



The geometric field of linearity of linear sets

Dibyayoti Jena¹ · Geertrui Van de Voorde¹

Received: 18 July 2021 / Revised: 17 November 2021 / Accepted: 12 January 2022 /

Published online: 3 February 2022

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Abstract

If an \mathbb{F}_q -linear set L_U in a projective space is defined by a vector subspace U which is linear over a proper superfield of \mathbb{F}_q , then all of its points have weight at least 2. It is known that the converse of this statement holds for linear sets of rank h in $\text{PG}(1, q^h)$ but for linear sets of rank $k < h$ the converse of this statement is in general no longer true. The first part of this paper studies the relation between the weights of points and the size of a linear set, and introduces the concept of the *geometric field of linearity* of a linear set. This notion will allow us to show the main theorem, stating that for particular linear sets without points of weight 1, the converse of the above statement still holds as long as we take the geometric field of linearity into account.

Keywords Linear set · Projective line · Weight of a point · Geometric field of linearity

Mathematics Subject Classification 51E20

1 Sets and their linearity

1.1 Linear sets

Linear sets in finite projective spaces are a generalization of subgeometries. They have attracted a lot of attention in the last few years. Their applications include construction of other mathematical objects like blocking sets [17], translation ovoids [15], KM-arcs [7] and caps [2]. They have also proven useful in the study of semifields [12] and rank metric codes [14, 19, 21, 24]. More details about linear sets can be found in [13, 18].

Communicated by G. Lunardon.

Dibyayoti Jena and Geertrui Van de Voorde are supported by the Marsden Fund Council administered by the Royal Society of New Zealand.

✉ Geertrui Van de Voorde
geertrui.vandevoorde@canterbury.ac.nz

¹ University of Canterbury, Christchurch, New Zealand

More formally, let \mathbb{F}_{q^h} denote the finite field of order q^h , where q is a prime power, and let V be an r dimensional vector space over \mathbb{F}_{q^h} . The vector space V can also be seen as an rh dimensional vector space over \mathbb{F}_q . An \mathbb{F}_q -linear set L of rank k in $\text{PG}(V) = \text{PG}(r - 1, q^h)$ is the set of points defined by a k -dimensional \mathbb{F}_q -subspace U of V in the following way:

$$L = L_U = \{ \langle u \rangle_{q^h} \mid u \in U \setminus \{0\} \}.$$

Here $\langle u \rangle_{q^h}$ denotes the point of $\text{PG}(r - 1, q^h)$ determined by the vector u (where u is seen as a vector in V). If W is an \mathbb{F}_{q^h} -vector subspace of dimension s of V , then $\langle W \rangle_{q^h}$ denotes the projective $(s - 1)$ -dimensional subspace corresponding to U in $\text{PG}(V)$. To avoid confusion, the subspace spanned by two subspaces, say S_1, S_2 , of $\text{PG}(r - 1, q^h)$ will be denoted by $\text{span}(S_1, S_2)$, and likewise, the vector subspace spanned by two vector spaces V_1, V_2 will be denoted by $\text{span}(V_1, V_2)$.

The \mathbb{F}_q -weight of a point $P = \langle u_P \rangle_{q^h}$ in a linear set L_U is defined to be the vector space dimension of the \mathbb{F}_q -subspace U_P of all the vectors determining P , i.e.

$$U_P = \{0\} \cup \{u \mid u \in U, \langle u \rangle_{q^h} = \langle u_P \rangle_{q^h}\}.$$

We will simply use the term *weight* of a point if the underlying field is clear. Note that we need to specify the underlying vector space U in the definition of the rank of a linear set and in the definition of the weight of a point in a linear set L_U ; this will become more clear later in this paper.

The isomorphism between $V = \mathbb{F}_{q^h}^r$ and $V = \mathbb{F}_q^{rh}$ induces a natural map ϕ , called the *field reduction map*, from $\text{PG}(r - 1, q^h)$ to $\text{PG}(rh - 1, q)$. The map ϕ takes points to $(h - 1)$ -dimensional subspaces, and in general $(n - 1)$ -dimensional subspaces in $\text{PG}(r - 1, q^h)$ to $(nh - 1)$ -dimensional subspaces of $\text{PG}(rh - 1, q)$. The images of points under ϕ form a *Desarguesian* $(h - 1)$ -spread \mathcal{D} of $\text{PG}(rh - 1, q)$, and every subfield \mathbb{F}_{q^s} of \mathbb{F}_{q^h} gives rise to a unique Desarguesian $(s - 1)$ -spread partitioning the elements of \mathcal{D} . Rank- k linear sets L_U can be viewed geometrically as the pre-image under ϕ of sets of elements of \mathcal{D} intersecting a fixed $(k - 1)$ -dimensional projective subspace $\langle U \rangle_q$ of $\text{PG}(rh - 1, q)$. The \mathbb{F}_q -weight of a point P , from this point of view, is one more than the dimension of $\phi(P) \cap \langle U \rangle_q$. It is well known (and it follows easily from the above point of view) that a Desarguesian spread is *normal*: if $M, N \in \mathcal{D}$, then the subspace $\langle M, N \rangle$ is partitioned by elements of \mathcal{D} .

The original point of view on linear sets was given by [15] using *projections*. Let $\Sigma^* = \text{PG}(k - 1, q^h)$, $\Sigma = \text{PG}(k - 1, q)$ be a canonical subgeometry of Σ^* , and $\Omega = \text{PG}(r - 1, q^h)$ be an $(r - 1)$ -dimensional subspace of Σ^* . For a $(k - r - 1)$ -dimensional subspace Π of Σ^* disjoint from Σ and Ω , the *projection map* $\mathcal{P}_{\Pi, \Omega} : \Sigma \rightarrow \Omega$ is defined by $\mathcal{P}_{\Pi, \Omega}(x) = \langle x, \Pi \rangle \cap \Omega$. The image of Σ under $\mathcal{P}_{\Pi, \Omega}$ is the *projection* of Σ from Π onto Ω . It was proven in [15, Theorems 1 and 2] that any linear set L of rank k in $\Omega = \text{PG}(r - 1, q^h)$ is either a canonical subgeometry of Ω or equivalent to a projection of $\Sigma = \text{PG}(k - 1, q)$, a canonical subgeometry of $\Sigma^* = \text{PG}(r - 1, q^h)$, from Π to Ω , where Π is a $(k - r - 1)$ -dimensional subspace of Σ^* disjoint from Σ and Ω . From this point of view, the \mathbb{F}_q -weight of a point P is one more than the dimension of the pre-image of P under the projection map (see also [21, Proposition 2.7]).

1.2 The geometric field of linearity of a linear set

In this paper, we will distinguish between sets being \mathbb{F}_{q^s} -linear sets, and having *geometric field of linearity* \mathbb{F}_{q^s} . We will explain the reason for this distinction in detail but start with the definitions as used in this paper.

Definition 1.1 An \mathbb{F}_q -linear set L_U , defined by an \mathbb{F}_q -vector space U , is an \mathbb{F}_{q^s} -linear set if U is an \mathbb{F}_{q^s} -vector space (i.e. if the set of vectors in U is also closed under taking \mathbb{F}_{q^s} -multiples.)

Definition 1.2 A strictly \mathbb{F}_{q^s} -linear set is an \mathbb{F}_{q^s} -linear set that is not an \mathbb{F}_{q^i} -linear set for any $i > s$.

We also say (as defined in [5]) that the *maximum field of linearity* of a strictly \mathbb{F}_{q^s} -linear set L_U is \mathbb{F}_{q^s} .

We introduce the following definitions in this paper:

Definition 1.3 An \mathbb{F}_q -linear set L_U has *geometric field of linearity* \mathbb{F}_{q^s} if there exists an \mathbb{F}_{q^s} -linear set L_W with $L_U = L_W$.

Definition 1.4 An \mathbb{F}_q -linear set L_U has *maximum geometric field of linearity* \mathbb{F}_{q^s} if s is the largest integer such that L_U has geometric field of linearity \mathbb{F}_{q^s} .

It follows from the definitions that the order of the maximum geometric field of linearity is at least the order of the maximum field of linearity of that set. The following example describes a strictly \mathbb{F}_q -linear set with a geometric field of linearity different from \mathbb{F}_q , showing that the notions of being \mathbb{F}_{q^s} -linear and having geometric field of linearity \mathbb{F}_{q^s} do not always coincide.

Example 1.5 Consider the set S of points of the subline $\text{PG}(1, q^3)$ contained in $\text{PG}(1, q^6)$. The set S equals L_U where $U = \{(a, b) \mid a, b \in \mathbb{F}_{q^3}\}$ and as such, we see that $S = L_U$ is an \mathbb{F}_{q^3} -linear set of rank 2 over \mathbb{F}_{q^3} and an \mathbb{F}_q -linear set of rank 6.

Now consider any vector subspace V of U of dimension 5. Then $L_U = L_V$ and L_V is an \mathbb{F}_q -linear set of rank 5.

Now V is not \mathbb{F}_{q^i} -linear for any $i > 1$, and hence, L_V is a strictly \mathbb{F}_q -linear set. But it is clear that L_V behaves as an \mathbb{F}_{q^3} -linear set (as it is in fact simply a subline $\text{PG}(1, q^3)$). And indeed, according to our definition, L_V has geometric field of linearity \mathbb{F}_{q^3} .

In many of the well-studied cases, there is no distinction between linear sets being \mathbb{F}_{q^s} -linear and having geometric field of linearity \mathbb{F}_{q^s} . In particular, we will see in Proposition 1.10 that this is the case for linear sets of rank h in $\text{PG}(1, q^h)$. For *simple* linear sets, the same holds true as seen in the following Remark.

Remark 1.6 A *simple* \mathbb{F}_q -linear set L_U in $\text{PG}(1, q^h)$ (or an \mathbb{F}_q -linear set of *class* 1 as introduced in [5]) has the property that if $L_U = L_V$, then $U = \lambda V$ for some $\lambda \in \mathbb{F}_{q^h}^*$. Since U is \mathbb{F}_{q^s} -linear if and only if λU is \mathbb{F}_{q^s} -linear, it immediately follows from the definition that for simple \mathbb{F}_q -linear sets, the maximum field of linearity and the maximum geometric field of linearity coincide.

The proof of Proposition 1.10 is essentially a corollary of the following theorem which discusses the number of directions determined by a function over a finite field. Note that the *graph* of a function f is defined as the subspace $\{(x, f(x)) \mid x \in \mathbb{F}_{q^h}\}$ of $(\mathbb{F}_{q^h})^2 \cong \text{AG}(2, q^h)$.

Theorem 1.7 ([1, 4]) *Let $f : \mathbb{F}_{q_0} \rightarrow \mathbb{F}_{q_0}$ be a function, q_0 a power of the prime p . Let N be the number of directions determined by f . Let $r = p^s$ be maximal such that any line with a direction determined by f that is incident with a point of the graph of f is incident with a multiple of r points of the graph of f . Then one of the following holds:*

1. $r = 1$ and $\frac{q_0+3}{2} \leq N \leq q_0 + 1$;

2. \mathbb{F}_r is a subfield of \mathbb{F}_{q_0} and $\frac{q_0}{r} + 1 \leq N \leq \frac{q_0-1}{r-1}$;
3. $r = q_0$ and $N = 1$.

Moreover, if $r > 2$, then the graph of f is \mathbb{F}_r -linear.

Remark 1.8 The previous theorem says that if $r > 2$, the point set of the graph of f is \mathbb{F}_r -linear, which the authors show by proving that f is an \mathbb{F}_r -linear map. It is clear that $\{(x, f(x)) \mid x \in \mathbb{F}_{q^h}\}$ is indeed \mathbb{F}_r -linear if and only if f is an \mathbb{F}_r -linear map.

In Proposition 2.3 of [5], the following is shown, using Theorem 1.7:

Result 1.9 Let L_U be an \mathbb{F}_q -linear set of $\text{PG}(W, q^h) = \text{PG}(1, q^h)$ of rank h . The maximum field of linearity of L_U is \mathbb{F}_{q^s} where $s = \min\{w_{L_U}(P) : P \in L_U\}$.

