Designs, Codes and Cryptography (2022) 90:1567–1583
https://doi.org/10.1007/s10623-022-01056-w

A method of constructing 2-resolvable t-designs for $t = 3, 4$

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Received: 6 October 2021 / Revised: 15 February 2022 / Accepted: 3 May 2022 /
Published online: 24 May 2022

Abstract
The paper introduces a method for constructing 2-resolvable t-designs for $t = 3, 4$. The main idea is based on the assumption that there exists a partition of a t-design into Steiner 2-designs. A remarkable property of the method is that it enables the construction of 2-resolvable t-designs with a large variety of block sizes. For $t = 4$, it is required that the Steiner 2-designs of the partition are projective planes and this case would also lead to a construction of 3-resolvable 5-designs. For instance, we show the existence of an infinite series of 3-resolvable 5-designs having $N = 5$ resolution classes with parameters 5-$(14+8m, 7, 10(9+8m)(1+m))$ for any $m \geq 0$ as a byproduct. Moreover, it turns out that the method is very effective, as it yields infinitely many 2-resolvable 3-designs. However, the question of simplicity of the constructed designs has not been yet investigated.

Keywords s-resolvable t-design · Steiner 2-design · Projective plane

Mathematics Subject Classification 05B05

1 Introduction

A t-(v, k, λ) design is called s-resolvable if it can be partitioned into s-(v, k, δ) designs with $s < t$. The interesting case is $s \geq 2$. Especially, the s-resolvability of the complete k-$(v, k, 1)$ design is known in the literature as a large set of an s-(v, k, δ) design. Large sets are an essential element in proving the existence of simple t-designs for arbitrarily large t which have been intensively studied over three decades, see for instance [1, 10–14]. By contrast, very little is known about s-resolvability of non-trivial t-designs, when $s > 1$, see [4, 15, 17–19]. We are interested in non-trivial t-designs having s-resolutions. By focussing on $s = 2$ we introduce a method of constructing 2-resolvable t-designs, for $t = 3, 4$. In essence, the method is based on the assumption that there exists a t-design which can be partitioned

Communicated by L. Teirlinck.

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into Steiner 2-designs, and for $t = 4$ it is further required that the Steiner 2-designs must be projective planes. Some examples among others satisfying the assumption can be found in large sets of 2-$(v, 3, 1)$ Steiner triple systems for $v \equiv 1, 3 \bmod 6$, $v \neq 7$, in partition of certain infinite classes of 3-$(v, 4, 1)$ Steiner quadruple systems into 2-$(v, 4, 1)$ designs, for $v = 2^{2m}$, $m \geq 2$, [4], and $v = 2p^n + 2$, $p \in \{7, 31, 127\}$ [15], or in large sets of the projective planes of order 3, i.e. a symmetric 2-(13, 4, 1) design, [6, 8]. It appears that the method is very effective, actually, when starting with examples above, it will provide a huge number of 2-resolvable 3-designs for a large variety of block sizes. Moreover, with suitable parameters for $t = 4$, we can also construct 4-$(2k + 1, k, \Lambda)$ designs having 2-resolutions and therefore they can be extended to 3-resolvable 5-$(2k+2, k+1, \Lambda)$ designs. For instance, the case corresponding to the projective plane of order 3 yields a 3-resolvable 5-(14, 7, 90) design, which in turn leads to the existence of an infinite series of 3-resolvable 5-designs having $N = 5$ resolution classes with parameters 5-$(14 + 8m, 7, 10(9 + 8m)(1 + m))$ for any $m \geq 0$ as a byproduct.

We recall a few basic definitions. A t-design, denoted by t-(v, k, λ), is a pair $(X, \mathcal{B})$, where X is a v-set of *points* and $\mathcal{B}$ is a collection of k-subsets of X, called *blocks*, such that every t-subset of X is a subset of exactly λ blocks of $\mathcal{B}$. A t-design is called *simple* if no two blocks are identical, otherwise, it is called *non-simple*. A t-$(v, k, 1)$ design is called a *Steiner t-design*. It can be shown by simple counting that a t-(v, k, λ) design is an s-(v, k, λ_s) design for $0 \leq s \leq t$, where $\lambda_s = \lambda\binom{v-s}{t-s}/\binom{k-s}{t-s}$. Since λ_s is an integer, necessary conditions for the parameters of a t-design are $\binom{k-s}{t-s} | \lambda\binom{v-s}{t-s}$ for $0 \leq s \leq t$. The smallest positive integer λ for which these necessary conditions are satisfied is denoted by $\lambda_{\text{min}}(t, k, v)$ or simply λ_{min}. If $\mathcal{B}$ is the set of all k-subsets of X, then $(X, \mathcal{B})$ is a t-$(v, k, \lambda_{\text{max}})$ design, called the *complete* design, where $\lambda_{\text{max}} = \binom{v-t}{k-t}$. If we take δ copies of the complete design, we obtain a t-$(v, k, \delta\binom{v-t}{k-t})$ design, to which we refer as a *trivial t-design*. Again a t-(v, k, λ) design $(X, \mathcal{B})$ is said to be *s-resolvable*, for $0 < s < t$, if its block set $\mathcal{B}$ can be partitioned into $N \geq 2$ classes $\mathcal{A}_1, \ldots, \mathcal{A}_N$ such that each $(X, \mathcal{A}_i)$ is an s-(v, k, δ) design for $i = 1, \ldots, N$. Each $\mathcal{A}_i$ is called an s-resolution class or simply a resolution class and the set of N classes is called an s-resolution of $(X, \mathcal{B})$. If the complete k-$(v, k, 1)$ design is t-resolvable, i.e. it can be partitioned into N disjoint t-(v, k, λ) designs, where $k > t$, then we say that there exists a *large set* of size N of t-designs denoted by $LS[N](t, k, v)$ or by $LS_\lambda(t, k, v)$ to emphasize the value λ.

For more information about s-resolvable t-designs with $1 < s < t$, see for instance [16–19]. It should be remarked that s-resolvable t-designs have been used in the construction of t-designs [16].

2 Description of the method

The details of the method are described in this section. Here, two elements are required.

1. Let $(X, \mathcal{B})$ be a 2-resolvable t-(v, k, λ) design, where each class is a 2-$(v, k, 1)$ design. Thus, there are $N = \lambda\frac{\binom{v-2}{t-2}}{\binom{k-2}{t-2}}$ resolution classes. Let $\mathcal{B}_1, \ldots, \mathcal{B}_N$ denote the resolution classes of $(X, \mathcal{B})$, so each $(X, \mathcal{B}_i)$ is a 2-$(v, k, 1)$ design. We call $(X, \mathcal{B})$ the *outer design*.
2. Let $(Y, \mathcal{C})$ be a t-$(\frac{v-1}{k-1}, \ell, \mu)$ design. We call $(Y, \mathcal{C})$ the *inner design*.

Consider a fixed resolution class $(X, \mathcal{B}_i)$. Let $Y = \{1, \ldots, \frac{v-1}{k-1}\}$ be the point set of the inner design. For a point $x \in X$, let $\mathcal{Y}_{i,x} = \{B_{i,x}^1, \ldots, B_{i,x}^{\frac{v-1}{k-1}}\}$ denote the set of $\frac{v-1}{k-1}$ blocks through x of $\mathcal{B}_i$, i.e. $B_{i,x}^j \in \mathcal{B}_i$ with $x \in B_{i,x}^j$, $1 \le j \le |Y|$. For a block $C \in \mathcal{C}$, define

$$D_{i,x}^C = \bigcup_{j \in C} B_{i,x}^j,$$

and

$$\mathcal{D}_{i,x} = \{D_{i,x}^C \mid C \in \mathcal{C}\}.$$

That is, block $D_{i,x}^C$ is formed by the union of blocks in $\mathcal{Y}_{i,x}$ indexed by C, and $\mathcal{D}_{i,x}$ is the set of μ_0 such blocks $D_{i,x}^C$. Further, define

$$\mathcal{D}_i = \bigcup_{x \in X} \mathcal{D}_{i,x},$$

and

$$\mathcal{D} = \bigcup_{i=1}^{N} \mathcal{D}_i.$$

Similarly, define

$$D_{i,x}^{*C} = \bigcup_{j \in C} B_{i,x}^j \setminus \{x\}, \quad \mathcal{D}_{i,x}^* = \{D_{i,x}^{*C} \mid C \in \mathcal{C}\},$$
$$\mathcal{D}_i^* = \bigcup_{x \in X} \mathcal{D}_{i,x}^*,$$

and

$$\mathcal{D}^* = \bigcup_{i=1}^{N} \mathcal{D}_i^*.$$

If $(X, \mathcal{D})$ or $(X, \mathcal{D}^*)$ forms a t-design, we call it the *constructed design*.

