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Point-missing s-resolvable t-designs: infinite series of 4-designs with constant index

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Abstract

The paper deals with t-designs that can be partitioned into s-designs, each missing a point of the underlying set, called point-missing s-resolvable t-designs, with emphasis on their applications in constructing t-designs. The problem considered may be viewed as a generalization of overlarge sets which are defined as a partition of all the $\binom{v+1}{k}$ k-sets chosen from a $(v+1)$-set X into $(v+1)$ mutually disjoint s-(v, k, δ) designs, each missing a different point of X. Among others, it is shown that the existence of a point-missing $(t-1)$-resolvable t-(v, k, λ) design leads to the existence of a t-$(v, k+1, \lambda')$ design. As a result, we derive various infinite series of 4-designs with constant index using overlarge sets of disjoint Steiner quadruple systems. These have parameters 4-$(3^n, 5, 5)$, 4-$(3^n+2, 5, 5)$ and 4-$(2^n+1, 5, 5)$, for $n \geq 2$, and were unknown until now. We also include a recursive construction of point-missing s-resolvable t-designs and its application.

Keywords Point-missing s-resolvable t-design · Overlarge set of s-designs

Mathematics Subject Classification 05B05

1 Introduction

The paper is concerned with point-missing s-resolutions of t-designs and applications thereof. In general, a partition of a t-(v, k, λ) design $(X, \mathcal{B})$ into mutually disjoint s-(w, k, δ) designs, $w \leq v$, $s < t$, is termed an s-resolution. If $w = v$, then $(X, \mathcal{B})$ is called s-resolvable; in particular, if $(X, \mathcal{B})$ is the complete k-$(v, k, 1)$ design, then an s-resolution of $(X, \mathcal{B})$ is called a *large set* of s-designs. If $w = v - 1$, then $(X, \mathcal{B})$ is called point-missing s-resolvable. A point-missing s-resolution of the complete k-$(v, k, 1)$ design is called an *overlarge set* of s-designs. Point-missing s-resolvability remains still sparsely investigated; however, several

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computational and theoretical works on the subject can be found in the literature [9, 13, 15, 16, 19, 20, 23]. Point-missing s-resolvability is complementarily related to what we call pencil-like s-resolvability for t-designs, and vice versa. As far as we know the first example of infinite series of non-trivial point-missing s-resolvable t-designs for $t \geq 4$ can be found in a paper of Alltop in 1972 [2], in which the author constructed a series of 4-$(2^n+1, 2^{n-1}, (2^{n-1}-3)(2^{n-2}-1))$ designs for $n \geq 4$ as the union of 2^n+1 mutually disjoint 3-$(2^n, 2^{n-1}, 2^{n-2}-1)$ designs. We prove theorems for constructing new t-designs from point-missing and pencil-like s-resolvable t-designs. By using these theorems for overlarge sets of disjoint Steiner quadruple systems with $v = 3^n - 1$ and $v = 3^n + 1$ points constructed by Teirlinck [23], including the already known case with $v = 2^n$, we derive various infinite series of 4-$(v+1, 5, 5)$ designs, which were unknown until now. It is worthy of note that no large sets of Steiner quadruple systems are constructed to date; however, large sets of Steiner 2-designs for $k = 4$ with $v = 13, 16$ points are known to exist [10, 12, 14]. We also show a recursive construction of point-missing s-resolvable t-designs and its application.

For the sake of clarity we include a few basic definitions. A t-design, denoted by t-(v, k, λ), is a pair $(X, \mathcal{B})$, where X is a v-set of *points* and $\mathcal{B}$ is a collection of k-subsets of X, called *blocks*, such that every t-subset of X is a subset of exactly λ blocks, and λ is called the *index* of the design. A t-design is called *simple* if no two blocks are identical, otherwise, it is called *non-simple*. A t-$(v, k, 1)$ design is called a *Steiner t-design*. For any point $x \in X$, let $\mathcal{B}_x = \{B\backslash\{x\} : x \in B \in \mathcal{B}\}$. Then $(X\backslash\{x\}, \mathcal{B}_x)$ is a $(t-1)$-$(v-1, k-1, \lambda)$ design, called a *derived design* of $(X, \mathcal{B})$. It can be shown by simple counting that a t-(v, k, λ) design is an s-(v, k, λ_s) design for $0 \leq s \leq t$, where $\lambda_s = \lambda\binom{v-s}{t-s}/\binom{k-s}{t-s}$. Since λ_s is an integer, necessary conditions for the parameters of a t-design are $\binom{k-s}{t-s}|\lambda\binom{v-s}{t-s}$ for $0 \leq s \leq t$. The smallest positive integer λ for which these necessary conditions are satisfied is denoted by $\lambda_{\min}(t, k, v)$ or simply $\lambda_{\min}$. If $\mathcal{B}$ is the set of all k-subsets of X, then $(X, \mathcal{B})$ is a t-$(v, k, \lambda_{\max})$ design, called the *complete* design, where $\lambda_{\max} = \binom{v-t}{k-t}$. If we take δ copies of the complete design, we obtain a t-$(v, k, \delta\binom{v-t}{k-t})$ design, which is referred to as a *trivial t-design*; otherwise, it is called a *non-trivial t-design*.

2 Point-missing *s*-resolvable *t*-designs

A t-(v, k, λ) design $(X, \mathcal{B})$ is said to be s-resolvable, for $0 < s < t$, if its block set $\mathcal{B}$ can be partitioned into $N \geq 2$ classes $\mathcal{B}_1, \ldots, \mathcal{B}_N$ such that each $(X, \mathcal{B}_i)$ is an s-(v, k, δ) design for $i = 1, \ldots, N$. Such a partition is called an *s-resolution* of $(X, \mathcal{B})$ and each $\mathcal{B}_i$ is called an s-resolution class or simply a resolution class, see e.g. [25, 26].

If the complete k-$(v, k, 1)$ design can be partitioned into N disjoint t-(v, k, λ) designs, where $N = \binom{v-t}{k-t}/\lambda$, then we say that there exists a *large set* of t-designs denoted by $LS[N](t, k, v)$ or by $LS_\lambda(t, k, v)$ to emphasize the value λ.

