Uniqueness of the inversive plane of order sixty-four

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Abstract

The uniqueness of the inversive plane of order sixty-four, up to isomorphism, is established.
Equivalently, it is shown that every ovoid of PG(3, 64) is an elliptic quadric.

Keywords Inversive planes · Ovoids · Ovals

Mathematics Subject Classification 51E20 · 51E21

1 Introduction

An **inversive plane** is an incidence structure of points and circles such that:

(i) (i) every three distinct points are incident with a unique circle;
(ii) (ii) given two points P, Q and a circle C on P (but not on Q), there is a unique circle D incident with both P and Q whose only common point with C is P;
(iii) (iii) there are at least four points;
(iv) (iv) there is a non-incident point,circle pair; and
(v) (v) every circle is incident with a non-empty set of points. (See [12, pp. 252–253].)

When the inversive plane **I** is finite, there is an integer $n \geq 2$, called the **order** of **I** such that **I** has $n^2 + 1$ points, **I** has $n^3 + n$ circles and every circle of **I** is incident with $n + 1$ points of **I**. In fact, a finite inversive plane of order n is exactly a $3 - (n^2 + 1, n + 1, 1)$-design ([12, pp. 252–254]).

An **ovoid** of PG(3, q) is a set Ω of $q^2 + 1$ points, no 3 collinear, if $q > 2$; if $q = 2$, it is a set of 5 points, no 4 coplanar. A **secant plane** to an ovoid Ω is a plane meeting Ω in more than one point. (A **tangent plane** is a plane meeting Ω in a unique point.) The incidence structure $\mathbf{I}(\Omega)$ of points of Ω and plane sections by secant planes is an inversive plane of order q ([12, 6.1.2]). A finite inversive plane is **egglike** if it is isomorphic to $\mathbf{I}(\Omega)$, for some ovoid Ω of PG(3, q). All known finite inversive planes are egglike. Moreover, given ovoids Ω_1, Ω_2 of PG(3, q). $\mathbf{I}(\Omega_1)$ is isomorphic to $\mathbf{I}(\Omega_2)$ if and only if there is a collineation g of PG(3, q) with $\Omega_1^g = \Omega_2$.

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In 1963, Dembowski proved that every inversive plane of even order is egglike [11], and hence has order a power of two.

There are two known families of finite inversive planes: $\mathbf{I}(\Omega)$, where Ω is an elliptic quadric of PG(3, q), and $\mathbf{I}(\Omega)$, where Ω is a Tits ovoid of PG(3, 2^{2e+1}), $e \geq 1$ [39] (and these are not isomorphic). The inversive planes $\mathbf{I}(\Omega)$, where Ω is an elliptic quadric of PG(3, q), are called **Miquelian** for, by a result of van der Waerden and Smid [41], they are characterised by the Theorem of Miquel [12, 6.1.5]. The bundle theorem is a similar configuration to Miquel's Theorem [12, pp. 255–256]. It is satisfied by every egglike inversive plane [12, 6.1.4]. In 1980, Kahn proved the converse, so: an inversive plane is egglike if and only if it satisfies the bundle theorem [17].

Here we show that every inversive plane of order sixty-four is Miquelian (Corollary 2 in Sect. 4). Previously known results along these lines are : uniqueness of the inversive planes of orders 2, 3, 4 [42], 5 [8,13,37], 7 [14,37], 9 [35] and 16 [1,21,22] and classification of the inversive planes of orders 8 [1,15,29] and 32 [23]. (For order 4, see also [34].)

Ovoids have been used to construct maximal arcs (and thereby partial geometries) [36], unitals [7,19] and generalised quadrangles [12, p. 304], [25, 3.1.2]. Thus our results have consequences for enumerating Buekenhout–Metz unitals, Thas maximal arcs and Tits generalised quadrangles, which we will not dwell on here.

2 Background results

An **oval** of PG(2, q) is a set of $q + 1$ points, no three collinear. A line l is **external, tangent, secant** to O accordingly as $|l \cap O|$ is 0, 1 or 2. An example of an oval is a nondegenerate conic. Many other ovals are known in characteristic two; see [27] for the most recent survey. A 1955 result of Segre shows that the situation in odd characteristic is in strong contrast to that in characteristic two.

