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On sets of subspaces with two intersection dimensions and a geometrical junta bound

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Abstract
In this article, constant dimension subspace codes whose codewords have subspace distance in a prescribed set of integers, are considered. The easiest example of such an object is a *junta* (Combin Probab Comput 18(1–2):107–122, 2009); i.e. a subspace code in which all codewords go through a common subspace. We focus on the case when only two intersection values for the codewords, are assigned. In such a case we determine an upper bound for the dimension of the vector space spanned by the elements of a non-junta code. In addition, if the two intersection values are consecutive, we prove that such a bound is tight, and classify the examples attaining the largest possible dimension as one of four infinite families.

Keywords Subspace codes · Rank codes · Extremal Combinatorics · Galois geometry

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1 Introduction and preliminaries

Let $\mathbb{V} = V(\mathbb{F})$ be a finite dimensional vector space over a (possibly finite) field $\mathbb{F}$, and let $k \in \mathbb{Z}^+$ be a positive integer and $\ell \in \mathbb{N}$, such that $\ell < k$. A $(k; \ell)$-*SCID* (Subspaces with Constant Intersection Dimension) in $\mathbb{V}$, is a set of k-dimensional subspaces of $\mathbb{V}$ (k-spaces in the following) pairwise intersecting in an ℓ-dimensional space [1]. The easiest way of constructing such an object is by considering a so-called ℓ-*sunflower*. Precisely, by taking a family $\mathcal{S}$ of k-spaces of $\mathbb{V}$ containing an ℓ-space V' and having no points in common outside of $\mathbb{V}'$. In the following, we will refer to V' as the *center* of the sunflower, and will call the elements of $\mathcal{S}$, the *petals* of $\mathcal{S}$.

Of course, not all ℓ-sunflowers with the same number of petals span a subspace of the same dimension in $\mathbb{V}$. A sunflower $\mathcal{S}$ is said to be of *maximal dimension* if among all sunflowers with the same number of petals, it spans a subspace of $\mathbb{V}$ of largest dimension.

In this article we focus on a natural generalization of the concept of $(k; \ell)$-SCID. More precisely, let $\ell_1, \ell_2, ..., \ell_v \in \mathbb{N}$ be non-negative integers such that $\ell_1, \ell_2, ..., \ell_v < k$. We give the following definition.

Definition 1.1 A set $\mathcal{S}$ of k-spaces of $\mathbb{V}$ is a $(k; \ell_1, \ell_2, ..., \ell_v)$-SPID (Subspaces with Pre-assigned Intersection Dimensions) if for each pair of distinct subspaces $\pi_i, \pi_j \in \mathcal{S}$, we have $\dim(\pi_i \cap \pi_j) \in \{\ell_1, \ell_2, ..., \ell_v\}$, and for each integer $\ell_m \in \{\ell_1, \ell_2, ..., \ell_v\}$, there exist at least two k-spaces in $\mathcal{S}$ such that $\dim(\pi_i \cap \pi_j) = \ell_m$.

For our purposes, we always suppose that the dimension of $\mathbb{V}$ to be large enough in order to assure the existence of such an object. Clearly, in the case when $v = 1$, we get back the definition of a $(k; \ell)$-SCID in $\mathbb{V}$.

The notion of ℓ-sunflower in $\mathbb{V}$ can also be naturally generalized. We say that a $(k; \ell_1, \ell_2, ..., \ell_v)$-SPID $\mathcal{S}$ is an ℓ-*junta* in $\mathbb{V}$, if all elements of $\mathcal{S}$ pass through a common ℓ-space of $\mathbb{V}$.

These geometric objects arise from a more general problem stated in [2,4,6], and recently gained a particular interest due to the fact that they provide *constant subspace codes*, which are a main tool in random network coding [7–9].

In this paper, we elaborate on finite $(k; \ell_1, \ell_2, ..., \ell_v)$-SPIDs, mainly focusing on the case when only two intersection values are assigned. Moreover, if $\tilde{\ell} = \min\{\ell_1, \ell_2, ..., \ell_v\}$, we will assume that $\dim(\mathbb{V}) \geq |\mathcal{S}|(k - \tilde{\ell}) + \tilde{\ell}$. Under this hypothesis, it is easy to see that $\mathcal{S}$ is an $\tilde{\ell}$-sunflower of $\mathbb{V}$ of maximal dimension if any element $\pi \in \mathcal{S}$ meets the subspace generated by all others precisely in the center. In this case, we determine an upper bound for the dimension of the vector space spanned by the elements of a non-junta code, providing the smallest intersection value is strictly larger than zero. In addition, if these two possible intersection values are consecutive integers, we prove that this bound is tight and classify the examples attaining the largest dimension as one of four infinite families.

Let $\mathcal{S} = \{\pi_1, \pi_2, \ldots, \pi_n\}$ be a $(k; \ell_1, \ell_2, \ldots, \ell_v)$-SPID. As in [1], for each $j \in \{1, ..., n\}$, the differences

$$\delta_j(\mathcal{S}) = \dim\langle \pi_1, \ldots, \pi_j \rangle - \dim\langle \pi_1, \ldots, \pi_{j-1} \rangle$$

will be an important arithmetic tool in order to prove our results. We underline here that we consider the span of the empty set as the null subspace; accordingly we put $\delta_1 = k$. Clearly, the values $\delta_j(\mathcal{S})$ depend on the labeling of the subspaces in $\mathcal{S}$. In the following, we will enclose these integers in an array, say $\delta(\mathcal{S}) = (\delta_1(\mathcal{S}), \ldots, \delta_n(\mathcal{S}))$. Regarding this array, we show the following fact which also will play a crucial role.

Proposition 1.2 *Let $k, t_1, t_2, \ldots, t_v \in \mathbb{Z}^+$ be integers such that $k \geq t_1 > t_2 > \cdots > t_v \geq 1$. Let $\mathcal{S} = \{\pi_1, \ldots, \pi_n\}$ be a $(k; k-t_1, k-t_2, \ldots, k-t_v)$-SPID in a vector space $\mathbb{V}$, with $n \geq 3$. Then there exists a permutation σ of the indices in the set $I_n = \{1, 2, \ldots, n\}$ such that*

$$t_1 = \delta_2(\mathcal{S}_\sigma) \geq \delta_3(\mathcal{S}_\sigma) \geq \ldots \geq \delta_n(\mathcal{S}_\sigma),$$

where $\mathcal{S}_\sigma = \{\pi_{\sigma(1)}, \ldots, \pi_{\sigma(n)}\}$, and

$$\delta_j(\mathcal{S}_\sigma) = \dim\langle \pi_{\sigma(1)}, \ldots, \pi_{\sigma(j)} \rangle - \dim\langle \pi_{\sigma(1)}, \ldots, \pi_{\sigma(j-1)} \rangle.$$

Proof Let $m \in \mathbb{Z}^+$ be the maximum integer for which there exist m k-spaces, $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_m}$, of $\mathcal{S}$, forming a $(k-t_1)$-sunflower of maximal dimension; obviously $m \geq 2$. Consider

$$\max_{\substack{1 \leq i \leq n \\ i \neq i_1, \ldots, i_m}} \dim(\pi_i \cap \langle \pi_h \mid h \neq i \rangle),$$

and denote by i_n an integer in $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$ such that

$$\dim(\pi_{i_n} \cap \langle \pi_h \mid h \neq i_n \rangle) = \max_{\substack{1 \leq i \leq n \\ i \neq i_1, \ldots, i_m}} \dim(\pi_i \cap \langle \pi_h \mid h \neq i \rangle).$$

Similarly, consider

$$\max_{\substack{1 \leq i \leq n \\ i \neq i_1, \ldots, i_m, i_n}} \dim(\pi_i \cap \langle \pi_h \mid h \neq i, i_n \rangle),$$

then there exists an integer, say $i_{n-1} \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m, i_n\}$, such that

$$\dim(\pi_{i_{n-1}} \cap \langle \pi_h \mid h \neq i_{n-1}, i_n \rangle) = \max_{\substack{1 \leq i \leq n \\ i \neq i_1, \ldots, i_m, i_n}} \dim(\pi_i \cap \langle \pi_h \mid h \neq i, i_n \rangle).$$

After $n - m$ steps, we obtain a sequence of indices $(i_{m+1}, \ldots, i_n)$.

