Correction to: CCZ equivalence of power functions

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Abstract

An error in the proof of Lemma 2.13 of my article [3] is corrected. We point out that the validity of the main result [3, Theorem 1.1] is not affected.

The correct form of [3, Lemma 2.13] is:

Lemma 1 *Let $0 < k < p^n - 1$, k not a p -power. Then:*

- (a) $C_{\text{Aut}(p_k)}(\mathcal{Z}_k) = \mathcal{Z}_k$.
- (b) *Either $\mathcal{N}_k$ is the common stabilizer of the spaces V and W in $\text{Lin}(p_k)$ or $n = 4b$, $k \equiv p^a(1 + p^b) \pmod{p^n - 1}$ for some integers a, b and cyclic groups of order $p^n - 1$ in $\text{Aut}(p_k)$ are conjugate to $\mathcal{Z}_k$.*

The author missed the second part of assertion (b) of Lemma 1. The second part of assertion (b) does not change the validity of the main result [3, Theorem 1.1]: To see this consider power functions p_k and p_ℓ on $\text{GF}(p^n)$ which are CCZ equivalent. One has to show that there exists an integer a such that $\ell \equiv p^a k \pmod{p^n - 1}$ or $k\ell \equiv p^a \pmod{p^n - 1}$ (in which case k is invertible modulo $p^n - 1$). If $n = 4b$ and $k \equiv p^a(1 + p^b) \pmod{p^n - 1}$ then by the second part of assertion (b) of Lemma 1 and [3, Lemma 2.23] we get $\ell \equiv p^c k \pmod{p^n - 1}$ as k is not invertible modulo $p^n - 1$. In particular the assertion of the main result holds. So one can assume from now on $k, \ell \not\equiv p^a(1 + p^b) \pmod{p^n - 1}$ and $\mathcal{N}_k$ is the common stabilizer of the spaces V and W in $\text{Lin}(p_k)$; i.e. [3, Lemma 2.13] can be applied in its original form and we can complete the proof of [3, Theorem 1.1] following the arguments of [3].

We recall the notation from [3]. Set $F = \text{GF}(p^n)$, p a prime, n a positive integer. Let k be a non-negative integer $< p^n - 1$. Then $p_k : F \rightarrow F'$ denotes the power function on F defined by $p_k(x) = x^k$ and F' stands for the subfield of F generated by the image of p_k . Set $U = V \oplus W$,

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$V = F \oplus 0$, $W = 0 \oplus F'$. Then U contains the graph $\Gamma(p_k) = \{(x, p_k(x)) \mid x \in F\}$ of p_k . The automorphism group of p_k (in the sense of CCZ equivalence) is the stabilizer of $\Gamma(p_k)$ in $\text{AGL}(U)$: $\text{Aut}(p_k) = \{\Phi \in \text{AGL}(U) \mid \Gamma(p_k)\Phi = \Gamma(p_k)\}$. Here $\text{AGL}(U)$ denotes the group of invertible affine transformations on U , i.e. a typical element of $\text{AGL}(U)$ has the form $\phi \circ \tau_u$ with $\phi \in \text{GL}(U)$ and τ_u is the translation $\tau_u : U \ni x \mapsto x + u \in U$. The group $\text{Lin}(p_k) = \text{Aut}(p_k) \cap \text{GL}(U)$ is the stabilizer of 0 in $\text{Aut}(p_k)$. For $0 \neq b \in F$ define $\mu_b \in \text{GL}(U)$ by $(x, y)\mu_b = (bx, b^k y)$. Define $\phi \in \text{GL}(U)$ by $(x, y)\phi = (x^p, y^p)$. Then the groups $\mathcal{Z}_k = \{\mu_b \mid 0 \neq b \in F\}$ and $\mathcal{N}_k = \mathcal{Z}_k \langle \phi \rangle$ are subgroups of $\text{Lin}(p_k)$.

Keeping the notation of [3] we revise the proof of part (b) and use the following lemma. Note, that we consider in this lemma *isomorphisms of $\text{GF}(p)G$ -modules* although these modules may be defined as *G -modules over larger fields*.

Lemma 2 *Let $G = \text{SL}(d, p^a)$, $d \geq 2$, $(d, p^a) \neq (2, 2), (2, 3)$ and denote by M the natural FG -module, $F = \text{GF}(p^a)$. Set $n = ad$ and let N be a simple $\text{GF}(p)G$ -module of dimension n . One of the following $\text{GF}(p)G$ -isomorphisms holds:*

- (i) N is isomorphic to M ,
- (ii) $d > 2$ and $N \simeq M^*$ (the dual module of M),
- (iii) $d = 2$, $a = 2b$, and N is isomorphic to a $\text{GF}(p^b)G$ -submodule R of $M \otimes M^{(p^b)}$, $\dim_{\text{GF}(p^b)} R = 4$.