Theorem 1 [31] *An oval of* PG(2, q), *q odd, is a conic.*

Using this result, the same year, Barlotti and Panella independently classified ovoids of PG(3, q), q odd.

Theorem 2 [2,24] *An ovoid of* PG(3, q), *q odd, is an elliptic quadric.*

Barlotti proved a little more (and Segre proved a slightly stronger result four years later, and gave a far more explicit statement).

Theorem 3 [2, Sect. 3] [33, Theorem V] *An ovoid of* PG(3, q), *q even, is an elliptic quadric if and only if every secant plane section is a conic.*

The best result in this direction is that of Brown from 2000, although we will not need it here.

Theorem 4 [4] *An ovoid of* PG(3, q), *q even. is an elliptic quadric if and only if some secant plane section is a conic.*

Earlier, in 1963, Dembowski had shown that inversive planes of even order arise from ovoids.

Theorem 5 [11] *An inversive plane of even order is egglike, and so has order a power of two.*

3 Hyperovals in PG(2, 64)

Theorem 6 *U.G. Mitchell 1910* [20] *The tangent lines to a conic in* PG(2, q), *q even, are concurrent.*

So the union of a conic of PG(2, q), q even, and the point of concurrency of its tangent lines is a hyperoval, which is called a **regular hyperoval**.

By 1957 [32], it had been shown that all hyperovals of PG(2, 2), PG(2, 4) and PG(2, 8) are regular, and irregular hyperovals of PG(2, q) had been constructed for $q = 2^h$, $h = 5$ and $h \geq 7$. Segre raised the question of existence of irregular hyperovals in PG(2, 16) and PG(2, 64). The next year, Lunelli and Sce [18] constructed irregular hyperovals in PG(2, 16). Nearly four decades passed before the other question was settled [28] by the construction of two irregular hyperovals in PG(2, 64), one with a group of order 60; the other with a group of order 15. The last of the hyperovals in PG(2, 64) was constructed the following year [30]; it has a group of order 12. The hyperovals with groups of orders 60 and 15 were generalised to the infinite families of Subiaco hyperovals in [10] in 1996. The hyperoval with a group of order 12 was generalised to the infinite family of Adelaide hyperovals in [9] in 2003. In 2019 [40], hyperovals of PG(2, 64) were classified by Vandendriessche.

Theorem 7 *There are four isomorphism classes of hyperovals in* PG(2, 64).

The regular hyperoval gives rise to two ovals, the conic and the point conic. The Subiaco hyperoval with a group of order 60 gives rise to 3 ovals; the Subiaco hyperoval with a group of order 15 gives rise to 6 ovals and the Adelaide hyperoval gives rise to 8 ovals.

Corollary 1 *There are nineteen isomorphism classes of ovals in* PG(2, 64).

In more detail, the regular hyperoval contains representatives of two isomorphism classes of ovals- the conic and the pointed conic; the Subiaco hyperoval with a group of order 60 contains representatives of three isomorphism classes of ovals, with groups of orders 60, 12 and 1; the Subiaco hyperoval with a group of order 15 contains representatives of six isomorphism classes of ovals, one with a group of order 15, one with a group of order 3 and four with a group of order 1; and the Adelaide hyperoval contains representatives of eight isomorphism classes of ovals, two with a group of order 12, one with a group of order 3 and five with a group of order 1.

4 Ovoids in PG(3, 64)

In order to prove this result, we follow the strategy established in [23].

The set of secant plane sections of an ovoid of PG(3, q) on a tangent line ℓ is a called a **pencil** with **carrier** ℓ.

Let O_1 and O_2 be ovals of PG(2, q), q even and let P be a point not on either oval, nor equal to either nucleus. Then O_1 and O_2 are compatible at P if they have the same nucleus, they have a point Q in common, the line PQ is a tangent line to each oval and every secant line to O_1 on P is external to O_2 (and hence every external line to O_1 on P is secant to O_2).

In order to keep the treatment as synthetic as possible, we lift some of the ideas of [5].

An **augmented fan** of ovals of PG(2, q) is a set F of ovals with common nucleus N and common point Q indexed by the points (other than P) of a conic C on P with nucleus N so

that $F = \{O_X : X \in C, X \neq P\}$ such that O_X and O_Y are compatible at $XY \cap PN$, for all $X \neq Y \in C \setminus \{P\}$. C is called the **augmenting conic** of F.