For $t = 3$, we show that $(X, \mathcal{D})$ and $(X, \mathcal{D}^*)$ are 3-designs. For $t = 4$, if each resolution class of the outer design is a symmetric 2-$(v, k, 1)$ design, i.e. a projective plane of order $(k-1)$ with $v = q^2+q+1, k = q+1$, we prove that $(X, \mathcal{D})$ and $(X, \mathcal{D}^*)$ will form 4-designs. Further, it is shown that $(X, \mathcal{D}_i)$ and $(X, \mathcal{D}_i^*)$ are 2-designs. Obviously, the construction method makes clear that the constructed designs $(X, \mathcal{D})$ and $(X, \mathcal{D}^*)$ are 2-resolvable, as they are the union of designs $(X, \mathcal{D}_i)$ and $(X, \mathcal{D}_i^*)$, respectively. In case $t = 4$ and for suitable parameters of the outer design, the constructed design can be extended to a 3-resolvable 5-design, as shown in the subsequent section. A further investigation shows that if the inner design is also 2-resolvable with L resolution classes, then the constructed design is 2-resolvable with NL resolution classes. A major advantage of the method is the fact that it enables us to construct 2-resolvable t-designs with a large variety of block sizes, because there is no restriction on the parameters of the inner designs.

3 2-Resolvable 3-designs

In this section we deal with the case $t = 3$. We prove that $(X, \mathcal{D})$ and $(X, \mathcal{D}_i)$ are 3-$(v, \ell(k-1)+1, \Lambda)$ and 2-$(v, \ell(k-1)+1, \delta)$ designs, respectively. Similarly, $(X, \mathcal{D}^*)$ and $(X, \mathcal{D}_i^*)$ are 3-$(v, \ell(k-1), \Lambda^*)$ and 2-$(v, \ell(k-1), \delta^*)$ designs. Thus, we need to determine $\Lambda, \delta, \Lambda^*, \delta^*$. Recall that we consider the complete 2-$(v, 2, 1)$ design as a t-$(v, 2, 0)$ design for $t \geq 3$.

3.1 $(X, \mathcal{D})$ and $(X, \mathcal{D}_i)$ designs

We use the notation as described in the construction method. In the first step we show that $(X, \mathcal{D}_i)$ is a 2-$(v, \ell(k-1)+1, \delta)$ design, and in the next step $(X, \mathcal{D})$ is a 3-$(v, \ell(k-1)+1, \Lambda)$ design.

Step 1 $(X, \mathcal{D}_i)$ is a 2-$(v, \ell(k-1)+1, \delta)$ design.

Recall that $(X, \mathcal{B}_i)$ is a 2-$(v, k, 1)$ design and $(Y, \mathcal{C})$ is a 3-$(\frac{v-1}{k-1}, \ell, \mu)$ design with $Y = \{1, \ldots, \frac{v-1}{k-1}\}$. As usual μ_1 (resp. μ_2) denote the number of blocks of $(Y, \mathcal{C})$ containing a point (resp. two points). For a given point $x \in X$, there are $|Y|$ blocks of $\mathcal{B}_i$, say $B_{i,x}^1, \ldots, B_{i,x}^{|Y|}$ containing x. Let $C = \{j_1, \ldots, j_\ell\} \subseteq Y$ be a block of $\mathcal{C}$. Then block $D_{i,x}^C \in \mathcal{D}_{i,x}$ is defined by $D_{i,x}^C = B_{i,x}^{j_1} \cup \cdots \cup B_{i,x}^{j_\ell}$. Now let $a, b \in X$, $a \neq b$. Let B be the unique block of $\mathcal{B}_i$ containing $\{a, b\}$. We distinguish two types of points of X, namely points $x \in B$ and points $x \in X \setminus B$. If $x \in B$, then B is one of the blocks $B_{i,x}^1, \ldots, B_{i,x}^{|Y|}$, thus by forming the blocks of $\mathcal{D}_{i,x}$ we see that block B is contained in μ_1 blocks of $\mathcal{D}_{i,x}$, consequently $\{a, b\}$ appears in μ_1 blocks D of $\mathcal{D}_{i,x}$. Thus k points of B contribute $k\mu_1$ blocks $D \supseteq \{a, b\}$. If $x \in X \setminus B$, then $\{a, x\}$ and $\{b, x\}$ determine two distinct blocks $\{a, x, \ldots\}$ and $\{b, x, \ldots\}$ of $B_{i,x}^1, \ldots, B_{i,x}^{|Y|}$. All the blocks $D \in \mathcal{D}_{i,x}$ containing $\{a, x, \ldots\}$ and $\{b, x, \ldots\}$ will contain $\{a, b\}$. So, there are μ_2 blocks D containing $\{a, x, \ldots\}$ and $\{b, x, \ldots\}$. Thus $\{a, b\}$ appears in μ_2 blocks D of $\mathcal{D}_{i,x}$. Hence, $(v-k)$ points of $x \in X \setminus B$ contribute $(v-k)\mu_2$ blocks $D \supseteq \{a, b\}$. Altogether it gives

$$\delta = k\mu_1 + (v-k)\mu_2.$$

Hence, $(X, \mathcal{D}_i)$ is a 2-$(v, \ell(k-1)+1, \delta)$ design.

Step 2 $(X, \mathcal{D})$ is a 3-$(v, \ell(k-1)+1, \Lambda)$ design.

Let $T = \{a, b, c\} \subseteq X$. Note that among the N resolution classes $\mathcal{B}_1, \ldots, \mathcal{B}_N$ of $(X, \mathcal{B})$ there are λ classes, say, $\mathcal{B}_1, \ldots, \mathcal{B}_\lambda$ having the property that each has a unique block containing T.

(i) We first focus on blocks D containing T constructed from classes $\mathcal{B}_1, \ldots, \mathcal{B}_\lambda$. Consider $\mathcal{B}_1$. Let B be its unique block containing T. Each point of B gives μ_1 blocks D containing T. Whereas, each point of $X \setminus B$ gives μ blocks D containing T. Thus class $\mathcal{B}_1$ contributes $k\mu_1 + (v-k)\mu$ blocks $D \supseteq T$. It follows that the classes $\mathcal{B}_1, \ldots, \mathcal{B}_\lambda$ together give $\lambda(k\mu_1 + (v-k)\mu)$ blocks $D \supseteq T$.

(ii) The remaining $N - \lambda = \lambda\frac{v-k}{k-2}$ classes $\mathcal{B}_{\lambda+1}, \ldots, \mathcal{B}_N$ of $(X, \mathcal{B})$ have the property that $|B \cap T| \leq 2$, for any block $B \in \mathcal{B}_i$, $i = \lambda+1, \ldots, N$. Consider $\mathcal{B}_{\lambda+1}$. Let $B_{ab} = \{a, b, x_2, \ldots, x_k\}$, $B_{ac} = \{a, c, y_2, \ldots, y_k\}$, and $B_{bc} = \{b, c, z_2, \ldots, z_k\}$ be three unique blocks in $\mathcal{B}_{\lambda+1}$ containing $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, respectively. Two types of points of X need to be distinguished

(I) $3(k-1)$ points of $B_{ab} \cup B_{ac} \cup B_{bc}$,
(II) $(v-3(k-1))$ points of $X \setminus B_{ab} \cup B_{ac} \cup B_{bc}$.

