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On the 430-cap of PG(6, 4) having two intersection sizes with respect to hyperplanes

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Abstract
Let $\mathcal{C}$ be a 430-cap of PG(6, 4) having two intersection sizes with respect to hyperplanes. We show that no hyperplane of PG(6, 4) intersects $\mathcal{C}$ in a Hill 78-cap. So if it can be shown that the Hill 78-cap of PG(5, 4) is projectively unique, then such a 430-cap does not exist, or equivalently, a two-weight $[430, 7]_{\mathbb{F}_4}$ linear code with dual weight at least 4, does not exist.

Keywords Uniformly packed code · 430-cap · Association scheme

Mathematics Subject Classification 51E22 · 94B05 · 05E30

1 Introduction

Uniformly packed codes generalise perfect codes, and the 1-error correcting examples have connections to strongly regular graphs, partial quadrangles, and two-character sets in finite projective spaces (see [3]). An e-error correcting code C is *uniformly packed* if spheres of radius $e + 1$ about codewords cover the whole space, and vectors at distance e from the C are in $\lambda + 1$ spheres while vectors at distance $e + 1$ from the code are in μ spheres. For the case that $e = 1$, a code C is 1-error correcting if and only if the dual code $C^\perp$ has two nonzero weights. If C has minimum distance at least 3, then $C^\perp$ is *projective*, and gives rise to a *two-character set* of a projective space: a set of points S such that there are only two values for the possible intersection size of a hyperplane with S.

There is also a connection with finite *partial quadrangles*. Partial quadrangles were introduced by Cameron [4] as a (finite) geometry of points and lines such that every two points are on at most one line, there are $s + 1$ points on a line, every point is on $t + 1$ lines and satisfying the following two important properties: (i) for every point P and every line ℓ not incident with P, there is at most one point on ℓ collinear with P; (ii) there is a constant μ such that for every pair of non-collinear points (X, Y) there are precisely μ points collinear with X and

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Y. The collinearity graph of a partial quadrangle is strongly regular. The only known partial quadrangles, that are not generalised quadrangles, are: triangle-free strongly regular graphs, obtained by removing points from a generalised quadrangle of order (s, s^2), or one of three *exceptional examples* arising from the linear representation of one of the Coxeter 11-cap of $\mathrm{PG}(4, 3)$, the Hill 56-cap of $\mathrm{PG}(5, 3)$ or the Hill 78-cap of $\mathrm{PG}(5, 4)$.

A *k-cap* of a projective space $\mathrm{PG}(n, q)$ is a set of k points with no three collinear. Calderbank [2] proved using number-theoretic arguments that if a partial quadrangle is a linear representation of a k-cap then $q \geq 5$ or it is isomorphic to the linear representation of one of the following: (i) an ovoid of $\mathrm{PG}(3, q)$; (ii) the Coxeter 11-cap of $\mathrm{PG}(4, 3)$; (iii) the Hill 56-cap of $\mathrm{PG}(5, 3)$; (iv) a 78-cap of $\mathrm{PG}(5, 4)$; (v) a 430-cap of $\mathrm{PG}(6, 4)$. Tzanakis and Wolfskill [8] proved that if $q \geq 5$, then the first case applies. It is still not known if case (v) occurs; that is, whether there is a 430-cap of $\mathrm{PG}(6, 4)$ such that every hyperplane intersects it in 78 or 110 elements. If a hyperplane intersects it in 78 elements, then it is a two-character 78-cap of $\mathrm{PG}(5, 4)$ (see Lemma 5.1). This leaves two open problems:

1. Does there exist a two-character 78-cap of $\mathrm{PG}(5, 4)$ projectively inequivalent to Hill's cap?
2. Does there exist a two-character 430-cap of $\mathrm{PG}(6, 4)$?

These problems are of interest to finite geometry and coding theory alike, and have been open for over 40 years, since at least [2]. We show in this note that a negative solution to the first problem implies a negative solution to the second problem.

