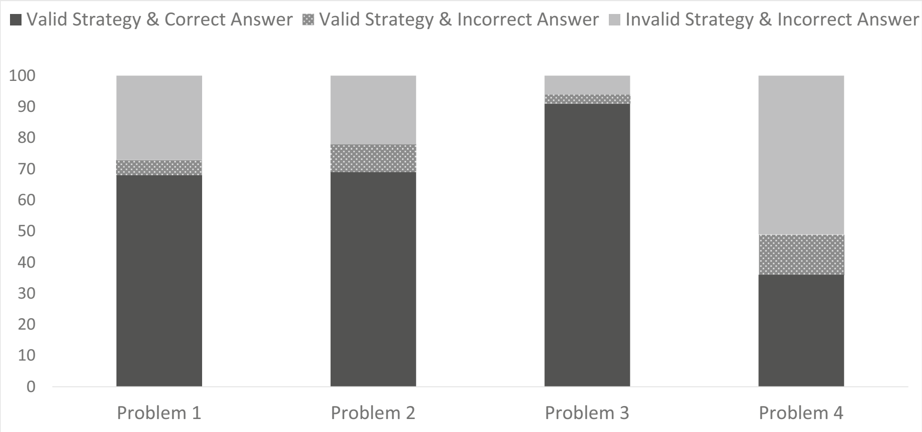

Fig. 2 Distribution of valid strategies and correct answers by word problem

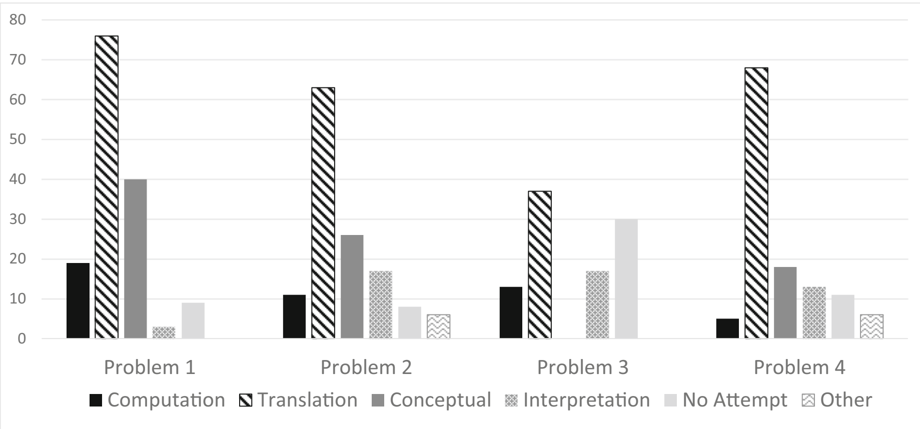

Fig. 3 Distribution of solutions by errors on the four-word problems

teachers who did not devise a valid strategy also made a conceptual error. Thus, this finding indicates that, although teachers' understanding of fraction concepts was an important contributor to teachers' devising a valid strategy, their strategic competence seemed to extend beyond their conceptual understanding.

5.2 Mathematization of the word problems

Table 4 lists how teachers mathematized each of the word problems (i.e., the strategies and representations they used, and how they dealt with quantities in a given problem). Although teachers varied in their use of specific strategies, depending on the problem, patterns seemed to emerge in teachers' mathematization of word problems in terms of how they dealt with presenting unknown quantities in their solutions. Specifically, teachers' mathematization of word problems across the four problems suggested that the most common way of mathematizing word problems was by focusing on *only the known quantities*.

Table 4 Distribution of correct solutions by strategy for the four-word problems

Strategy	Correct solution by strategy, %			
	Word problem1	Word problem2	Word problem3	Word problem4
Fraction operations				
Addition/subtraction combinations				
Unknown quantity = whole − known quantity 1 − known quantity 2	10	4	0	0
Known quantity 1 + unknown quantity = whole − known quantity 2	3	0	0	0
Whole = known quantity 1 + known quantity 2 + unknown quantity	20	20	0	8
Unknown quantity = whole − (known quantity 1 + known quantity 2)	34	41	0	0
Repeated addition/subtraction	0	0	58	0
Fraction multiplication	0	31	5	35
Fraction division	0	2	12	3
Fraction concepts				
Unit fraction	0	39	0	15
Proportional reasoning	0	11	0	18
Setting up proportion	0	10	0	5
Pictorial/direct modeling	17	13	26	28
Algebraic notations	18	16	4	21
Other	3	8	5	3