Each point of type (I) gives μ_2 blocks $D \supseteq T$. Hence points of type (I) contribute $3(k-1)\mu_2$ blocks $D \supseteq T$.
Each point of type (II) gives μ blocks $D \supseteq T$. Hence points of type (II) contribute $(v-3(k-1))\mu$ blocks $D \supseteq T$.
It follows that all $N-\lambda$ classes $\mathcal{B}_{\lambda+1}, \ldots, \mathcal{B}_N$ contribute

$$(N-\lambda)(3(k-1)\mu_2 + (v-3(k-1))\mu)$$

blocks $D \supseteq T$.
Hence, Cases (i) and (ii) together show that

$$\Lambda = \lambda(k\mu_1 + (v-k)\mu) + (N-\lambda)(3(k-1)\mu_2 + (v-3(k-1))\mu).$$

Thus $(X, \mathcal{D})$ is a 3-design.
To compute the values of Λ and δ in terms of v, k, λ, ℓ, μ we have to separate two cases: $\ell = 2$ and $\ell = 3$.
$\ell = 2$:
In this case the inner design is the 2-$(\frac{v-1}{k-1}, 2, 1)$ design, which is considered as a degenerated 3-design with $\mu = 0$, $\mu_2 = 1$ and $\mu_1 = \frac{v-k}{k-1}$. Therefore

$$\begin{aligned}
\delta &= k\mu_1 + (v-k)\mu_2 \\
&= k\frac{v-k}{k-1} + (v-k) \\
&= (v-k)\frac{(2k-1)}{(k-1)},
\end{aligned}$$

and

$$\begin{aligned}
\Lambda &= \lambda(k\mu_1 + (v-k)\mu) + (N-\lambda)(3(k-1)\mu_2 + (v-3(k-1))\mu) \\
&= \lambda(k\frac{v-k}{k-1}) + \lambda\frac{v-k}{k-2}(3(k-1)) \\
&= \lambda(v-k)\frac{(2k-1)(2k-3)}{(k-1)(k-2)}.
\end{aligned}$$

$\ell \geq 3$:
The inner design with parameters 3-$(\frac{v-1}{k-1}, \ell, \mu)$ will give $\mu_2 = \mu\frac{(v-2k+1)}{(k-1)(\ell-2)}$ and $\mu_1 = \mu\frac{(v-k)(v-2k+1)}{(k-1)^2(\ell-1)(\ell-2)}$. Replacing μ_2 and μ_1 by their values in the formulas for δ and Λ and so simplifying we obtain

$$\begin{aligned}
\delta &= k\mu_1 + (v-k)\mu_2 \\
&= k\mu\frac{(v-k)(v-2k+1)}{(k-1)^2(\ell-1)(\ell-2)} + (v-k)\mu\frac{(v-2k+1)}{(k-1)(\ell-2)} \\
&= \frac{(v-k)(v-2k+1)}{(k-1)^2(\ell-1)(\ell-2)}\mu(k\ell-\ell+1),
\end{aligned}$$

and

$$\Lambda = \lambda(k\mu_1 + (v-k)\mu) + \lambda\frac{(v-k)}{(k-2)}(3(k-1)\mu_2 + (v-3(k-1))\mu)$$

$$= \lambda(k\mu\frac{(v-k)(v-2k+1)}{(k-1)^2(\ell-1)(\ell-2)} + (v-k)\mu)$$
$$+\lambda\frac{(v-k)}{(k-2)}(3(k-1)\mu\frac{(v-2k+1)}{(k-1)(\ell-2)} + (v-3(k-1))\mu)$$
$$= \frac{(v-k)(v-2k+1)}{(k-1)^2(k-2)(\ell-1)(\ell-2)}\lambda\mu((k-1)^2\ell^2-1).$$

3.1.1 The case with 2-resolvable inner designs

We further study the resolvability of the constructed designs when the inner designs are 2-resolvable. Suppose that the inner 3-$(\frac{v-1}{k-1}, \ell, \mu)$ design $(Y, \mathcal{C})$ is 2-resolvable with L resolution classes. Let $\mathcal{C}_1, \dots, \mathcal{C}_L$ be the L classes of $(Y, \mathcal{C})$. Then

$$(Y, \mathcal{C}) = (Y, \mathcal{C}_1) \cup \cdots \cup (Y, \mathcal{C}_L),$$

where each $(Y, \mathcal{C}_i)$ is a 2-$(\frac{v-1}{k-1}, \ell, \frac{\mu_2}{L})$ design, and $\mu_2 = \mu\frac{v-2k+1}{(k-1)(\ell-2)}$. It follows that

$$(X, \mathcal{D}_i) = (X, \mathcal{E}_i^{(1)}) \cup \cdots \cup (X, \mathcal{E}_i^{(L)}).$$

This is because the 2-$(v, \ell(k-1)+1, \delta)$ design $(X, \mathcal{D}_i)$ constructed from $(X, \mathcal{B}_i)$ and $(Y, \mathcal{C})$ in Step 1 is the union of L disjoint 2-$(v, \ell(k-1)+1, \frac{\delta}{L})$ designs $(X, \mathcal{E}_i^{(j)})$, $j = 1, \dots, L$. Each $(X, \mathcal{E}_i^{(j)})$ is the 2-design constructed from $(X, \mathcal{B}_i)$ and $(Y, \mathcal{C}_j)$.

As a result, the constructed design $(X, \mathcal{D})$ is 2-resolvable with NL resolution classes, and each class is a 2-$(v, \ell(k-1)+1, \frac{\delta}{L})$ design.

3.2 $(X, \mathcal{D}^*)$ and $(X, \mathcal{D}_i^*)$ designs

To show that $(X, \mathcal{D}^*)$ and $(X, \mathcal{D}_i^*)$ are designs, a very similar proof as above is to be employed, therefore it will be omitted. The results show that $(X, \mathcal{D}_i^*)$ is a 2-$(v, \ell(k-1), \delta^*)$ design with

$$\delta^* = (k-2)\mu_1 + (v-k)\mu_2,$$

and $(X, \mathcal{D}^*)$ is a 3-$(v, \ell(k-1), \Lambda^*)$ design with

$$\Lambda^* = \lambda((k-3)\mu_1 + (v-k)\mu) + \lambda\frac{(v-k)}{(k-2)}(3(k-2)\mu_2 + (v-3(k-1))\mu).$$

Putting the explicit values of μ_1, μ_2, both δ^* and Λ^* are expressed in terms of v, k, λ, ℓ, μ as shown in the next theorem.

The resolvability of $(X, \mathcal{D}^*)$ and $(X, \mathcal{D}_i^*)$ is the same as that of $(X, \mathcal{D})$ and $(X, \mathcal{D}_i)$.

We summarize the results in the following theorem.

Theorem 3.1 *Assume that the following designs exist.*

(i) *A* 2-*resolvable* 3-(v, k, λ) *design* $(X, \mathcal{B})$ *having* $N = \lambda\frac{v-2}{k-2}$ *resolution classes and each class is a* 2-$(v, k, 1)$ *design.*
(ii) *A* 3-$(\frac{v-1}{k-1}, \ell, \mu)$ *design* $(Y, \mathcal{C})$.

Then there exist 2-*resolvable* 3-$(v, (k-1)\ell+1, \Lambda)$ *and* 3-$(v, (k-1)\ell, \Lambda^*)$ *designs* $(X, \mathcal{D})$ *and* $(X, \mathcal{D}^*)$, *with* N *resolution classes, where each class is a* 2-$(v, (k-1)\ell+1, \delta)$ *and* 2-$(v, (k-1)\ell, \delta^*)$ *design, respectively.*

(i) *For* $\ell = 2$,

$$\Lambda = \lambda(v-k)\frac{(2k-1)(2k-3)}{(k-1)(k-2)}, \quad \delta = (v-k)\frac{(2k-1)}{(k-1)},$$
$$\Lambda^* = 2\lambda(v-k)\frac{(2k-3)}{(k-1)}, \quad \delta^* = (v-k)\frac{(2k-3)}{(k-1)}.$$

(ii) *For* $\ell \geq 3$,

$$\Lambda = \frac{(v-k)(v-2k+1)}{(k-1)^2(k-2)(\ell-1)(\ell-2)}\lambda\mu((k-1)^2\ell^2-1),$$
$$\delta = \frac{(v-k)(v-2k+1)}{(k-1)^2(\ell-1)(\ell-2)}\mu(k\ell-\ell+1),$$
$$\Lambda^* = \frac{(v-k)(v-2k+1)}{(k-1)^2(k-2)(\ell-1)(\ell-2)}\lambda\mu((\ell(k-1)-1)(\ell(k-1)-2)),$$
$$\delta^* = \frac{(v-k)(v-2k+1)}{(k-1)^2(\ell-1)(\ell-2)}\mu(k\ell-\ell-1).$$

Further, if $(Y, \mathcal{C})$ *is 2-resolvable with* L *resolution classes, then* $(X, \mathcal{D})$ *and* $(X, \mathcal{D}^*)$ *are 2-resolvable with* NL *resolution classes and each class is a* 2-$(v, (k-1)\ell+1, \frac{\delta}{L})$ *and* 2-$(v, (k-1)\ell, \frac{\delta^*}{L})$ *design, respectively.*

To illustrate the effectiveness of Theorem 3.1 we show a concrete example. For the outer design take a 3-(64, 4, 1) design [4] which is partitioned into $N = 31$ Steiner 2-(64, 4, 1) designs. The inner design can be chosen from all possible 3-$(21, \ell, \mu)$ designs with the following parameters.