Let σ be a permutation of the indices $\{1, \ldots, n\}$, fixing the set $\{i_1, i_2, \ldots, i_m\}$ and such that $\sigma(j) = i_j$, for every $j = m+1, \ldots, n$. Consider $\mathcal{S}_\sigma = \{\pi_{\sigma(1)}, \ldots, \pi_{\sigma(n)}\}$, we will show that

$$\delta_{j+1}(\mathcal{S}_\sigma) \leq \delta_j(\mathcal{S}_\sigma) \quad \text{for all} \quad j = 2, \ldots, n-1.$$

First of all, $\delta_j(\mathcal{S}_\sigma) \leq t_1$, for each $j = 2, \ldots, n$; indeed

$$\delta_j(\mathcal{S}_\sigma) = k - \dim(\pi_{\sigma(j)} \cap \langle \pi_{\sigma(1)}, \ldots, \pi_{\sigma(j-1)} \rangle) \leq k - \dim(\pi_{\sigma(j)} \cap \pi_{\sigma(1)}) \leq t_1.$$

Also, since $\pi_{\sigma(1)}, \ldots, \pi_{\sigma(m)}$ form a $(k-t_1)$-sunflower of maximal dimension, then we have $\delta_j(\mathcal{S}_\sigma) = t_1$, with $2 \leq j \leq m$. Note that

$$\dim(\pi_{i_{j+1}} \cap \langle \pi_h \mid h \neq i_{j+1}, \ldots, i_n \rangle) \geq \dim(\pi_{i_j} \cap \langle \pi_h \mid h \neq i_j, \ldots, i_n \rangle)$$

for all $m+1 \leq j \leq n-1$, because otherwise we would have

$$\begin{aligned} \dim(\pi_{i_{j+1}} \cap \langle \pi_h \mid h \neq i_{j+1}, \ldots, i_n \rangle) &< \dim(\pi_{i_j} \cap \langle \pi_h \mid h \neq i_j, \ldots, i_n \rangle) \\ &\leq \dim(\pi_{i_j} \cap \langle \pi_h \mid h \neq i_j, i_{j+2}, \ldots, i_n \rangle), \end{aligned}$$

which is a contradiction by the definition of i_{j+1}. Then

$$\delta_{j+1}(\mathcal{S}_\sigma) = k - \dim(\pi_{i_{j+1}} \cap \langle \pi_h \mid h \neq i_{j+1}, \dots, i_n \rangle) \\ \leq k - \dim(\pi_{i_j} \cap \langle \pi_h \mid h \neq i_j, \dots, i_n \rangle) = \delta_j(\mathcal{S}_\sigma).$$

This concludes the proof. □

In other terms, it is always possible to sort k-spaces in $\mathcal{S}$ in such a way that the associated array $\delta(\mathcal{S})$, is non-increasing (see also [1, Theorem 2]).

Remark 1.3 We note explicitly that for a $(k; k-t)$-SCID $\mathcal{S} = \{\pi_1, \dots, \pi_n\} \subset \mathbb{V}$, we have

$$\delta(\mathcal{S}) := (\delta_1, \delta_2, \dots, \delta_n) = (k, t, \dots, t) \tag{1}$$

if and only if $\mathcal{S}$ is a $(k-t)$-sunflower of maximal dimension. The necessary condition is in fact trivial. While, regarding the sufficiency we may observe that since $\delta_n(\mathcal{S}) = t$ and

$$\pi_n \cap \langle \pi_1, \dots, \pi_{n-1} \rangle \supseteq \pi_n \cap \pi_i,$$

we get for each $i \in \{1, ..., n-1\}$, $\pi_n \cap \pi_i = V'$, where $V' = \pi_n \cap \langle \pi_1, \dots, \pi_{n-1} \rangle$. This implies that $\mathcal{S}$ is a $(k-t)$-sunflower. Finally, by using Grassmann's formula, it is easy to show that $\mathcal{S}$ is of maximal dimension.

2 A junta-property bound for $(k; k-t_1, k-t_2)$-SPIDs

In this section, we restrict our discussion to the case where only two values for the intersection dimensions are possible.

We start by showing a result which appears as a quite natural generalization of [1, Theorem 2] to $(k; k-t_1, k-t_2)$-SPID, with $k - t_1 \neq 0$.

Theorem 2.1 *Let $k, t_1, t_2 \in \mathbb{Z}^+$ such that $k > t_1 > t_2 \geq 2$. Let $\mathcal{S}$ be a $(k; k-t_1, k-t_2)$-SPID in $\mathbb{V}$, with $n = |\mathcal{S}| \geq 3$. If $\dim\langle \mathcal{S} \rangle \geq k + (t_1 - 1)(n-1) + 2$, then $\mathcal{S}$ is a $(k-t_1)$-junta.*

Proof Let $\delta(\mathcal{S})$ be any non-increasing array associated with $\mathcal{S}$. In particular, arguing as in the proof of Proposition 1.2, we can choose as first m spaces, $m \geq 2$, those forming a $(k-t_1)$-sunflower of maximal dimension. By Remark 1.3, the integer m is the largest index for which $\delta_m = t_1$. Let V' be the center of the sunflower formed by $\pi_1, \dots, \pi_m$. Hence, we get $\dim V' = k - t_1$. Assume that $\mathcal{S}$ is not a $(k-t_1)$-junta, so we can find a subspace $\pi_r \in \mathcal{S} \setminus \{\pi_1, \dots, \pi_m\}$ not containing V'. We denote $k - t_1 - \dim(\pi_r \cap V')$ by ε; hence, $\varepsilon \geq 1$. Also, in the quotient vector space $\Pi = \langle \mathcal{S} \rangle / (V' \cap \pi_r)$, we have that $\dim_\Pi \pi_r = t_1 + \varepsilon$, and that $\dim_\Pi(\pi_r \cap \pi_i) \in \{\varepsilon, \varepsilon + t_1 - t_2\}$, for each $1 \leq i \leq m$. Also, the subspaces $(\pi_r \cap \pi_i)/(V' \cap \pi_r)$ of Π, with $1 \leq i \leq m$, are linearly independent. Hence,

$$\delta_r = \dim \pi_r - \dim(\langle \pi_1, \pi_2, \dots, \pi_{r-1} \rangle \cap \pi_r) \leq \dim_\Pi \pi_r - \dim_\Pi \langle \pi_1 \cap \pi_r, \dots, \pi_m \cap \pi_r \rangle \\ = t_1 + \varepsilon - \sum_{i=1}^{m} \dim_\Pi(\pi_r \cap \pi_i) \leq t_1 + \varepsilon - m \cdot \varepsilon \leq t_1 - m + 1.$$

Since $\delta(\mathcal{S}) = (\delta_1, \ldots, \delta_n)$ is non-increasing, it is easy to see that

$$\begin{aligned} \dim\langle\mathcal{S}\rangle &= \sum_{i=1}^{n}\delta_i = k + \sum_{i=2}^{m}\delta_i + \sum_{i=m+1}^{r-1}\delta_i + \sum_{i=r}^{n}\delta_i \\ &\leq k + (m-1)t_1 + (r-m-1)(t_1-1) + (n-r+1)(t_1-m+1) \\ &= k + (n-1)(t_1-1) - (n-r)(m-2) + 1 \\ &\leq k + (n-1)(t_1-1) + 1, \end{aligned} \tag{2}$$

which proves the theorem. □

Remark 2.2 We point out here that unlike what happens for SCIDs, in general the bound stated above is not tight. For instance, with the same notation as used in Theorem 2.1; if $t_1 > t_2 \geq 2$ and there exists an integer s such that $r > s > m$ with $\delta_s \leq t_2$, we can slightly improve on the upper bound stated in Theorem 2.1. In fact, if this is the case we can repeat the proof of Theorem 2.1, and by re-writing Inequality (2), we get

$$\begin{aligned} \dim\langle\mathcal{S}\rangle &= \sum_{i=1}^{n}\delta_i = k + \sum_{i=2}^{m}\delta_i + \sum_{i=m+1}^{s-1}\delta_i + \sum_{i=s}^{r-1}\delta_i + \sum_{i=r}^{n}\delta_i \\ &\leq k + (m-1)t_1 + (s-m-1)(t_1-1) + (r-s)t_2 + (n-r+1)(t_1-m+1) \\ &= k + (n-1)(t_1-1) - (n-r)(m-2) - (r-s)(t_1-t_2-1) + 1 \\ &\leq k + (n-1)(t_1-1) - (t_1-t_2) + 2. \end{aligned} \tag{3}$$

This possibility can be realised if the first $r-1$ spaces form a $(k-t_1)$-junta with $\dim(\pi_s \cap \pi_j) = k - t_2$ for some $j \in \{1, \ldots, s-1\}$. In what follows, we exhibit a concrete example.