Using this terminology, we can restate the Plane Equivalent Theorem without using coordinates.

Theorem 8 [Plane Equivalent Theorem] [23, Theorem 2.1] *An ovoid Ω in* PG$(3, q)$, *q even, is equivalent to an augmented fan* $\{O_X : X \in C, X \neq Q\}$ *of ovals of* PG$(2, q)$. *Moreover, for every tangent line ℓ each pencil with carrier ℓ gives rise to such a set and for each plane π of the pencil there is a parameterisation of the planes π_s of the pencil, $s \in GF(q)$ such that $\pi_0 = \pi$, a parameterisation of the points P_s of the augmenting conic, other than the common point, and there is a homography M_s taking $\pi_s \cap \Omega$ to O_{P_s} and the carrier line of the pencil to the line QN.*

We note that the Plane Equivalent Theorem [23, Theorem 2.1] was proved in [26] and [16] (independently), and first published in [21]. (As it was applied, in [5,6,22,23], it was (slightly) modified into a more useful form, and it is in that form that it is stated here, rather than the original form of [26].)

Theorem 9 [23, Theorem 3.1] *Let O be an oval of* PG$(2, q)$. *If an oval equivalent to O appears as a plane section of an ovoid of* PG$(3, q)$, *then for every tangent line ℓ of O, there is an augmented fan of ovals containing O. Furthermore, for every point P on ℓ off O, not the nucleus of O, one of the ovals of this fan is compatible with O at P.*

Lemma 1 [23, Lemma 3.2] *Let (O_1, P_1) and (O_2, P_2) be two oval-point pairs. If there is a collineation g such that $gP_2 = P_1$ and gO_2 is compatible with O_1 at P_1, then g maps the lines through P_2 secant to O_2 to the lines through P_1 external to O_1, and the one line through P_2 tangent to O_2 to the one line through P_1 tangent to O_1.*

Now the lines through any point P can be parameterised by a single non-homogeneous parameter in the following fashion: select two lines, say $[a_1, b_1, c_1]$ and $[a_2, b_2, c_2]$ and give them parameters ∞ and 0, respectively. Then the line $[\lambda a_1 + a_2, \lambda b_1 + b_2, \lambda c_1 + c_2]$ also passes through P and will be labelled λ. The lines through any point are thus parameterised with $GF(q) \cup \infty$, in the same way as PG$(1, q)$ is parameterised when (x, y) is given parameter x/y (with $1/0 = \infty$).

Any collineation g such that $gP_2 = P_1$ maps the lines through P_2 to the lines through P_1 and induces an element $g' \in P\Gamma L(2, q)$ on the corresponding sets of parameters. Returning to the situation of the two oval-point pairs (O_1, P_1) and (O_2, P_2) we can insist that the two tangent lines involved both be assigned the parameter ∞. The stabiliser of ∞ in $P\Gamma L(2, q)$ is the group $A\Gamma L(1, q)$. Therefore a collineation g such that $gP_2 = P_1$ and gO_2 is compatible with O_1 at P_1 induces an element $g' \in A\Gamma L(1, q)$ that maps the set of parameters of the external lines to O_2 through P_2 to the set of parameters of the secant lines to O_1 through P_1. The next theorem provides a converse to this result.

Theorem 10 [23, Theorem 3.3] *Let (O_1, P_1) and (O_2, P_2) be two oval-point pairs, with the lines through P_1 and P_2 parameterised in such a way that the tangent lines to the two ovals receive the parameter ∞. Let S_1 and E_2 be the sets of parameters of the secant lines to O_1 through P_1 and the external lines to O_2 through P_2, respectively. Then there exists a collineation g such that such that $gP_2 = P_1$ and gO_2 is compatible with O_1 at P_1 if and only if there is an element $g' \in A\Gamma L(1, q)$ such that $g'E_2 = S_1$.*

Now it is straightforward to see which ovals can appear together in an ovoid. For every oval-point pair (O, P), we parameterise the lines through P as described above and determine the *local secant parameter set L* as the set of parameters of the secant lines. Then two oval-point pairs (O_1, P_1) and (O_2, P_2) with local secant parameter sets L_1 and L_2 are said to *match* provided that there is some element $g' \in A\Gamma L(1, q)$ such that $g'L_1 = \{x \in GF(q) \colon x \notin L_2\}$. Two ovals can only appear together in the same pencil of an ovoid if they both have points at which the corresponding two oval-point pairs match.