If the maximum field of linearity is \mathbb{F}_q , then the rank of L_U as an \mathbb{F}_q -linear set is uniquely defined, i.e. for each \mathbb{F}_q -subspace V of W , if $L_V = L_W$, then $\dim_q(V) = h$.

The proof of the following proposition goes along the same lines as the proof of Result 1.9.

Proposition 1.10 Let L_U be an \mathbb{F}_q -linear set of rank h in $\text{PG}(1, q^h)$. Then L_U is a strictly \mathbb{F}_{q^s} -linear set if and only if L_U has maximum geometric field of linearity \mathbb{F}_{q^s} .

Proof Let L_U be an \mathbb{F}_q -linear set of rank h in $\text{PG}(1, q^h)$, where U is strictly \mathbb{F}_{q^s} -linear. Every \mathbb{F}_q -linear set of rank h in $\text{PG}(1, q^h)$ can be mapped by an element of $\text{PGL}(1, q^h)$ to a linear set not containing the point $\langle(0, 1)\rangle_{q^h}$, and hence, be written as a set of points L_U with $U = \{(x, f(x)) \mid x \in \mathbb{F}_{q^h}\}$ for some \mathbb{F}_q -linear map f which is strictly \mathbb{F}_{q^s} -linear.

Since U is strictly \mathbb{F}_{q^s} -linear we have that every vector line through two vectors of U contains a multiple of q^s points of U but there is no $i > s$ such that every vector line through two vectors of U contains a multiple of q^i points of U . By Theorem 1.7 we have that the number of directions determined by f , and hence, the number of points in L_U is contained in $[q^{h-s} + 1, \frac{q^h-1}{q^s-1}]$. Assume that there exists an $L_V = L_U$ such that V is $\mathbb{F}_{q^{s'}}$ -linear for some $s' > s$.

If the rank of V is h , then $L_V = \{\langle(x, g(x))\rangle_{q^h} \mid x \in \mathbb{F}_{q^h}\}$ for some $\mathbb{F}_{q^{s'}}$ -linear map g . Similarly, by Theorem 1.7 the number of points in L_V is then contained in $[q^{h-s'} + 1, \frac{q^h-1}{q^{s'}-1}]$. It follows that $s = s'$, a contradiction. If the rank of V is larger than h , then L_V is the entire line $\text{PG}(1, q^h)$, a contradiction. Now assume that V has rank smaller than h . Since V is $\mathbb{F}_{q^{s'}}$ -linear, the rank of V is a multiple of s' , so it follows that the rank of V is at most $h - s'$.

This means that L_V has at most $\frac{q^{h-s'}-1}{q-1}$ points, and we know that the number of points in L_U is contained in $[q^{h-s} + 1, \frac{q^h-1}{q^s-1}]$. Since $s' > s$, this is a contradiction. This argument shows that if an \mathbb{F}_q -linear set L_U of rank h is strictly \mathbb{F}_{q^s} -linear, its maximum field of linearity is \mathbb{F}_{q^s} .

Vice versa, suppose that L_U is an \mathbb{F}_q -linear set of rank h with maximum field of linearity \mathbb{F}_{q^s} , that is, such that there exists some $L_V = L_U$ where V is \mathbb{F}_{q^s} -linear. The above argument shows that if L_V has rank h , then U is indeed \mathbb{F}_{q^s} -linear. So suppose that L_V has rank smaller than h , then we know that the rank of V is at most $h - s$ and the number of points in L_V , and hence, in L_U is at most $\frac{q^{h-s}-1}{q-1}$. Since U has rank h , this implies, as above, that U is (at least) \mathbb{F}_{q^s} -linear. Since the maximum field of linearity of L_U is \mathbb{F}_{q^s} it follows that U is strictly \mathbb{F}_{q^s} -linear. □

Remark 1.11 Care needs to be taken when using the terminology on the linearity of linear sets as used by Sziklai in [22]: we believe that the maximum field of linearity as used by the author should correspond to our definition of maximum geometric field of linearity. In particular, the author conjectures in [22] that a linear set with rank $t + 1$ in $\text{PG}(2, p^h)$ with “maximum field of linearity \mathbb{F}_{p^e} ” (between quotation marks) has at least $(p^e)^t + (p^e)^{t-1} + 1$ points.

Now consider, similar to Example 1.5, the point set L_U in $\text{PG}(2, q^4)$ defined by the \mathbb{F}_{q^2} -vector space $U = \{(a, b, c) \mid a, b, c \in \mathbb{F}_{q^2}\}$. We see that L_U determines a Baer subplane $\text{PG}(2, q^2)$ and has $q^4 + q^2 + 1$ points. Now $L_U = L_V$ with $V = \{(a, b, c) \mid a \in \mathbb{F}_q, b, c \in \mathbb{F}_{q^2}\}$ where L_V has rank 5 and has maximum field of linearity \mathbb{F}_q . Since $q^4 + q^2 + 1 < q^4 + q^3 + 1$ this example violates the lower bound predicted by the above conjecture. However, this linear set has geometric field of linearity \mathbb{F}_{q^2} , and we believe this is what the author intended when writing the “maximum field of linearity”. Elsewhere in the paper, the (*geometric*) *exponent* of a point P (in a blocking set B in $\text{PG}(2, p^h)$) is defined as the maximum integer e such that all lines through P meet the set B in $1 \pmod{p^e}$ points. Furthermore, it is shown that for a blocking set where all points have exponent e , \mathbb{F}_{p^e} is a subfield of \mathbb{F}_{p^h} and conjectured that the set B is an \mathbb{F}_{p^e} -linear set. Note that in our example above, every line through a point P of L_V meets L_V in $1 \pmod{p^2}$ points, again indicating that \mathbb{F}_{p^2} is the intended “maximum field of linearity”, rather than the field \mathbb{F}_q that follows from the definition. We choose the terminology for *geometric* field of linearity for its similarity with the distinction Sziklai makes between the *algebraic* and *geometric* exponent of a point set.

1.3 The minimum size of a linear set and points of weight at least two

We now link the field of linearity of a linear set with the weights of its points. The following lemma is easy to see.

Lemma 1.12 *Let L_U be an \mathbb{F}_{q^s} -linear set, then all points have \mathbb{F}_q -weight at least s .*

Proof The weight of $P = \langle u_P \rangle_{q^h}$ is the \mathbb{F}_q -dimension of the space

$$U_P = \{0\} \cup \{u \mid u \in U, \langle u \rangle_{q^h} = \langle u_P \rangle_{q^h}\}.$$

Since U is closed under \mathbb{F}_{q^s} -multiplication, for any $u \in U$, also $\beta u \in U$ where $\beta \in \mathbb{F}_{q^s}$, and since $\langle u \rangle_{q^h} = \langle \beta u \rangle_{q^h}$, we see that U_P is at least s -dimensional. \square

The converse of Lemma 1.12 is not true as seen from the example below.

Example 1.13 Consider again Example 1.5. We see that $S = L_U$ is an \mathbb{F}_{q^3} -linear set of rank 2 over \mathbb{F}_{q^3} and an \mathbb{F}_q -linear set of rank 6. Every point of L_U has weight 3, as predicted by Lemma 1.12. Every point of L_V still has weight at least 2, but L_V is not an \mathbb{F}_{q^2} -linear, nor \mathbb{F}_{q^3} -linear set (since V is a strictly \mathbb{F}_q -linear vector space of dimension 5). However L_V has geometric field of linearity \mathbb{F}_{q^3} . The main theorem of this paper will show that this behaviour is not a coincidence.

The research of this paper was originally motivated by the link with the minimum size of a linear set, which we will now introduce.

First note that the following lower bound immediately follows from Result 1.7.

Corollary 1.14 *A strictly \mathbb{F}_q -linear set of rank h in $\text{PG}(1, q^h)$ has at least $q^{h-1} + 1$ points.*

Recently, in [6, Theorem 3.7], the lower bound $q^{h-1} + 1$ for linear sets of rank h in $\text{PG}(1, q^h)$ was generalised as follows:

Result 1.15 *A linear set of rank k in $\text{PG}(1, q^h)$ containing at least one point of weight one has at least $q^{k-1} + 1$ points.*

Example 1.16 Consider again the linear set S from Example 1.5. We see that S is a set of $q^3 + 1$ points, but when considering S as L_U , then it is a linear set of rank 6 all its points having weight at least 3. When considering S as L_V , where $V = \{(a, b) \mid a \in \mathbb{F}_q, b \in \mathbb{F}_{q^3}\}$, then L_V is a linear set of rank 4 with one point of weight 3 (namely $\langle(0, 1)\rangle_{q^6}$) and q^3 points of weight one. We see that L_V indeed reaches the lower bound for linear sets of rank 4 containing at least one point of weight 1. This example also shows that we cannot simply remove the hypothesis that there is a point of weight one in Result 1.15.

Even though every linear set L_V can be written as a linear set L_U containing at least one point of weight 1 (by taking a subspace U of V of the correct dimension), the statement of Result 1.15 makes clear why we are interested in linear sets without points of weight one.

We also see that for a rank- $k \leq h$ linear set in $\text{PG}(1, q^h)$ to have a size lower than $q^{k-1} + 1$, all the points in it must have weight at least 2. Up to our knowledge, the only constructions of \mathbb{F}_q -linear sets with all points of weight at least 2 are obtained by considering linear sets that have geometric field of linearity \mathbb{F}_{q^s} (as done in Example 1.5).

It follows from Result 1.9 that for linear sets of rank $k = h$ in $\text{PG}(1, q^h)$ this is the only way to obtain sets with only points of weight at least 2.

But we have seen in Example 1.13 that we cannot hope that the converse of Lemma 1.12 holds true in general. We will show that the following weaker version holds for a particular class of examples: if all their points have weight more than 1, then they have geometric field of linearity \mathbb{F}_{q^s} for some $s > 1$.

More precisely, we will show the following (see Theorem 3.12).

Main Theorem *If L is an \mathbb{F}_q -linear set of rank k , $4 \leq k \leq h$, in $\text{PG}(1, q^h)$ with one point P of weight $w \geq 2$ and all other points of the same weight $k - w \geq 2$ then L has geometric field of linearity \mathbb{F}_{q^s} with $s \mid w$, $s > 1$, $s \mid h$ and $s \geq k - w$.*

Note that we are not claiming that \mathbb{F}_{q^s} is the *maximum geometric field of linearity* of the set, see Remark 3.10.

2 Linear sets as projections

2.1 Π -lines and their type

Recall that a rank- k linear set L in $\Omega = \text{PG}(r - 1, q^h)$ can be viewed as a projection of $\Sigma = \text{PG}(k - 1, q)$, a canonical subgeometry of $\Sigma^* = \text{PG}(k - 1, q^h)$, from a subspace $\Pi = \text{PG}(k - r - 1, q^h)$ disjoint from Σ and Ω . From now on, the notations $L, \Sigma, \Sigma^*, \Omega$ and Π will refer to these unless stated otherwise.

Furthermore, when we consider a point $P \in \text{PG}(s, q)$ we will assume $P = \langle \mathbf{v} \rangle_{q^h} = \langle (\lambda_1, \lambda_2, \dots, \lambda_{s+1}) \rangle_{q^h}$ where $\lambda_1, \dots, \lambda_{s+1} \in \mathbb{F}_q$, i.e. the vector \mathbf{v} representing the point P in the subgeometry Σ must have all coordinates in \mathbb{F}_q . In what follows, all points will be considered as points in Σ^* so we will drop the subscript $_{q^h}$ if there is no ambiguity.

The following definitions will be helpful in providing more structure to the linear sets that we are going to explore.

Definition 2.1 A line ℓ of Σ is said to be a Π -line if its extension in Σ^* intersects Π . A point of Π lying on the extension of a Π -line is a point of rank 2.

Definition 2.2 (see also [8]) Let Q be a point of Π of rank 2 lying on the extension of a Π -line ℓ containing two points $P_1 = \langle v_1 \rangle$ and $P_2 = \langle v_2 \rangle$, where $P_1, P_2 \in \Sigma$, then $Q = \langle v_1 - \alpha v_2 \rangle$ for some $\alpha \in \mathbb{F}_{q^h} \setminus \mathbb{F}_q$. We say that Q has type S_α , where

$$S_\alpha := \left\{ \frac{a\alpha + b}{c\alpha + d} \mid a, b, c, d \in \mathbb{F}_q, ad \neq bc \right\}.$$

We also say that the line ℓ is of type S_α .