1. 2-(21, 2, 1),
2. 3-(21, 3, 1),
3. 3-$(21, 4, m6)$, $1 \leq m \leq 3$
4. 3-$(21, 5, m3)$, $1 \leq m \leq 51$
5. 3-$(21, 6, m4)$, $1 \leq m \leq 204$
6. 3-$(21, 7, m15)$, $1 \leq m \leq 204$
7. 3-$(21, 8, m84)$, $1 \leq m \leq 102$
8. 3-$(21, 9, m42)$, $1 \leq m \leq 442$
9. 3-$(21, 10, m72)$, $1 \leq m \leq 442$.

The existence of 3-$(21, \ell, \mu)$ designs above for $4 \leq \ell \leq 10$ can be found in [7]. For each value of m for which a 3-$(21, \ell, \mu)$ design exists, the parameters of 2-resolvable 3-$(64, 3\ell+1, \Lambda)$ and 3-$(64, 3\ell, \Lambda^*)$ designs for $\ell = 2, \ldots, 10$, and their corresponding 2-designs in the resolution constructed from Theorem 3.1 are as follows.

(i) 3-$(64, 7, 70 \times 5)$, 2-$(64, 7, 70 \times 2)$,
3-$(64, 6, 20 \times 10)$, 2-$(64, 6, 20 \times 5)$,
(ii) 3-$(64, 10, 380 \times 20)$, 2-$(64, 10, 20 \times 5)$,
3-$(64, 9, 190 \times 28)$, 2-$(64, 9, 10 \times 8)$,
(iii) 3-$(64, 13, 95m \times 286)$, 2-$(64, 13, 95m \times 52)$,
3-$(64, 12, 380m \times 55)$, 2-$(64, 12, 380m \times 11)$, $\quad 1 \leq m \leq 3$,
(iv) 3-$(64, 16, 304m \times 35)$, 2-$(64, 16, 304m \times 5)$,
3-$(64, 15, 133m \times 65)$, 2-$(64, 15, 133m \times 10)$, $\quad 1 \leq m \leq 51$,
(v) 3-$(64, 19, 38m \times 323)$, 2-$(64, 19, 38m \times 38)$,
3-$(64, 18, 76m \times 136)$, 2-$(64, 18, 76m \times 17)$, $\quad 1 \leq m \leq 204$,

(vi) 3-$(64, 22, 380m \times 110)$, 2-$(64, 22, 380m \times 11)$,
3-$(64, 21, 190m \times 190)$, 2-$(64, 21, 190m \times 20)$, $1 \le m \le 204$,
(vii) 3-$(64, 25, 95m \times 2300)$, 2-$(64, 25, 95m \times 200)$,
3-$(64, 24, 760m \times 253)$, 2-$(64, 24, 760m \times 23)$, $1 \le m \le 102$,
(viii) 3-$(64, 28, 2660m \times 39)$, 2-$(64, 28, 2660m \times 3)$,
3-$(64, 27, 95m \times 975)$, 2-$(64, 27, 95m \times 78)$, $1 \le m \le 442$,
(ix) 3-$(64, 31, 1178m \times 145)$, 2-$(64, 31, 38m \times 310)$,
3-$(64, 30, 76m \times 2030)$, 2-$(64, 30, 76m \times 145)$, $1 \le m \le 442$.

Remarks 3.1 1. Observe that all values of Λ and Λ^* of the constructed designs above are really small. For example, by taking a 3-$(21, 9, 42)$ design as the inner design, the parameters of the 3-$(64, 27, \Lambda^*)$ constructed design become 3-$(64, 27, 95 \times 975)$, compared with its general parameters 3-$(64, 27, m^* \times 975)$, where $1 \le m^* \le 60961764003119$. Even if the complete 3-$(21, 9, 442 \times 42)$ design is used, the corresponding constructed design will be of parameters 3-$(64, 27, 41990 \times 975)$, showing that $m^* = 41990 \ll 60961764003119$ is still quite small.

2. The constructed 3-$(64, 10, 380 \times 20)$ and 3-$(64, 9, 190 \times 28)$ designs under (ii) are 2-resolvable with $NL = 31.19 = 589$ resolution classes each, this is because the 3-$(21, 3, 1)$ inner design can be partitioned into $L = 19$ Steiner 2-$(21, 3, 1)$ designs.
Other examples are the 3-$(64, 13, 95m \times 286)$ and 3-$(64, 12, 380m \times 55)$ designs with $m = 3$ from (iii). Here, the inner design is the complete 3-$(21, 4, 3 \times 6)$ design, which again can be partitioned into $L = 19$ disjoint 2-$(21, 4, 9)$ designs. Thus, both 3-$(64, 13, 95 * 3 \times 286)$ and 3-$(64, 12, 380 * 3 \times 55)$ designs are 2-resolvable with $NL = 589$ resolution classes.
In general, when the inner design is the complete 3-$(21, \ell, \binom{18}{\ell-3})$ design, we may employ the knowledge of large sets $LS_L(2, \ell, 21)$ to obtain further refinement of the resolution for the constructed design. For instance, there are $LS_{17}(2, \ell, 21)$ for $\ell = 5, 6, 7, 8$, thus the constructed designs under (iv), (v), (vi), (vii) have $NL = 31.17 = 527$ resolution classes.

The following corollaries show some applications of Theorem 3.1. It is a well-known result that there exists an $LS_{\nu_{\min}}(2, 3, v)$ for $v \ne 7$. In particular, if $v \equiv 1, 3 \pmod 6$, then $\nu_{\min} = 1$, i.e. the 3-$(v, 3, 1)$ design can be partitioned into $N = (v - 2)$ disjoint 2-$(v, 3, 1)$ designs. Take the 3-$(v, 3, 1)$ design as the outer design. Take the 2-$(\frac{v-1}{2}, 2, 1)$ and 3-$(\frac{v-1}{2}, 3, 1)$ design as the inner design. Again, in the second case the 3-$(\frac{v-1}{2}, 3, 1)$ design is 2-resolvable with $L = \frac{v-5}{2\nu_{\min}}$ resolution classes, each class is a 2-$(\frac{v-1}{2}, 3, \nu_{\min})$. Now applying Theorem 3.1 we have the following result.

Corollary 3.2 *Let $\nu_{\min} = \nu_{\min}(2, 3, \frac{v-1}{2})$, where v is an integer such that $v \equiv 1, 3 \pmod 6$, $v \ne 7$. Let $N = (v - 2)$ and $L = \frac{v-5}{2\nu_{\min}}$. Then*

(i) *There exists a 2-resolvable* 3-$(v, 5, \frac{15}{2}(v - 3))$ *design having $N = (v - 2)$ resolution classes, each class is a* 2-$(v, 5, \frac{5}{2}(v - 3))$ *design.*
(ii) *There exists a 2-resolvable* 3-$(v, 7, \frac{35}{8}(v - 3)(v - 5))$ *design having NL resolution classes, each class is a* 2-$(v, 7, \frac{7}{4}\nu_{\min}(v - 3))$ *design.*
(iii) *There exists a 2-resolvable* 3-$(v, 6, \frac{5}{2}(v - 3)(v - 5))$ *design having NL resolution classes, each class is a* 2-$(v, 6, \frac{5}{4}\nu_{\min}(v - 3))$ *design.*

For $n \ge 2$ there is a 2-resolvable 3-$(2^{2n}, 4, 1)$ design with $N = 2^{2n-1} - 1$ resolution classes and each class is a 2-$(2^{2n}, 4, 1)$ design, see [4]. Take this design as the outer design.

Now any 3-$(\frac{2^{2n}-1}{3}, \ell, \mu)$ design can be used as the inner design. Thus it produces innumerable 2-resolvable 3-designs with a large variety of block sizes. As an example, the next corollary shows the results for the first two cases with $\ell = 2, 3$, i.e. the inner design is the 2-$(\frac{2^{2n}-1}{3}, 2, 1)$ and 3-$(\frac{2^{2n}-1}{3}, 3, 1)$ design. Again, note that the 3-$(\frac{2^{2n}-1}{3}, 3, 1)$ design can be partitioned into $L = \frac{2^{2n}-7}{3\nu_{\min}}$ classes of 2-$(\frac{2^{2n}-1}{3}, 3, \nu_{\min})$ designs.