5 The computational results

For each of the 19 ovals of PG(2, 64), the group stabilising each oval was computed, along with the orbits of this stabiliser on non-nucleus points off the ovals, and a list of all possible oval-point pairs was made.

The conic has a group of order 1,572,480 which has one orbit on points not on the hyperoval containing the conic; the pointed conic has a group of order 24,192 which has two orbits on points not on the hyperoval containing the pointed conic; the oval O with a group of order 60 has 76 orbits on points not on the hyperoval containing O; the oval O with a group of order 15 has 275 orbits on points not on the hyperoval containing O; the three ovals O with a group of order 12 have 351 orbits on points not on the hyperoval containing O; the two ovals O with groups of order 3 have 1375 orbits on points not on the hyperoval containing O; the ten ovals O with groups of order 1 have 4095 orbits on points not on the hyperoval containing O.

After this computation, there remains a list of 45107 oval-point pairs. For each oval-point pair, the local secant parameter set L was computed, and a canonical representative of the orbit of L under $A\Gamma L(1, 64)$ was stored (the representative was the lexicographically least local secant parameter set in the orbit). For each oval-point pair, a canonical representative of the orbit of the complement in $GF(64)$ of L was also stored. Two oval-point pairs match if and only if the canonical representative of the local secant parameter set of the first is actually equal to the canonical representative of the complement of the local secant parameter set of the second. All possible matches between oval-point pairs were thus determined.

Of the 19 ovals, 18 had points at which they do not match with anything at all; hence, none of these 18 ovals can be a secant plane section of an ovoid. The remaining oval is a conic. Hence, **every plane section of an ovoid of** PG(3, 64) **is a conic**.

Theorem 11 *Every ovoid of* PG(3, 64) *is an elliptic quadric.*

Proof This follows from the results above and Theorem 3.   □

6 Consequences

Theorem 12 *Every inversive plane of order* 64 *is Miquelian.*

Proof This follows from Theorems 11 and 5.   □

Corollary 2 *There is a unique inversive plane of order 64.*

Proof This follows from the result of van der Waerden and Smid mentioned in the introduction and the fact that any two elliptic quadrics of PG(3, q) are equivalent under a collineation of PG(3, q).   □

Corollary 3 *There is a unique* $3 - (4097, 65, 1)$ *design.*

Proof This is a reformulation of Corollary 2 in the language of design theory.   □

Corollary 4 *A symplectic translation plane of order* 4096 *with kernel containing* $GF(64)$ *is Desarguesian.*

Proof By applying Theorem 11 and [33, Theorem III], it follows that a symplectic spread of PG(3, 64) is regular, since the symplectic generalised quadrangle $W(q)$ is self-dual [25, 3.2.1] in characteristic 2.   □

A point P of a generalised quadrangle of order (s, t) is **3-regular** if

$$|\{P, Q, R\}^{\perp\perp}| = s + 1,$$

whenever Q and R are points that are not collinear, and both not collinear with P. The points and lines contained in an elliptic quadric of PG(5, q) form a generalised quadrangle of order (q, q^2), called $Q(5, q)$ [25, Sect. 3.1]. Our work yields two characterisations of this generalised quadrangle for $q = 64$.

Corollary 5 *A generalised quadrangle of order* (64, 4096) *with a 3-regular point is isomorphic to* $Q(5, 64)$.

Proof This is a consequence of Theorem 11 and [25, 5.3.1,3.2.4].   □

Corollary 6 *A generalised quadrangle of order* (64, 4096) *with a subquadrangle isomorphic to* $W(64)$ *is isomorphic to* $Q(5, 64)$.

Proof This is a consequence of Theorem 11 and [38, Theorem 7.1].   □

A generalised quadrangle has **Property (G) at a pair** $\{P, P'\}$ of collinear points if, every triple $\{P, Q, R\}$ of points, with P, Q, R pairwise non-collinear and $P' \in \{P, Q, R\}^{\perp}$, is 3-regular.

Corollary 7 *A generalised quadrangle of order* (64, 4096) *that satisfies Property (G) at a pair of distinct collinear points is the dual of a flock generalised quadrangle.*

Proof This is a consequence of Theorem 11 and [3, Theorem 25].   □

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