In the most general form, the concept of point-missing s-resolvability of a t-(v, k, λ) design can be defined as follows.

Definition 2.1 Let $(X, \mathcal{B})$ be a t-(v, k, λ) design and let $1 \leq s \leq t-1$. $(X, \mathcal{B})$ is called point-missing s-resolvable, if the block set $\mathcal{B}$ can be partitioned into mutually disjoint s-$(v-1, k, \delta)$ designs, each missing a point of X.

However, Definition 2.1 is equivalent to a definition that describes point-missing resolutions with more exact details. We now give an explanation.

Let $X = \{x_1, \ldots, x_v\}$ and let $X_i = X \setminus \{x_i\}$, $i = 1, \ldots, v$. Let m_i denote the number of s-$(v-1, k, \delta)$ designs $(X_i, \mathcal{B}_i)$ missing x_i in the resolution. First we show that any $x_i \in X$ is a missing point of an s-design $(X_i, \mathcal{B}_i)$. More precisely, let $Y \subseteq X$ be the subset of X such that there is no design $(X_i, \mathcal{B}_i)$ missing point x_i, when $x_i \in Y$. Assume that $Y \neq \emptyset$. Then the blocks of $\mathcal{B}$ can be written as follows.

$$\mathcal{B} = \bigcup_{x_h \in X \setminus Y} m_h \mathcal{B}_h, \text{ where } \; m_h \mathcal{B}_h := \underbrace{\mathcal{B}_h \cup \cdots \cup \mathcal{B}_h}_{m_h \text{ times}}.$$

Consider two given points $x_i \in Y$ and $x_j \in X \setminus Y$. Since $x_i \in Y$, there is no s-design $(X_i, \mathcal{B}_i)$ missing x_i. Thus x_i appears in each design $(X_h, \mathcal{B}_h)$, where $x_h \in X \setminus Y$, therefore x_i appears in $\sum_{x_h \in X \setminus Y} m_h \delta_1$ times in the blocks of $\mathcal{B}$, where $\delta_1 = \delta \frac{\binom{v-2}{s-1}}{\binom{k-1}{s-1}}$. Whereas the point $x_j \in X \setminus Y$ appears in $\sum_{x_h \in X \setminus \{Y \cup \{x_j\}\}} m_h \delta_1$ times in the blocks of $\mathcal{B}$, which is a contradiction if $Y \neq \emptyset$. Further, we show that $m_1 = \cdots = m_v$. W.l.o.g., assume by contradiction that $m_1 \neq m_2$. Then the number of blocks containing x_1 (resp. x_2) is then $\sum_{x \in X \setminus \{x_1\}} m_x \delta_1 = m_2 \delta_1 + \sum_{i=3}^{v} m_i \delta_1$ (resp. $\sum_{x \in X \setminus \{x_2\}} m_x \delta_1 = m_1 \delta_1 + \sum_{i=3}^{v} m_i \delta_1$). Since $m_2 \delta_1 + \sum_{i=3}^{v} m_i \delta_1 = m_1 \delta_1 + \sum_{i=3}^{v} m_i \delta_1$, we have $m_2 \delta_1 = m_1 \delta_1$, or equivalently $m_2 = m_1$, contradicting the assumption. Thus we must have $m_1 = \cdots = m_v$.

The discussion above suggests an equivalent formulation of Definition 2.1 as follows.

Definition 2.2 Let $(X, \mathcal{B})$ be a t-(v, k, λ) design and let $1 \leq s < t$ be an integer. $(X, \mathcal{B})$ is said to be point-missing s-resolvable, if there is an integer $m \geq 1$ such that the following hold.

1. $\mathcal{B} = \mathcal{B}_{x_1} \cup \cdots \cup \mathcal{B}_{x_v}$, where $X = \{x_1, \ldots, x_v\}$,
2. $\mathcal{B}_x = \mathcal{B}_x^1 \cup \cdots \cup \mathcal{B}_x^m$, each $(X \setminus \{x\}, \mathcal{B}_x^j)$ is an s-$(v-1, k, \delta)$ design, $j = 1, \ldots, m$, and m is called the multiplicity of the point x.

If $m = 1$, $(X, \mathcal{B})$ is simply called point-missing s-resolvable. Moreover, if $m > 1$, then $(X \backslash \{x\}, \mathcal{B}_x)$ is an s-$(v-1, k, m\delta)$ design. Therefore, $(X, \mathcal{B})$ again is a union of v mutually disjoint s-$(v-1, k, m\delta)$ design, each missing a different point of X. Hence, in general, when we speak of point-missing s-resolvable t-designs we mean $m = 1$.

If the complete k-$(v, k, 1)$ design can be partitioned into v mutually disjoint s-$(v-1, k, \delta)$ designs (i.e. point-missing s-resolvable), then we have an *overlarge set* of s-$(v-1, k, \delta)$ designs.

Lemma 2.1 *Let $(X, \mathcal{B})$ be a point-missing s-resolvable t-(v, k, λ) design and assume that each point in the resolution has multiplicity m. Then*

$$\delta = \lambda \binom{v-s}{t-s} \Big/ \binom{k-s}{t-s} m(v-s).$$

In particular, if the complete t-$(v, t, 1)$ design is point-missing $(t-1)$-resolvable, then the designs in the resolution are Steiner $(t-1)$-$(v-1, t, 1)$ designs.