Theorem 1.1 *Let $\mathcal{C}$ be a* 430*-cap of* $\mathrm{PG}(6, 4)$ *having two intersection sizes with respect to hyperplanes. Then no hyperplane of* $\mathrm{PG}(6, 4)$ *intersects $\mathcal{C}$ in a cap projectively equivalent to the Hill* 78*-cap.*

The basic argument proceeds as follows. Suppose H is the Hill 78-cap of $\mathrm{PG}(5, 4)$ and embed $\mathrm{PG}(5, 4)$ as a hyperplane Π of $\mathrm{PG}(6, 4)$. Let $\mathcal{Q}$ be the partial quadrangle arising from the linear representation of H, and let Γ be its collinearity graph. Then Γ is a strongly regular graph with parameters $(4096, 234, 2, 14)$. Now the affine points are the points of $\mathcal{Q}$, and the affine lines meeting Π in a point of H are the lines of $\mathcal{Q}$. Let $\mathcal{C}$ be a 430-cap of $\mathrm{PG}(6, 4)$ containing H. So the affine points $\bar{\mathcal{C}} := \mathcal{C}\backslash\Pi$ of $\mathcal{C}$ form a set of points of size 352 of $\mathcal{Q}$ such that every line of $\mathcal{Q}$ intersects it in at most one point. Moreover, $\bar{\mathcal{C}}$ forms a *Delsarte coclique* for Γ; a coclique that has size attaining the Delsarte/Hoffman bound. We will show that Γ does not have a Delsarte coclique, which then shows that the Hill 78-cap does not extend to a 430-cap of $\mathrm{PG}(6, 4)$. To do this, we take the Schurian scheme for the automorphism group of Γ, which is a 9-class fission scheme for the natural 2-class scheme arising from Γ. We then use another 2-class fusion of this Schurian scheme to yield information on the inner distribution of a putative Delsarte coclique.

2 Some background

Let Ω be a set, and let $A_0, A_1, \ldots, A_d$ be symmetric $\{0, 1\}$-matrices with rows and columns indexed by Ω. Then $\mathcal{A} = (\Omega, \{A_0, A_1, \ldots, A_d\})$ is a d-class *association scheme* if the following conditions hold:

1. A_0 is the identity matrix I,
2. $\sum_{i=0}^{d} A_i$ is the matrix with every entry equal to 1,
3. There exist constants p_{ij}^k depending only on i, j, and k, such that $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$.

The matrices $A_0, A_1, \ldots, A_d$ are the *adjacency matrices* of $\mathcal{A}$, and indeed, each A_i is the adjacency matrix of an undirected graph. A strongly regular graph Δ is essentially equivalent to a 2-class association scheme, where A_1 and A_2 are the adjacency matrices for Δ and its complement.

It is well known that $\mathbb{R}^\Omega$ decomposes into $d+1$ simultaneous eigenspaces for the adjacency matrices of $\mathcal{A}$. Moreover, there are projection matrices $E_0, E_1, \ldots, E_d$ (the *minimal idempotents*) onto each of these eigenspaces, such that

$$E_i = \sum_{j=0}^{d} Q_{ji} A_j,$$

where Q is called the *matrix of dual eigenvalues*. If C is a subset of Ω, then its *inner distribution* is the vector $a = (a_0, a_1, \ldots, a_d)$ defined by

$$a_i = \frac{1}{|C|} \mathbb{1}_C A_i \mathbb{1}_C^\top.$$

where we use $\mathbb{1}_C$ to denote the characteristic function of C in Ω. If Q is the matrix of dual eigenvalues of $\mathcal{A}$, then

$$(aQ)_j = \frac{|\Omega|}{|C|} \mathbb{1}_C E_j \mathbb{1}_C^\top$$

for all $j \geq 0$. The vector aQ is sometimes known as the *MacWilliams transform* of C, and it follows from the fact that the E_j are positive semidefinite, that each entry of aQ is non-negative.

The *dual degree set* of C is the set of nonzero indices j for which the j-th coordinate of its MacWilliams transform is nonzero. Two subsets of Ω are *design-orthogonal* if their dual degree sets are disjoint. In this case, we have the following elementary result, due at least to Roos.

Theorem 2.1 ([7, Corollary 3.3]) *If $S, T \subset \Omega$ are design-orthogonal, then* $|S \cap T| = \dfrac{|S| \cdot |T|}{|\Omega|}$.