It follows from [8, Lemma 2.2] that the above notion of type is well-defined: for a point Q of rank 2, if $Q = \langle v_1 - \alpha v_2 \rangle = \langle v'_1 - \alpha' v'_2 \rangle$, then $\alpha' = \frac{a\alpha + b}{c\alpha + d}$ for some $a, b, c, d \in \mathbb{F}_q, ad \neq bc$. Vice versa, if $Q = \langle v_1 - \alpha' v_2 \rangle$ with $\alpha' \in S_\alpha$, then there exist v'_1, v'_2 such that $Q = \langle v'_1 - \alpha v'_2 \rangle$.

Note that the set S_α is simply the orbit of α under the natural action of $\text{PGL}(2, q)$ on the elements of $\mathbb{F}_{q^h} \setminus \mathbb{F}_q$. The following lemma then easily follows.

Lemma 2.3 If $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = 2$, then $|S_\alpha| = q^2 - q$. If $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] > 2$, then $|S_\alpha| = q^3 - q$.

Proof Let $\alpha \in \mathbb{F}_{q^h}$ with $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = 2$ and let α' be in $\mathbb{F}_q(\alpha) \setminus \mathbb{F}_q$. Note that, since $\mathbb{F}_q(\alpha)$ is 2-dimensional over \mathbb{F}_q , $|\mathbb{F}_q(\alpha) \setminus \mathbb{F}_q| = q^2 - q$ and $\alpha' = \frac{a\alpha + b}{c\alpha + d}$, for some $a \neq 0 \in \mathbb{F}_q, b \in \mathbb{F}_q$. Hence, $\alpha' = \frac{a\alpha + b}{c\alpha + d}$ for $c = 0, d = 1$. Since $ad - bc \neq 0, \alpha' \in S_\alpha$, the first statement follows.

Secondly, let $\alpha \in \mathbb{F}_{q^h}$ with $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] > 2$. Since $|\text{PGL}(2, q)| = q^3 - q, S_\alpha$ has at most $q^3 - q$ elements. It is easy to see that if $\frac{a\alpha + b}{c\alpha + d} = \frac{a'\alpha + b'}{c'\alpha + d'}$ for different elements of $\text{PGL}(2, q)$ defined by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$, then α satisfies a non-vanishing quadratic equation with coefficients in \mathbb{F}_q , a contradiction. Hence, $|S_\alpha| = q^3 - q$. □

Note that since the sets S_α are orbits of elements in $\mathbb{F}_{q^h} \setminus \mathbb{F}_q$, the different sets S_β partition $\mathbb{F}_{q^h} \setminus \mathbb{F}_q$. We also see that if $\alpha' \in S_\alpha$, then $\lambda\alpha' \in S_\alpha$ for all $\lambda \in \mathbb{F}_q^*$. In what follows, we will consider the elements of S_α up to scalar multiples in \mathbb{F}_q^* .

Definition 2.4 Consider the cosets of \mathbb{F}_q^* in the set S_α and let \bar{S}_α denote this set of cosets:

$$\bar{S}_\alpha = \{ \{ \lambda\alpha' \mid \lambda \in \mathbb{F}_q^* \} \mid \alpha' \in S_\alpha \}.$$

It follows from Lemma 2.3 that $|\bar{S}_\alpha| = q$ if $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = 2$, and $|\bar{S}_\alpha| = q^2 + q$ otherwise.

2.2 Π -lines and points of weight at least 2

Recall that every Π -line in Σ extends to a line in Σ^* containing precisely one point of rank 2 in Π . Vice versa, a rank-2 point lies on the extension of a unique Π -line: since Π -lines are contained in the subgeometry Σ , we see that if (the extensions in Σ^* of) two Π -lines intersect, they do so in Σ . We conclude that there is a bijection between the set of rank-2 points in Π and the set of Π -lines in Σ .

Furthermore, every Π -line ℓ gives rise to a unique point of weight at least 2 in the linear set L : all the points of ℓ are projected from Π onto the same point, say P , of Ω . We also say

that ℓ corresponds to the point P . More generally, a point of weight w in the linear set that is obtained from the projection of Σ corresponds to a $(w - 1)$ -dimensional subspace H of Σ ; all the lines in H are clearly Π -lines, but we will see that not all of them have the same type.

We conclude that there is a surjective mapping from the set of Π -lines to the set of points in L of weight more than 1. For more details about relation between rank-2 points and points of weight at least 2, see e.g. [21].

Finally, if $\alpha' \in S_\alpha$, then α' can be expressed as a rational function of α , so $\mathbb{F}_q(\alpha) = \mathbb{F}_q(\alpha')$. This ensures that the statement of the following lemma is well-defined.

Lemma 2.5 *Let $P = \langle \mathbf{v} \rangle \in \Sigma$ be a point lying on a Π -line ℓ of type S_α . If $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = 2$, then ℓ is the unique Π -line through P of type S_α . If $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] > 2$, then there are at most $q + 1$ Π -lines of type S_α through P . Moreover, if there are $q + 1$ Π -lines of type S_α through P then for each $\gamma \in S_\alpha$ there is precisely one point $\langle \mathbf{u}_\gamma \rangle \in \Sigma$ such that $\langle \mathbf{v} - \gamma \mathbf{u}_\gamma \rangle \in \Pi$.*

Proof We first show that every point different from P on a Π -line through P gives rise to a unique element of \bar{S}_α , that is, to a unique coset of \mathbb{F}_q^* in S_α . Consider a point $Q = \langle \mathbf{u} \rangle$, $Q \neq P$, in Σ lying on a Π -line of type S_α through P , then by definition there is some $\alpha_1 \in S_\alpha$ such $R = \langle \mathbf{v} - \alpha_1 \mathbf{u} \rangle$ is a point of Π . Now $\alpha_1 \in S_\alpha$ lies in a unique element of \bar{S}_α . In order to see that the map sending Q to the coset of α_1 is well-defined, note that the only vectors \mathbf{u}' such that $Q = \langle \mathbf{u}' \rangle = \langle \mathbf{u} \rangle$ are of the form $\mathbf{u}' = \lambda \mathbf{u}$ for some $\lambda \in \mathbb{F}_q^*$. The point R is the unique point of Π on the line PQ . We see that $R = \langle \mathbf{v} - \alpha_2 \mathbf{u}' \rangle$, with $\alpha_2 = \frac{1}{\lambda} \alpha_1$ and hence, α_1 and α_2 indeed define the same element of \bar{S}_α .

We will now show that the above mapping takes different points on Π -lines of type S_α through P to different elements of \bar{S}_α . To this end, consider two different points, say $\langle \mathbf{u}_1 \rangle$ and $\langle \mathbf{u}_2 \rangle$ in Σ and suppose that $R_1 = \langle \mathbf{v} - \alpha' \mathbf{u}_1 \rangle$ and $R_2 = \langle \mathbf{v} - \lambda \alpha' \mathbf{u}_2 \rangle$ are points of Π . If $R_1 = R_2$, then it easily follows that $\langle \mathbf{u}_1 \rangle = \langle \mathbf{u}_2 \rangle$, a contradiction since we started with different points in Σ . If $R_1 \neq R_2$, we see that the line $R_1 R_2$ intersects Σ in the point $\langle \mathbf{u}_1 - \lambda \mathbf{u}_2 \rangle$, again a contradiction since Σ and Π are disjoint.

It follows that the number of points on Π -lines of type S_α through P is at most $|\bar{S}_\alpha|$. Since every line through P contains q points, different from P , it follows from Definition 2.4 that the number of Π -lines is at most 1 if $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = 2$ and at most $q + 1$ if $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] > 2$.

Now assume that there are precisely $q + 1$ Π -lines with type contained in S_α through P . Let $\beta_1, \dots, \beta_{|\bar{S}_\alpha|}$ be a set of representatives of each coset in \bar{S}_α . It follows from the above reasoning and the pigeonhole principle that each of the points on Π -lines through P is of the form $\langle \mathbf{v} - \beta_i \mathbf{u} \rangle$ and moreover, all β_i occur when describing these points. Now consider an element $\gamma \in S_\alpha$ and let β_j be the coset representative of γ , then $\gamma = \lambda \beta_j$ for some $\lambda \in \mathbb{F}_q^*$. The statement now follows from the observation that $\langle \mathbf{v} - \gamma \mathbf{u}' \rangle$ with $\mathbf{u}' = \frac{1}{\lambda} \mathbf{u}$ is the same point as $\langle \mathbf{v} - \beta_j \mathbf{u} \rangle$. □

Lemma 2.6 *Let H be a $(w - 1)$ -dimensional subspace of Σ corresponding to a point of weight $w \geq 3$ in $\Omega = \text{PG}(r - 1, q^h)$. If H contains at least $q^{w-1} + q^{w-2} + \dots + 1$ Π -lines of type S_α then the following hold:*

- (i) H contains precisely $q^{w-1} + q^{w-2} + \dots + 1$ Π -lines of type S_α
- (ii) $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = s > 2$ and $s \mid w$.

Proof (i) We count pairs (P, L) where P is a point of H and L is a Π -line of type S_α through P . Let a be the average number of Π -lines of type S_α through a point of H and b be the number of Π -lines of type S_α . Then

$$(q + 1)b = a|H|.$$

Now $|H| = q^{w-1} + q^{w-2} + \dots + 1$ and by Lemma 2.5, $a \leq q + 1$, and hence, $b \leq q^{w-1} + q^{w-2} + \dots + 1$. It follows that if $b \geq q^{w-1} + q^{w-2} + \dots + 1$, then indeed $b = q^{w-1} + q^{w-2} + \dots + 1$ and every point of H lies on exactly $q + 1$ Π -lines of type S_α .

(ii) Lemma 2.5 confirms that if every point of H lies on exactly $q + 1$ Π -lines of type S_α , we have that $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = s > 2$, and moreover, that for any point $P = \langle \mathbf{v} \rangle$ in H and any $\beta \in S_\alpha$, there exists a unique point $\langle \mathbf{u}_\beta \rangle \in H$ such that $\langle \mathbf{v} - \beta \mathbf{u}_\beta \rangle \in \Pi$.

Now let $Q = \langle \mathbf{v}_1 \rangle$ be an arbitrary point in H . Then, since $\alpha \in S_\alpha$, there exists a unique point $\langle \mathbf{v}_2 \rangle \in H$ such that $\langle \mathbf{v}_1 - \alpha \mathbf{v}_2 \rangle \in \Pi$. Similarly, given $\langle \mathbf{v}_2 \rangle$, there exists a unique point $\langle \mathbf{v}_3 \rangle \in H$ such that $\langle \mathbf{v}_2 - \alpha \mathbf{v}_3 \rangle \in \Pi$ and so on. Thus, the point $Q = \langle \mathbf{v}_1 \rangle$ gives rise in a unique way to an ordered set of points $\langle \mathbf{v}_2 \rangle, \langle \mathbf{v}_3 \rangle, \dots$ such that $\langle \mathbf{v}_i - \alpha \mathbf{v}_{i+1} \rangle \in \Pi$. We will show that the points $\langle \mathbf{v}_1 \rangle, \langle \mathbf{v}_2 \rangle, \dots$ obtained in this way are precisely the $\frac{q^s - 1}{q - 1}$ points forming an $(s - 1)$ -dimensional subspace of H .

First note that the extension of H to Σ^* intersects Π in a $(w - 2)$ -dimensional subspace H' of Π . As the points $\langle \mathbf{v}_1 - \alpha \mathbf{v}_2 \rangle, \langle \mathbf{v}_2 - \alpha \mathbf{v}_3 \rangle, \dots$ lie in H' , there can be a maximum of $w - 1$ such points whose corresponding vectors are linearly independent over \mathbb{F}_q . Suppose that the vectors $\mathbf{v}_1 - \alpha \mathbf{v}_2, \mathbf{v}_2 - \alpha \mathbf{v}_3, \dots, \mathbf{v}_{t-1} - \alpha \mathbf{v}_t$ are independent but that $\mathbf{v}_t - \alpha \mathbf{v}_{t+1}$ is dependent of the previous vectors. This implies that the points $\langle \mathbf{v}_1 - \alpha \mathbf{v}_2 \rangle, \langle \mathbf{v}_2 - \alpha \mathbf{v}_3 \rangle, \dots, \langle \mathbf{v}_{t-1} - \alpha \mathbf{v}_t \rangle$ span a $(t - 2)$ -dimensional subspace of H' containing the point $\langle \mathbf{v}_t - \alpha \mathbf{v}_{t+1} \rangle$.