We denote by $M^{(p^c)}$ the Galois conjugate of a module M under the field automorphism $x \mapsto x^{p^c}$.

Proof Set $F_0 = \text{End}_{\text{GF}(p)G} N$. Then $F_0 \simeq \text{GF}(p^b)$ is a finite field (Schur's Lemma). By [1, (25.8)] N is an absolutely irreducible $F_0 G$ -module. By [5, Theorem 2.1] $\dim_{F_0} N \geq d^{a/(a,b)}$. So $ad = n = \dim_{\text{GF}(p)} N \geq bd^{a/(a,b)}$. Two possibilities can occur: $a = b$ or $d = 2$, $a = 2b$. Set $N^F = F \otimes_{F_0} N$ which is simple as a FG -module. If $a = b$ then $\dim_F N^F = d$ and by [6, Theorem 2.2] assertion (i) or (ii) holds as Galois conjugates of FG -modules are isomorphic as $\text{GF}(p)G$ -modules. Case $a = 2b$ yields assertion (iii): Now $d = 2$ and $\dim_F N^F = 4$ so that [6, Theorem 2.2] implies $N^F \simeq M \otimes M^{(p^c)}$ (as $\text{GF}(p)G$ -modules) for some c . However N^F as a $F_0 G$ -module is the direct sum of two isomorphic, simple modules (which we call R) and $(N^F)^{(p^b)} \simeq N^F$. This forces $b = c$. $\square$

Proof of (b) Let L be the stabilizer of the spaces V and W in $\text{Lin}(p_k)$. Let $\pi : L \rightarrow \text{GL}(V)$ be the projection of L into $\text{GL}(V)$. Then $\ker \pi = 1$ by [3, Lemma 2.4]. By [4] $L \simeq \text{Im } \pi \simeq \Gamma L(d, p^a)$, $ad = n$, and V is the natural FL -module, $F = \text{GF}(p^a)$, viewed as a module over $\text{GF}(p)$. The group L contains the normal subgroups $\text{SL}(d, p^a) \simeq L_0 \leq L_1 \simeq \text{GL}(d, p^a)$. The first assertion of (b) holds if $d = 1$.

So assume $d > 1$. If the projection of L_0 into $\text{GL}(W)$ is trivial, then p_k is constant (L_0 is transitive on $V - \{0\}$) and $k = 0$, a contradiction.

So let the projection of L_0 into $\text{GL}(W)$ be nontrivial. We consider V and W as $\text{GF}(p)L_0$ -modules and distinguish the three cases of Lemma 2.

Assume, that $V \simeq W$ as $\text{GF}(p)L_0$ -modules. By [3, Lemma 2.3] the nontrivial L_0 -orbits consist of the nontrivial elements of an n -space and k would be a p -power by [3, Lemma 2.11], a contradiction. Assume next, that $d > 2$ and that the modules V and W are dual to each other. Then the stabilizer of a vector on V does not fix any nontrivial vector in W , i.e. this group does not have the orbit $B = \Gamma(p_k) - \{0\}$ of length $p^n - 1$ in $U - (V \cup W)$, a contradiction.

Finally, we assume (notation as in Lemma 2) $d = a/b = 2$, $V \simeq M$, and $W \simeq R$ as $\text{GF}(p)L_0$ -modules. *This case was missed in the proof of [3, Lemma 2.13].* Set $F_0 = \text{GF}(p^b)$.

We know that V is the natural module of L_1 and in the obvious way one extend the F_0L_0 -module $W \simeq M \otimes M^{(p^b)}$ and even R to a F_0L_1 -module. Note, that if L_1 acts in this form on W there exist orbits of length $p^n - 1$ in $U - (V \cup W)$ (the stabilizer in L_1 of a nontrivial vector of V fixes in W a nontrivial vector too).