Corollary 3.3 *Let* $\nu_{\min} = \nu_{\min}(2, 3, \frac{(2^{2n}-1)}{3})$, $n \geq 2$. *Let* $N = (2^{2n-1} - 1)$ *and* $L = \frac{2^{2n}-7}{3\nu_{\min}}$. *Then*

- (i) *There exists a 2-resolvable* 3-$(2^{2n}, 7, \frac{35}{6}(2^{2n}-4))$ *design having* N *resolution classes, each class is a* 2-$(2^{2n}, 7, \frac{7}{3}(2^{2n}-4))$ *design.*
- (ii) *There exists a 2-resolvable* 3-$(2^{2n}, 6, \frac{10}{3}(2^{2n}-4))$ *design having* N *resolution classes, each class is a* 2-$(2^{2n}, 6, \frac{5}{3}(2^{2n}-4))$ *design.*
- (iii) *There exists a 2-resolvable* 3-$(2^{2n}, 10, \frac{20}{9}(2^{2n}-4)(2^{2n}-7))$ *design having* NL *resolution classes, each class is a* 2-$(2^{2n}, 10, \frac{5}{3}\nu_{\min}(2^{2n}-4))$ *design.*
- (iv) *There exists a 2-resolvable* 3-$(2^{2n}, 9, \frac{14}{9}(2^{2n}-4)(2^{2n}-7))$ *design having* NL *resolution classes, each class is a* 2-$(2^{2n}, 9, \frac{4}{3}\nu_{\min}(2^{2n}-4))$ *design.*

4 2-Resolvable 4-designs

This section deals with the case, where the designs in the resolution of the outer design are symmetric 2-$(v, k, 1)$ designs, i.e. each resolution class is a projective plane of parameters 2-$(q^2+q+1, q+1, 1)$. Obviously, $(X, \mathcal{D}_i)$ and $(X, \mathcal{D}_i^*)$ are 2-designs, as shown in the previous section. We prove that $(X, \mathcal{D})$ and $(X, \mathcal{D}^*)$ are 4-designs.

4.1 4-$(v, \ell(k-1)+1, \Lambda)$ design $(X, \mathcal{D})$

Again use the notation as described in the construction method. We omit the proof that $(X, \mathcal{D}_i)$ is a 2-$(v, \ell(k-1)+1, \delta)$ design, as it is the same as that in the previous section. Here, we focus on the proof in the main step that $(X, \mathcal{D})$ is a 4-$(v, \ell(k-1)+1, \Lambda)$ design.
Main step $(X, \mathcal{D})$ is a 4-$(v, \ell(k-1)+1, \Lambda)$ design.

To simplify the writing we temporarily keep the parameters 2-$(v, k, 1)$ for the symmetric design of the resolution, and will replace them with 2-$(q^2+q+1, q+1, 1)$ at the end of the proof.

Let $T = \{a, b, c, d\} \subseteq X$. With respect to T, there are three types of resolution classes:
(i) Classes having a unique block B containing T,
(ii) Classes having a unique block B with $|B \cap T| = 3$,
(iii) Classes having only blocks B with $|B \cap T| \leq 2$.

The number of classes of type (i) is λ, of type (ii) $4(\lambda\frac{v-3}{k-3} - \lambda) = 4\lambda\frac{v-k}{k-3}$. The remaining $N - (4\lambda\frac{v-k}{k-3} + \lambda)$ classes are of type (iii). So, w.l.o.g., we may assume that $\mathcal{B}_1, \ldots, \mathcal{B}_\lambda$ are classes of type (i) and $\mathcal{B}_{\lambda+1}, \ldots, \mathcal{B}_{\lambda+4\lambda\frac{v-k}{k-3}}$ classes of type (ii).

- (i) Consider class $\mathcal{B}_1$ of type (i). Let B be its unique block containing T. Each point of B gives μ_1 blocks D containing T. Whereas, each point of $X \setminus B$ gives μ blocks D containing T. Thus class $\mathcal{B}_1$ produces $k\mu_1 + (v-k)\mu$ blocks D. It follows that the classes $\mathcal{B}_1, \ldots, \mathcal{B}_\lambda$ together give $\lambda(k\mu_1 + (v-k)\mu)$ blocks $D \supseteq T$.

(ii) Each of $\mathcal{B}_{\lambda+1}, \ldots, \mathcal{B}_{\lambda+4\lambda\frac{v-k}{k-3}}$ classes of type (*ii*) has a unique block B with $|B \cap T| = 3$. Consider class $\mathcal{B}_{\lambda+1}$. There are four 3-subsets of T. So, w.l.o.g., we assume that $B \cap T = \{a, b, c\}$. Let $B := B_{abc} = \{a, b, c, u_3, \ldots, u_k\}$, $B_{da} = \{d, a, x_2, \ldots, x_k\}$, $B_{db} = \{d, b, y_2, \ldots, y_k\}$, and $B_{dc} = \{d, c, z_2, \ldots, z_k\}$ be the four unique blocks in $\mathcal{B}_{\lambda+1}$ containing $\{a, b, c\}$, $\{d, a\}$, $\{d, b\}$ and $\{d, c\}$, respectively. In $\mathcal{B}_{\lambda+1}$, the contribution to blocks $D \supseteq T$ depends on three distinct point types of X, that are the following.

(I) k points of B_{abc}. These points produce $k\mu_2$ blocks $D \supseteq T$.
(II) $1 + 3(k-2) = 3k-5$ points of B_{da}, B_{db} and B_{dc} different from a, b, c. These points give $(3k-5)\mu_3$ blocks $D \supseteq T$.
(III) $(v-4k+5)$ points of $X \setminus B_{abc} \cup B_{da} \cup B_{db} \cup B_{dc}$. These points produce $(v-4k+5)\mu$ blocks $D \supseteq T$.

So, class $\mathcal{B}_{\lambda+1}$ gives $(k\mu_2 + (3k-5)\mu_3 + (v-4k+5)\mu)$ blocks $D \supseteq T$. It follows that for all four 3-subsets $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ of T, the $4\lambda\frac{v-k}{k-3}$ classes of type (*ii*) produce $4\lambda\frac{v-k}{k-3}(k\mu_2 + (3k-5)\mu_3 + (v-4k+5)\mu)$ blocks $D \supseteq T$ in total.

(iii) Consider the remaining $N - (4\lambda\frac{v-k}{k-3} + \lambda)$ classes of type (*iii*). Let $\mathcal{B}_j$ be such a class. Since $(X, \mathcal{B}_j)$ is a 2-$(v, k, 1)$ projective plane, and $|B \cap T| \leq 2$ for any block $B \in \mathcal{B}_j$, the 6 pairs of points of $T = \{a, b, c, d\}$ are on 6 unique blocks.

$$\begin{aligned}
B_{ab} &= \{a, b, x_3, x_4, \ldots, x_k\}, \\
B_{cd} &= \{c, d, x_3, y_4, \ldots, y_k\}, \\
B_{ad} &= \{a, d, x_3', x_4', \ldots, x_k'\}, \\
B_{bc} &= \{b, c, x_3', y_4', \ldots, y_k'\}, \\
B_{ac} &= \{a, c, x_3'', x_4'', \ldots, x_k''\}, \\
B_{bd} &= \{b, d, x_3'', y_4'', \ldots, y_k''\}.
\end{aligned}$$

These blocks partition the points of X in 3 types.

(I) $(6k-14)$ points:

$$a, b, c, d, x_4, \ldots, x_k, y_4, \ldots, y_k, x_4', \ldots, x_k', y_4', \ldots, y_k', x_4'', \ldots, x_k'', y_4'', \ldots, y_k''.$$

These points give $(6k-14)\mu_3$ blocks $D \supseteq T$.
(II) 3 points: x_3, x_3', x_3''. These points give $3\mu_2$ blocks $D \supseteq T$.
(III) $(v-6k+11)$ points of $X \setminus (B_{ab} \cup B_{cd} \cup B_{ad} \cup B_{bc} \cup B_{ac} \cup B_{bd})$. These points produce $(v-6k+11)\mu$ blocks $D \supseteq T$.