The *outer distribution* B of S is the $|\Omega| \times d$ matrix, with rows indexed by Ω and columns indexed by the R_i, defined by

$$B_{x,i} = |\{y \in S : (x, y) \in R_i\}| = \mathbb{1}_{\{x\}} A_i \mathbb{1}_S^\top.$$

A transitive group G acting on Ω is *generously transitive* if for any distinct pair (α, β) of elements of Ω, there is some $g \in G$ such that $\alpha^g = \beta$ and $\beta^g = \alpha$. If a finite group G acts generously transitively on a set Ω, then the orbits of G on unordered pairs of Ω give rise to an association scheme: a *Schurian association scheme*. An association scheme is called a *translation scheme* if there an abelian group of automorphisms acting regularly on its vertices. If there is an ordering of the relations and minimal idempotents such that the matrix of eigenvalues P is equal to the matrix of dual eigenvalues Q, then we say the association scheme is *formally dual*. We refer the reader to [6] or [9] for more information on association schemes.

3 A Schurian scheme and some interesting subsets

The following *cyclotomic* construction of Γ can be found as [3, Example FE3]. Let z be a primitive element of $\mathbb{F}_{4^6}$. Let O be $\langle z^{35}\rangle \cup \langle z^{35}\rangle z^7$. Then Γ is isomorphic to the Cayley graph $\mathrm{Cay}(V, O)$ where V is the additive group of $\mathbb{F}_{4^6}$, and it is a strongly regular graph with parameters (4096, 234, 2, 14). Note that we can also view O as the set of underlying vectors of the Hill 78-cap (n.b., $234 = 3 \times 78$), represented as elements of $\mathbb{F}_{4^6}$.

Some of the details below were aided by computer, and in particular, the AssociationScheme package [1] in GAP. The code can be found in the Appendix. The automorphisms of Γ are generated by the translations (addition by elements of V), multiplication by z^{35}, and the map $\rho : x \mapsto z^{42}x^4$ (of order 6). So $\mathrm{Aut}(\Gamma)$ is isomorphic to $C_2^{12} : (C_{117} : C_6)$. Indeed, the stabiliser of 0 is generated by z^{35} and ρ, and these automorphisms act on O. Moreover, $\mathrm{Aut}(\Gamma)$ acts generously transitively on the points of Γ.

Let $v := 4096$, the number of vertices of Γ. Take the Schurian association scheme $\mathcal{A}$ for $\mathrm{Aut}(\Gamma)$, which is a fission scheme for the 2-class association scheme $\mathcal{G}$ associated to the original strongly regular graph Γ. Then the valencies of $\mathcal{A}$ are (in order) 1, 117, 234, 234, 351, 351, 702, 702, 702, 702 with R_2 being the adjacency relation for Γ (and the R_i are indexed with $i \in \{0, \dots, 9\}$). In fact, $\mathcal{A}$ is a translation scheme and it is formally dual. The matrix P of eigenvalues, and the matrix Q of dual eigenvalues, for $\mathcal{A}$ are:

$$P = Q = \begin{bmatrix} 1 & 117 & 234 & 234 & 351 & 351 & 702 & 702 & 702 & 702 \\ 1 & -27 & 10 & 10 & 15 & 63 & 30 & 30 & -66 & -66 \\ 1 & 5 & -22 & 10 & 15 & -33 & 30 & 30 & 30 & -66 \\ 1 & 5 & 10 & -22 & 15 & -33 & 30 & 30 & -66 & 30 \\ 1 & 5 & 10 & 10 & 47 & -1 & -34 & -34 & -2 & -2 \\ 1 & 21 & -22 & -22 & -1 & 31 & -2 & -2 & -2 & -2 \\ 1 & 5 & 10 & 10 & -17 & -1 & 30 & -34 & -2 & -2 \\ 1 & 5 & 10 & 10 & -17 & -1 & -34 & 30 & -2 & -2 \\ 1 & -11 & 10 & -22 & -1 & -1 & -2 & -2 & 30 & -2 \\ 1 & -11 & -22 & 10 & -1 & -1 & -2 & -2 & -2 & 30 \end{bmatrix}$$

We note that there is an involution acting as automorphisms on the association scheme. It is induced by the following semilinear map of order 12:

$$\tau : x \mapsto z^{14}x^2$$

and it interchanges relations of the association scheme in the following way: $R_2^\tau = R_3$, $R_6^\tau = R_7$, and $R_8^\tau = R_9$.