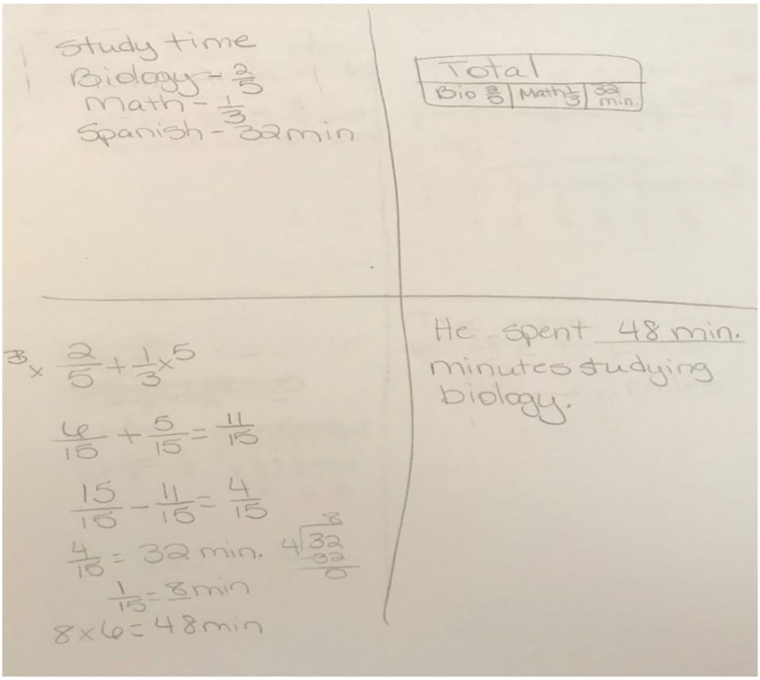

Fig. 4 Sample solution involving dealing with only known quantities to find an answer to problem 2

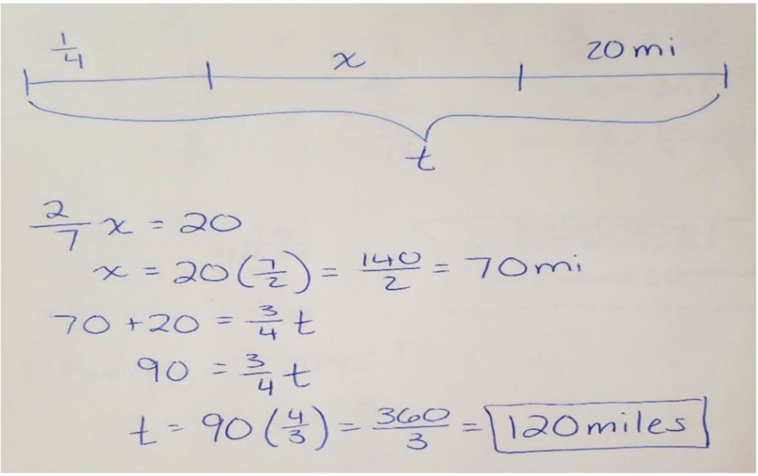

Fig. 5 Sample solution involving dealing with known and unknown quantities to find an answer to problem 4

In the first three problems, many strategies dealt only with the known quantities, such as by adding or subtracting the known quantities. For instance, like the strategy used for the first problem, more than half of the teachers tended to add the two known quantities and find the remaining quantity either mentally or by subtracting from the whole in the second question (see Fig. 4). Similarly, the majority of teachers dealt only with known quantities when solving the third word problem (i.e., by adding or subtracting the known quantity until they found the answer).

Although dealing with only known quantities when mathematizing word problems could lead to a correct answer for the first three problems, when the solution required teachers to deal with an unknown quantity, as in the fourth problem, their success rate dropped. Indeed, the strategy most commonly used by teachers who answered the fourth problem was attending to the unknown quantity in their mathematical representations of the problem (Fig. 5). Note that the highest percentage of algebraic notations and pictorial representations was used in solving the fourth problem. It is important to point out that using pictorial representations or algebraic notation to represent the unknown quantity along with the known quantities was also the second most common approach teachers used when mathematizing the first two problems.

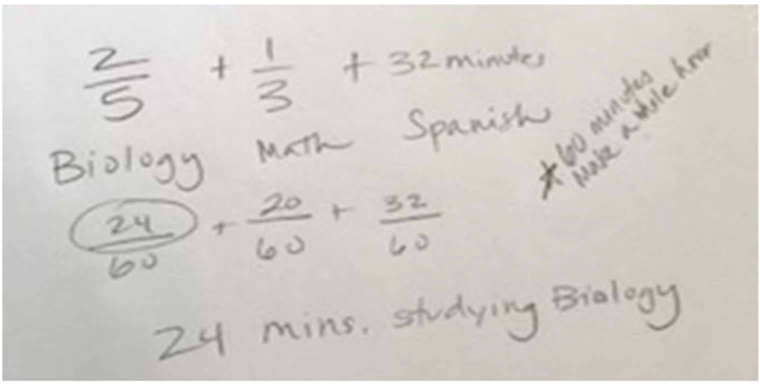

Fig. 6 Sample conceptual error made by using an incorrect referent whole in solving problem 2

Teachers' knowledge of fraction concepts seemed to be associated with their mathematization of word problems. For instance, teachers used informal strategies such as using proportional reasoning and unit fractions to solve the problems rather than formal strategies, such as setting up a proportion or using multiplication or division. Similarly, teachers' understanding of referent units seemed to influence their mathematization. Specifically, 26% who made an error in their solutions to the second problem used an incorrect referent whole for the problem (see Fig. 6). The same kind of error (i.e., using the same referent unit for fractions) was also observed in teachers' solutions to the fourth problem.