Altogether $\mathcal{D}_j$ has $3\mu_2 + (6k-14)\mu_3 + (v-6k+11)\mu$ blocks $D \supseteq T$. Hence the $N - (4\lambda\frac{v-k}{k-3} + \lambda)$ classes of type (*iii*) produce

$$(N - (4\lambda\frac{v-k}{k-3} + \lambda))(3\mu_2 + (6k-14)\mu_3 + (v-6k+11)\mu)$$

blocks $D \supseteq T$.

In summary, cases (*i*), (*ii*), (*iii*) together yield

$$\begin{aligned}
\Lambda = \lambda(k\mu_1 + (v-k)\mu) &+ 4\lambda\frac{v-k}{k-3}(k\mu_2 + (3k-5)\mu_3 + (v-4k+5)\mu) \\
&+ (N - (4\lambda\frac{v-k}{k-3} + \lambda))(3\mu_2 + (6k-14)\mu_3 + (v-6k+11)\mu).
\end{aligned}$$

Putting $v = q^2 + q + 1$, $k = q + 1$, $N = \lambda \frac{(q^2+q-1)(q^2+q-2)}{(q-1)(q-2)}$, $\mu_i = \mu \frac{\binom{q+1-i}{4-i}}{\binom{\ell-i}{4-i}}$, $i = 1, 2, 3$, we find that

(1) for $\ell = 2$,

$$\Lambda = \frac{2\lambda q}{(q-2)}(4q^2 - 1), \quad \delta = q(2q+1),$$

(2) for $\ell = 3$,

$$\Lambda = \frac{\lambda q}{2(q-2)}(9q^2 - 1)(3q - 2), \quad \delta = \frac{q(q-1)}{2}(3q+1),$$

(3) for $\ell \geq 4$,

$$\Lambda = \frac{\lambda \mu q}{(\ell-1)(\ell-2)(\ell-3)}(q^2\ell^2 - 1)(q\ell - 2), \quad \delta = \frac{q(q-1)(q-2)(q\ell+1)}{(\ell-1)(\ell-2)(\ell-3)}\mu.$$

The 4-design $(X, \mathcal{D})$ is 2-resolvable with N resolutions classes, because it is the union of 2-designs $(X, \mathcal{D}_i)$s. Further, if the inner design $(Y, \mathcal{C})$ is also 2-resolvable with L resolution classes, then the same argument as above shows that $(X, \mathcal{D})$ is 2-resolvable with NL resolution classes.

4.2 4-(v, $\ell(k-1)$, Λ^*) design (X, $\mathcal{D}^*$)

Again, this case may be handled in a similar manner as that of $(X, \mathcal{D})$, and therefore we will omit the proof, despite the fact that several tiresome calculations for Λ^* have to be carefully carried out.

We record the results for both cases in the following theorem.

Theorem 4.1 *Assume that the following designs exist.*

(1) *A* 4-$(q^2 + q + 1, q + 1, \lambda)$ *design* $(X, \mathcal{B})$ *that can be partitioned into* $N = \lambda \frac{(q^2+q-1)(q^2+q-2)}{(q-1)(q-2)}$ *symmetric* 2-$(q^2 + q + 1, q + 1, 1)$ *designs, i.e. projective planes.*
(2) *A* 4-$(q + 1, \ell, \mu)$ *design* $(Y, \mathcal{C})$.

Then there exist 2-resolvable 4-$(q^2 + q + 1, q\ell + 1, \Lambda)$ *and* 4-$(q^2 + q + 1, q\ell, \Lambda^*)$ *designs* $(X, \mathcal{D})$ *and* $(X, \mathcal{D}^*)$ *with N resolution classes, where each class is a* 2-$(q^2+q+1, q\ell+1, \delta)$ *and a* 2-$(q^2 + q + 1, q\ell, \delta^*)$ *design, respectively,*

(i) *For* $\ell = 2$,

$$\Lambda = \frac{2\lambda q}{(q-2)}(4q^2 - 1), \quad \delta = q(2q+1),$$
$$\Lambda^* = \frac{2\lambda q}{(q-2)}(2q - 1)(2q - 3), \quad \delta^* = q(2q-1).$$

(ii) *For* $\ell = 3$,

$$\Lambda = \frac{\lambda q}{2(q-2)}(9q^2 - 1)(3q - 2), \quad \delta = \frac{q(q-1)}{2}(3q+1),$$
$$\Lambda^* = \frac{3\lambda q}{2(q-2)}(3q - 1)(3q - 2)(q - 1), \quad \delta^* = \frac{q(q-1)}{2}(3q-1).$$

(iii) *For* $\ell \geq 4$,

$$\Lambda = \frac{\lambda\mu q}{(\ell-1)(\ell-2)(\ell-3)}(q^2\ell^2-1)(q\ell-2),$$
$$\delta = \frac{q(q-1)(q-2)(q\ell+1)}{(\ell-1)(\ell-2)(\ell-3)}\mu,$$
$$\Lambda^* = \frac{\lambda\mu q}{(\ell-1)(\ell-2)(\ell-3)}(q\ell-1)(q\ell-2)(q\ell-3),$$
$$\delta^* = \frac{q(q-1)(q-2)(q\ell-1)}{(\ell-1)(\ell-2)(\ell-3)}\mu.$$

Further, if $(Y, \mathcal{C})$ *is 2-resolvable with* L *resolution classes, then* $(X, \mathcal{D})$ *and* $(X, \mathcal{D}^*)$ *are 2-resolvable with* NL *resolution classes and each class is a* 2-$(q^2+q+1, q\ell+1, \frac{\delta}{L})$ *and a* 2-$(q^2+q+1, q\ell, \frac{\delta^*}{L})$ *design, respectively.*

We illustrate Theorem 4.1 by showing the following examples. Let $q = 2^m$, $m \geq 5$ odd. Consider two infinite classes of 4-designs with parameters 4-$(q+1, 5, 5)$ and 4-$(q+1, 6, 10)$. The first one can be found in [2] and the second in [5]. All these designs are 3-resolvable with $L = \frac{(q-2)}{6}$ resolution classes. Each resolution class of the 4-$(q+1, 5, 5)$ designs is a 3-$(q+1, 5, 15)$ design, which is also a 2-$(q+1, 5, 5(q-1))$ design. Further, each resolution class of the 4-$(q+1, 6, 10)$ designs is a 3-$(q+1, 6, 20)$ design, which is also a 2-$(q+1, 6, 5(q-1))$ design. Taking these 4-$(q+1, 5, 5)$ and 4-$(q+1, 6, 10)$ designs as the inner design $(Y, \mathcal{C})$ and applying Theorem 4.1 we obtain the following result.

Corollary 4.2 *Let* $q = 2^m$, $m \geq 5$ *odd and let* $L = \frac{(q-2)}{6}$. *Assume that there exists a* 4-$(q^2+q+1, q+1, \lambda)$ *design that can be partitioned into* $N = \lambda\frac{(q^2+q-1)(q^2+q-2)}{(q-1)(q-2)}$ *projective planes of order* q. *Then there exist 2-resolvable* 4-$(q^2+q+1, q\ell+1, \Lambda)$ *and* 4-$(q^2+q+1, q\ell, \Lambda^*)$ *designs* $(X, \mathcal{D})$ *and* $(X, \mathcal{D}^*)$ *with* $NL = \lambda\frac{(q^2+q-1)(q+2)}{6}$ *resolution classes, where classes are* 2-$(q^2+q+1, q\ell+1, \frac{\delta}{L})$ *and* 2-$(q^2+q+1, q\ell, \frac{\delta^*}{L})$ *designs* $(X, \mathcal{E}_i)$ *and* $(X, \mathcal{E}_i^*)$, *respectively.*

(i) $(X, \mathcal{D})$: 4-$(q^2+q+1, 5q+1, \Lambda)$, $\Lambda = \frac{5\lambda q}{24}(5q+1)(5q-1)(5q-2)$,
$(X, \mathcal{E}_i)$: 2-$(q^2+q+1, 5q+1, \frac{\delta}{L})$, $\frac{\delta}{L} = \frac{5}{4}q(q-1)(5q+1)$,
(ii) $(X, \mathcal{D}^*)$: 4-$(q^2+q+1, 5q, \Lambda^*)$, $\Lambda^* = \frac{5\lambda q}{24}(5q-1)(5q-2)(5q-3)$,
$(X, \mathcal{E}_i^*)$: 2-$(q^2+q+1, 5q, \frac{\delta^*}{L})$, $\frac{\delta^*}{L} = \frac{5}{4}q(q-1)(5q-1)$,
(iii) $(X, \mathcal{D})$: 4-$(q^2+q+1, 6q+1, \Lambda)$, $\Lambda = \frac{\lambda q}{6}(6q+1)(6q-1)(6q-2)$,
$(X, \mathcal{E}_i)$: 2-$(q^2+q+1, 6q+1, \frac{\delta}{L})$, $\frac{\delta}{L} = q(q-1)(6q+1)$,
(iv) $(X, \mathcal{D}^*)$: 4-$(q^2+q+1, 6q, \Lambda^*)$, $\Lambda^* = \frac{\lambda q}{6}(6q-1)(6q-2)(6q-3)$,
$(X, \mathcal{E}_i^*)$: 2-$(q^2+q+1, 6q, \frac{\delta^*}{L})$, $\frac{\delta^*}{L} = q(q-1)(6q-1)$.