There are some unions of relations in $\mathcal{A}$ that yield interesting graphs.

(i) R_2 is the original strongly regular graph Γ. Moreover, the non-principal minimal idempotents for Γ are $\sum_{j\in\{1,3,4,6,7,8\}} E_j$ and $\sum_{j\in\{2,5,9\}} E_j$, where E_j is the j-th minimal idempotent for $\mathcal{A}$.

(ii) $R_2 \cup R_7 \cup R_8$ yields a strongly regular Cayley graph $\mathcal{F}$ that will feature in our proof of Theorem 1.1. The elements of the subfield $\mathbb{F}_{4^3}$ form a maximal clique of $\mathcal{F}$.

Below we list some interesting (Delsarte) designs for $\mathcal{A}$. We denote by V_j the j-th eigenspace for $\mathcal{A}$, for which the minimal idempotent E_j projects to.

3.1 Example 1: a subfield design

Consider the elements U of V that lie in the subfield $\mathbb{F}_{4^3}$. It turns out that the inner distribution of U is $(1, 0, 9, 0, 0, 0, 0, 27, 27, 0)$, and so its MacWilliams transform is $(64, 0, 576, 0, 0, 0, 0, 1728, 1728, 0)$. Therefore, $\mathbb{1}_U \in V_0 \perp V_2 \perp V_7 \perp V_8$.

3.2 Example 2: a Delsarte coclique

The complement of Γ is k-regular with $k = 3861$, and it has least eigenvalue $\tau := -11$. The *Delsarte bound* for the size of a coclique of Γ is then $1 - k/\tau = 352$. Suppose there exists a coclique S of Γ of size 352. Then S is a *Delsarte coclique*, and $\mathbb{1}_S E = 0$ where $E := \sum_{j\in\{1,3,4,6,7,8\}} E_j$ (see [6, Corollary 3.7.2]). (Note that $P_{i2} = -22$ for $i \in \{2, 5, 9\}$.) Recall that if $E := \sum_{j\in\{1,3,4,6,7,8\}} E_j$, then $\mathbb{1}_S E = 0$. Consider E_j where $j \in \{1, 3, 4, 6, 7, 8\}$. Then $E_j = EE_j$ and so $\mathbb{1}_S E_j = \mathbb{1}_S EE_j = 0$. So we have $(aQ)_j = 0$ for $j \in \{1, 3, 4, 6, 7, 8\}$, or in other words,

$$\mathbb{1}_S \in V_0 \perp V_2 \perp V_5 \perp V_9.$$

If we apply the involution τ, we find that $\mathbb{1}_{S^\tau} \in V_0 \perp V_3 \perp V_5 \perp V_8$.

Now consider the vector $v_P := 22\mathbb{1}_{\{P\}} + \mathbb{1}_{P^\perp}$ where P is a point of $\mathcal{Q}$, and $P^\perp$ is the set of points adjacent to P. Notice that $v_P = \mathbb{1}_{\{P\}}(A_2 + 22I)$ and so $v_P E_j = 0$ for $j \in \{2, 5, 9\}$. In particular, v_P is design-orthogonal[1] to S and so $\mathbb{1}_S \cdot v_P = 22$. It follows that $|P^\perp \cap S|$ is equal to 22 when $P \notin S$, but equal to 0 when $P \in S$.

4 Proof of Theorem 1.1

Proof Let S be a Delsarte coclique for Γ and let B be the outer distribution of S. By a theorem of Delsarte [5, Theorem 3.1], for all vertices x of Γ, and for all $j \in \{1, 3, 4, 6, 7, 8\}$,

$$\sum_{i=0}^{d} \frac{P_{ji}}{P_{0i}} B_{x,i} = 0. \tag{1}$$

Fix an element $x \notin S$. Recall that there are 22 elements of S adjacent to x (see Sect. 3.2 above), and so we can write $B_{x,i} = (0, y_1, 22, y_3, y_4, y_5, y_6, y_7, y_8, 330 - y_1 - y_3 - y_4 - y_5 - y_6 - y_7 - y_8)$ for some y_i. Then, we have the following equations (arising from (1)):