Proof By assumption, we have

$$\mathcal{B} = \bigcup_{x \in X} \{\mathcal{B}_x^1 \cup \cdots \cup \mathcal{B}_x^m\},$$

where $(X \setminus \{x\}, \mathcal{B}_x^i)$ is an s-$(v-1, k, \delta)$ design. Let $S = \{x_1, \ldots, x_s\} \subseteq X$. Then S does not appear in any block of $\mathcal{B}_{x_j}^i$, for $j = 1, \ldots, s$ and $i = 1, \ldots, m$. Further, S appears in each $\mathcal{B}^i_{x_j}$ with $j \neq 1, \ldots, s$, exactly δ times. Thus S appears $m(v-s)\delta$ times in the blocks of $\mathcal{B}$. On the other hand, the number of blocks in $\mathcal{B}$ containing S is $\lambda_s = \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}\lambda$. Therefore $\lambda_s = m(v-s)\delta$ and thus $\delta = \frac{\lambda_s}{m(v-s)}$, as desired. □

Recall that the complement of an s-resolvable t-design is again s-resolvable. However, it is not true with a point-missing s-resolvable t-design. Let $X := \{x_1, \ldots, x_v\}$ and let $X_i := X \setminus \{x_i\}$, $i = 1, \ldots, v$. To simplify the typing we write: if $Y \subseteq X$, then $\overline{Y} := X \setminus Y$, whereas if $Y \subseteq X_i$, then $\widetilde{Y} := X_i \setminus Y$. Let $(X, \mathcal{D})$ be a point-missing s-resolvable t-design with parameters t-(v, k, λ) and let $(X, \overline{\mathcal{D}})$ be its complement which has parameters t-$(v, v-k, \overline{\lambda})$, where $\overline{\lambda} = \lambda\binom{v-k}{t}/\binom{k}{t}$. Let $\mathcal{D} = \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_v$ be a partition of $\mathcal{D}$ into v point-missing s-resolution classes, where $(X_i, \mathcal{D}_i)$ is an s-$(v-1, k, \delta)$ design, for $i = 1, \ldots, v$. The complement of $(X_i, \mathcal{D}_i)$ (within X_i) is an s-$(v-1, v-1-k, \widetilde{\delta})$ design $(X_i, \widetilde{\mathcal{D}_i})$ with $\widetilde{\delta} = \delta\binom{v-1-k}{s}/\binom{k}{s}$. So, we have $\overline{\mathcal{D}} = \overline{\mathcal{D}}_1 \cup \cdots \cup \overline{\mathcal{D}}_v = (\{x_1\} \cup \widetilde{\mathcal{D}}_1) \cup \cdots \cup (\{x_v\} \cup \widetilde{\mathcal{D}}_v)$, where $\{x_i\} \cup \widetilde{\mathcal{D}_i} = \{\{x_i\} \cup \widetilde{D} \mid \widetilde{D} \in \widetilde{\mathcal{D}_i}\}$. Thus, $\overline{\mathcal{D}_i} = (\{x_i\} \cup \widetilde{\mathcal{D}_i})$ is not an s-design, but rather a "pencil". Hence, the decomposition of $(X, \overline{\mathcal{D}})$ suggests the following definition.

Definition 2.3 Let $X = \{x_1, \ldots, x_v\}$ and denote $X_i := X \setminus \{x_i\}$, $i = 1, \ldots, v$. Let $(X, \mathcal{B})$ be a t-(v, k, λ) design. If for some $x_i \in X$ there exists an s-$(v-1, k-1, \delta)$ design $(X_i, \mathcal{B}_i)$ for $1 \leq s < t$, then we call $\{x_i\} \cup \mathcal{B}_i = \{\{x_i\} \cup \widetilde{B} \mid \widetilde{B} \in \widetilde{\mathcal{B}_i}\} \subseteq \widetilde{\mathcal{B}}$ an s-pencil of $(X, \mathcal{B})$. If $\mathcal{B} = (\{x_1\} \cup \mathcal{B}_1) \cup \cdots \cup (\{x_v\} \cup \mathcal{B}_v)$, where $(X_i, \mathcal{B}_i)$ is an s-$(v-1, k-1, \delta)$ design, then $(X, \mathcal{B})$ is said to be pencil-like s-resolvable.

As observed above, the complement of a point-missing s-resolvable t-design is a pencil-like s-resolvable t-design. Conversely, it is straightforward to check that the complement of a pencil-like s-resolvable t-design is a point-missing s-resolvable t-design. Hence the notion of point-missing s-resolvability and that of pencil-like s-resolvability are complementary equivalent. We record this fact in the following lemma.

Lemma 2.2 *A t-design is point-missing s-resolvable if and only if its complement is pencil-like s-resolvable.*

The next theorem shows a relation between certain classes of t-designs and point-missing $(t-1)$-resolvable t-designs, in terms of derived designs.

Theorem 2.3 *Let $(X, \mathcal{B})$ be a simple t-(v, k, λ) design with $|B \cap B'| \leq k-2$ for any two distinct blocks $B, B' \in \mathcal{B}$. Then there exists a simple point-missing $(t-1)$-resolvable t-$(v, k-1, (k-t)\lambda)$ design $(X, \mathcal{D})$. In particular, if $(X, \mathcal{B})$ is a Steiner t-$(v, t+1, 1)$ design, then there exists an overlarge set of Steiner $(t-1)$-$(v-1, t, 1)$ designs.*

Proof For a given point $x \in X$ consider the derived design $(X\setminus\{x\}, \mathcal{B}_x)$ at x with parameters $(t-1)$-$(v-1, k-1, \lambda)$. Here $\mathcal{B}_x = \{B\setminus\{x\} \mid x \in B,\ B \in \mathcal{B}\}$. Define $\mathcal{D} = \bigcup_{x\in X} \mathcal{B}_x$. We claim that $(X, \mathcal{D})$ is a t-$(v, k-1, (k-t)\lambda)$ design. Let $T = \{x_1, \ldots, x_t\} \subseteq X$. Then there are λ blocks of $\mathcal{B}$, say, $B_1, \ldots, B_\lambda$ containing T. Each B_i, $i = 1, \ldots, \lambda$, gives rise to a set $\mathbb{D}_i = \{D = B_i\setminus\{x\} \mid x \in B_i\setminus T\} \subseteq \mathcal{D}$ having $(k-t)$ blocks D containing T. Thus there are $(k-t)\lambda$ blocks $D \in \mathcal{D}$ containing T in total, as desired. The simplicity of $(X, \mathcal{D})$ is a consequence of the property: $|B \cap B'| \leq k-2$, $B, B' \in \mathcal{B}$, $B \neq B'$, which can be seen as follows. Let D, D' be two blocks of $\mathcal{D}$. If $D, D' \in \mathcal{B}_x$ for some $x \in X$, then $D \neq D'$, since $(X\setminus\{x\}, \mathcal{B}_x)$ is the derived design at x. If $D \in \mathcal{B}_x$ and $D' \in \mathcal{B}_y$ with $x \neq y$, then again $D \neq D'$. This is because if $D = D'$, then the two blocks $B = D\cup\{x\}$ and $B' = D'\cup\{y\}$ of $\mathcal{B}$