Let $k, t_1, t_2 \in \mathbb{Z}^+$ such that $k > t_1 > t_2 + 1 > 2$ and consider $t_1 - t_2 + 1 \leq m \leq \min\{t_1 + 1, n - 1\}$. Let $V', X, N_1, \ldots, N_m, M_{m+1}, \ldots, M_{s-1}$ and $P_s, \ldots, P_{n-1}$ be linearly independent subspaces of $\mathbb{V}$ such that

(a) $\dim V' = k - t_1$,
(b) $\dim X = t_1 - m + 1$,
(c) $\dim N_i = t_1$, for $i = 1, \ldots, m$,
(d) $\dim M_j = t_1 - 1$, for $j = m+1, \ldots, s-1$,
(e) $\dim P_\ell = t_2$, for $\ell = s, \ldots, n-1$.

Let $A_i = \{a_{i1}, \ldots, a_{i,t_1-t_2}\}$ be a set of linearly independent 1-spaces in N_i, for $i = 1, \ldots, m$, $|A_i| = t_1 - t_2$, and we choose in A_i a 1-space, for example a_{i1}. Now, let $b_{m+1}, \ldots, b_{s-1}$ be distinct 1-spaces in $\langle a_{11}, \ldots, a_{m1}\rangle \setminus \{a_{11}, \ldots, a_{m1}\}$ (where $\frac{q^m-1}{q-1} \geq s - m - 1$ when $\mathbb{V}$ is a vector space over the Galois field of order q, $\mathbb{F}_q$) and let W be a $(k - t_1 - 1)$-space in V'. Then we define the k-spaces $\pi_1, \ldots, \pi_n$ as follows.

- $\pi_1 = \langle V', N_1\rangle, \pi_2 = \langle V', N_2\rangle, \ldots, \pi_m = \langle V', N_m\rangle$,
- $\pi_{m+1} = \langle V', b_{m+1}, M_{m+1}\rangle, \ldots, \pi_{s-1} = \langle V', b_{s-1}, M_{s-1}\rangle$,
- $\pi_s = \langle V', Q_s, P_s\rangle, \ldots, \pi_{n-1} = \langle V', Q_{n-1}, P_{n-1}\rangle$,
- $\pi_n = \langle W, a_{11}, \ldots, a_{m1}, X\rangle$,

where $Q_s, \ldots, Q_{n-1}$ are $(t_1 - t_2)$-spaces equal to $\langle A_i\rangle$, for some $i \in \{1, \ldots, m\}$. It is easy to verify that

(i) $\pi_i \cap \pi_j = V'$, with $i, j = 1, \ldots, s-1$,
(ii) $\pi_i \cap \pi_j = V'$, with $i = m+1, \ldots, s-1$ and $j = s, \ldots, n-1$,

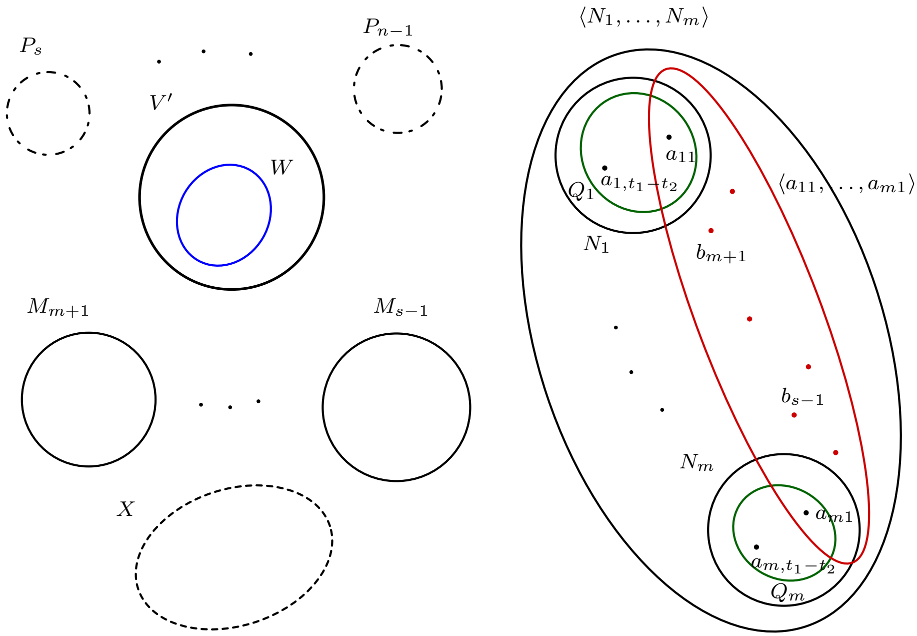

Fig. 1 The $(k; k - t_1, k - t_2)$-SPID described in Remark 2.2

(iii) $\dim(\pi_i \cap \pi_j) \in \{k - t_1, k - t_2\}$, for $i = 1, \ldots, m$ and $j = s, \ldots, n$.

Hence, $\mathcal{S} = \{\pi_1, \ldots, \pi_n\}$ is a set of n distinct k-spaces pairwise meeting either in a space of dimension $k - t_1$ or of dimension $k - t_2$, i.e. a $(k; k - t_1, k - t_2)$-SPID. Also, it is clear that $\mathcal{S}$ is not a $(k - t_1)$-junta. Now, we have that

$$\langle \mathcal{S} \rangle = \langle \pi_1, \ldots, \pi_n \rangle = \langle V', N_1, \ldots, N_m, M_{m+1}, \ldots, M_{s-1}, P_s, \ldots, P_{n-1}, X \rangle.$$

So, by hypothesis,

$$\begin{aligned} \dim\langle \mathcal{S} \rangle &= k + (m-1)t_1 + (s-m-1)(t_1-1) + (n-s)t_2 + (t_1 - m + 1) \\ &= k + (n-1)(t_1-1) - (n-s)(t_1-t_2-1) + 1 \\ &\leq k + (n-1)(t_1-1) - (t_1-t_2) + 2. \end{aligned}$$

We find that the array $\delta(\mathcal{S})$ corresponding to such a SPID is as follows:

$$\delta(\mathcal{S}) = (k, \underbrace{t_1, \ldots, t_1}_{m-1 \text{ times}}, \underbrace{t_1 - 1, \ldots, t_1 - 1}_{s-m-1 \text{ times}}, \underbrace{t_2, \ldots, t_2}_{n-s \text{ times}}, t_1 - m + 1).$$

In the following, we will show that if in addition we ask that the two possible values for the dimensions of the intersection between elements of the SPID are consecutive integers, then the bound in Theorem 2.1 is sharp. Towards this aim, we put beforehand the following result.

Proposition 2.3 *Let $t_1, t_2 \in \mathbb{Z}^+$ with $k > t_1 > t_2 \geq 2$. Let $\mathcal{S}$ be a $(k; k - t_1, k - t_2)$-SPID in a vector space $\mathbb{V}$, with $n = |\mathcal{S}| \geq 3$, such that* $\dim\langle \mathcal{S} \rangle = k + (n-1)(t_1-1) + 1$. *Also let $\delta(\mathcal{S})$ be any non-increasing array associated with $\mathcal{S}$.*
Then, there is no $(k - t_1)$-sunflower of maximal dimension with at least three petals in $\mathcal{S}$, if

and only if

$$\delta(\mathcal{S}) = (k, t_1, t_1 - 1, \ldots, t_1 - 1). \tag{4}$$

Moreover, if $\mathcal{S} = \{\pi_1, \pi_2, \ldots, \pi_n\}$ *has associated non-increasing array* $\delta(\mathcal{S})$ *like in (4), then any permutation* σ*, fixing* π_1 *and* π_2*, does not change* $\delta(\mathcal{S})$.

Proof The necessity is obvious because if any such a $\delta(\mathcal{S})$ is like in (4), then, by Proposition 1.2 and by Remark 1.3, $\mathcal{S}$ can not contain a $(k - t_1)$-sunflower of maximal dimension with at least three petals.