5.3 Differences in the mathematization of word problems between teachers with strong strategic competence and those with weak strategic competence

Here, we report how teachers with strong strategic competence mathematized word problems compared with teachers with weak strategic competence (See Fig. 7). We found, based on the chi-squared tests and adjusted p-values for multiple comparisons, that a significantly higher percentage of teachers with strong strategic competence used pictorial representations, algebraic notation, proportional reasoning, unit fractions, and fraction multiplication, as compared with those with less strategic competence ($p < 0.001$). In addition, a higher percentage of teachers with strong strategic competence devised a strategy by presenting both the known and unknown quantities in their solutions, as compared with those with weak strategic competence ($p < 0.001$).

Recall that the way teachers mathematized the word problem was related to how they dealt with the known and unknown quantities. In fact, 47% of the teachers with strong strategic competence dealt with the unknown quantity along with the known quantities, whereas only 24% of those with weak strategic competence dealt with the unknown quantities in one of the problems. Additionally, teachers with strong strategic competence used pictorial representations and algebraic notation more often than those with weak strategic competence. Of the teachers with strong strategic competence, 74% incorporated pictorial representations, whereas this rate was only 35% among teachers with weak strategic competence. Fifty-five percent of the teachers with strong strategic competence used algebraic notation in at least one of the problems, whereas only 19% of the teachers with weak strategic competence used algebraic notation in at least one of their solutions.

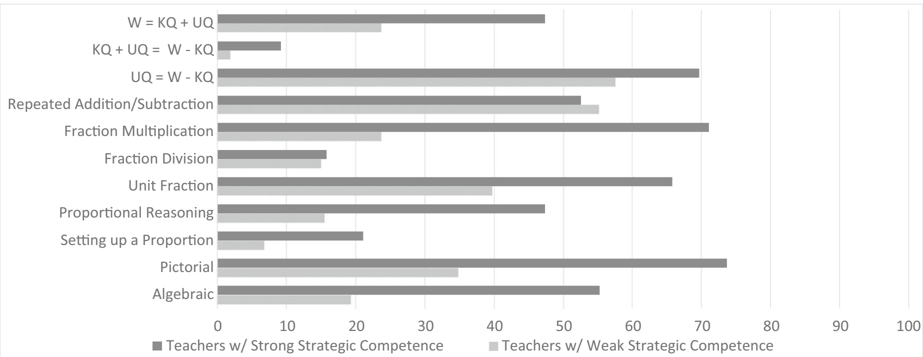

Fig. 7 Percentage of teachers with strong or weak strategic competence that used particular strategies for any of the four problems

6 Discussion

This study focused on one important yet overlooked indicator of teachers' mathematical knowledge: strategic competence. Our motivation was based on the NRC's (2001) assertion that mathematical proficiency requires strategic competence in addition to other aspects of mathematical knowledge, such as procedural knowledge and conceptual understanding. Our study provided evidence for the role strategic competence played in solving word problems. Specifically, we found that the three manifestations of strategic competence, as adapted from the NRC's (2001) definition (i.e., devising a valid solution strategy, mathematizing the word problem, and executing a solution strategy to solve the problem correctly), were useful for distinguishing teachers who are more or less successful in solving word problems.

Of the three components of strategic competence, we found that devising a valid strategy (regardless of what solution strategy was chosen) was a precursor to teachers' strategic competence. Our analysis of both correct and incorrect solution strategies along with correct and incorrect final answers confirmed this result. The majority of teachers who devised a valid strategy arrived at a correct answer. In addition, across all four-word problems, the most common error preventing teachers from arriving at a correct answer was a translation error (i.e., not devising a valid strategy). One implication of our findings is that teachers may need more opportunities in teacher education or professional development programs to enhance their skill in devising solution strategies for multistep word problems.

Our findings also underscored that the way in which teachers mathematized word problems, especially how they dealt with unknown quantities and the use of representations, was an important component of their strategic competence. A statistically higher percentage of teachers with strong strategic competence dealt with both known and unknown quantities in their strategies compared with teachers with weak strategic competence; the latter seemed to deal only with the given quantities. We contend that the fourth problem, which could not be solved simply by focusing on the given quantities, was particularly challenging for this reason. We believe teachers may need more learning opportunities involving word problems that require them to represent both unknown quantities in their solution strategies so that they may enhance their skills in mathematizing word problems.