Under the condition of Corollary 4.2 we may find more infinite classes of 2-resolvable 4-designs by using the inner design $(Y, \mathcal{C})$ as 3-resolvable 4-$(q+1, k, \lambda)$ designs for $k = 8, 9$ in [5, 17].

We include a further application of Theorem 4.1. In [12] Teirlinck proves that an $LS_{\nu_{\min}}(3, 4, n)$ exists if $n \equiv 0 \pmod 3$. Let q be a prime power such that $q \equiv 2 \pmod 3$. Take the 4-$(q+1, 4, 1)$ design as the inner design, which is the union of L disjoint 3-$(q+1, 4, \nu_{\min})$ designs. Thus $L = \frac{q-2}{\nu_{\min}}$. Notice that a 3-$(q+1, 4, \nu_{\min})$ design is also a 2-$(q+1, 4, \nu_{\min}\frac{q-1}{2})$ design. Now applying Theorem 4.1 gives the following result.

Corollary 4.3 *Let q be a prime power such that $q \equiv 2 \pmod 3$. Let $\nu_{\min} = \nu_{\min}(3, 4, q+1)$ and let $L = \frac{q-2}{\nu_{\min}}$. Assume that there exists a 4-$(q^2+q+1, q+1, \lambda)$ design that can be partitioned into $N = \lambda\frac{(q^2+q-1)(q^2+q-2)}{(q-1)(q-2)}$ projective planes of order q. Then there exist 2-resolvable 4-$(q^2+q+1, 4q+1, \Lambda)$ and 4-$(q^2+q+1, 4q, \Lambda^*)$ designs $(X, \mathcal{D})$ and $(X, \mathcal{D}^*)$ with $NL = \frac{\lambda}{\nu_{\min}}(q^2+q-1)(q+2)$ resolution classes, where classes are 2-$(q^2+q+1, 4q+1, \frac{\delta}{L})$ and 2-$(q^2+q+1, 4q, \frac{\delta^*}{L})$ designs $(X, \mathcal{E}_i)$ and $(X, \mathcal{E}_i^*)$, respectively.*

(i) $(X, \mathcal{D})$: 4-$(q^2+q+1, 4q+1, \Lambda)$, $\quad \Lambda = \frac{\lambda q}{6}(4q-1)(4q+1)(4q-2)$,
$(X, \mathcal{E}_i)$: 2-$(q^2+q+1, 4q+1, \frac{\delta}{L})$, $\quad \frac{\delta}{L} = \nu_{\min}\frac{q(q-1)(4q+1)}{6}$,

(ii) $(X, \mathcal{D}^*)$: 4-$(q^2+q+1, 4q, \Lambda^*)$, $\quad \Lambda^* = \frac{\lambda q}{6}(4q-1)(4q-2)(4q-3)$,
$(X, \mathcal{E}_i^*)$: 2-$(q^2+q+1, 4q, \frac{\delta^*}{L})$, $\quad \frac{\delta^*}{L} = \nu_{\min}\frac{q(q-1)(4q-1)}{6}$.

5 5-Designs

Let us take a close look at the constructed design $(X, \mathcal{D}^*)$ with parameters 4-$(q^2+q+1, q\ell, \Lambda^*)$ in Theorem 4.1, when q is odd. Observe that if the inner design $(Y, \mathcal{C})$ is a 4-$(q+1, \frac{q+1}{2}, \mu)$ design, then the parameters of $(X, \mathcal{D}^*)$ become 4-$(q^2+q+1, \frac{q(q+1)}{2}, \Lambda^*)$. In this case, $(X, \mathcal{D}^*)$ can be extended to a 5-$(q^2+q+2, \frac{q(q+1)}{2}+1, \Lambda^*)$ design, by a theorem of Alltop [2, 3], which is described as follows.

Let $(X, \mathcal{B})$ be a t-$(2k+1, k, \lambda)$ design with t even, and let $\infty \notin X$. Define

$$\mathcal{B}^+ = \{B \cup \{\infty\} \mid B \in \mathcal{B}\},$$
$$\mathcal{B}^- = \{X \setminus B \mid B \in \mathcal{B}\}.$$

Then $(X \cup \{\infty\}, \mathcal{B}^+ \cup \mathcal{B}^-)$ is a $(t+1)$-$(2k+2, k+1, \lambda)$ design.

We prove the following lemma.

Lemma 5.1 *Let $(X, \mathcal{B})$ be a t-$(2k+1, k, \lambda)$ design with t even. Let $(X \cup \{\infty\}, \mathcal{B}^+ \cup \mathcal{B}^-)$ be its $(t+1)$-$(2k+2, k+1, \lambda)$ extending design. Assume that $(X, \mathcal{B})$ is s-resolvable with N resolution classes; each class is an s-$(2k+1, k, \delta)$ design.*

(i) *If s is even, then the extending design is $(s+1)$-resolvable with N resolution classes, each class is an $(s+1)$-$(2k+2, k+1, \delta)$ design.*
(ii) *If s is odd, then the extending design is s-resolvable with N resolution classes, each class is an s-$(2k+2, k+1, \delta\frac{2k+2-s}{k+1-s})$ design.*

Proof Let $\mathcal{B}_1, \dots, \mathcal{B}_N$ be the N resolution classes of $(X, \mathcal{B})$, where each $(X, \mathcal{B}_i)$ is an s-$(2k+1, k, \delta)$ design and $\delta = \frac{\lambda_s}{N}$.

(i) s even. Applying the Alltop theorem, we find

$$\mathcal{B}^+ = \mathcal{B}_1^+ \cup \dots \cup \mathcal{B}_N^+,$$
$$\mathcal{B}^- = \mathcal{B}_1^- \cup \dots \cup \mathcal{B}_N^-.$$

Hence

$$\mathcal{B}^+ \cup \mathcal{B}^- = (\mathcal{B}_1^+ \cup \mathcal{B}_1^-) \cup \dots \cup (\mathcal{B}_N^+ \cup \mathcal{B}_N^-).$$

Each $(X \cup \{\infty\}, \mathcal{B}_i^+ \cup \mathcal{B}_i^-)$ is an $(s+1)$-$(2k+2, k+1, \delta)$ design, for $i = 1, \dots, N$. Thus, $(X \cup \{\infty\}, \mathcal{B}^+ \cup \mathcal{B}^-)$ is $(s+1)$-resolvable.

(ii) s odd. Each class $(X, \mathcal{B}_i)$ is an s-$(2k+1, k, \delta)$ design. Thus, $(X, \mathcal{B}_i)$ may be considered as an $(s-1)$-$(2k+1, k, \delta_{s-1})$ design with $(s-1)$ even and $\delta_{s-1} = \delta \frac{2k+1-(s-1)}{k-(s-1)}$. Again, applying the Alltop theorem shows that the extending design $(X \cup \{\infty\}, \mathcal{B}^+ \cup \mathcal{B}^-)$ is s-resolvable, and each resolution class is an s-$(2k+2, k+1, \delta \frac{2k+2-s}{k+1-s})$ design.

□

Thus, starting with an inner design $(Y, \mathcal{C})$ of parameters 4-$(q+1, \frac{q+1}{2}, \mu)$ for q odd and applying Lemma 5.1 we find that the constructed design $(X, \mathcal{D}^*)$ in Theorem 4.1 is extended to a 3-resolvable 5-design $(X \cup \{\infty\}, \mathcal{D}^{*+} \cup \mathcal{D}^{*-})$.

We state the result in the following theorem.