would have $|B \cap B'| = k-1$, a contradiction. In addition, if $(X, \mathcal{B})$ is a Steiner t-$(v, t+1, 1)$ design, then $(X, \mathcal{D})$ becomes the complete t-$(v, t, 1)$ design. In other words, the set of v distinct $(t-1)$-$(v-1, t, 1)$ derived designs of $(X, \mathcal{B})$ forms an overlarge set. □

Remark 2.1 1. The proof of Theorem 2.3 shows that the constructed t-$(v, k-1, (k-t)\lambda)$ design is not simple, if there are two blocks $B, B' \in \mathcal{B}$ with $|B \cap B'| = k-1$.
2. It should be stressed that the set of v distinct derived designs of a Steiner t-$(v, k, 1)$ design with $k > t+1$ in Theorem 2.3 will not form an overlarge set of $(t-1)$-$(v-1, k-1, 1)$ designs, but rather a point-missing $(t-1)$-resolution of a t-$(v, k-1, (k-t))$ design.

The following corollary is an immediate consequence of Theorem 2.3.

Corollary 2.4 *Assume that there exists a Steiner* t-$(v, k, 1)$ *design. Then there exists a point-missing* $(t-1)$*-resolvable* t-$(v, k-1, k-t)$ *design.*

The case $k = t+1$ of Corollary 2.4 is known as examples of overlarge sets of Steiner designs, see [23]. Thus, if there exists a Steiner t-$(v, t+1, 1)$ design, then there exists a point-missing $(t-1)$-resolvable t-$(v, t, 1)$ design, i.e. an overlarge set of Steiner $(t-1)$-$(v-1, t, 1)$ designs. Note that the converse of this statement is not true, i.e. if there exists an overlarge set of Steiner $(t-1)$-$(v-1, t, 1)$ designs, it is not necessarily true that a Steiner t-$(v, t+1, 1)$ design exists. For example, Östergård and Pottonen [17] have shown that a Steiner 4-(17, 5, 1) design does not exist. Nevertheless, there exists an overlarge set of Steiner 3-(16, 4, 1) designs, see [23]. And crucially, Teirlinck [23] has shown that there are overlarge sets of Steiner 3-$(v, 4, 1)$ designs for $v = 3^n - 1, n \geq 2$ and $v = 3^n + 1, n \geq 1$, despite the fact that only a finite number of Steiner 4-$(v, 5, 1)$ designs are hitherto known.

The general case $k \geq t+2$ is interesting, since Theorem 2.3 provides a point-missing $(t-1)$-resolvable t-$(v, k-1, k-t)$ design, which is not a complete design. Examples about this case can be seen, for instance, from Steiner 5-(24, 8, 1) and 5-(28, 7, 1) designs. Here we obtain point-missing 4-resolvable 5-(24, 7, 3) and 5-(28, 6, 2) designs, where designs in the resolution are Steiner 4-(23, 7, 1) and 4-(27, 6, 1) designs, respectively. Similarly, there are point-missing 3-resolvable 4-(23, 6, 3) and 4-(27, 5, 2) designs having Steiner 3-(22, 6, 1) and 3-(26, 5, 1) designs in the resolution, respectively.

As a further application of Theorem 2.3, we consider the infinite series of 4-$(q+1, 6, 10)$ designs with $q = 2^n$, $n \geq 5$ and $\gcd(n, 6) = 1$, [8], having the property that any two blocks of the designs intersect in at most 4 points. Thus we have the following result.

Corollary 2.5 *Let* $q = 2^n$*,* $n \geq 5$ *and* $\gcd(n, 6) = 1$*. Then there exists a point-missing 3-resolvable* 4-$(q+1, 5, 20)$ *design having a* 3-$(q, 5, 10)$ *design in the resolution.*

Corollary 3.3 shows an interesting example of 4-designs that are 3-resolvable, and point-missing 3-resolvable as well.

3 Constructions of *t*-designs from point-missing (*t* − 1)-resolvable *t*-designs

Recall that Lemma 2.2 shows a natural connection between point-missing and pencil-like s-resolvability via the complement action. However, we observe that point-missing $(t-1)$-resolvable t-designs may be used to construct pencil-like $(t-1)$-resolvable t-designs which are not related to the complementary connection, as shown in the following theorem.

Theorem 3.1 *Let $(X, \mathcal{B})$ be a point-missing $(t-1)$-resolvable t-(v, k, λ) design with $(t-1)$-$(v-1, k, \delta)$ designs in the resolution. Then there is a pencil-like $(t-1)$-resolvable t-$(v, k+1, t\delta+\lambda)$ design $(X, \mathcal{B}^*)$. If $|B \cap B'| \leq k-2$ for any two distinct blocks $B, B' \in \mathcal{B}$, then $(X, \mathcal{B}^*)$ is simple. Further, if there are two blocks $B, B' \in \mathcal{B}$ with $|B \cap B'| = k-1$, then the simplicity of $(X, \mathcal{B}^*)$ depends on the structure of the resolution.*