We find that

$$\mathbf{v}_t - \alpha \mathbf{v}_{t+1} = \xi_1(\mathbf{v}_1 - \alpha \mathbf{v}_2) + \dots + \xi_{t-1}(\mathbf{v}_{t-1} - \alpha \mathbf{v}_t) \tag{1}$$

for some $\xi_1, \dots, \xi_{t-1} \in \mathbb{F}_{q^h}$. It follows that \mathbf{v}_{t+1} is an \mathbb{F}_{q^h} -linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_t$ but since these vectors have all have entries in \mathbb{F}_q , we have that $\mathbf{v}_{t+1} = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \dots + \lambda_t \mathbf{v}_t$ for some $\lambda_1, \dots, \lambda_t \in \mathbb{F}_q$. Note that the vectors $\mathbf{v}_1, \dots, \mathbf{v}_t$ are linearly independent: if they were dependent, the points $\langle \mathbf{v}_1 \rangle, \dots, \langle \mathbf{v}_t \rangle$ would be contained in a subspace of dimension $t - 2$ of H , which in turns yields that the points $\mathbf{v}_1 - \alpha \mathbf{v}_2, \mathbf{v}_2 - \alpha \mathbf{v}_3, \dots, \mathbf{v}_{t-1} - \alpha \mathbf{v}_t$ are contained in a $(t - 3)$ -dimensional subspace of H' , a contradiction.

From (1) we find that

$$\begin{aligned} -\alpha \lambda_1 &= \xi_1 \\ -\alpha \lambda_2 &= \xi_2 - \alpha \xi_1 \\ -\alpha \lambda_3 &= \xi_3 - \alpha \xi_2 \\ &\vdots \\ -\alpha \lambda_{t-1} &= \xi_{t-1} - \alpha \xi_{t-2} \\ 1 - \alpha \lambda_t &= -\alpha \xi_{t-1}. \end{aligned}$$

Eliminating ξ_1, \dots, ξ_{t-1} from the above system of equations yields that α must satisfy the equation

$$\alpha^t \lambda_1 + \alpha^{t-1} \lambda_2 + \dots + \alpha \lambda_t = 1.$$

As $\mathbb{F}_q(\alpha) = \mathbb{F}_{q^s}$ is a subfield of \mathbb{F}_{q^h} , and as α satisfies a degree- t polynomial, we must have $s \leq t \leq w$ and $s \mid h$.

We now show that $s = t$. In order to do so, let the minimal polynomial of $\alpha \in \mathbb{F}_{q^h}$ be

$$\alpha^s v_1 + \alpha^{s-1} v_2 + \dots + \alpha v_s = 1. \tag{2}$$

Consider a point $Q = \langle \mathbf{w}_1 \rangle$ of H , then we have seen before that there are unique points $\langle \mathbf{w}_2 \rangle, \dots, \langle \mathbf{w}_s \rangle$ such that $\langle \mathbf{w}_1 - \alpha \mathbf{w}_2 \rangle, \langle \mathbf{w}_2 - \alpha \mathbf{w}_3 \rangle, \dots, \langle \mathbf{w}_{s-1} - \alpha \mathbf{w}_s \rangle$ are points of H' . Furthermore, the above argument shows that $\langle \mathbf{w}_1 \rangle, \dots, \langle \mathbf{w}_s \rangle$ span an $(s - 1)$ -dimensional subspace of H . Denote this $(s - 1)$ -space by H_Q .

Note that

$$\begin{aligned} & \langle \mathbf{w}_1 - \alpha \mathbf{w}_2 + \alpha(\mathbf{w}_2 - \alpha \mathbf{w}_3) + \alpha^2(\mathbf{w}_3 - \alpha \mathbf{w}_4) + \dots + \alpha^{s-2}(\mathbf{w}_{s-1} - \alpha \mathbf{w}_s) \rangle \\ & = \langle \mathbf{w}_1 - \alpha^{s-1} \mathbf{w}_s \rangle \end{aligned}$$

is a point of H' . Similarly, it easily follows that the points $\langle \mathbf{w}_2 - \alpha^{s-2} \mathbf{w}_s \rangle, \dots, \langle \mathbf{w}_{s-1} - \alpha \mathbf{w}_s \rangle$ also lie in H' . This implies that the point

$$\begin{aligned} & \langle \mathbf{w}_s - \alpha(v_1 \mathbf{w}_1 + v_2 \mathbf{w}_2 + \dots + v_s \mathbf{w}_s) \rangle \\ & = \langle (\alpha^s v_1 + \alpha^{s-1} v_2 + \dots + \alpha v_s) \mathbf{w}_s - \alpha(v_1 \mathbf{w}_1 + v_2 \mathbf{w}_2 + \dots + v_s \mathbf{w}_s) \rangle \\ & = \langle \alpha v_1 (\alpha^{s-1} \mathbf{w}_s - \mathbf{w}_1) + \alpha v_2 (\alpha^{s-2} \mathbf{w}_s - \mathbf{w}_2) + \dots + \alpha v_{s-1} (\alpha \mathbf{w}_s - \mathbf{w}_{s-1}) \rangle \end{aligned}$$

is also contained in H' , where we have used (2) in the first equality. Since

$$\langle \mathbf{w}_s - \alpha(v_1 \mathbf{w}_1 + v_2 \mathbf{w}_2 + \dots + v_s \mathbf{w}_s) \rangle \in H'$$

and the point $\langle \mathbf{w}_{s+1} \rangle$ such that $\langle \mathbf{w}_s - \alpha \mathbf{w}_{s+1} \rangle \in \Pi$ is unique, we obtain that

$$\langle \mathbf{w}_{s+1} \rangle = \langle v_1 \mathbf{w}_1 + v_2 \mathbf{w}_2 + \dots + v_s \mathbf{w}_s \rangle,$$

which is clearly contained in H_Q .

Recall that $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_s$ form a basis for the s -dimensional \mathbb{F}_q -vector space determining H_Q . Hence, there is a unique \mathbb{F}_q -linear map ϕ which satisfies $\phi(\mathbf{w}_i) = \mathbf{w}_{i+1}$ for $i = 1, \dots, s - 1$ and $\phi(\mathbf{w}_s) = v_1 \mathbf{w}_1 + v_2 \mathbf{w}_2 + \dots + v_s \mathbf{w}_s$. With respect to the basis $\mathbf{w}_1, \dots, \mathbf{w}_s$, this map is represented by the matrix

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & v_1 \\ 1 & 0 & \dots & 0 & v_2 \\ 0 & 1 & \dots & 0 & v_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & v_s \end{bmatrix}.$$

From the definition of ϕ , we see that for $1 \leq i \leq s$, $\langle \mathbf{w}_i - \phi(\mathbf{w}_i) \rangle$ is a point of Π . But since ϕ is an \mathbb{F}_q -linear map, we have that for a point $\mathbf{w} = \mu_1 \mathbf{w}_1 + \dots + \mu_s \mathbf{w}_s$, with $\mu_i \in \mathbb{F}_q$,

$$\begin{aligned} \langle \mathbf{w} - \phi(\mathbf{w}) \rangle & = \langle \mu_1 \mathbf{w}_1 + \dots + \mu_s \mathbf{w}_s - \phi(\mu_1 \mathbf{w}_1 + \dots + \mu_s \mathbf{w}_s) \rangle \\ & = \langle \mu_1 \mathbf{w}_1 + \dots + \mu_s \mathbf{w}_s - \mu_1 \phi(\mathbf{w}_1) - \dots - \mu_s \phi(\mathbf{w}_s) \rangle \\ & = \langle \mu_1 (\mathbf{w}_1 - \phi(\mathbf{w}_1)) + \dots + \mu_s (\mathbf{w}_s - \phi(\mathbf{w}_s)) \rangle \end{aligned}$$

is a point of Π too. In other words, for each point $\langle \mathbf{w} \rangle \in H_Q$, $\langle \phi(\mathbf{w}) \rangle$ is the unique point \mathbf{w}' such that $\langle \mathbf{w} - \alpha \mathbf{w}' \rangle$ in Π . Since $\phi(H_Q) = H_Q$ we are done. (In fact, A is the companion matrix of the minimal polynomial for $1/\alpha$ and the map ϕ generates a Singer cycle on the points of H_Q).

It is clear from the construction that if for two points $Q, R \in H$, the spaces H_Q and H_R have a point in common, then H_Q and H_R coincide. Moreover, since every point S determines a subspace H_S , we find that H is partitioned by the $(s - 1)$ -spaces $H_P, P \in H$. The existence of this $(s - 1)$ -spread in the $(w - 1)$ -space H shows that $s \mid w$. □

Remark 2.7 It follows from the construction of Desarguesian spreads from Singer cycles that the $(s - 1)$ -spread of H , constructed in Lemma 2.6, is a Desarguesian spread (see e.g. [9, 23]). The fact that the spread is normal (which also implies Desarguesian if $w > 2s$) is easy to see geometrically: the extension of two different $(s - 1)$ -spread elements H_P and H_Q of H intersect Π in two disjoint $(s - 2)$ -subspaces, say H'_P and H'_Q . As for any point $R \in \text{span}(H_P, H_Q)$ the space H'_R in Π is a subspace of $\text{span}(H'_P, H'_Q)$, the spread element H_R must be in $\text{span}(H_P, H_Q)$.

2.3 Constructing Π -lines

Lemma 2.6 tells us what happens when there are many Π -lines of the same type. The following lemma provides a way to find such lines. The first part of this lemma was already proven for linear sets in $\text{PG}(1, q^5)$ in [8].

Lemma 2.8 *If there are three disjoint Π -lines ℓ_1, ℓ_2 and ℓ_3 in a 3-dimensional subspace H of Σ , whose extensions in Σ^* intersect Π in collinear points, contained in a line M of Π , then all the lines in the unique regulus \mathcal{R} defined by ℓ_1, ℓ_2, ℓ_3 are Π -lines of the same type.*

Moreover if there exists one more line ℓ in H , disjoint from all the lines of \mathcal{R} , whose extension in Σ^ is contained in M , then H is partitioned by Π -lines of the same type S_α with $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = 2$.*

Vice versa, if a 3-space H contains 3 disjoint Π -lines ℓ_1, ℓ_2, ℓ_3 of the same type S_α whose extensions in Σ^ are collinear, and $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = 2$, then the space H is partitioned by Π -lines of type S_α .*

Proof Let $\ell_1, \dots, \ell_{q+1}$ be the $q + 1$ lines of the regulus \mathcal{R} in H defined by ℓ_1, ℓ_2, ℓ_3 . Assume that the extensions of ℓ_1, ℓ_2, ℓ_3 , say $\bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3$, are contained in some line M of Π . Let \mathcal{R}' be the $q^h + 1$ lines of the regulus defined by $\bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3$ and let m_1, m_2, m_3 be transversals to \mathcal{R} . We then see that $\bar{m}_1, \bar{m}_2, \bar{m}_3$ are transversals to \mathcal{R}' in Σ . Furthermore, M intersects $\bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3$ so M is a transversal to \mathcal{R}' too. Since $\ell_i, 1 \leq i \leq q + 1$ intersects m_1, m_2, m_3 we have that $\bar{\ell}_i$ intersects $\bar{m}_1, \bar{m}_2, \bar{m}_3$, so $\bar{\ell}_i$ is an element of \mathcal{R}' . Since M is a transversal to \mathcal{R}' , we conclude that all lines $\bar{\ell}_i$ intersect the line M .

Now consider the points $\ell_i \cap m_1 = P_i, \ell_i \cap m_2 = Q_i$ and $\bar{\ell}_i \cap M = S_i, i = 1, \dots, q + 1$.