Regarding sufficiency, clearly we have $\dim(\pi_1 \cap \pi_2) = k - t_1$ and by hypothesis in any non-increasing array the largest index m for which $\delta_m = t_1$, is 2. Now, if $\delta_n \le t_1 - 2$, then

$$\begin{aligned}\dim\langle\mathcal{S}\rangle &= \sum_{i=1}^{n} \delta_i = k + t_1 + \sum_{i=3}^{n-1} \delta_i + \delta_n \le k + t_1 + (n-3)(t_1 - 1) + t_1 - 2\\ &= k + (n-1)(t_1 - 1),\end{aligned}$$

a contradiction. Hence, $(\delta_1, \delta_2, \ldots, \delta_n) = (k, t_1, t_1 - 1, \ldots, t_1 - 1)$.
Now, we show that any permutation of the k-spaces in $\mathcal{S}$, fixing π_1 and π_2, does not change the array (4). First of all, we notice that

$$\dim(\pi_j \cap \langle\pi_1, \pi_2\rangle) = k - t_1 + 1, \quad \text{for all } j = 3, \ldots, n.$$

Indeed, for $3 \le j \le n$,

$$\begin{aligned}k - t_1 &\le \dim(\pi_j \cap \pi_1) \le \dim(\pi_j \cap \langle\pi_1, \pi_2\rangle)\\ &\le \dim(\pi_j \cap \langle\pi_1, \pi_2, \ldots, \pi_{j-1}\rangle) = k - t_1 + 1.\end{aligned}$$

If $\dim(\pi_j \cap \langle\pi_1, \pi_2\rangle) = k - t_1$, then

$$\pi_j \cap \pi_1 = \pi_j \cap \langle\pi_1, \pi_2\rangle = \pi_j \cap \pi_2.$$

Consequently, we also have $\pi_j \cap \langle\pi_1, \pi_2\rangle = \pi_1 \cap \pi_2$. This implies that π_1, π_2, π_j form a $(k - t_1)$-sunflower of maximal dimension; in fact, $\dim\langle\pi_1, \pi_2, \pi_j\rangle = k + 2t_1$ and, eventually applying the same procedure as in the proof of Proposition 1.2, we would get a non-increasing array $\delta(\mathcal{S})$ with $\delta_2 = \delta_3 = t_1$; a contradiction. Hence, $\dim(\pi_j \cap \langle\pi_1, \pi_2\rangle) = k - t_1 + 1$.

Nevertheless, since any non-increasing array $\delta(\mathcal{S})$ is as in (4), we have

$$\begin{aligned}k - t_1 + 1 &= \max_{3 \le i \le n} \dim(\pi_i \cap \langle\pi_h \mid h \ne i\rangle) \ge \dim(\pi_j \cap \langle\pi_h \mid h \ne j\rangle)\\ &\ge \dim(\pi_j \cap \langle\pi_1, \pi_2\rangle) = k - t_1 + 1.\end{aligned}$$

Hence, for any $I \subset I_n = \{1, \ldots, n\}$, with $1, 2 \in I$ and $j \notin I$,

$$\dim(\pi_j \cap \langle\pi_h \mid h \in I\rangle) = k - t_1 + 1.$$

Now, let σ be any permutation of I_n such that $\sigma(1) = 1$ and $\sigma(2) = 2$, then

$$\begin{aligned}\delta_j(\mathcal{S}_\sigma) &= k - \dim(\pi_{\sigma(j)} \cap \langle\pi_{\sigma(1)}, \pi_{\sigma(2)}, \ldots, \pi_{\sigma(j-1)}\rangle)\\ &= k - \dim(\pi_{\sigma(j)} \cap \langle\pi_1, \pi_2, \pi_{\sigma(3)}, \ldots, \pi_{\sigma(j-1)}\rangle) = t_1 - 1,\end{aligned}$$

for all $j = 3, \ldots, n$. □

Next, we exhibit four families of $(k; k - t, k - t + 1)$-SPIDs which are not $(k - t)$-juntas, and such that $\dim\langle\mathcal{S}\rangle = k + (n-1)(t-1) + 1$.

In the following, we will denote by $\delta'(\mathcal{S})$ any non-increasing array obtained as described in Proposition 1.2.

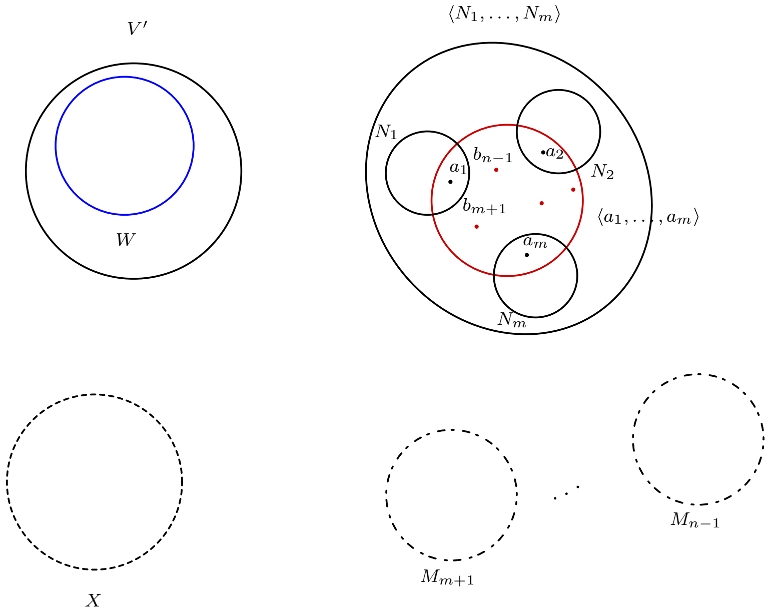

Fig. 2 The $(k; k-t, k-t+1)$-SPID described in Class I

2.1 SPIDs with $\delta' = (k, t, \ldots, t, t-1, \ldots, t-1, t+1-m)$

Let $t \in \mathbb{Z}^+$ such that $2 \le t \le k-1$. Let $m \in \mathbb{Z}^+$ be a positive integer such that $m > 2$. We provide a class of $(k; k-t, k-t+1)$-SPIDs with non-increasing array

$$\delta'(\mathcal{S}) = (k, \underbrace{t, \ldots, t}_{m-1 \text{ times}}, \underbrace{t-1, \ldots, t-1}_{n-m-1 \text{ times}}, t+1-m).$$

- **Class I**

Let $2 < m \le \min\{t+1, n-1\}$. Let V', X, $N_1, \ldots, N_m$ and $M_{m+1}, \ldots, M_{n-1}$ be linearly independent subspaces of $\mathbb{V}$ such that $\dim V' = k-t$, $\dim X = t-m+1$, $\dim N_i = t$ for $i = 1, \ldots, m$, and $\dim M_j = t-1$ for $j = m+1, \ldots, n-1$ (Fig. 2). Let $a_1, \ldots, a_m$ be 1-spaces in $N_1, \ldots, N_m$, respectively. Also, let $b_{m+1}, \ldots, b_{n-1}$ be 1-spaces in $\langle a_1, \ldots, a_m \rangle$ such that either

(i) at least two of them are the same 1-space, or
(ii) at least one of them is equal to a_i, with $i \in \{1, \ldots, m\}$.

Let W be a $(k-t-1)$-space in V'. Then we define the k-spaces $\pi_1, \ldots, \pi_n$ as follows.

- $\pi_1 = \langle V', N_1 \rangle, \pi_2 = \langle V', N_2 \rangle, \ldots, \pi_m = \langle V', N_m \rangle$,
- $\pi_{m+1} = \langle V', M_{m+1}, b_{m+1} \rangle, \ldots, \pi_{n-1} = \langle V', M_{n-1}, b_{n-1} \rangle$,
- $\pi_n = \langle W, a_1, \ldots, a_m, X \rangle$.

By Requests (i) and (ii), it is clear that the pairwise intersection of distinct spaces π_i and π_j, $i, j = 1, \ldots, n-1$, either is the $(k-t)$-space V' or it is a $(k-t+1)$-space containing V'. Moreover, since each of the spaces $\pi_1, \ldots, \pi_{n-1}$ contains a unique 1-space from the set $\{a_1, \ldots, a_m, b_{m+1}, \ldots, b_{n-1}\}$ (note that by Properties (i) and (ii), some of the 1-spaces could be equal), we have $\dim(\pi_n \cap \pi_i) = k-t$, for all $i = 1, \ldots, n-1$. Hence, the set

$\mathcal{S} = \{\pi_1, \ldots, \pi_n\}$ is a set of n distinct k-spaces pairwise meeting in a space of dimension $k-t$ or $k-t+1$. Also, since not all pairwise intersections equal the same $(k-t)$-space, $\mathcal{S}$ is not a $(k-t)$-junta. The set $\{a_1, \ldots, a_m, b_{m+1}, \ldots, b_{n-1}\}$ is contained in $\langle N_1, \ldots, N_m \rangle$ and $W \subset V'$. Then

$$\langle \mathcal{S} \rangle = \langle \pi_1, \ldots, \pi_n \rangle = \langle V', N_1, \ldots, N_m, M_{m+1}, \ldots, M_{n-1}, X \rangle.$$

Clearly, since V', X, $N_1, \ldots, N_m$ and $M_{m+1}, \ldots, M_{n-1}$ are linearly independent spaces of $\mathbb{V}$, we have that

$$\begin{aligned} \dim\langle \mathcal{S} \rangle &= k - t + m \cdot t + (n-1-m) \cdot (t-1) + t - m + 1 \\ &= k + (n-1)(t-1) + 1. \end{aligned}$$

Arguing as in Proposition 1.2, we find

$$\delta'(\mathcal{S}) = (k, \underbrace{t, \ldots, t}_{m-1 \text{ times}}, \underbrace{t-1, \ldots, t-1}_{n-m-1 \text{ times}}, t+1-m).$$