Any F_0L_1 -module whose restriction to L_0 is isomorphic to $M \otimes M^{(p^b)}$ is of the form $(M \otimes M^{(p^b)}) \otimes \chi$ where $\chi : L_1/L_0 \rightarrow F_0^*$ is a linear character. If $W \simeq R$ as a $\text{GF}(p)L_1$ is a subrepresentation of $(M \otimes M^{(p^b)}) \otimes \chi$ then $U - (V \cup W)$ contains L_1 -orbits of length $p^n - 1$ only if $\chi = 1$. This shows $W \subset M \otimes M^{(p^b)}$ as a $\text{GF}(p)L_1$ -module. Let ζ be a primitive element in F . Then μ_ζ has on $F \otimes V$ the eigenvalues $\zeta, \zeta^p, \dots, \zeta^{p^{n-1}}$, on $F \otimes W$ the eigenvalues $\zeta^k, \zeta^{pk}, \dots, \zeta^{p^{n-1}k}$ and on $M \otimes M^{(p^b)}$ the eigenvalues $\zeta^{1+p^c}, \zeta^{p(1+p^b)}, \dots, \zeta^{p^{n-1}(1+p^b)}$. Hence $k = p^a(1 + p^b)$ for some a , i.e. p_k is a quadratic function. Set $v(c) = \beta(c) \circ \tau_{(c, p_k(c))} \in \text{AGL}(U)$, $c \in F$, where $\beta(c) \in \text{GL}(U)$ is defined by $(x, y)\beta(c) = (x, p_k(x + c) + p_k(x) + y)$. Then $\mathcal{M} = \{v(c) \mid c \in F\}$ is an elementary abelian p -group, that acts regularly on $\Gamma(p_k)$. A computation shows that $\mathcal{Z}_k$ normalizes $\mathcal{M}$. So $\mathcal{M} \cdot \mathcal{Z}_k \leq G = \text{Aut}(p_k)$ acts doubly transitive on $\Gamma(p_k)$. The socle of G is either an elementary abelian subgroup of order p^n or is a nonabelian simple group.

Suppose first that the socle $\mathcal{M}_1$ of G is elementary abelian. Then $C_{\mathcal{M}_1}(\mathcal{M}) \neq 1$ is invariant under $\mathcal{Z}_k$ showing $C_{\mathcal{M}_1}(\mathcal{M}) = \mathcal{M}_1$ and therefore $\mathcal{M} = \mathcal{M}_1$. Now $G = \mathcal{M} \cdot \text{Lin}(p_k)$ and as $L \leq \text{Lin}(p_k)$ we deduce from a Theorem of Hering (see [6, Appendix]) $\text{GL}(e, p^c) \leq \text{Lin}(p_k) \leq \Gamma L(e, p^c)$, $e \geq d$, $ec = n$. The well known Theorem of Zsigmondy implies that cyclic groups of order $p^n - 1$ of $\text{GL}(n, p)$ are Singer groups and [4] shows that cyclic groups of order $p^n - 1$ of a subgroup of $\text{GL}(n, p)$ form a conjugacy class in this subgroup. Therefore a cyclic subgroup of $G/\mathcal{M}$ of this order is conjugate to $\mathcal{Z}_k\mathcal{M}/\mathcal{M}$ and the Schur–Zassenhaus Theorem in turn implies the second assertion of (b).

Suppose now that the socle S of G is a nonabelian simple group. Then S is listed in [2, Theorem 5.3]. Note, that S has prime power degree and a point stabilizer contains a group isomorphic to $\text{SL}(2, p^a)$. A quick inspection of the examples shows $S \simeq \text{Alt}(p^n)$. As $p^n \geq 16$ the degree of a minimal nontrivial projective representation of $\text{Alt}(p^n)$ is $\geq p^n - 2$ (see [7, 8]). But the image of S under the epimorphism $\text{AGL}(U) \rightarrow \text{GL}(U)$ has a nontrivial representation of degree $2n$, a contradiction. $\square$

References

1. Aschbacher M.: Finite Group Theory. Cambridge University Press, Cambridge (2000).
2. Cameron P.: Primitive permutation groups and finite simple groups. Bull. Lond. Math. Soc. **13**, 1–22 (1981).
3. Dempwolff U.: CCZ equivalence of power functions. Des. Codes Cryptogr. **86**, 665–692 (2018).
4. Kantor W.: Linear groups containing a Singer cycle. J. Algebra **62**, 232–234 (1980).
5. Liebeck M.: On the orders of maximal subgroups of the finite classical groups. Proc. Lond. Math. Soc. (3) **50**, 426–446 (1985).
6. Liebeck M.: The affine permutation groups of rank three. Proc. Lond. Math. Soc. (3) **54**, 477–516 (1987).
7. Wagner A.: The faithful linear representation of least degree of S_n and A_n over a field of characteristic 2. Math. Z. **151**, 127–137 (1976).
8. Wagner A.: An observation on the degrees of projective representations of the symmetric and alternating group over an arbitrary field. Arch. Math. (Basel) **29**, 583–589 (1977).