Proof Let $X = \{1, \ldots, v\}$. For $i \in X$ denote $(X\backslash\{i\}, \mathcal{B}_i)$ the $(t-1)$-$(v-1, k, \delta)$ design in the point-missing $(t-1)$-resolution. Define $\mathcal{B}_i^* = \{i\} \cup \mathcal{B}_i = \{\{i\} \cup B \mid B \in \mathcal{B}_i\}$, for $i = 1, \ldots, v$, and $\mathcal{B}^* = \bigcup_{i \in X} \mathcal{B}_i^*$. We claim that $(X, \mathcal{B}^*)$ is a pencil-like $(t-1)$-resolvable t-$(v, k+1, t\delta+\lambda)$ design. Let $T = \{i_1, \ldots, i_t\} \subseteq X$. Consider a resolution class $\mathcal{B}_j$ with $j \in T$. Since $(X\backslash\{j\}, \mathcal{B}_j)$ is a $(t-1)$-$(v-1, k, \delta)$ design, it follows that $\{i_1, \ldots, i_t\}\backslash\{j\}$ is contained in δ blocks of $\mathcal{B}_j$. Therefore $\{j\} \cup \{i_1, \ldots, i_t\}\backslash\{j\} = \{i_1, \ldots, i_t\}$ is contained in δ blocks of $\mathcal{B}_j^*$. Thus $\mathcal{B}_{i_1}^*, \ldots, \mathcal{B}_{i_t}^*$ together have $t\delta$ blocks containing T. Further, the $(v-t)$ resolution classes $\mathcal{B}_j$ with $j \notin T$ have λ blocks containing T. Therefore the $(v-t)$ classes $\mathcal{B}_j^*$ with $j \notin T$ together have λ blocks containing T. It follows that $(X, \mathcal{B}^*)$ is a t-$(v, k+1, t\delta+\lambda)$ design. Assume that $|B \cap B'| \leq k-2$ for any two distinct blocks $B, B' \in \mathcal{B}$. Let $B^*, B'^* \in \mathcal{B}^*$ be the two corresponding blocks of B and B'. If $B^*, B'^* \in \mathcal{B}_i^*$, then $B^* = \{i\} \cup B$ and $B'^* = \{i\} \cup B'$, so $B^* \neq B'^*$, since $B \neq B'$. The other case is that $B^* \in \mathcal{B}_i^*$ and $B'^* \in \mathcal{B}_j^*$ for $i \neq j$, thus $B^* = \{i\} \cup B$, $B'^* = \{j\} \cup B'$, where $B \in \mathcal{B}_i$ and $B' \in \mathcal{B'}_j$. Since $|B \cap B'| \leq k-2$, we have $B^* \neq B'^*$. Thus $(X, \mathcal{B}^*)$ is simple. □

The next theorem may be viewed as the reverse of Theorem 3.1.

Theorem 3.2 *Let $(X, \mathcal{B})$ be a pencil-like $(t-1)$-resolvable t-(v, k, λ) design with $(t-1)$-$(v-1, k-1, \delta)$ designs in the resolution. Then there is a point-missing $(t-1)$-resolvable t-$(v, k-1, \lambda-t\delta)$ design $(X, \mathcal{B}^*)$. If $|B \cap B'| \leq k-2$ for any two distinct blocks $B, B' \in \mathcal{B}$, then $(X, \mathcal{B}^*)$ is simple. Further, if there are two blocks $B, B' \in \mathcal{B}$ with $|B \cap B'| = k-1$, then the simplicity of $(X, \mathcal{B}^*)$ depends on the structure of the pencil-like $(t-1)$-resolution.*

Proof Let $X = \{1, \ldots, v\}$. For $i \in X$ denote $(X\backslash\{i\}, \mathcal{B}_i)$ the $(t-1)$-$(v-1, k-1, \delta)$ design in the pencil-like $(t-1)$-resolution of $(X, \mathcal{B})$. We have $\mathcal{B} = (\{1\} \cup \mathcal{B}_1) \cup \cdots \cup (\{v\} \cup \mathcal{B}_v)$ Define $\mathcal{B}^* = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_v$. We claim that $(X, \mathcal{B}^*)$ is a t-$(v, k-1, \lambda-t\delta)$ design, which is point-missing $(t-1)$-resolvable. Let $T = \{i_1, \ldots, i_t\} \subseteq X$. Then T is contained in λ blocks of $(X, \mathcal{B})$, which are distributed in v classes of the pencil-like $(t-1)$-resolution. Note that T is contained in δ blocks of $(\{i_j\} \cup \mathcal{B}_{i_j})$, for $i_j \in T$, so T is contained in $t\delta$ blocks of $(\{i_1\} \cup \mathcal{B}_{i_1}) \cup \cdots \cup (\{i_t\} \cup \mathcal{B}_{i_t})$ (i.e., T is not contained in any block of $\mathcal{B}_{i_1} \cup \cdots \cup \mathcal{B}_{i_t}$). The remaining $(v-t)$ classes $\{(\{1\} \cup \mathcal{B}_1) \cup \cdots \cup (\{v\} \cup \mathcal{B}_v)\}\backslash\{(\{i_1\} \cup \mathcal{B}_{i_1}) \cup \cdots \cup (\{i_t\} \cup \mathcal{B}_{i_t})\}$ of $(X, \mathcal{B})$ will have $\lambda - t\delta$ blocks containing T. Moreover, if T is contained in a block $\{j\} \cup B \in (\{j\} \cup \mathcal{B}_j)$, $j \in \{1, \ldots, v\}\backslash T$, then T is contained in $B \in \mathcal{B}_j$. Hence, $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_v$ will have $\lambda - t\delta$ blocks containing T and $(X, \mathcal{B}^*)$ is point-missing $(t-1)$-resolvable. Assume that $|B \cap B'| \leq k-2$ for any two distinct blocks $B, B' \in \mathcal{B}$. Obviously, the two corresponding blocks $B^*, B'^* \in \mathcal{B}^*$ are distinct. Thus $(X, \mathcal{B}^*)$ is simple. □