Lemma 2.4 *Let $\mathcal{S}$ be a $(k; k-t, k-t+1)$-SPID of $\mathbb{V}$, where $2 \leq t \leq k-1$, such that $\mathcal{S}$ is not a $(k-t)$-junta, with $|\mathcal{S}| = n \geq 3$. If $\dim\langle \mathcal{S} \rangle = k + (n-1)(t-1) + 1$ and there exists a $(k-t)$-sunflower of maximal dimension with at least three petals in $\mathcal{S}$, then $\mathcal{S}$ belongs to Class I.*

Proof Since $\dim\langle \mathcal{S} \rangle = k + (n-1)(t-1) + 1$, from the proof of Theorem 2.1 we get

$$\sum_{i=1}^{n} \delta_i = k + (n-1)(t_1 - 1) - (n-r)(m-2) + 1.$$

Moreover, since this implies that $(n-r)(m-2) = 0$ and $m \geq 3$, necessarily $r = n$. Then,

$$\delta'(\mathcal{S}) = (k, \underbrace{t, \ldots, t}_{m-1 \text{ times}}, \underbrace{t-1, \ldots, t-1}_{n-m-1 \text{ times}}, t+1-m). \tag{5}$$

Consider $\mathcal{S}' = \{\pi_1, \ldots, \pi_{n-1}\}$. Since $\dim\langle \mathcal{S}' \rangle = k + (n-2)(t-1) + m - 1 \geq k + (n-2)(t-1) + 2$, then, by Theorem 2.1, we have that $\mathcal{S}'$ is a $(k-t)$-junta. Let V' be the common $(k-t)$-space through which the k-spaces $\pi_1, \ldots, \pi_{n-1}$ pass, and denote $k - t - \dim(\pi_n \cap V')$ by ε. Since $\mathcal{S}$ is not a junta, $\varepsilon \geq 1$; indeed, by the proof of Theorem 2.1, necessarily $\varepsilon = 1$. Let W denote the $(k-t-1)$-subspace $\pi_n \cap V'$. Furthermore, we note the first k-spaces form a sunflower of maximal dimension since $\delta_2 = \cdots = \delta_m = t$. Hence, there exist t-spaces $N_1, \ldots, N_m$, with $i = 1, \ldots, m$, such that $N_1, \ldots, N_m$, V' are linearly independent, and $\pi_i = \langle V', N_i \rangle$. Also, by hypothesis, there exist at least two k-spaces in $\mathcal{S}$ such that they meet in a $(k-t+1)$-space. We first show that

$$\dim(\pi_n \cap \pi_j) = k - t, \qquad \text{for all } j \in \{1, \ldots, n-1\}.$$

For this purpose, suppose by way of contradiction that there exists a $j \in \{1, \ldots, n-1\}$ such that $\dim(\pi_n \cap \pi_j) = k - t + 1$; we may distinguish between two cases:

(a) $j \in \{1, \ldots, m\}$. Then there are two 1-spaces a_{j_1} and a_{j_2} in $\pi_n \cap \pi_j$ not in V', and there is at least another 1-space $a_i \in \pi_n \cap \pi_i$, for all $i \in \{1, \ldots, m\} \setminus \{j\}$ not in V'. Without loosing any generality, we may choose the N_i's in such a way that $a_1 \in N_1, \ldots, a_m \in N_m$ and $\langle a_{j_1}, a_{j_2} \rangle \subseteq N_j$. Hence,

$$\pi_n \cap \langle \pi_1, \ldots, \pi_{n-1} \rangle \supseteq \langle W, a_1, \ldots, a_{j-1}, a_{j_1}, a_{j_2}, a_{j+1}, \ldots, a_m \rangle,$$

obtaining that

$$t-m+1=\delta_n \le k-\dim\langle W, a_1, \ldots, a_{j-1}, a_{j_1}, a_{j_2}, a_{j+1}, \ldots, a_m\rangle = t-m,$$

a contradiction.

(b) $j \in \{m+1, \ldots, n-1\}$. Since, by Point (a), $\dim(\pi_n \cap \pi_i) = k-t$ for every $i = 1, \ldots, m$, π_n contains the 1-spaces $a_1 \in \pi_1, \ldots, a_m \in \pi_m$, meeting V' trivially. Furthermore, since $\dim(\pi_n \cap \pi_j) = k-t+1$, there must be two 1-spaces $a', a'' \in \pi_n \cap \pi_j$ not in V' and such that $\langle a', a''\rangle \cap W$ is trivial. Also, the subspace $\langle a', a''\rangle$ can not be contained in $\langle V', a_1, \ldots, a_m\rangle$, otherwise we would have

$$\pi_j \cap \langle V', a_1, \ldots, a_m\rangle \supseteq \langle V', a', a''\rangle,$$

and, consequently,

$$t-1=\delta_j \le k-\dim\langle V', a', a''\rangle = t-2.$$

Moreover, $\langle a', a''\rangle$ meets $\langle V', a_1, \ldots, a_m\rangle$ in a 1-space, otherwise

$$\pi_n \cap \langle \pi_1, \ldots, \pi_{n-1}\rangle \supseteq \langle W, a_1, \ldots, a_m, a', a''\rangle$$

obtaining again $t-m+1=\delta_n \le t-m-1$. However, if $b \in \langle a', a''\rangle \setminus \langle V', a_1, \ldots, a_m\rangle$, then

$$t-m+1=\delta_n \le k-\dim\langle W, a_1 \ldots, a_m, b\rangle = t-m;$$

which is again a contradiction.

Hence, definitely $\dim(\pi_n \cap \pi_j) = k-t$, for each $j \in \{1, \ldots, n-1\}$.
Now, since $\delta_n = t-m+1$ and π_n intersects V' in the $(k-t-1)$-dimensional subspace W, we get that the k-space π_n may be realised as follows

$$\pi_n = \langle W, a_1, \ldots, a_m, X\rangle,$$

for suitable points $a_1 \in N_1, \ldots, a_m \in N_m$ and X a $(t-m+1)$-dimensional subspace such that $V', N_1, \ldots, N_m, X$ are linearly independent.

Since $\pi_n \cap \pi_j$, $j = m+1, \ldots, n-1$, is a $(k-t)$-space contained in $\langle W, a_1, \ldots, a_m\rangle$, there must exist a 1-space b_j in $\langle a_1, \ldots, a_m\rangle \setminus W$ such that $\pi_n \cap \pi_j = \langle W, b_j\rangle$ otherwise

$$t-m+1=\delta_n = k-\dim(\pi_n \cap \langle \pi_1, \ldots, \pi_{n-1}\rangle) \le k-\dim\langle W, a_1, \ldots, a_m, b_j\rangle = t-m,$$

a contradiction. Moreover, since $\delta_j = t-1$, it is immediate that for any $j = m+1, \ldots, n-1$, we have $\pi_j = \langle V', b_j, M_j\rangle$, with M_j a $(t-1)$-space and such that $V', N_1, \ldots, N_m, M_{m+1}, \ldots, M_{n-1}$ and X are linearly independent. Note explicitly that if $\dim(\pi_i \cap \pi_j) = k-t+1$, with $i, j \in \{m+1, \ldots, n-1\}$, then $b_i = b_j$. Indeed, let $\pi_i \cap \pi_j = \langle V', a'\rangle$. This space is contained in $\langle \pi_1, \pi_2, \ldots, \pi_m\rangle$, since if $a' \notin \langle \pi_1, \pi_2, \ldots, \pi_m\rangle$, assuming $j > i$, we have

$$\pi_j \cap \langle \pi_1, \ldots, \pi_i\rangle \supseteq \langle V', b_j, a'\rangle,$$

obtaining $\delta_j \le t-2$. So, $\langle V', a'\rangle \subseteq \langle \pi_1, \ldots, \pi_m\rangle$. Now, since $\pi_i \cap \langle \pi_1, \pi_2, \ldots, \pi_m\rangle = \langle V', b_i\rangle$ and $\pi_j \cap \langle \pi_1, \pi_2, \ldots, \pi_m\rangle = \langle V', b_j\rangle$ have dimension $k-t+1$ and

$$\pi_i \cap \pi_j = \pi_i \cap \pi_j \cap \langle \pi_1, \pi_2, \ldots, \pi_m\rangle,$$

we get that both $\langle V', b_i\rangle$ and $\langle V', b_j\rangle$ are equal to $\pi_i \cap \pi_j$. Now, since we assumed $j > i$, if $b_i \neq b_j$ then again we would have $t-1=\delta_j \le t-2$, which is not the case. Suppose that