A comparison of how word problems were mathematized between teachers with strong strategic competence or weak strategic competence also highlighted the importance of using representations in solving word problems. We found that a greater percentage of teachers with strong strategic competence used more pictorial representations and algebraic notations compared with those with weak strategic competence. Our findings are similar to those by Stylianou and Silver (2004) in their comparison of expert mathematicians' and undergraduate students' use of representations. They found that experts used visual representations more frequently and more widely than did novices when solving problems. We argue that the use of representations is particularly helpful in how teachers deal with unknown quantities to find solutions. Note that teachers who were successful in solving the fourth problem were more likely to use representations to include unknown quantities in their solution strategies. Although determining the importance of teachers' use of representations to deal with unknown quantities in their teaching is beyond the scope of this study, we believe further research is needed, particularly because prior work has documented that students struggle with algebraic notation (e.g., MacGregor & Stacey, 1997). Given that teachers may teach the strategies they naturally use to solve problems (Fisher, 1988), it is possible that students may not be learning to how to represent unknown quantities by using algebraic notation in multistep word problems.

Not surprisingly, our findings indicate arriving at a correct answer (i.e., the third indicator of strategic competence) depended closely on whether teachers first devised a valid strategy (i.e., the first indicator of strategic competence). After teachers devised a valid strategy, they were generally successful in executing that strategy. The differences in these two indicators of strategic competence appeared to be more in the kinds of errors associated with these indicators. Specifically, our analysis indicated that teachers usually made computational or interpretation errors during the execution of a valid strategy. On the other hand, conceptual errors were associated with teachers not devising a valid solution strategy. We believe that further research is needed to examine how these three indicators of strategic competence are related to different aspects of teachers' knowledge. Recall that students' low success rate in solving multistep word problems is concerning (e.g., Mullis et al., 2016). Further research is needed to determine how these three manifestations of teachers' strategic competence play out and their role in students' development of strategic competence.

Note also that our findings did not seem be in alignment with prior work. We believe these discrepancies are related to the study population and to methodological differences. For instance, none of the teachers in the study by Lee (2017) used repeated subtraction or unit rate strategies, whereas teachers in the present research used such strategies frequently. We argue that unlike preservice teachers, in-service teachers have more opportunities to develop strategic competence by learning through their teaching, the curriculum materials, or professional development (e.g., Copur-Gencturk, 2015, Copur-Gencturk et al., 2019; Hiebert et al., 2007). As a related point, we did not restrict teachers to using particular representations or strategies, which may have affected how they represented the problem mathematically.

The errors teachers made in this study were different from those reported in prior studies. This could again be due to differences in the study population as well as in prior use of one-step problems versus our use of multistep word problems (e.g., Adu-Gyamfi et al., 2019; Lee, 2017). Unlike in the majority of studies, we used multistep word problems; therefore, the errors teachers made when solving one-step word problems may have been different from the kinds of errors they made when solving multistep problems. We found that translation errors (i.e., not devising a valid solution strategy) were possibly due to the larger number of concepts and processes involved in multistep problems, which may have created more opportunities to capture teachers' devising valid strategies for solving word problems.

The discrepancy between the percentages of teachers solving word problems was related to the aforementioned issues along with the number of problems used in the present study. Recall that prior work has focused heavily on the use of a single word problem. Specifically, by asking teachers to solve a set of problems rather than only one, as in many other studies, we were able to provide a different depiction of teachers' success rate in solving word problems correctly. Recall that more than 90% of the teachers solved one of the problems (i.e., the third problem) correctly, whereas only one third of the same sample answered another question correctly. This highlights how capturing teachers' strategic competence may require a set of questions with varying degrees of content and difficulty.

7 Concluding remarks

Teachers' mathematical knowledge is an important component of mathematics teaching and students' learning. Although several prior studies have focused on teachers' mathematical knowledge, relatively less attention has been given to teachers' strategic competence, specifically their strategic competence for word problems. Given that word problems are an important component of the mathematics curriculum, yet are difficult for students to learn, understanding the role of teachers' strategic competence in relation to students' developing strategic competence for word problems could reveal more insights into why students may be struggling with word problems and how teachers' strategic competence could play a role in it.

Overall, this study contributes to the literature by providing evidence for different manifestations of teachers' strategic competence. We showed that teachers' strategic competence is highly contingent on whether they can devise a valid solution strategy and how they deal with known and unknown quantities in a given word problem. Thus, we contend that more attention needs to be given to understanding the development of teachers' strategic competence and the role of strategic competence in their teaching and students' learning.

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