The simplicity of $(X, \mathcal{B}^*)$ in Theorem 3.1 in the case that there are two blocks $B, B' \in \mathcal{B}$ with $|B \cap B'| = k-1$ remains a main open question. In fact, examples for simple as well as non-simple $(X, \mathcal{B}^*)$ do exist in this case. We illustrate the situation with two explicit examples. First, consider the unique Steiner 3-(8, 4, 1) design $(X, \mathcal{B})$. By applying Lemma 2.2 we have

$$
\begin{aligned}
\mathcal{B}_0 &= 123 \;\; 345 \;\; 256 \;\; 136 \;\; 467 \;\; 157 \;\; 237 \\
\mathcal{B}_1 &= 024 \;\; 235 \;\; 456 \;\; 036 \;\; 057 \;\; 267 \;\; 347 \\
\mathcal{B}_2 &= 014 \;\; 135 \;\; 346 \;\; 056 \;\; 167 \;\; 037 \;\; 457 \\
\mathcal{B}_3 &= 125 \;\; 246 \;\; 045 \;\; 016 \;\; 567 \;\; 027 \;\; 147 \\
\mathcal{B}_4 &= 012 \;\; 236 \;\; 035 \;\; 156 \;\; 067 \;\; 137 \;\; 257 \\
\mathcal{B}_5 &= 123 \;\; 034 \;\; 146 \;\; 026 \;\; 367 \;\; 017 \;\; 247 \\
\mathcal{B}_6 &= 234 \;\; 145 \;\; 025 \;\; 013 \;\; 357 \;\; 047 \;\; 127 \\
\mathcal{B}_7 &= 356 \;\; 046 \;\; 015 \;\; 126 \;\; 023 \;\; 134 \;\; 245
\end{aligned}
$$

Thus the block set $\mathcal{D} = \bigcup_{x \in X} \mathcal{B}_x$ is the union of derived designs of $(X, \mathcal{B})$ at all points of $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Here $\mathcal{B}_0, \ldots, \mathcal{B}_7$ form an overlarge set of Steiner 2-(7, 3, 1) designs. It is easy to check that the resulting 3-(8, 4, 4) design $(X, \mathcal{B}^*)$ is not simple, more precisely each block is repeated 4 times. The second example is chosen from the set of 11 non-isomorphic of overlarge sets for 2-(7, 3, 1) designs [18]. The following representation is taken from [15].

$$
\begin{aligned}
\mathcal{B}'_0 &= 123 \;\; 145 \;\; 167 \;\; 247 \;\; 256 \;\; 346 \;\; 357 \\
\mathcal{B}'_1 &= 026 \;\; 035 \;\; 047 \;\; 234 \;\; 257 \;\; 367 \;\; 456 \\
\mathcal{B}'_2 &= 015 \;\; 037 \;\; 046 \;\; 136 \;\; 147 \;\; 345 \;\; 567 \\
\mathcal{B}'_3 &= 014 \;\; 025 \;\; 067 \;\; 127 \;\; 156 \;\; 246 \;\; 457 \\
\mathcal{B}'_4 &= 016 \;\; 023 \;\; 057 \;\; 125 \;\; 137 \;\; 267 \;\; 356 \\
\mathcal{B}'_5 &= 017 \;\; 024 \;\; 036 \;\; 126 \;\; 134 \;\; 237 \;\; 467 \\
\mathcal{B}'_6 &= 013 \;\; 027 \;\; 045 \;\; 124 \;\; 157 \;\; 235 \;\; 347 \\
\mathcal{B}'_7 &= 012 \;\; 034 \;\; 056 \;\; 135 \;\; 146 \;\; 236 \;\; 245
\end{aligned}
$$

It is straightforward to check that $(X, \mathcal{B}'^*)$ forms a simple 3-(8, 4, 4) design.

The examples indicate an involved problem of deciding the simplicity of $(X, \mathcal{B}^*)$, when $(X, \mathcal{B})$ has two blocks B and B' with $|B \cap B'| = k - 1$. The most interesting case for this situation, as mentioned in Theorem 2.3, is overlarge sets of disjoint Steiner $(t-1)$-$(v, t, 1)$ designs, i.e. the complete t-$(v+1, t, 1)$ design is point-missing $(t-1)$-resolvable having Steiner $(t-1)$-$(v, t, 1)$ designs in the resolution classes. Teirlinck [23] has shown that overlarge sets for Steiner 3-$(3^n - 1, 4, 1)$ and 3-$(3^n + 1, 4, 1)$ designs for $n \geq 2$ exist, including the known overlarge sets of Steiner 3-$(2^n, 4, 1)$ designs. By using these results we obtain the following infinite series of 4-designs with constant index as a corollary of Theorem 3.1.

Corollary 3.3 *There exist infinite series of pencil-like* 3*-resolvable* 4*-designs with the following parameters:*

1. 4-$(2^n + 1, 5, 5)$ *for* $n \geq 2$,
2. 4-$(3^n, 5, 5)$ *for* $n \geq 2$,
3. 4-$(3^n + 2, 5, 5)$ *for* $n \geq 2$.

Remark 3.1 It should be remarked that for all the designs in Corollary 3.3 we have $\lambda_{\min} = 1$ or 5. More precisely,

$$\lambda_{\min} = 5 \begin{cases} \text{for } v = 2^n + 1, & \text{and } n \equiv 3 \pmod 4, \\ \text{for } v = 3^n, & \text{and } n \equiv 2 \pmod 4, \\ \text{for } v = 3^n + 2, & \text{and } n \equiv 3 \pmod 4. \end{cases}$$

Note that Alltop [1] has constructed infinite series of simple 4-$(2^n + 1, 5, 5)$ designs for n odd and $n \geq 5$; thus the first series extends the point number to all possible values of n.