Let $P_1 = \langle \mathbf{u}_1 \rangle, Q_1 = \langle \mathbf{v}_1 \rangle, P_2 = \langle \mathbf{u}_2 \rangle, Q_2 = \langle \mathbf{v}_2 \rangle$. Note that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2$ are linearly independent over \mathbb{F}_{q^h} . Without loss of generality (since $\text{PGL}(4, q)$ acts transitively on the frames of H), we may assume that the point $\langle \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{v}_1 + \mathbf{v}_2 \rangle$ is a point of ℓ_3 . It then follows that the points $P_i, i = 3, \dots, q + 1$ are of the form $\langle \mathbf{u}_1 + \lambda_i \mathbf{u}_2 \rangle$ and Q_i of the form $\langle \mathbf{v}_1 + \lambda_i \mathbf{v}_2 \rangle$ where $\lambda_i \in \mathbb{F}_q^*$. We have that $S_1 = \langle \mathbf{u}_1 - \alpha \mathbf{v}_1 \rangle$ for some $\alpha \in \mathbb{F}_{q^h} \setminus \mathbb{F}_q$ and $S_2 = \langle \mathbf{u}_2 - \alpha' \mathbf{v}_2 \rangle$ for some $\alpha' \in \mathbb{F}_{q^h} \setminus \mathbb{F}_q$. Expressing that $\bar{\ell}_i$ intersects m_1, m_2 and M yields that there are $\xi_1, \xi_2, \psi \in \mathbb{F}_{q^h}$ such that

$$\xi_1(\mathbf{u}_1 - \alpha \mathbf{v}_1) + \xi_2(\mathbf{u}_2 - \alpha' \mathbf{v}_2) = \mathbf{u}_1 + \lambda_i \mathbf{u}_2 - \psi_i(\mathbf{v}_1 + \lambda_i \mathbf{v}_2).$$

Since $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linearly independent over \mathbb{F}_{q^h} , it follows that $\xi_1 = 1, \xi_2 = \lambda_i, \psi_i = \alpha$ and $\alpha' = \alpha$. It follows that $\psi_i = \alpha' = \alpha$. So all lines ℓ_i are of the same type S_α .

Now assume that there is an additional line ℓ in Σ , not contained in \mathcal{R} , whose extension $\bar{\ell}$ contains a point of M ; it follows from the first part that ℓ is a Π -line of type S_α . Thus there exist points $\langle \mathbf{u} \rangle$ and $\langle \mathbf{v} \rangle$ in ℓ with $\mathbf{u} = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{v}_1 + \lambda_3 \mathbf{u}_2 + \lambda_4 \mathbf{v}_2, \mathbf{v} = \mu_1 \mathbf{u}_1 + \mu_2 \mathbf{v}_1 + \mu_3 \mathbf{u}_2 + \mu_4 \mathbf{v}_2$ such that $\langle \mathbf{u} - \alpha \mathbf{v} \rangle \in M$ i.e.

$$\mathbf{u} - \alpha \mathbf{v} = \xi_1(\mathbf{u}_1 - \alpha \mathbf{v}_1) + \xi_2(\mathbf{u}_2 - \alpha \mathbf{v}_2). \tag{3}$$

Solving (3) by equating the coefficients of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1$ and \mathbf{v}_2 , we get

$$\mu_1\alpha^2 + (\mu_2 - \lambda_1)\alpha - \lambda_2 = 0 \tag{4}$$

and

$$\mu_3\alpha^2 + (\mu_4 - \lambda_3)\alpha - \lambda_4 = 0. \tag{5}$$

As $\alpha \notin \mathbb{F}_q, \alpha$ satisfies a degree-2 equation given by (4) or (5). Thus $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = 2$.

Now let the minimal polynomial of α over \mathbb{F}_q be

$$a\alpha^2 + b\alpha - 1 = 0. \tag{6}$$

As the Eqs. (4), (5) and (6) are multiples of each other, we must have $\mu_1 = a\lambda_2, \mu_2 = b\lambda_2 + \lambda_1, \mu_3 = a\lambda_4$ and $\mu_4 = b\lambda_4 + \lambda_3$.

We now see that a point $\langle \mathbf{u}' \rangle$ with $\mathbf{u}' = v_1\mathbf{u}_1 + v_2\mathbf{v}_1 + v_3\mathbf{u}_2 + v_4\mathbf{v}_2$ in H lies on a line of type S_α : consider the point $\langle \mathbf{v}' \rangle$ with $\mathbf{v}' = av_2\mathbf{u}_1 + (bv_2 + v_1)\mathbf{v}_1 + av_4\mathbf{u}_2 + (bv_4 + v_3)\mathbf{v}_2$, then

$$\begin{aligned} \langle \mathbf{u}' - \alpha\mathbf{v}' \rangle &= \langle v_1(\mathbf{u}_1 - \alpha\mathbf{v}_1) + v_2((1 - b\alpha)\mathbf{v}_1 - a\alpha\mathbf{u}_1) \\ &\quad + v_3(\mathbf{u}_2 - \alpha\mathbf{v}_2) + v_4((1 - b\alpha)\mathbf{v}_2 - a\alpha\mathbf{u}_2) \rangle \\ &= \langle v_1(\mathbf{u}_1 - \alpha\mathbf{v}_1) + v_2(a\alpha^2\mathbf{v}_1 - a\alpha\mathbf{u}_1) \\ &\quad + v_3(\mathbf{u}_2 - \alpha\mathbf{v}_2) + v_4(a\alpha^2\mathbf{v}_2 - a\alpha\mathbf{u}_2) \rangle \\ &= \langle (v_1 - a\alpha v_2)(\mathbf{u}_1 - \alpha\mathbf{v}_1) + (v_3 - a\alpha v_4)(\mathbf{u}_2 - \alpha\mathbf{v}_2) \rangle \in M. \end{aligned}$$

Hence every point in H is contained in a Π -line of type S_α . By Lemma 2.5 every point can lie on at most one Π -line of type S_α . Therefore H must be partitioned by Π -lines of type S_α .

Vice versa, suppose now that the 3-space H contains 3 disjoint Π -lines ℓ_1, ℓ_2, ℓ_3 of the same type S_α with $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = 2$ such that their extension meets Π in M . It follows from the first part of the proof that we can take points $\langle \mathbf{u}_1 \rangle, \langle \mathbf{v}_1 \rangle$ on ℓ_1 and $\langle \mathbf{u}_2 \rangle, \langle \mathbf{v}_2 \rangle$ on ℓ_2 such that the line M contains the points $S_1 = \langle \mathbf{u}_1 - \alpha\mathbf{v}_1 \rangle$ and $S_2 = \langle \mathbf{u}_1 - \alpha\mathbf{v}_2 \rangle$. Writing the minimal polynomial for α as $a\alpha^2 + b\alpha - 1 = 0$, we find again (as above) that for any point $\langle \mathbf{u}' \rangle$ with $\mathbf{u}' = v_1\mathbf{u}_1 + v_2\mathbf{v}_1 + v_3\mathbf{u}_2 + v_4\mathbf{v}_2$ we have that the point $\langle \mathbf{v}' \rangle$ with $\mathbf{v}' = av_2\mathbf{u}_1 + (bv_2 + v_1)\mathbf{v}_1 + av_4\mathbf{u}_2 + (bv_4 + v_3)\mathbf{v}_2$ satisfies $\langle \mathbf{u}' - \alpha\mathbf{v}' \rangle \in M$. And hence, the space H is indeed partitioned by Π -lines of type S_α . □

3 Linear sets on a line with all points of weight at least two

3.1 Finding a subfield

We now turn our attention towards linear sets contained in a line, i.e. $\Omega = \text{PG}(1, q^h)$ when the linear set L of rank $k, 4 \leq k \leq h$, in Ω has one point of weight $k - 2$ and all others of weight 2. In Lemma 3.1 and Theorem 3.3 we prove the existence of a proper subfield of \mathbb{F}_{q^h} and later, we will prove that L has this field as geometric field of linearity.

Lemma 3.1 *If L is a rank-4 linear set in $\text{PG}(1, q^h), h \geq 4$, with all points of weight 2, then \mathbb{F}_{q^h} contains the subfield \mathbb{F}_{q^2} (i.e. h is even).*

Proof A rank-4 linear set L in $\Omega = \text{PG}(1, q^h)$ can be viewed as a projection of $\Sigma = \text{PG}(3, q)$, a canonical subgeometry of $\Sigma^* = \text{PG}(3, q^h)$, from $\Pi = \text{PG}(1, q^h)$ onto Ω , where Π and

Ω are in Σ^* with Π disjoint from Σ and Ω . In this case all the $q^2 + 1$ weight-2 points in L correspond to different Π -lines of rank 2 each meeting the line Π in a point. Using Lemma 2.8 we see that all Π -lines are of type S_α with $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = 2$. □

Remark 3.2 It will follow from our more general approach that the linear sets considered in Lemma 3.1 have geometric field of linearity \mathbb{F}_{q^2} (see Main Theorem). In this particular case however, it is not too hard to see that the set L is in fact \mathbb{F}_{q^2} -linear and isomorphic to $\text{PG}(1, q^2)$. A different way to deduce this fact is to use a characterization of sublines (e.g. given in [20, Theorem 1.5]): a set of $q^2 + 1$ points in $\text{PG}(1, q^h)$ is a subline $\text{PG}(1, q^2)$ if and only if it is closed under taking \mathbb{F}_q -sublines determined by any three points of the set.

Theorem 3.3 *If L is a linear set of rank k , $5 \leq k \leq h$, in $\text{PG}(1, q^h)$ with one point of weight $k - 2$ and all other points of weight 2 then \mathbb{F}_{q^h} must contain a proper subfield \mathbb{F}_{q^s} with $s \mid k - 2, s > 1$.*

Proof Recall that L can be viewed as a projection of $\Sigma = \text{PG}(k - 1, q)$, a canonical subgeometry of $\Sigma^* = \text{PG}(k - 1, q^h)$, from $\Pi = \text{PG}(k - 3, q^h)$ onto $\Omega = \text{PG}(1, q^h)$, where Π and Ω are subspaces of Σ^* , with Π disjoint from Σ and Ω . We have seen that each of the weight-2 points determines a unique Π -line and the weight- $(k - 2)$ point corresponds to a $(k - 3)$ -dimensional subspace H of Σ . Denote the set of Π -lines corresponding to the weight-2 points by \mathcal{S} . Note that the Π -lines in \mathcal{S} partition the set of points in $\Sigma \setminus H$. Also note that the extension of H to Σ^* intersects Π in a hyperplane H' of Π .

Let ℓ_1 and ℓ_2 be different lines from \mathcal{S} and let $J = \text{span}(\ell_1, \ell_2)$ be the 3-dimensional subspace of Σ spanned by ℓ_1 and ℓ_2 . Since ℓ_1 and ℓ_2 are Π -lines, the extension $\bar{J} = \text{span}(\ell_1, \ell_2)$ of J in Σ^* contains the points $\bar{\ell}_1 \cap \Pi$ and $\bar{\ell}_2 \cap \Pi$. If $\dim(\bar{J} \cap \Pi) > 1$, then all the points in J would be projected onto a single point in Ω , a contradiction since ℓ_1 and ℓ_2 correspond to different points of L . We conclude that $\bar{J} \cap \Pi$ is a line M . Since $\dim(\Sigma) \geq 5$, $\dim(J) = 3$ and $\text{span}(J, H) = \Sigma$, $J \cap H$ is a line ℓ . The lines ℓ_1, ℓ_2 and ℓ define a unique regulus \mathcal{R} in J . Note that ℓ is contained in the $(k - 3)$ -space H , and hence, its extension $\bar{\ell}$ meets the $(k - 4)$ -space H' in a point which then necessarily lies on M .

By Lemma 2.8 all the Π -lines in \mathcal{R} are of the same type, say S_α . Repeating the same argument for any pair ℓ_1, ℓ'_2 of lines in \mathcal{S} , we see that all the Π -lines in \mathcal{S} are of type S_α .