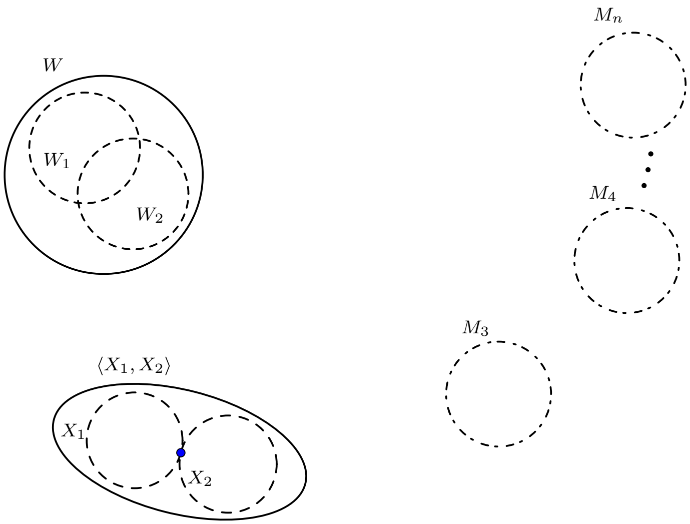

Fig. 3 The $(k; k-t, k-t+1)$-SPIDs described in Class II

there exist $i \in \{1,\dots,m\}$ and $j \in \{m+1,\dots,n-1\}$ such that $\dim(\pi_i \cap \pi_j) = k-t+1$, then there exists a 1-space $a' \in N_i$ such that

$$\langle V', a'\rangle = \pi_i \cap \pi_j \subseteq \pi_j \cap \langle \pi_1,\dots,\pi_m\rangle = \langle V', b_j\rangle.$$

Hence, $b_j \in \langle V', a'\rangle$ and, since $\delta_j = t-1$, $b_j \in \langle a_1,\dots,a_m\rangle \cap N_i$ otherwise

$$t-1 = \delta_j = k - \dim(\pi_j \cap \langle \pi_1,\dots,\pi_{j-1}\rangle) \le k - \dim\langle V', a', b_j\rangle = t-2;$$

this implies that $b_j = a_i$. Note explicitly that a k-space π_j in $\mathcal{S}$, with $j \in \{m+1,\dots,n-1\}$, can meet at most one π_i, with $i \in \{1,\dots,m\}$, in a $(k-t+1)$-space. Finally, it is possible that in $\mathcal{S}$ there exists a k-space π_j, with $j \in \{m+1,\dots,n-1\}$, that intersects π_i, with $i \in \{1,\dots,m\}$, and π_h, with $h \in \{m+1,\dots,n-1\}$, in two $(k-t+1)$-spaces. From previous results, $b_h = b_j = a_i$. So, $\mathcal{S}$ belongs to Class I. □

2.2 SPIDs with $\delta' = (k, t, t-1, t-1, \dots, t-1)$

• **Class II**
Choose integers $n \ge 3$ and k, t such that $2 \le t \le k-1$. Let W be a $(k-t+1)$-subspace of $\mathbb{V}$, and X_1, X_2 t-spaces such that $\dim\langle X_1, X_2\rangle = 2t-1$. Moreover, consider $M_3,\dots,M_n$ $(t-1)$-subspaces of $\mathbb{V}$ such that W, $\langle X_1, X_2\rangle$, $M_3,\dots,M_n$ are linearly independent. Let W_1 and W_2 be two $(k-t)$-spaces in W (Fig. 3). Then we define the sets $\pi_1,\dots,\pi_n$ as follows:

- $\pi_1 = \langle W_1, X_1\rangle$, $\pi_2 = \langle W_2, X_2\rangle$,
- $\pi_3 = \langle W, M_3\rangle$, ..., $\pi_n = \langle W, M_n\rangle$.

Now, since $\dim(X_1 \cap X_2) = 1$, $\pi_1 \cap \pi_2$ is a $(k-t)$-space. Moreover, these two spaces meet other ones either in W_1 or in W_2, and $\{\pi_3,\dots,\pi_n\}$ is a $(k-t+1)$-sunflower with center W. Clearly,

$$\langle \mathcal{S}\rangle = \langle \pi_1,\dots,\pi_n\rangle = \langle W, X_1, X_2, M_3,\dots,M_n\rangle.$$

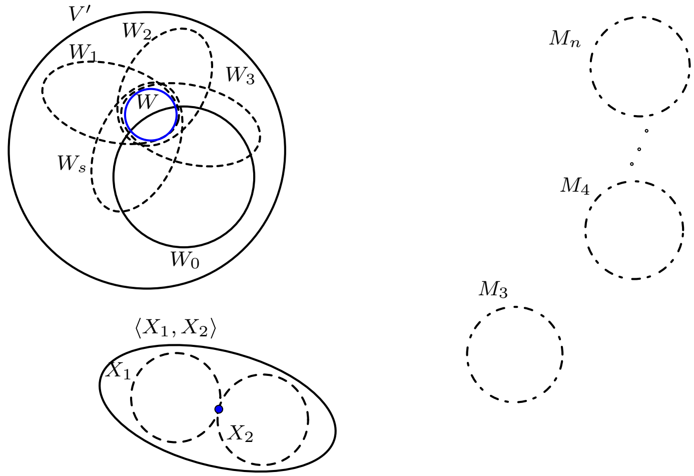

Fig. 4 The $(k; k-t, k-t+1)$-SPIDs described in Class III

Since W, $\langle X_1, X_2\rangle$, $M_3, \ldots, M_n$ are linearly independent, we find that

$$\begin{aligned}\dim\langle\mathcal{S}\rangle &= k-t+1+2t-1+(n-2)\cdot(t-1)\\ &= k+(n-1)(t-1)+1.\end{aligned}$$

Again arguing as in Proposition 1.2, we get

$$\delta'(\mathcal{S}) = (k, t, t-1, \ldots, t-1).$$

We observe that particular examples in this class contain $(k-t+1)$-sunflowers of maximal dimension, but do not contain $(k-t)$-sunflowers of maximal dimension. Nonetheless, they are $(k-t-1)$-juntas.

- **Class III**

Choose integers $n \geq 3$, $2 \leq s < n$ and k, t such that $2 \leq t \leq k-1$. Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$ which is either infinite or else a finite field $\mathbb{F}$ of order q with q a prime power such that $q+1 \geq s$. Let V', $\langle X_1, X_2\rangle$, $M_3, \ldots, M_n$ be linearly independent subspaces of $\mathbb{V}$ such that $\dim V' = k-t+2$, and $\dim X_1 = t$, $\dim X_2 = t-1$ with $\dim(X_1 \cap X_2) = 1$ and $\dim M_i = t-1$, for $i = 3, \ldots, n$. Let $W_0, W_1, \ldots, W_s$ be distinct $(k-t+1)$-spaces in V' such that $W_1, \ldots, W_s$ go through a $(k-t)$-space W (Fig. 4), and W_0 does not pass through W. We define the sets

$$\begin{aligned}\pi_1 &= \langle W, X_1\rangle,\ \pi_2 = \langle W_0, X_2\rangle,\\ \pi_3 &= \langle W_1, M_3\rangle, \ldots, \pi_{m_1} = \langle W_1, M_{m_1}\rangle,\\ \pi_{m_1+1} &= \langle W_2, M_{m_1+1}\rangle, \ldots, \pi_{m_2} = \langle W_2, M_{m_2}\rangle,\\ &\ldots\\ \pi_{m_{s-1}+1} &= \langle W_s, M_{m_{s-1}+1}\rangle, \ldots, \pi_n = \langle W_s, M_n\rangle.\end{aligned}$$

Clearly, $\mathcal{S} = \{\pi_1, \ldots, \pi_n\}$ is a $(k; k-t, k-t+1)$-SPID which is not a $(k-t)$-junta and

$$\langle\mathcal{S}\rangle = \langle\pi_1, \ldots, \pi_n\rangle = \langle V', X_1, X_2, M_3, \ldots, M_n\rangle.$$

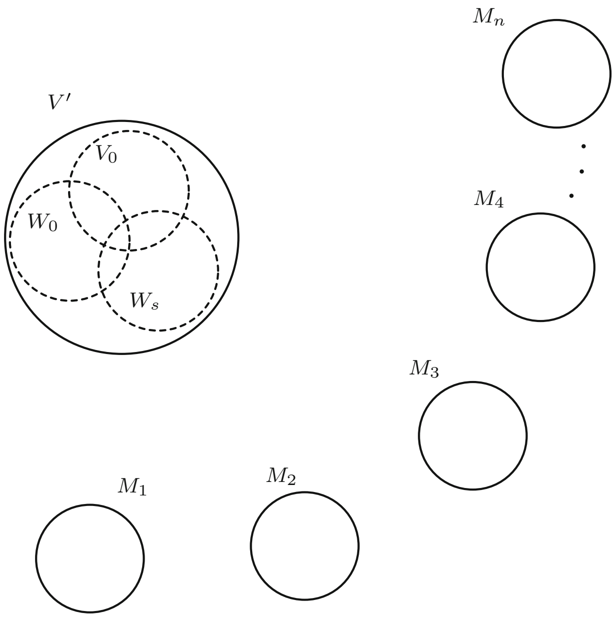

Fig. 5 The $(k; k-t, k-t+1)$-SPIDs described in Class IV

Since V', $\langle X_1, X_2\rangle$, $M_3, \ldots, M_n$ are linearly independent, we find that

$$\begin{aligned}\dim\langle \mathcal{S}\rangle &= k-t+2+2t-2+(n-2)(t-1)\\ &= k+(n-1)(t-1)+1.\end{aligned}$$

Also in this case we have $\delta'(\mathcal{S}) = (k, t, t-1, \ldots, t-1)$.