j	Equation
1	$220 + y_1 - (y_3 + y_4 + 2y_5 + y_6 + y_7) = 0$
3	$y_3 + y_5 + y_8 = 110$
4	$y_1 + y_3 + 3y_4 - y_6 - y_7 = 0$
6	$y_1 + y_3 + y_6 - y_4 - y_7 = 0$
7	$y_1 + y_3 + y_7 - y_4 - y_6 = 0$
8	$2(y_1 + y_3) - y_8 = 0$

[1] Note that design-orthogonality extends to weighted subsets in a straight-forward way.

These equations reduce: $2y_4 = y_6 = y_7 = y_8$, $y_3 = 110 - y_5 - y_8$, $y_1 = y_5 + \frac{3}{2}y_8 - 110$. So

$$B_{x,i} = (0, y_5 + \tfrac{3}{2}y_8 - 110, 22, 110 - y_5 - y_8, \tfrac{1}{2}y_8, y_5, y_8, y_8, y_8, 330 - y_5 - 4y_8).$$

Let a be the inner distribution of S. Now A_2 is the adjacency matrix of Γ, and so $\mathbb{1}_S A_2 \mathbb{1}_S^\top = 0$. Hence we can write the inner distribution of S as $a = (1, x_1, 0, x_3, x_4, x_5, x_6, x_7, x_8, 351 - x_1 - x_3 - x_4 - x_5 - x_6 - x_8)$, where the x_i are indeterminate. Now multiply by Q to yield the MacWilliams transform of S:

$$\begin{aligned} aQ = 32(11,\ & \tfrac{1}{2}(-x_1 + x_3 + x_4 + 2x_5 + x_6 + x_7 - 234), \quad x_1 + x_3 + x_4 + x_6 + x_7 + x_8 - 234, \\ & 117 - x_3 - x_5 - x_8, \quad \tfrac{1}{2}(x_1 + x_3 + 3x_4 - x_6 - x_7), \quad 2x_1 - x_3 + x_5, \\ & x_1 + x_3 - x_4 + x_6 - x_7, \quad x_1 + x_3 - x_4 - x_6 + x_7, \quad -2(x_1 + x_3) + x_8, \\ & 351 - 3x_1 - x_4 - x_5 - x_6 - x_7 - x_8). \end{aligned}$$

Recall that if $E := \sum_{j \in \{1,3,4,6,7,8\}} E_j$, then $\mathbb{1}_S E = 0$. Consider E_j where $j \in \{1, 3, 4, 6, 7, 8\}$. Then $E_j = EE_j$ and so $\mathbb{1}_S E_j = \mathbb{1}_S E E_j = 0$. So we have $(aQ)_j = 0$ for $j \in \{1, 3, 4, 6, 7, 8\}$, and hence

$$2x_4 = x_6 = x_7 = x_8, \quad x_3 = 117 - x_5 - x_8, \quad x_5 = x_1 - \tfrac{3}{2}x_8 + 117.$$

Therefore,

$$\begin{aligned} a &= \left(1, x_1, 0, \frac{x_8}{2} - x_1, \frac{x_8}{2}, x_1 - \frac{3x_8}{2} + 117, x_8, x_8, x_8, -x_1 - \frac{5x_8}{2} + 234\right), \\ aQ &= (352, 0, 64(2x_8 - 117), 0, 0, 32(117 + 4x_1 - 2x_8), 0, 0, 0, 64(117 - 2x_1 - x_8)). \end{aligned}$$