Now fix a Π -line $\ell \in \mathcal{S}$ of type S_α . Consider the set of lines $\mathcal{T} = \{\text{span}(\ell, \ell') \cap H \mid \ell' \in \mathcal{S} \setminus \{\ell\}\}$. First suppose that each of the spaces $\text{span}(\ell, \ell'), \ell' \in \mathcal{S} \setminus \{\ell\}$, contains at most $q - 1$ lines of $\mathcal{S} \setminus \{\ell\}$, then \mathcal{T} is a set of at least $q^{k-3} + q^{k-4} + \dots + 1$ different Π -lines of type S_α in H . By Lemma 2.6 \mathbb{F}_{q^h} must have a subfield \mathbb{F}_{q^s} with $s > 2, s \mid k - 2$ and we are done.

So suppose that there is a line $\ell' \in \mathcal{S} \setminus \{\ell\}$ such that $\text{span}(\ell, \ell')$ contains more than $q - 1$ lines of $\mathcal{S} \setminus \{\ell\}$. Then by Lemma 2.8 we will have $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = 2$, and hence, \mathbb{F}_{q^2} is a subfield of \mathbb{F}_{q^h} . We also find that $\text{span}(\ell, \ell') \setminus (H \cap \text{span}(\ell, \ell'))$ is partitioned by lines of \mathcal{S} . Furthermore, again invoking Lemma 2.8, since $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = 2$, we find that each 3-space spanned by ℓ and a line ℓ'' of $\mathcal{S} \setminus \{\ell\}$ is partitioned into $q^2 + 1$ Π -lines of type S_α , one of which is ℓ and one of which is the intersection of $\text{span}(\ell, \ell'')$ with H . Hence, we find that $q^2 - 1$ needs to divide $q^{k-2} - 1$, which in turn implies that $2 \mid k - 2$. □

Remark 3.4 If $s = 2$ in Theorem 3.3 and ℓ is of type S_α , then $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = 2$ and all the Π -lines in \mathcal{T} are of type S_α . By Lemma 2.5, every point in H lies in at most one Π -line of type S_α , and it is easy follows that each of the points in H lies on exactly one Π -line of type S_α . The line spread determined by Π -lines of type S_α of H is a normal spread. This follows easily from the construction: for $m_1, m_2 \in \mathcal{T}, \ell \in \mathcal{S}$, the subspace $\text{span}(\ell, m_1, m_2)$ is partitioned by elements of $\mathcal{S} \cup \mathcal{T}$.

3.2 (Re)constructing linear sets

In the following construction we use the projection point of view to construct an \mathbb{F}_q -linear set which will later prove to be the unique way of describing linear sets satisfying the hypotheses used in Lemma 3.1 and Theorem 3.3. We show that these linear sets have geometric field of linearity \mathbb{F}_{q^s} .

We use the standard vectors $\{\mathbf{e}_i \mid i = 1, \dots, k\}$, where $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ is a vector of length k with 1 in the i th position and 0 elsewhere, as the basis of V , the k -dimensional vector space over \mathbb{F}_q defining $\Sigma = \text{PG}(V) = \text{PG}(k - 1, q)$. We use the same set of vectors as the basis of V' , a k -dimensional vector space over \mathbb{F}_{q^h} defining $\Sigma^* = \text{PG}(V') = \text{PG}(k - 1, q^h)$, thus Σ is a canonical subgeometry of Σ^* .

Construction 3.5 Let $\Sigma^* = \text{PG}(k - 1, q^h) = \text{span}_{q^h}(\langle \mathbf{e}_1 \rangle, \dots, \langle \mathbf{e}_k \rangle)$, $4 \leq k \leq h$. Let Σ be the canonical subgeometry embedded in Σ^* , given by all points $\langle u \rangle_{q^h}$, where $u \in \text{span}_q(\mathbf{e}_1, \dots, \mathbf{e}_k)$. Suppose that $k = rs + 2$ and $s \mid h$. Partition the ordered set $\{\mathbf{e}_1, \dots, \mathbf{e}_k\} = \{\mathbf{e}_1, \dots, \mathbf{e}_{rs}, \mathbf{e}_{rs+1}, \mathbf{e}_{rs+2}\}$ into $r + 1$ parts, r of which have size $s \geq 2$ and are called A_1, \dots, A_r , and are given by the ordered sets

$$A_i = \{\mathbf{e}_{(i-1)s+1}, \dots, \mathbf{e}_{is}\} = \{\mathbf{e}_{i,1}, \dots, \mathbf{e}_{i,s}\}$$

and the last part is of size 2 given by the set

$$B = \{\mathbf{e}_{rs+1}, \mathbf{e}_{rs+2}\}.$$

Consider $\alpha \in \mathbb{F}_{q^h} \setminus \mathbb{F}_q$ generating a degree- s extension of \mathbb{F}_q (i.e. $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = s$). With each $A_i, i = 1, \dots, r$, we associate the $(s - 2)$ -dimensional subspace Π_i of Σ^* given by

$$\Pi_i = \text{span}(\langle \mathbf{e}_{i,1} - \alpha \mathbf{e}_{i,2} \rangle, \langle \mathbf{e}_{i,2} - \alpha \mathbf{e}_{i,3} \rangle, \dots, \langle \mathbf{e}_{i,s-1} - \alpha \mathbf{e}_{i,s} \rangle).$$

Let $\Pi = \text{PG}(k - 3, q^h)$ be the subspace

$$\Pi := \text{span}(\Pi_1, \Pi_2, \dots, \Pi_r, \langle \mathbf{e}_{1,s} - \beta_1 \mathbf{e}_{2,s} \rangle, \langle \mathbf{e}_{2,s} - \beta_2 \mathbf{e}_{3,s} \rangle, \dots, \langle \mathbf{e}_{r-1,s} - \beta_{r-1} \mathbf{e}_{r,s} \rangle, \langle \mathbf{e}_{rs+1} - \alpha \mathbf{e}_{rs+2} \rangle)$$

with $\beta_1, \dots, \beta_{r-1} \in \mathbb{F}_{q^h} \setminus \mathbb{F}_{q^s}$ such that Π is a $(k - 3)$ -space disjoint from Σ and Ω .

Finally, let $\Omega = \text{PG}(1, q^h)$ be the subspace

$$\Omega := \text{span}(\langle \mathbf{e}_{r,s} \rangle, \langle \mathbf{e}_{rs+2} \rangle).$$

The projection of Σ from Π onto Ω in Construction 3.5 is an \mathbb{F}_q -linear set, and we will show now that it has geometric field of linearity \mathbb{F}_{q^s} .

Lemma 3.6 *The linear set L obtained from Construction 3.5 has geometric field of linearity \mathbb{F}_{q^s} .*

Proof By the construction any point $\langle (\lambda_1, \dots, \lambda_k) \rangle_{q^h} = \langle (\lambda_{1,1}, \dots, \lambda_{r,s}, \lambda_{rs+1}, \lambda_{rs+2}) \rangle_{q^h}$ in Σ is projected onto the point $\langle (0, \dots, 0, \chi_1, 0, \chi_2) \rangle_{q^h}$ in Ω where

$$\begin{aligned} \chi_1 &= \beta_1 \beta_2 \dots \beta_{r-1} (\alpha^{s-1} \lambda_{1,1} + \alpha^{s-2} \lambda_{1,2} + \dots + \lambda_{1,s}) \\ &\quad + \beta_2 \beta_3 \dots \beta_{r-1} (\alpha^{s-1} \lambda_{2,1} + \alpha^{s-2} \lambda_{2,2} + \dots + \lambda_{2,s}) \\ &\quad + \dots \\ &\quad + \beta_{r-1} (\alpha^{s-1} \lambda_{r-1,1} + \alpha^{s-2} \lambda_{r-1,2} + \dots + \lambda_{r-1,s}) \\ &\quad + (\alpha^{s-1} \lambda_{r,1} + \alpha^{s-2} \lambda_{r,2} + \dots + \lambda_{r,s}) \end{aligned}$$

and

$$\chi_2 = \lambda_{r_s+2} + \alpha \lambda_{r_s+1}.$$

Thus the linear set L consists of all points $\langle \chi_1 \mathbf{e}_{r,s} + \chi_2 \mathbf{e}_{r_s+2} \rangle_{q^h}$ with

$$(\chi_1, \chi_2) = (\beta_1 \beta_2 \dots \beta_{r-1} f_1(\alpha) + \beta_2 \beta_3 \dots \beta_{r-1} f_2(\alpha) + \dots + f_r(\alpha), \lambda_{r_s+2} + \alpha \lambda_{r_s+1}),$$

where $f_i(\alpha)$ is an arbitrary polynomial with degree at most $s - 1$ and coefficients in \mathbb{F}_q and $\lambda_{r_s+1}, \lambda_{r_s+2} \in \mathbb{F}_q$. Recall that any element of \mathbb{F}_{q^s} can be represented as a polynomial in α with coefficients in \mathbb{F}_q and degree at most $s - 1$ and vice versa, each such polynomial defines an elements of \mathbb{F}_{q^s} . It follows that

$$(\chi_1, \chi_2) = (\beta_1 \beta_2 \dots \beta_{r-1} \gamma_1 + \beta_2 \beta_3 \dots \beta_{r-1} \gamma_2 + \dots + \gamma_r, \lambda_{r_s+2} + \alpha \lambda_{r_s+1})$$

where $\gamma_i, 1 \leq i \leq r$, is an arbitrary element of \mathbb{F}_{q^s} and $\lambda_{r_s+1}, \lambda_{r_s+2} \in \mathbb{F}_q$. Now consider the set of points M of the form $\langle \chi'_1 \mathbf{e}_{r,s} + \chi'_2 \mathbf{e}_{r_s+2} \rangle_{q^h}$ where

$$(\chi'_1, \chi'_2) = (\beta_1 \beta_2 \dots \beta_{r-1} \gamma_1 + \beta_2 \beta_3 \dots \beta_{r-1} \gamma_2 + \dots + \gamma_r, \gamma_{r+1}),$$

where $\gamma_i \in \mathbb{F}_{q^s}, 1 \leq i \leq r + 1$. Since the set of vectors of the form (χ'_1, χ'_2) is closed under addition and \mathbb{F}_{q^s} -multiplication, the set M is an \mathbb{F}_{q^s} -linear set of \mathbb{F}_{q^s} -rank $r + 1$ (and an \mathbb{F}_q -linear set of rank $r_s + s$). Note that, as an \mathbb{F}_{q^s} -linear set, M contains points of \mathbb{F}_{q^s} -weight 1, and hence, M is not an \mathbb{F}_{q^s} -linear set for any $i > 1$. It is clear that every point in L also belongs to M , so $L \subseteq M$. Moreover, if P is a point of M , then either $P = \langle \mathbf{e}_{r,s} \rangle_{q^h}$ which also belongs to L , or $P = \langle (\beta_1 \beta_2 \dots \beta_{r-1} \gamma'_1 + \beta_2 \beta_3 \dots \beta_{r-1} \gamma'_2 + \dots + \gamma'_r) \mathbf{e}_{r,s} + 1 \mathbf{e}_{r_s+2} \rangle_{q^h}$, where $\gamma'_i = \gamma_i / \gamma_{r+1}$, which then clearly belongs to L too. We conclude that $L = M$, and since M is an \mathbb{F}_{q^s} -linear set, L has geometric field of linearity \mathbb{F}_{q^s} . \square

Remark 3.7 The linear set L in Lemma 3.6 contains $q^{r_s} + 1$ points: one corresponds to $\langle \mathbf{e}_{r,s} \rangle$ and the other q^{r_s} correspond to the different choices of the polynomials f_1, \dots, f_r (or equivalently, the choice of elements $\gamma_1, \dots, \gamma_r$). The \mathbb{F}_q -weight of the point $\langle \mathbf{e}_{r,s} \rangle$ is $l_s = k - 2$ and every point other point in L is of weight 2. We have shown that $L = M$, where M is an \mathbb{F}_{q^s} -linear set, forming an l -club: it has $(q^s)^r$ points of \mathbb{F}_{q^s} -weight 1 and 1 point of \mathbb{F}_{q^s} -weight l . We also see that if $l = 1$ and $r = 1$, then L is a set of $q^s + 1$ points (with $s = k - 2$) which is the set of points of an \mathbb{F}_{q^s} -subline of $\text{PG}(1, q^h)$.