Examples in this class may contain $(k-t)$-sunflowers not of maximal dimension and $(k-t+1)$-sunflowers of maximal dimension.

• **Class IV**

Choose integers $n \geq 3$, $2 \leq s < n$ and k, t such that $2 \leq t \leq k-1$. Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$ which is either infinite or else a finite field $\mathbb{F}$ of order q with q a prime power such that $\frac{q^{k-t+2}-1}{q-1} \geq s+2$. Let $V', M_1, \ldots, M_n$ be linearly independent subspaces of $\mathbb{V}$ such that $\dim V' = k-t+2$, and $\dim M_i = t-1$, for $i = 1, \ldots, n$. Let $V_0, W_0, W_1, \ldots, W_s$ be $s+2$ $(k-t+1)$-spaces in V' such that they do not go through the same $(k-t)$-space, with $W_1, \ldots, W_s$ distinct (Fig. 5) (which in the case $\mathbb{V}$ is a vector space over a finite field of order q, exist for the above assumption on q). We define the sets

$$\begin{aligned}\pi_1 &= \langle V_0, M_1\rangle,\ \pi_2 = \langle W_0, M_2\rangle,\\ \pi_3 &= \langle W_1, M_3\rangle, \ldots,\ \pi_{m_1} = \langle W_1, M_{m_1}\rangle,\\ \pi_{m_1+1} &= \langle W_2, M_{m_1+1}\rangle, \ldots,\ \pi_{m_2} = \langle W_2, M_{m_2}\rangle,\\ &\ldots\\ \pi_{m_{s-1}+1} &= \langle W_s, M_{m_{s-1}+1}\rangle, \ldots,\ \pi_n = \langle W_s, M_n\rangle.\end{aligned}$$

Clearly, the set $\mathcal{S}$ is a $\{k; k-t, k-t+1\}$-SPID such that it is not a $(k-t)$-junta and

$$\langle \mathcal{S}\rangle = \langle \pi_1, \ldots, \pi_n\rangle = \langle V', M_1, M_2, M_3, \ldots, M_n\rangle.$$

Since $V', M_1, M_2, M_3, \ldots, M_n$ are linearly independent, we find that

$$\dim\langle\mathcal{S}\rangle = k - t + 2 + n\cdot(t-1) = k + (n-1)(t-1) + 1,$$

and $\delta'(\mathcal{S}) = (k, t, t-1, \ldots, t-1)$.

Examples in this last class may contain $(k-t+1)$-sunflowers of maximal dimension and $(k-t)$-sunflowers not of maximal dimension.

3 Tightness of the junta-property bound for $(k; k-t, k-t+1)$-SPIDs

We will prove the following classification result.

Theorem 3.1 *Let $\mathcal{S}$ be a $(k; k-t, k-t+1)$-SPID in a vector space $\mathbb{V}$, with $|\mathcal{S}| = n \geq 3$ and $2 \leq t \leq k-1$. If the dimension of $\langle\mathcal{S}\rangle$ is $k + (n-1)(t-1) + 1$, then $\mathcal{S}$ is either a $(k-t)$-junta or $\mathcal{S}$ is one of the examples described in Classes I, II, III or IV.*

First, we state the following lemma.

Lemma 3.2 *Let $\mathcal{S}$ be a $(k; k-t, k-t+1)$-SPID $(2 \leq t \leq k-1)$ in a vector space $\mathbb{V}$ such that $n = |\mathcal{S}| \geq 3$ and $\mathcal{S}$ is not a $(k-t)$-junta.*
If $\dim\langle\mathcal{S}\rangle = k+(n-1)(t-1)+1$ and there is not a $(k-t)$-sunflower of maximal dimension with at least three petals in $\mathcal{S}$, then $\mathcal{S}$ is equivalent to one of the examples described in Classes II, III, or IV.

Proof By Propositions 1.2 and 2.3, we may sort k-subspaces in $\mathcal{S}$ in such a way that

$$(\delta_1, \delta_2, \ldots, \delta_n) = (k, t, t-1, \ldots, t-1). \tag{6}$$

Also, arguing as in the proof of Proposition 2.3, we get that $\dim(\pi_4 \cap \langle\pi_1, \pi_2, \pi_3\rangle) = \dim(\pi_4 \cap \langle\pi_1, \pi_2\rangle) = k - t + 1$, and hence

$$\pi_4 \cap \pi_3 \subseteq \pi_4 \cap \langle\pi_1, \pi_2, \pi_3\rangle = \pi_4 \cap \langle\pi_1, \pi_2\rangle \subseteq \langle\pi_1, \pi_2\rangle.$$

Eventually rearranging the spaces $\pi_3, \ldots, \pi_n$ in $\mathcal{S}$, we can repeat the previous argument, getting

$$\pi_i \cap \pi_j \subseteq \langle\pi_1, \pi_2\rangle,$$

for all distinct π_i and π_j, with $i, j \in \{3, \ldots, n\}$. Moreover, $\dim(\pi_i \cap \langle\pi_1, \pi_2\rangle) = \dim(\pi_j \cap \langle\pi_1, \pi_2\rangle) = k - t + 1$.

Now, define in $\mathcal{S}' = \{\pi_3, \ldots, \pi_n\}$, the following binary relation

$$\pi_i \sim \pi_j \iff \pi_i \cap \langle\pi_1, \pi_2\rangle = \pi_j \cap \langle\pi_1, \pi_2\rangle,$$

for $i, j = 3, \ldots, n$.
Clearly, $\sim$ is an equivalence relation on $\mathcal{S}'$. The k-spaces of an equivalence class meet $\langle\pi_1, \pi_2\rangle$ in the same $(k-t+1)$-space. In this way, we get s distinct $(k-t+1)$-dimensional spaces in $\langle\pi_1, \pi_2\rangle$, say $W_1, \ldots, W_s$, where s is a given integer, $1 \leq s \leq n-2$, pairwise intersecting in a $(k-t)$-space. Indeed, let π_i and π_j be k-spaces of $\mathcal{S}'$ in different equivalence classes. Then, by the proof of Proposition 2.3, $\dim(\pi_i \cap \langle\pi_1, \pi_2\rangle) = \dim(\pi_j \cap \langle\pi_1, \pi_2\rangle) = k-t+1$,

$$\pi_i \cap \langle\pi_1, \pi_2\rangle = W_\ell \text{ and } \pi_j \cap \langle\pi_1, \pi_2\rangle = W_m,$$

for some distinct $\ell, m \in \{1, \dots, s\}$. Hence,

$$\pi_i \cap \pi_j = \pi_i \cap \pi_j \cap \langle \pi_1, \pi_2 \rangle = W_\ell \cap W_m.$$

Since $k-t \leq \dim(\pi_i \cap \pi_j) = \dim(W_\ell \cap W_m)$ and W_ℓ, W_m are distinct $(k-t+1)$-subspaces,

$$\dim(W_\ell \cap W_m) = k - t.$$

Now, since the relation $\sim$ induces a partition $J_1, J_2, \dots, J_s$ on the elements of the index set $\{3, 4, \dots, n\}$, by Proposition 2.3, we can label appropriately the elements of $\mathcal{S}$, obtaining

$$\begin{aligned} \pi_{j_1} &= \langle W_1, M_{j_1} \rangle \quad \text{with } j_1 \in J_1, \\ \pi_{j_2} &= \langle W_2, M_{j_2} \rangle \quad \text{with } j_2 \in J_2, \\ &\dots \\ \pi_{j_s} &= \langle W_s, M_{j_s} \rangle \quad \text{with } j_s \in J_s, \end{aligned}$$

where the elements in the set $\{M_{j_h} : j_h \in J_h,\ h \in \{1, 2, \dots, s\}\}$, are certain linearly independent $(t-1)$-spaces in $\mathbb{V}$.