We now take a different fusion scheme yielding a strongly regular graph $\mathcal{F}$. Let $A = \sum_{i \in \{2,7,8\}} A_i$ where the A_i are adjacency matrices of $\mathcal{A}$, ordered according to the matrix P above. Then A is the adjacency matrix of a strongly regular graph $\mathcal{F}$ with parameters $(4096, 1638, 662, 650)$. The matrix of eigenvalues for $\mathcal{F}$ is

$$P_{\mathcal{F}} = \begin{bmatrix} 1 & 1638 & 2457 \\ 1 & 38 & -39 \\ 1 & -26 & 25 \end{bmatrix}$$

and the matrix of dual eigenvalues $Q_{\mathcal{F}}$ is exactly the same as $P_{\mathcal{F}}$. From the inner distribution a for S, it follows that $|\mathcal{F}(v) \cap S| = 2x_8$ for all $v \in S$, where $\mathcal{F}(v)$ denotes the neighbourhood of v in $\mathcal{F}$. From the outer distribution of S, it follows that $|\mathcal{F}(v) \cap S| = 2y_8$ for all $v \notin S$. Therefore,

$$\mathbb{1}_S A = 2x_8 \mathbb{1}_S + 2y_8(\mathbb{1} - \mathbb{1}_S)$$

where $\mathbb{1}$ is the 'all ones' vector, and so $(2x_8 - 2y_8 - 1638)\mathbb{1}_S + 2y_8\mathbb{1}$ is an eigenvector for A. In particular, $(2x_8 - 2y_8 - 1638)\mathbb{1}_S + 2y_8\mathbb{1}$ is annihilated by one of the non-principal idempotents D of $\mathcal{F}$, and so $\mathbb{1}_S D = 0$ as $\mathbb{1} D = 0$. The inner distribution for S, with respect to $\mathcal{F}$, is $a_{\mathcal{F}} := (1, 2x_8, 351 - 2x_8)$ and therefore, its MacWilliams transform is

$$a_{\mathcal{F}} Q_{\mathcal{F}} = (352, 64(2x_8 - 117), 32(351 - 4x_8)).$$

Since $\mathbb{1}_S$ is annihilated by one of the non-principal minimal idempotents, $(a_{\mathcal{F}} Q_{\mathcal{F}})_j = 0$ for either $j = 1$ or $j = 2$. So there are two cases to consider.

Case 1: $(a_{\mathcal{F}} Q_{\mathcal{F}})_1 = 0$ Here we have $x_8 = 117/2$ and so

$$a = \left(1, x_1, 0, \tfrac{117}{4} - x_1, \tfrac{117}{4}, \tfrac{117}{4} + x_1, \tfrac{117}{2}, \tfrac{117}{2}, \tfrac{117}{2}, \tfrac{351}{4} - x_1\right),$$
$$aQ = (352, 0, 0, 0, 0, 128x_1, 0, 0, 0, 32(117 - 4x_1)).$$

This implies that S is design-orthogonal to the subfield design given in Sect. 3.1. So by Roos' Theorem 2.1,

$$|S \cap \mathbb{F}_{4^3}| = \frac{|S||\mathbb{F}_{4^3}|}{|\Gamma|} = \frac{352 \cdot 64}{4096} = \frac{11}{2}.$$

This is a contradiction as $|S \cap \mathbb{F}_{4^3}|$ is an integer.

Case 2: $(a_{\mathcal{F}} Q_{\mathcal{F}})_2 = 0$ Here we have $x_8 = 351/4$ and

$$a = \left(1, x_1, 0, \tfrac{351}{8} - x_1, \tfrac{351}{8}, x_1 - \tfrac{117}{8}, \tfrac{351}{4}, \tfrac{351}{4}, \tfrac{351}{4}, \tfrac{117}{8} - x_1\right),$$
$$aQ = \left(352, 0, 3744, 0, 0, 32(4x_1 - \tfrac{117}{2}), 0, 0, 0, 32(\tfrac{117}{2} - 4x_1)\right).$$

In particular, $aQ \geq 0$ implies that $x_1 = \frac{117}{8}$ and hence

$$aQ = (352, 0, 3744, 0, 0, 0, 0, 0, 0, 0)$$

and we have $\mathbb{1}_S \in V_0 \perp V_2$. So S is design-orthogonal to S^τ, and so by Roos' Theorem 2.1,

$$|S \cap S^\tau| = \frac{|S||S^\tau|}{|\Gamma|} = \frac{352 \cdot 352}{4096} = \frac{121}{4}.$$

This is a contradiction as $|S \cap S^\tau|$ is an integer.