It is very likely that many non-isomorphic series of 4-designs with parameters given in Corollary 3.3 will exist, which are simple as well as non-simple, due to the fact that the number of non-isomorphic overlarge sets of 3-$(v, 4, 1)$ will strongly increase as v is getting large. In particular, it is important to decide whether the 4-designs in Corollary 3.3 are simple or not. As an observation we take a close look at the first design in each of the 4-$(3^n, 5, 5)$ and 4-$(3^n + 2, 5, 5)$ series. These are 4-$(9, 5, 5)$ and 4-$(11, 5, 5)$ designs, corresponding to $n = 2$. Note that each 4-$(9, 5, 5)$ design is simple, since its complement is the complete 4-$(9, 4, 1)$ design (otherwise, we would have a non-simple 4-$(9, 4, 1)$ design, which is impossible). In fact, this can also be verified directly by checking the two non-isomorphic overlarge sets of 3-$(8, 4, 1)$ designs given in [9], yielding 4-$(9, 5, 5)$ designs. Note also that 4-$(9, 5, 5)$ is the parameters of the second design in the 4-$(2^n + 1, 5, 5)$ series. The case of 4-$(11, 5, 5)$ designs is quite different. We have inspected the complete list of 21 non-isomorphic overlarge sets of 3-$(10, 4, 1)$ designs as shown in [20] and found that they all yield non-simple 4-$(11, 5, 5)$ designs.

For the ease of the reader, we include a table of known infinite series of t-designs with constant index for $t \geq 4$ (Table 1).

Theorem 3.4 *There exists a pencil-like* 3*-resolvable* 4-$(2^n + 1, 7, \frac{70}{3}(2^n - 5))$ *design for* $n \geq 5$ *and* $\gcd(n, 6) = 1$.

Proof Each 4-$(2^n + 1, 6, 10)$ design $(X, \mathcal{B})$ with $n \geq 5$ and $\gcd(n, 6) = 1$ in [8] has the property that $|B \cap B'| \leq 4$ for any two distinct blocks $B, B' \in \mathcal{B}$. Its complement is a 4-$(2^n + 1, 2^n - 5, \frac{2}{3}\binom{2^n-5}{4})$ design $(X, \bar{\mathcal{B}})$ having block intersections at most $(2^n - 3)$. By Theorem 2.3 there is a point-missing 3-resolvable 4-$(2^n + 1, 2^n - 6, (2^n - 9)\frac{2}{3}\binom{2^n-5}{4})$ design $(X, \bar{\mathcal{D}})$. Again, the complement of $(X, \bar{\mathcal{D}})$ is pencil-like 3-resolvable 4-$(2^n + 1, 7, \frac{70}{3}(2^n - 5))$ design, as desired. □

By applying Theorem 3.2 to the point-missing 3-resolvable 4-$(2^n + 1, 2^{n-1}, (2^{n-1} - 3)(2^{n-2} - 1))$ design $(X, \mathcal{B})$ of Alltop [2], we obtain an interesting result. Namely, we prove that there is a point-missing 3-resolvable design $(X, \mathcal{B}^*)$ with the same parameters as $(X, \mathcal{B})$ and disjoint from $(X, \mathcal{B})$ (recall that any two distinct blocks $B, B' \in \mathcal{B}$ have $|B \cap B'| \leq 2^{n-1} - 2$). Let $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_v$ be a partition of $\mathcal{B}$ into point-missing 3-resolution classes, i.e. each $(X_i, \mathcal{B}_i)$ is a 3-$(2^n, 2^{n-1}, 2^{n-2} - 1)$ design with $X_i = X \backslash \{i\}$. Consider $(X, \bar{\mathcal{B}})$ as the complement of $(X, \mathcal{B})$. So, $(X, \bar{\mathcal{B}})$ has parameters 4-$(2^n + 1, 2^{n-1} + 1, (2^{n-1} + 1)(2^{n-2} - 1))$ and is pencil-like 3-resolvable. Here, $\bar{\mathcal{B}} = (\{1\} \cup \tilde{\mathcal{B}}_1) \cup \cdots \cup (\{v\} \cup \tilde{\mathcal{B}}_v)$, where $\tilde{\mathcal{B}}_j$ is the complement of $\mathcal{B}_j$ in X_j, and $(X_j, \tilde{\mathcal{B}}_j)$ is a 3-$(2^n, 2^{n-1}, 2^{n-2} - 1)$ design, for $j = 1, \ldots, v$. The proof of Theorem 3.2 shows that $(X, \tilde{\mathcal{B}}^*)$ with $\tilde{\mathcal{B}}^* = \tilde{\mathcal{B}}_1 \cup \cdots \cup \tilde{\mathcal{B}}_v$, is point-missing 3-resolvable with $(X_j, \tilde{\mathcal{B}}_j)$ as the design in the resolution. Clearly, $(X, \mathcal{B})$ and $(X, \tilde{\mathcal{B}}^*)$ are disjoint and they have the same parameters. Further, the 4-design $(X, \mathcal{B} \cup \tilde{\mathcal{B}}^*)$ can be extended to a 5-design. Thus we have the following theorem.

Table 1 Known infinite series of t-designs with constant index for $t \geq 4$

No	t-(v, k, λ)	Conditions	(Non-)simplicity	References
1	4-$(2^n + 1, 5, 5)$	$n \geq 5$ odd	Simple	[1]
2	4-$(4^n + 1, 5, 2)$	$n \geq 2$	Non-simple	[3]
3	4-$(2^n + 1, 5, 5)$	$n \geq 4$	?	Corollary 3.3
4	4-$(3^n, 5, 5)$	$n \geq 3$	?	Corollary 3.3
5	4-$(3^n + 2, 5, 5)$	$n \geq 3$	?	Corollary 3.3
6	4-$(2^n + 1, 5, \lambda)$	$\lambda \in \{20, 25\}$, $\gcd(n, 6) = 1$	Simple	Corollary 2.5, [8]
7	4-$(60u + 4, 5, 60)$	$\gcd(u, 60) = 1$ or 2	Simple	[22]
8	4-$(2^n + 1, 6, 10)$	$n \geq 5$ odd	Simple	[5]
9	4-$(2^n + 1, 6, \lambda)$	$\lambda \in \{60, 70, 90, 100, 150, 160\}$	Simple	[4]
		$\gcd(n, 6) = 1$		
10	4-$(2^n + 1, 8, 35)$	$\gcd(n, 6) = 1$	Simple	[4]
11	4-$(2^n + 1, 9, \lambda)$	$\lambda \in \{84, 63, 147\}$, $\gcd(n, 6) = 1$	Simple	[4, 6]
12	5-$(2^n + 2, 6, 15)$	$n \geq 3$	Non-simple	[11]
13	5-$(2^n, 6, 3)$	$n \geq 3$	Non-simple	[7]
14	7-$(2^n, 8, 45)$	$n \geq 6$	Non-simple	[7]
15	t-$(v, t + 1, (t + 1)!^{2t+1})$	$v \equiv t \pmod{(t + 1)!^{2t+1}}$	Simple	
		$v \geq t + 1$		[21]