Remark 3.8 Recall from the introduction that every \mathbb{F}_q -linear set can be described as the set of spread elements of the Desarguesian spread \mathcal{D} in $\text{PG}(2h - 1, q)$ intersecting some fixed subspace. On the one hand, the \mathbb{F}_q -linear set L of Construction 3.5 can be seen as those elements of \mathcal{D} meeting a fixed $(k - 1)$ -space π , where there is one element, say σ_P , of \mathcal{D} meeting π in a $(k - 3)$ -space μ and the other elements of \mathcal{D} meet π in a line or are disjoint from π . On the other hand, we have shown in Lemma 3.6 that L can be seen as the set of elements of the Desarguesian spread \mathcal{D} which meet a fixed $(k + s - 3)$ -space π' , where π' contains π and such that $\sigma_P \cap \pi = \sigma_P \cap \pi'$. Hence, there is one element, σ_P meeting π' in a $(k - 3)$ -space and all other elements of \mathcal{D} either meet π' in an $(s - 1)$ -dimensional space or are disjoint from π' . The set of $(s - 1)$ -dimensional subspaces of π' obtained as the intersections of elements of \mathcal{D} with π' are contained in the unique Desarguesian $(s - 1)$ -subspread of \mathcal{D} (see also [23]).

3.3 The proof of the main theorem

The following proposition forms the base case for our main theorem.

Proposition 3.9 *If L is a rank-4 \mathbb{F}_q -linear set in $\text{PG}(1, q^h)$, $h \geq 4$, with all points of weight 2, then L is an \mathbb{F}_{q^2} -linear set $\cong \text{PG}(1, q^2)$. If L is a linear set of rank k , $5 \leq k \leq h$, in $\text{PG}(1, q^h)$ with one point of weight $k - 2$ and all other points of weight 2 then L must have geometric field of linearity \mathbb{F}_{q^s} with $s \mid k - 2$, $s > 1$, and $s \mid h$.*

Proof We prove the statement by showing that any linear set satisfying the conditions of Lemma 3.1 or Theorem 3.3 admits a choice of basis vectors so that it can be obtained from Construction 3.5.

If a linear set satisfies the conditions of Theorem 3.1 then similarly to Lemma 2.8 we can take $\Sigma = \langle \text{span}(\mathbf{u}_1, \mathbf{v}_1, \mathbf{u}_2, \mathbf{v}_2) \rangle_q$ such that $\Pi = \text{span}(\langle \mathbf{u}_1 - \alpha \mathbf{v}_1 \rangle, \langle \mathbf{u}_2 - \alpha \mathbf{v}_2 \rangle)$. Now mapping the basis vectors $\mathbf{u}_1, \mathbf{v}_1, \mathbf{u}_2, \mathbf{v}_2$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ respectively we immediately see that L is obtained from Construction 3.5 with $k = 4, p = 1, s = 2$. Thus by Lemma 3.6 this set has geometric field of linearity \mathbb{F}_{q^2} , and since it can be written as an \mathbb{F}_{q^2} -linear set with $q^2 + 1$ points, we find that it is indeed an \mathbb{F}_{q^2} -subline.

If a linear set satisfies the conditions of Theorem 3.3 then we have $\Sigma = \text{span}(H, \ell)$, where ℓ is a Π -line corresponding to a point of weight 2, so there exist vectors \mathbf{w}_{k-1} and \mathbf{w}_k and an element $\alpha \in \mathbb{F}_{q^h} \setminus \mathbb{F}_q$ such that $\ell = \text{span}(\langle \mathbf{w}_{k-1} \rangle, \langle \mathbf{w}_k \rangle)$ and $\langle \mathbf{w}_{k-1} - \alpha \mathbf{w}_k \rangle \in \Pi$. Let $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = s$. If $s > 2$ then it follows from Remark 2.7 and Lemma 2.6, that H can be partitioned by $(s - 1)$ -dimensional subspaces which form a Desarguesian spread and that we can write $H = \text{span}(H_1, \dots, H_r)$ where $H_i = \text{span}(\langle \mathbf{w}_{i,1} \rangle, \langle \mathbf{w}_{i,2} \rangle, \dots, \langle \mathbf{w}_{i,s} \rangle)$ such that $\langle \mathbf{w}_{i,1} - \alpha \mathbf{w}_{i,2} \rangle, \langle \mathbf{w}_{i,2} - \alpha \mathbf{w}_{i,3} \rangle, \dots, \langle \mathbf{w}_{i,s-1} - \alpha \mathbf{w}_{i,s} \rangle \in \Pi$.

By possibly relabeling the vectors $\mathbf{w}_{i,1}, \mathbf{w}_{i,2}, \mathbf{w}_{i,s}$ whose corresponding points span each H_i , we can make sure that the set of r vectors $\mathbf{w}_{j,s}, j = 1, \dots, r$ are linearly independent. The line joining $\langle \mathbf{w}_{i,s} \rangle$ and $\langle \mathbf{w}_{i+1,s} \rangle, i = 1, \dots, r - 1$ is a Π line, say of type β_i such that $\langle \mathbf{w}_{i,s} - \beta_i \mathbf{w}_{i+1,s} \rangle \in \Pi$. It follows that $H \cap \Pi$ equals $\text{span}(\Pi_1, \Pi_2, \dots, \Pi_r, \langle \mathbf{w}_{1,s} - \beta_1 \mathbf{w}_{2,s} \rangle, \langle \mathbf{w}_{2,s} - \beta_2 \mathbf{w}_{3,s} \rangle, \dots, \langle \mathbf{w}_{r-1,s} - \beta_{r-1} \mathbf{w}_{r,s} \rangle)$. It follows that

$$\begin{aligned} \Pi = \text{span}(\Pi_1, \Pi_2, \dots, \Pi_r, \langle \mathbf{w}_{1,s} - \beta_1 \mathbf{w}_{2,s} \rangle, \langle \mathbf{w}_{2,s} - \beta_2 \mathbf{w}_{3,s} \rangle, \dots, \\ \langle \mathbf{w}_{r-1,s} - \beta_{r-1} \mathbf{w}_{r,s} \rangle, \langle \mathbf{e}_{rs+1} - \alpha \mathbf{e}_{rs+2} \rangle). \end{aligned}$$

Now mapping the basis vectors $\mathbf{w}_{1,1}, \dots, \mathbf{w}_{r,s}, \mathbf{w}_{k-1}, \mathbf{w}_k$ to $\mathbf{e}_1, \dots, \mathbf{e}_k$ we see that Π and Σ are as in Construction 3.5. Finally, we know that Ω can be taken arbitrarily but disjoint from Π , so we may take Ω to be $\Omega := \text{span}(\langle \mathbf{e}_{r,s} \rangle, \langle \mathbf{e}_{rs+2} \rangle)$. Note that the points $\langle \mathbf{e}_{r,s} \rangle$ and $\langle \mathbf{e}_{rs+2} \rangle$ lie in subspaces that are projected onto different points which means that Ω is indeed disjoint from Π .

By Lemma 3.6 this set has geometric field of linearity \mathbb{F}_{q^s} .

If $s = 2$ in Theorem 3.3, then by Remark 3.4, H is partitioned by Π -lines of type S_α into a Desarguesian spread. Thus we can write $H = \text{span}(\ell_1, \dots, \ell_r)$, with $\ell_i = \text{span}(\langle \mathbf{w}_{i,1} \rangle, \langle \mathbf{w}_{i,2} \rangle)$ and $\langle \mathbf{w}_{i,1} - \alpha \mathbf{w}_{i,2} \rangle \in \Pi$. Arguing as above, the linear set is obtained by Construction 3.5.

By Lemma 3.6 this set has geometric field of linearity \mathbb{F}_{q^2} . □

Remark 3.10 It remains to indicate why we do not draw the conclusion that our set has maximum geometric field of linearity \mathbb{F}_{q^s} . First note that the parameter s we find in the proof is determined by the type S_α of the lines corresponding to the points of weight 2 in L_U .

When we take a subline $L = \text{PG}(1, q^{k-2})$ in $\text{PG}(1, q^h)$, then the maximum geometric field of linearity is clearly $\mathbb{F}_{q^{k-2}}$. We can write this set as L_U where U is an \mathbb{F}_q -vector space

of rank $2k - 4$ which is an $\mathbb{F}_{q^{k-2}}$ -linear vector space. Then $\mu = \langle U \rangle$ is a $(2k - 5)$ -dimension projective subspace of $\text{PG}(2h - 1, q)$, partitioned by $(k - 3)$ -spaces, each corresponding to a point of L_U . This forms a Desarguesian $(k - 3)$ -spread, say \mathcal{D}' , of μ . Let H be one of the $(k - 3)$ -dimensional spread elements of \mathcal{D}' . Now for every proper divisor s of $k - 2$, we find a unique $(s - 1)$ -subspread of \mathcal{D}' (see e.g. [9, 23]), and the Π -lines of type S_α where $[\mathbb{F}_q(\alpha) : \mathbb{F}_q] = s$ through a point of H are entirely contained in an element of this $(s - 1)$ -subspread \mathcal{D}' . Vice versa, every line contained in a spread element of \mathcal{D}' is a Π -line of type S_β where $[\mathbb{F}_q(\beta) : \mathbb{F}_q] = s$.

This means that if we take a subspace V of U spanned by H and a line contained in one of the induced elements of the $(s - 1)$ -subspread in μ , we will obtain that $L_U = L_V$ where L_V satisfies the hypothesis of our theorem. But we will draw the conclusion that L_V has geometric field of linearity \mathbb{F}_{q^s} , whereas the maximum field of linearity is $\mathbb{F}_{q^{k-2}}$. If we start however from a subspace V' determined by H and any line in a spread element of \mathcal{D}' which is not contained in any of the induced subspreads, then the corresponding field element α generates $\mathbb{F}_{q^{k-2}}$ and our theorem will lead to the conclusion that $L_{V'}$ has geometric field of linearity $\mathbb{F}_{q^{k-2}}$.

Finally note that if $ls = k - 2 \mid h$, an \mathbb{F}_{q^s} -linear set of size $q^{k-2} + 1$ does not necessarily have maximum geometric field of linearity $\mathbb{F}_{q^{k-2}}$. If a linear set of size $q^{k-2} + 1$ has maximum geometric field of linearity $\mathbb{F}_{q^{k-2}}$, then it is necessarily a subline $\text{PG}(1, q^{k-2})$; but not all \mathbb{F}_{q^s} -linear sets of size $q^{ls} + 1$ are $\mathbb{F}_{q^{k-2}}$ -sublines. The easiest case to see this is for \mathbb{F}_q -linear sets of size $q^2 + 1$ in $\text{PG}(1, q^4)$: some of those are $\text{PG}(1, q^2)$ -sublines while others are not (see also [11, page 9]).

Corollary 3.11 *Let L_U be an \mathbb{F}_q -linear $(k - 2)$ -club of rank $k - 1$ in $\text{PG}(1, q^h)$, that is, L_U has one point of weight $k - 2$ and all others of weight 1. If $L_U = L_V$ for some V such that U is a proper subspace of V , then L_U has maximum field of linearity \mathbb{F}_{q^s} for some $s > 1$.*

Proof We switch to the representation of linear sets in terms of Desarguesian spreads. Let ϕ be the field reduction map $\phi : \text{PG}(1, q^h) \rightarrow \text{PG}(2h - 1, q)$ and let \mathcal{D} denote the $(h - 1)$ -spread determined by ϕ .

Then L_U corresponds to the set of elements of \mathcal{D} meeting the $(k - 2)$ -dimensional projective subspace $\pi = \langle U \rangle$ in at least a point. There is one element, say S , of \mathcal{D} intersecting π in a $(k - 3)$ -dimensional subspace while q^{k-2} other elements of \mathcal{D} intersect π in a point. Our hypothesis is that there is a projective subspace $\mu = \langle V \rangle$ containing π intersecting exactly the same elements of \mathcal{D} which intersect π . Let $\nu = \langle W \rangle$ be a subspace of projective dimension $k - 1$ such that $U < W \leq V$. Since $L_U = L_V$ this implies that $L_U = L_W = L_V$. If S would intersect ν in a subspace of dimension $k - 2$, then L_W would have size $q^{k-1} + 1$, a contradiction since L_U has size $q^{k-2} + 1$. Hence, S meets ν in $\pi \cap S$, which is a subspace of dimension $k - 3$. Since L_W has size $q^{k-2} + 1$, it follows that every element of \mathcal{D} intersecting π , necessarily intersects ν in a line: the projective dimension of the intersection of a spread element with the $(k - 1)$ -dimensional space ν cannot exceed 1 since S meets ν in a subspace of dimension $k - 3$, and $q^{k-2}(q + 1) + \frac{q^{k-2}-1}{q-1}$ is the number of points in the $(k - 1)$ -dimensional projective subspace ν . Hence, we have that $L_U = L_W$, where L_W is a linear set of rank k containing one point of weight $k - 2$ and all others of weight 2; the conclusion now follows from Proposition 3.9. □

We are now ready to prove the main theorem.