Both cases lead to a contradiction, and so there is no Delsarte coclique. □

5 Conclusion

The existence of a two-character 430-cap would not only yield a new 78-cap of PG(5, 4), but it would yield a two-character cap (see Lemma 5.1), and hence a new uniformly packed code with parameters [78, 6] (over $\mathbb{F}_4$). So indeed, if it can be shown that the Hill 78-cap of PG(5, 4) is the only two-character cap in this space (up to projectivity), then a two-character 430-cap does not exist. The following is well-known, but the author cannot find it in print, and so it is proved here.

Lemma 5.1 *Let $\mathcal{C}$ be a* 430*-cap of* PG(6, 4)*, such that every hyperplane intersects it in 78 or 110 elements. Let Σ be a hyperplane intersecting $\mathcal{C}$ in 78 elements. Then $\Sigma \cap \mathcal{C}$ is a cap (of a copy of* PG(5, 4)*) such that every hyperplane of Σ intersects it in 14 or 22 elements.*

Proof Let μ_i be the number of elements of $\Sigma \cap \mathcal{C}$ in the i-th hyperplane of Σ. Then the usual counting arguments show that

$$\sum_i \mu_i = 78 \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix}_{\mathbb{F}_4} = 78 \times 341,$$

$$\sum_i \mu_i(\mu_i - 1) = 78 \cdot 77 \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix}_{\mathbb{F}_4} = 78 \times 6545,$$

$$\sum_i \mu_i(\mu_i - 1)(\mu_i - 2) = 78 \cdot 77 \cdot 76 \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{\mathbb{F}_4} = 78 \times 122892.$$

Take a hyperplane H of Σ. Then it is on five hyperplanes, each meeting the 430-cap $\mathcal{C}$ in at most 110 points. Since each point of the cap is in at least one of these hyperplanes (n.b., the span of a point and H is a hyperplane of PG(6, 4)), we have

$$4(110 - |H \cap \mathcal{C}|) + 78 \geq 430$$

and hence $|H \cap \mathcal{C}| \leq 22$. So $(\mu_i - 14)^2(22 - \mu_i) \geq 0$ for all i. From the displayed equations above, we have

$$\begin{aligned}\sum_i (\mu_i - 14)^2(22 - \mu_i) &= \sum_i (-\mu_i(\mu_i - 1)(\mu_i - 2) + 47\mu_i(\mu_i - 1) - 763\mu_i + 4312) \\ &= 78\,(-122892 + 47 \times 6545 - 763 \times 341 + 75460) \\ &= 0.\end{aligned}$$

Therefore, $\mu_i \in \{14, 22\}$ as required. □

We remark that the largest coclique of Γ that we have been able to find by computation has size 119, but no well established technique that bounds the size of a coclique seemed to eliminate 352 cocliques immediately. This includes eigenvalue bounds, spherical code bounds, the No-Homomorphism Lemma, and the Clique-Adjacency polynomial.

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Appendix A: GAP code

The following GAP code was used for computing the matrix of eigenvalues of our translation scheme $\mathcal{A}$.

```
LoadPackage("associationscheme");
z := Z(4^6);
z35 := z^35;
# Make sure the ordering of the relations is the way we want it
reps := [z^21, z^0, z^14, z^41, z^1, z^3, z^5, z^2, z^4];;
```

```
# First find orbit reps of order 6 element.
reps6 := [];;
for r in reps do
  o := [r];
  for i in [1..6] do
    Append(o, List(o, t -> z^42 * t^4));
  od;
  Add(reps6, o);
od;
# Now find orbits of z^35 element, and collate.
orbs := List(reps6, t -> Union(Orbits(Group(z^35),t,OnRight)));;
# Compute the relation matrix of the association scheme
relmat := NullMat(4^6, 4^6);;
vv := AsList(GF(4^6));;
for i in [1..Size(vv)] do
    for j in [i+1..Size(vv)] do
        k := First([1..Size(orbs)], u -> vv[i]-vv[j] in orbs[u]);
        relmat[i][j] := k;
        relmat[j][i] := k;
    od;
od;
A := AssociationSchemeNC(relmat);
P := MatrixOfEigenvalues(A);; Display(P);
# We can reorder the minimal idempotents with the command
#   ReorderMinimalIdempotents
# so that Q=P.
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