Theorem 3.5 *Let $n \geq 4$. Then*

1. *There exists a simple point-missing* 3*-resovable* $4\text{-}(2^n+1, 2^{n-1}, 2(2^{n-1}-3)(2^{n-2}-1))$ *design,*
2. *There exists a simple* $5\text{-}(2^n+2, 2^{n-1}+1, 2(2^{n-1}-3)(2^{n-2}-1))$ *design.*

4 A construction of point-missing *s*-resolvable *t*-designs

In this section we show that the recursive construction of t-designs in [24] can be extended to a construction of point-missing s-resolvable t-designs. More precisely, we prove the following theorem.

Theorem 4.1 *Assume that there exists a point-missing s-resolvable t-(v, k, λ) design having s-$(v-1, k, \delta)$ designs in its resolution. If $v\lambda_0(\lambda_0 - \lambda_1) < \binom{v}{k}$, then there exists a point-missing s-resolvable t-$(v+1, k, (v+1-t)\lambda)$ design having s-$(v, k, (v-s)\delta)$ designs in its resolution.*

Proof Assume that $(Y, \mathcal{D})$ is a point-missing s-resolvable t-(v, k, λ) design. Let $X = \{1, \ldots, v+1\}$ and denote $X_j = X \backslash \{j\}$ for $j = 1, \ldots, v+1$. Let $(X_j, \mathcal{B}^{(j)})$ be a copy of $(Y, \mathcal{D})$ defined on X_j. If $v\lambda_0(\lambda_0 - \lambda_1) < \binom{v}{k}$, then by Theorem A in [24] there are $(v+1)$ mutually disjoint $\mathcal{B}^{(1)}, \ldots, \mathcal{B}^{(v+1)}$ and they form a t-$(v+1, k, (v+1-t)\lambda)$ design $(X, \mathcal{B})$, where

$$\mathcal{B} = \bigcup_{j=1}^{v+1} \mathcal{B}^{(j)}.$$

We prove that $(X, \mathcal{B})$ is point-missing s-resolvable. Denote the partition of $(X_j, \mathcal{B}^{(j)})$ into point-missing s-resolution classes by

$$\mathcal{B}^{(j)} = \overbrace{\mathcal{C}_1^{(j)} \cup \cdots \cup \mathcal{C}_{j-1}^{(j)} \cup \mathcal{C}_{j+1}^{(j)} \cup \cdots \cup \mathcal{C}_{v+1}^{(j)}}^{v},$$

with $(X_{i,j}, \mathcal{C}_i^{(j)})$ as an s-$(v-1, k, \delta)$ design, where $X_{i,j} = X_j \backslash \{i\}$ and $i \in X_j$. For each point $j \in X$ define

$$\mathcal{C}_j = \overbrace{\mathcal{C}_j^{(1)} \cup \mathcal{C}_j^{(2)} \cup \cdots \cup \mathcal{C}_j^{(j-1)} \cup \mathcal{C}_j^{(j+1)} \cup \cdots \cup \mathcal{C}_j^{(v+1)}}^{v}.$$

We claim that $(X_j, \mathcal{C}_j)$ is an s-$(v, k, (v-s)\delta)$ design. Let $S = \{j_1, \ldots, j_s\} \subseteq X_j$. Then S will not appear in the blocks of $\mathcal{C}_j^{(j_1)}, \mathcal{C}_j^{(j_2)}, \ldots, \mathcal{C}_j^{(j_s)}$. Hence S appears in $(v-s)$ block sets $\mathcal{C}_j^{(i)}$, for $i \neq j_1, \ldots, j_s$. In other words, S is contained in the blocks of $\mathcal{C}_j$ exactly $(v-s)\delta$ times, which proves the claim. Further, since

$$\mathcal{B} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{v+1},$$

$(X, \mathcal{B})$ is point-missing s-resolvable with $\mathcal{C}_1, \ldots, \mathcal{C}_{v+1}$ as resolution classes. Note that the value of δ can be computed in terms of $t,\ v,\ k,\ \lambda$ by using Lemma 2.1. □

As an application of Theorem 4.1 consider the infinite series of 4-designs $(X, \mathcal{D})$ constructed by Alltop in [2]. $(X, \mathcal{D})$ has parameters $4\text{-}(2^n+1, 2^{n-1}, (2^{n-1}-3)(2^{n-2}-1))$,

$n \geq 4$, and is point-missing 3-resolvable with 3-$(2^n, 2^{n-1}, 2^{n-2}-1)$ designs in its resolution. For $n \geq 5$ the condition $v\lambda_0(\lambda_0 - \lambda_1) < \binom{v}{k}$ is satisfied, therefore Theorem 4.1 gives the following corollary.

Corollary 4.2 *For $n \geq 5$, there exists an infinite series of simple point-missing 3-resolvable* 4-$(2^n+2, 2^{n-1}, (2^n-2)(2^{n-1}-3)(2^{n-2}-1))$ *designs. The parameters of the 3-designs in the resolution are* 3-$(2^n+1, 2^{n-1}, (2^n-2)(2^{n-2}-1))$.

5 Conclusion

The paper deals with point-missing s-resolvable t-designs with emphasis on their use in constructing t-designs. Among others, we show the existence of infinite series of 4-$(v, 5, 5)$ designs with $v = 2^n+1,\ 3^n,\ 3^n+2$ for $n \geq 2$. It remains an open question about the simplicity of the designs in these series. We also present a recursive construction of point-missing s-resolvable t-designs including an application.

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