Theorem 5.2 *Let q be an odd positive integer. Assume that there is a 2-resolvable 4-$(q^2+q+1, q+1, \lambda)$ design with $N = \lambda \frac{(q^2+q-1)(q^2+q-2)}{(q-1)(q-2)}$ resolution classes, each class is a symmetric 2-$(q^2+q+1, q+1, 1)$ design. Assume that there is also a 4-$(q+1, \frac{q+1}{2}, \mu)$ design. Then there is a 3-resolvable 5-$(q^2+q+2, \frac{q(q+1)}{2}+1, \Lambda^*)$ design with N resolution classes; each class is a 3-$(q^2+q+2, \frac{q(q+1)}{2}+1, \delta^*)$ design, where Λ^* and δ^* are as follows.*

(i) *For $q = 3$,*

$$\Lambda^* = \frac{2\lambda q}{(q-2)}(2q-1)(2q-3) = 90,$$
$$\delta^* = q(2q-1) = 15.$$

(ii) *For $q = 5$,*

$$\Lambda^* = \frac{3\lambda q}{2(q-2)}(3q-1)(3q-2)(q-1) = \lambda 1820,$$
$$\delta^* = \frac{q(q-1)}{2}(3q-1) = 140.$$

(iii) *For $q \geq 7$,*

$$\Lambda^* = \frac{\lambda \mu q}{(q-3)(q-5)}(q+2)(q^2+q-4)(q^2+q-6),$$
$$\delta^* = \frac{4q(q-1)(q^2-4)}{(q-3)(q-5)}\mu.$$

Further, if the 4-$(q+1, \frac{q+1}{2}, \mu)$ design is 2-resolvable with L resolution classes, then the 5-$(q^2+q+2, \frac{q(q+1)}{2}+1, \Lambda^*)$ design is 3-resolvable with NL resolution classes and each class is a 3-$(q^2+q+2, \frac{q(q+1)}{2}+1, \frac{\delta^*}{L})$ design.

In 1978, Magliveras conjectured that there will exist a large set of projective planes of order q for $q \geq 3$, provided q is the order of a projective plane. This conjecture is still an unsettled problem, except for $q = 3$, [8]. The main assumption of Theorems 4.1 and 5.2 is the existence of a 2-resolvable 4-$(q^2+q+1, q+1, \lambda)$ design as the outer design, whose resolution classes are projective planes of order q. In particular, if we take the complete 4-$(q^2+q+1, q+1, \binom{q^2+q-3}{q-3})$ design as the outer design, then the assumption is equivalent to the existence of a large set of projective planes of order q. To further clarify Theorems 4.1 and 5.2 we focus on this special case.

Consider case (i) with $q = 3$ of Theorem 5.2. The outer design becomes the 4-(13, 4, 1) design, which can be partitioned into $N = 55$ symmetric 2-(13, 4, 1) designs by [6] and [8].

Applying Theorem 4.1 with the 2-(4, 2, 1) inner design yields a 2-resolvable 4-(13, 6, 90) design with $N = 55$ resolution classes, where each class is a 2-(13, 6, 15) design. By Theorem 5.2, this 4-design is extendable to a 3-resolvable 5-(14, 7, 90) design with the same number of resolution classes and each class is a 3-(14, 7, 15) design. Note that both 4-(13, 6, 90) and 5-(14, 7, 90) designs are not simple, since the complete 4-(13, 6, $\lambda_{\max}$) and 5-(14, 7, $\lambda_{\max}$) design will have $\lambda_{\max} = 36$. However, they are also non-trivial, since 90 is not a multiple of 36. It should be remarked that the designs in both resolutions are simple. This is an interesting fact that we want to record in the following corollary.

Corollary 5.3 (i) *There is a non-trivial* 2*-resolvable* 4-(13, 6, 90) *design with repeated blocks having* $N = 55$ *resolution classes, where each class is a simple* 2-(13, 6, 15) *design.*
(ii) *There is a non-trivial* 3*-resolvable* 5-(14, 7, 90) *design with repeated blocks having* $N = 55$ *resolution classes, where each class is a simple* 3-(14, 7, 15) *design.*

Case (ii) with $q = 5$ displays another feature of Theorem 5.2. Assume that there is a partition of a 4-(31, 6, λ) outer design into projective planes of order 5. If $\lambda = \lambda_{\max} = 117 \times 3$, the constructed design will have parameters 5-(32, 16, 16380×39). Note that the index of this 5-design is much less than that of its corresponding complete 5-(32, 16, 334305×39) design. By contrast, if $\lambda = \lambda_{\min} = 3$, the index of the corresponding 5-(32, 16, Λ^*) constructed design would be drastically reduced to $\Lambda^* = 140 \times 39$. Further, since the 3-(6, 3, 1) inner design is 2-resolvable with $L = 2$ resolution classes, the number of 3-resolution classes of the constructed design is $NL = \frac{\lambda}{3}406$.

For some small values of q, for example $q = 7, 9, 11$, we may use the large sets $LS_5(2, 4, 8)$, $LS_{14}(2, 5, 10)$, $LS_{42}(2, 6, 12)$ for the inner designs. Thus, if there would exist a partition of 4-(q^2+q+1, $q+1$, λ) design into projective planes of order $q = 7, 9, 11$, then Theorems 5.2 would yield 3-resolvable 5-designs having parameters 5-(58, 29, $\lambda 63 \times 325$), 5-(92, 46, $\lambda 198 \times 903$), 5-(134, 67, $\frac{\lambda}{3}2002 \times 2016$) with $NL = \lambda 495,\ \lambda 1958,\ \frac{\lambda}{3}23842$ resolution classes, respectively.

6 An infinite series of 3-resolvable 5-designs derived from the 5-(14, 7, 90) design

In this short excursus we will focus on the 3-resolvable 5-(14, 7, 90) design in Corollary 5.3 and explain how to create an infinite series of 3-resolvable 5-designs from this single design. For the reader's convenience we include here a result in a recent paper by the author [19].

Corollary 6.1 (Corollary 3.4 [19]) *Suppose that there exists an s-resolvable t-(v, k, λ) design with N resolution classes such that $z = \frac{\lambda}{\binom{v-t}{k-t}} = \frac{Nu}{n}$, where u, n are positive integers. If there exists an $LS[n](k-2, k-1, v-1)$, then there exists an s-resolvable t-$(v + m(v-k+1), k, z\binom{v-t+m(v-k+1)}{k-t})$ design with N resolution classes for any $m \geq 0$.*

Observe the main fact of Corollary 6.1: it states that one can construct an infinite series of s-resolvable t-designs from a single t-design and a single large set. Now we will apply this recursive construction to the 5-(14, 7, 90) design in Corollary 5.3. As the design is 3-resolvable with 55 resolution classes, it is especially 3-resolvable with $N = 5$ resolution classes. The expression $z = \frac{\lambda}{\binom{v-t}{k-t}} = \frac{Nu}{n}$ becomes $z = \frac{5}{2}$, which implies that $n = 2$. Further, since an $LS[2](5, 6, 13)$ exists [9], there exists a 3-resolvable 5-$(14+8m, 7, 10(9+$

$8m)(1+m))$ design having $N = 5$ resolution classes for any $m \geq 0$ by Corollary 6.1. This design is obviously nonsimple, since the 5-$(14+8m, 7, \lambda_{\max})$ design will have $\lambda_{\max} = 4(9+8m)(1+m)$, however it is nontrivial, since $10(9+8m)(1+m)$ is not a multiple of $\lambda_{\max}$. We record the result in the following theorem.

Theorem 6.2 *There exists a 3-resolvable nonsimple and nontrivial* 5-$(14+8m, 7, 10(9+8m)(1+m))$ *design having* $N = 5$ *resolution classes for any* $m \geq 0$.

Moreover, it should be noted that there are at least two non-isomorphic series of 3-resolvable 5-$(14+8m, 7, 10(9+8m)(1+m))$ designs in Theorem 6.2 due to the existence of two non-isomorphic large sets $LS[55](2, 4, 13)$ as proven by Kolotoğlu and Magliveras [8].

7 Conclusion

The paper presents a method for constructing 2-resolvable t-designs for $t = 3, 4$ based on the assumption that there exists a partition of a t-design into Steiner 2-designs. The case $t = 4$ corresponds to partitioning a 4-design into projective planes. Especially, if the order of the projective planes is odd, it also enables to construct 3-resolvable 5-designs with a largest possible block size. In general, the method appears to be very effective, as it yields infinitely many 2-resolvable 3-designs with a large variety of blocks sizes. A study of simplicity of the constructed designs remains a challenging problem.

Funding Open Access funding enabled and organized by Projekt DEAL.

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