Theorem 3.12 *If L is a linear set of rank k , $4 \leq k \leq h$, in $\text{PG}(1, q^h)$ with one point P of weight $k - w \geq 2$ and all other points of the same weight $w \geq 2$ then L has geometric field of linearity \mathbb{F}_{q^s} with $s \mid k - w$, $s > 1$, $s \mid h$, and $s \geq w$.*

Proof We again switch to the representation of linear sets in terms of Desarguesian spreads, where ϕ denotes the field reduction map.

Let $L = L_U$ be the linear set defined by the $(k - 1)$ -dimensional projective subspace $\sigma = \langle U \rangle$ of $\text{PG}(2h - 1, q)$ and let P be the point of weight $k - w$. Then $\phi(P) \cap \sigma = \sigma_P$, where σ_P is a $(k - w - 1)$ -dimensional subspace of $\text{PG}(2h - 1, q)$. Let $Q \neq P$ be a point of L_U , then $\phi(Q) \cap \sigma = \sigma_Q$ is $(w - 1)$ -dimensional since Q has weight w . Now consider a $(k - w + 1)$ -dimensional subspace σ' of σ containing σ_P (which exists since $w \geq 2$), then $\sigma' = \langle V \rangle$ with $V < U$. This implies that $L_V \subseteq L_U$. Let $R \neq P$ be a point of L_U , then $\sigma_R = \phi(R) \cap \sigma$ is a $(w - 1)$ -dimensional subspace disjoint from σ_P . Since σ' is a $(k - w + 1)$ -dimensional subspace containing σ_P , it follows that $\phi(R)$ meets σ' in a line, say ℓ_R , and hence, $L_U \subseteq L_V$. We conclude that $L_U = L_V$ and that all points, different from P , have weight 2 in L_V .

This implies that L_V is an \mathbb{F}_q -linear set of rank $k - w + 2$ with one point of weight $k - w$ and all other points of weight 2. Hence by Proposition 3.9 we see that L_V , and hence also L_U , has geometric field of linearity \mathbb{F}_{q^s} with $s \mid k - w$.

The only thing left to prove is that L_U has geometric field of linearity $\mathbb{F}_{q^{s'}}$ for some $s' \geq w$. In the proof of Proposition 3.9, we have deduced that $L_V = L_W$, where W is \mathbb{F}_{q^s} -linear, has \mathbb{F}_q -dimension $k - w + s$ and $\langle W \rangle$ is spanned by σ_P and an $(s - 1)$ -dimensional subspace τ of $\phi(R)$ containing the line ℓ_R . Recall that σ_R is $(w - 1)$ -dimensional. Hence, if τ contains σ_R , then $s - 1 \geq w - 1$ and we are done.

So suppose that there is a point of σ_R , say S , not contained in τ . Consider a line m through S and a point, say T , of ℓ_R . Note that $\text{span}(\sigma_P, T)$ is a $(k - w)$ -dimensional projective subspace $\langle U' \rangle$ with $U' \leq U$ such that $L_U = L_{U'}$, in fact, $L_{U'}$ is a $(k - w)$ -club of rank $k - w + 1$.

Let V' be a $(k - w + 2)$ -dimensional vector subspace such that $\langle V' \rangle = \text{span}(\sigma_P, m)$. Since $V' < U$, by the same reasoning as above for L_U and L_V , we again obtain that $L_U = L_{V'}$, and that $L_{V'}$ has one point of weight $k - w$ and all others of weight 2. It follows that $L_{V'}$ has geometric field of linearity $\mathbb{F}_{q^{s'}}$ for some $s' \mid k - w$. Furthermore, again as before, m is contained in an $(s' - 1)$ -dimensional subspace τ' of $\phi(R)$ such that there exists an $\mathbb{F}_{q^{s'}}$ -linear subspace W' of \mathbb{F}_q -dimension $k - w + s'$ with $\langle W' \rangle = \text{span}(\sigma_P, \tau')$. It follows that $L_{V'} = L_{W'}$.

The $(s - 1)$ -space τ through T is an element of the unique Desarguesian $(s - 1)$ -subspread of \mathcal{D} while the $(s' - 1)$ -space τ' through T is an element of the unique Desarguesian $(s' - 1)$ -subspread of \mathcal{D} . Since $\tau \neq \tau'$, we have that $s \neq s'$, and furthermore, $\text{span}(\tau, \tau')$ is contained in an element of the unique Desarguesian $(t - 1)$ -spread of \mathcal{D} where $t = \text{lcm}(s, s')$. Let τ'' be this $(t - 1)$ -space and consider the subspace $X = \text{span}(\sigma_P, \tau'')$. Since $\sigma_P = \phi(P) \cap \langle W \rangle$ and W is \mathbb{F}_{q^s} -linear, we see that the subspace σ_P is partitioned by $(s - 1)$ -dimensional subspaces of the $(s - 1)$ -subspread and similarly, by elements of the $(s' - 1)$ -subspread. This implies that σ_P is partitioned by elements of the $(t - 1)$ -subspread too, and it follows from the construction of $\langle X \rangle$ that X is \mathbb{F}_{q^t} -linear. This implies that $\langle X \rangle$ meets every element of \mathcal{D} in (at least) a $(t - 1)$ -space, and since $\langle X \rangle$ is $(k - w + t - 1)$ -dimensional and contains the $(k - w - 1)$ -space σ_P , L_X contains precisely $q^{k-w} + 1$ points. Since X contains U' , we have that $L_{U'} \subseteq L_X$, and as both have the same size, $L_{U'} = L_X$. We conclude that $L_{U'} = L_U$ has geometric field of linearity \mathbb{F}_{q^t} .

If $t < w$, then there is a point R' of σ_P , not contained in $span(\tau, \tau')$ and we can repeat the reasoning above to find that R' lies in an element of the Desarguesian $(s'' - 1)$ -subspread τ'' with $s'' \neq t$, L_U can be written as $L_{X'}$ where X' is $\mathbb{F}_{q^{t'}}$ -linear with $t' = lcm(t, s'')$. Eventually, we will find that all points of σ_P are contained in an element τ'' of a Desarguesian $(t'' - 1)$ -spread and that $L_U = L_{X''}$ where X'' is $\mathbb{F}_{q^{t''}}$ -linear. Since σ_P is contained in τ'' , we find that so $t'' - 1 \geq w - 1$ and we are done. \square

4 Conclusion

In this paper we have shown that if a rank- k \mathbb{F}_q -linear set on a line $PG(1, q^h)$, $k \leq h$, has one point of weight $k - w$ and all others of weight w , then L has geometric field of linearity \mathbb{F}_{q^s} for some $1 < s$, $s \mid k - w$, $s \geq w$. As indicated in Remark 3.7, these linear sets can be viewed as \mathbb{F}_{q^s} -linear clubs or \mathbb{F}_{q^s} -sublines, which are in some sense the ‘easiest’ types of linear sets.

Main open problem The larger question of whether the same conclusion can be obtained for all linear sets without points of weight one, remains unsolved. Although we believe that this should indeed be the case, the methods developed in this paper are insufficient to tackle this question.

The case that we have studied in this paper has the advantage that the field of linearity we are looking for (say \mathbb{F}_{q^s}) necessarily has s dividing $k - w$, where $k - w$ is the weight of the unique point, different from all the ones of weight w appearing in the set. In general, if we take an \mathbb{F}_{q^s} -linear set L_U , where U has rank ls and consider a subspace V of U of dimension at least $ls - s + 2$, all points of L_V will have weight at least 2. However, there is no longer an obvious way to deduce the value of s from the weights of the points in L_V .

A geometric point of view using subspreads Geometrically speaking, the belief we expressed above is equivalent to the following: whenever a subspace μ meets all elements of a Desarguesian spread \mathcal{D} in either 0 points or in at least a line, the partition of μ induced by the elements of \mathcal{D} consists of parts, each of which are contained in some fixed Desarguesian subspread of \mathcal{D} (induced by a subfield). We have shown in this paper that the above statement indeed holds true if the partition consists of one $(k - w - 1)$ -space and all other parts are $(w - 1)$ -spaces.

The rank of the linear set We have given examples of linear sets whose rank is not uniquely defined, in the sense that such a linear set can be written as L_U and L_V with U, V subspaces of different dimensions. If $L_U = L_V$ with $k = \dim(U) < l = \dim(V)$, then all points of L_V will have weight at least 2: if not, then L_V has a point of weight 1 and hence, $|L_V| \geq q^{l-1} + 1$ by Result 1.15. But since $|L_U| \leq \frac{q^k - 1}{q - 1} < q^{l-1} + 1$ this cannot happen. As indicated above, we believe that the fact that L_V has only points of weight at least 2 should imply that L_U has maximum geometric field of linearity \mathbb{F}_{q^s} for some $s > 1$.

A polynomial point of view Considerable effort has been done in recent years to investigate linear sets from a (linearised) polynomial point of view – every rank- h linear set in $PG(1, q^h)$ can be described through such a polynomial as a set determined by the points $\langle(x, f(x))\rangle$, where x ranges in \mathbb{F}_{q^h} . For a recent result linking linearised polynomials with the weights of the point sets they define, see [3].

As indicated in the introduction (see Results 1.9), for linear sets of rank h in $PG(1, q^h)$, it is not only known that if they contain only points of weight at least 2, the linear set have geometric field of linearity \mathbb{F}_{q^s} . They are, in fact, already \mathbb{F}_{q^s} -linear themselves. While it

is possible to describe a linear set of rank $k < h$ as a set $\langle(x, f(x))\rangle$ where x ranges in a subspace of \mathbb{F}_{q^h} , it seems that the fact that x does not range in the entire field makes a polynomial approach (much) more complicated.

In [10], the authors study the direction problem for point sets of size less than q and prove a statement which resembles Theorem 1.7. However, we have seen that we cannot hope to extend the result of Proposition 1.9 to all linear sets. This is in part indicated by the fact that in the more general theorem of [10], the authors have to take into account two different parameters, s and t . These parameters will essentially correspond to the algebraic and geometric field of linearity of the point sets if we look at the directions determined by an \mathbb{F}_q -linear map defined on a subspace of \mathbb{F}_{q^h} .

Linear sets with complementary weights Recently (after the current paper was submitted), the paper [16] appeared in which the authors study a general form for linear sets with complementary weights: those are linear sets of rank $s + t$ containing a point of weight s and a point of weight t . The linear sets discussed in our current paper belong to this category. Their most explicit, polynomial based, general form of linear sets is for the case $h = s + t$ in $\text{PG}(1, q^h)$, which is precisely the case for which we know our conjecture is true. It is however conceivable that the methods of that paper allow us to say something about linear sets with complementary weights of rank $< h$, all whose points have weight more than one. In particular, in [16, Theorem 4.1], the authors give equivalent conditions for a linear set with complementary weights to have two points with weight more than one and all others of weight one; the conditions depend on the (algebraic) behaviour of two \mathbb{F}_q -subspaces S and T of \mathbb{F}_{q^h} spanning the subspace underlying the linear set. It may be possible to derive a similar algebraic condition on S and T such that the linear set defined by the span of S and T has only points of weight more than one. If our conjecture is correct, the conditions on S and T would have to be boiled down to the existence of some $s > 1$ and \mathbb{F}_{q^s} -linear subspaces S', T' of \mathbb{F}_{q^h} such that $S \subseteq S'$ and $T \subseteq T'$.

Funding Open Access funding enabled and organized by CAUL and its Member Institutions.

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