We divide the remainder of the proof in two steps:

(1) First, we look at the case where all elements in $\mathcal{S}'$ meet $\langle \pi_1, \pi_2 \rangle$ in the same $(k-t+1)$-space, say W. It is clear that for $3 \leq j \leq n$ and $i = 1, 2$, $\pi_i \cap \pi_j = \pi_i \cap W$. Indeed, since $W = \pi_j \cap \langle \pi_1, \pi_2 \rangle$; we have

$$\pi_i \cap \pi_j \subseteq W.$$

Hence, $\pi_i \cap \pi_j \subseteq W \cap \pi_i$. On the other hand, since

$$W = \pi_j \cap \langle \pi_i, \pi_2 \rangle \subseteq \pi_j,$$

then we also have $W \cap \pi_i \subseteq \pi_j \cap \pi_i$. Next we show that

$$\dim(\pi_i \cap \pi_j) = k - t,$$

for $3 \leq j \leq n$ and $i = 1, 2$. To this aim, suppose that either the space π_1 or π_2 contains W ($W \nsubseteq \pi_1 \cap \pi_2$, since $\dim(\pi_1 \cap \pi_2) = k - t$). For instance, let π_1 contain W. Then, $\pi_2 \cap W = \pi_1 \cap \pi_2$; in fact, we have that $\pi_1 \cap \pi_2 \supseteq \pi_2 \cap W = \pi_2 \cap \pi_j$ with $j \in \{3, ..., n\}$. But then $\mathcal{S}$ is a $(k-t)$-junta; a contradiction. Hence, $\pi_1 \cap W = W_1$ and $\pi_2 \cap W = W_2$ are $(k-t)$-spaces, and they are distinct otherwise $\mathcal{S}$ is again a $(k-t)$-junta. Precisely, they are two hyperplanes of W. We denote by W' the $(k-t-1)$-space of W in which they meet, and choose a basis of $\mathbb{V}$ in such a way that the following happens

$$\pi_1 \cap W = \langle W', a_1 \rangle \quad \text{and} \quad \pi_2 \cap W = \langle W', a_2 \rangle,$$

with a_1, a_2 distinct 1-spaces in $W_1 \setminus W'$ and $W_2 \setminus W'$, with W', a_1, a_2 linearly independent. Then, there also exist two t-spaces X_1 and X_2, having a 1-space in common and such that

$$\pi_1 = \langle W', a_1, X_1 \rangle, \quad \pi_2 = \langle W', a_2, X_2 \rangle.$$

This finally means that $\mathcal{S}$ is one of the examples in Class II.

(2) Now, we suppose that $s \geq 2$. In this case, $W_1, \dots, W_s$ are $(k-t+1)$-spaces pairwise intersecting in a $(k-t)$-space. Hence, by [3, Sect. 9.3], either

(a) they have a $(k-t)$-space in common, or
(b) they lie in a $(k-t+2)$-space V'.

Note explicitly that for $s = 2$, (a) and (b) are equivalent. If $s \geq 3$, we will show that

$$\dim\langle W_1, W_2, \ldots, W_s\rangle = k - t + 2, \tag{7}$$

which is equivalent to prove that, for all $1 \leq h \leq s$,

$$W_h \subseteq \langle W_1, W_2\rangle. \tag{8}$$

Suppose that $W_1, W_2, \ldots, W_s$ go through a $(k-t)$-space in $\langle \pi_1, \pi_2\rangle$ and let $\pi_{j_1}, \pi_{j_2}, \pi_{j_h}$ be k-spaces belonging to different equivalence classes with respect to $\sim$, such that

$$\pi_{j_1} = \langle W_1, M_{j_1}\rangle \qquad \pi_{j_2} = \langle W_2, M_{j_2}\rangle \qquad \pi_{j_h} = \langle W_h, M_{j_h}\rangle.$$

Since there is not a sunflower of maximal dimension with at least three petals contained in $\mathcal{S}$, we have

$$\dim(\pi_{j_h} \cap \langle \pi_{j_1}, \pi_{j_2}\rangle) \geq k - t + 1.$$

Then, by applying Grassmann's Formula, we obtain

$$\begin{aligned} k - t + 1 \leq \dim(\pi_{j_h} \cap \langle \pi_{j_1}, \pi_{j_2}\rangle) &= 2k + t - \dim\langle W_1, W_2, W_h, M_{j_1}, M_{j_2}, M_{j_h}\rangle \\ &= 2k + t - 3(t-1) - (\dim W_h + \dim\langle W_1, W_2\rangle - \dim(W_h \cap \langle W_1, W_2\rangle)). \end{aligned}$$

This implies $\dim(W_h \cap \langle W_1, W_2\rangle) \geq k - t + 1$ and hence we get property (8). So, all $(k-t+1)$-spaces $W_1, \ldots, W_s$ lie in a $(k-t+2)$-space, say V'.
Obviously, V' is contained in $\langle \pi_1, \pi_2\rangle$ and

$$k - t \leq \dim(\pi_i \cap V') \leq k - t + 1, \qquad \text{for } i = 1, 2. \tag{9}$$

Indeed, since for $i \in \{1, 2\}$ and for any $j \in \{3, \ldots, n\}$,

$$\pi_i \cap \pi_j = \pi_i \cap \pi_j \cap \langle \pi_1, \pi_2\rangle = \pi_i \cap W_h \subseteq \pi_i \cap V',$$

for some $h \in \{1, \ldots, s\}$, then the first inequality in (9) follows.
On the other hand, if $\dim(\pi_i \cap V') \geq k - t + 2$, for $i = 1$ or 2, then V' is contained either in π_1 or in π_2 (not in both since $\dim(\pi_1 \cap \pi_2) = k - t$). Without loss of generality, we can suppose that V' is contained in π_1. Then

$$\pi_2 \cap \pi_{j_h} = \pi_2 \cap W_h \subseteq \pi_2 \cap V' \subseteq \pi_1 \cap \pi_2,$$

for $h \in \{1, \ldots, s\}$. This implies that $\pi_1 \cap \pi_2$ is contained in all elements of $\mathcal{S}$ and hence it is a $(k-t)$-junta. Furthermore, $\pi_1 \cap V'$ and $\pi_2 \cap V'$ are distinct subspaces. Indeed,

($\diamond$) if $\pi_1 \cap V' = \pi_2 \cap V'$ and it is a $(k-t+1)$-space, then $\pi_1 \cap \pi_2$ is a $(k-t+1)$-space, a contradiction;

($\diamond\diamond$) if $\pi_1 \cap V' = \pi_2 \cap V'$ is a $(k-t)$-space, since for $i = 1, 2$ and $h = 1, \ldots, s$, $\pi_i \cap W_h$ has dimension at least $k - t$ and $W_h \subseteq V'$, we have that

$$\pi_1 \cap W_h = \pi_1 \cap V' = \pi_2 \cap V' = \pi_2 \cap W_h.$$

This implies that $\pi_1 \cap \pi_2$ is contained in all elements of $\mathcal{S}$; again this is not the case.

Now, let W_h be a $(k-t+1)$-space with $1 \leq h \leq s$, then

$$\begin{aligned} k - t &= \dim(\pi_1 \cap \pi_2) \geq \dim(\pi_1 \cap \pi_2 \cap W_h) \\ &\geq \dim(\pi_1 \cap W_h) + \dim(\pi_2 \cap W_h) - \dim W_h \geq 2(k-t) - k + t - 1 = k - t - 1. \end{aligned} \tag{10}$$

By taking into account Inequalities (9) and (10), the discussion may be reduced to one of the following three cases:

(i) $\dim(\pi_1 \cap V') = \dim(\pi_2 \cap V') = k - t$ (and $\dim(\pi_1 \cap \pi_2 \cap V') = k - t - 1$).
(ii) π_1 and π_2 meet V' in subspaces with different dimensions.
(iii) $\pi_1 \cap V'$ and $\pi_2 \cap V'$ are two hyperplanes of V'.

Case (i): We shall show that for all $3 \leq j \leq n$,

$$\pi_j \cap \langle \pi_1, \pi_2 \rangle = \langle \pi_1 \cap V', \pi_2 \cap V' \rangle. \tag{11}$$

Since $\pi_j \cap \langle \pi_1, \pi_2 \rangle \subseteq V'$, and π_1 and π_2 meet V' in a $(k - t)$-space,

$$\pi_j \cap \pi_1 = \pi_1 \cap V' \text{ and } \pi_j \cap \pi_2 = \pi_2 \cap V' ,$$

obtaining that

$$\pi_j \cap \langle \pi_1, \pi_2 \rangle \supseteq \langle \pi_j \cap \pi_1, \pi_j \cap \pi_2 \rangle = \langle \pi_1 \cap V', \pi_2 \cap V' \rangle.$$

However, since they both are $(k - t)$-spaces in V', we obtain equality stated in (11). Hence, every π_j, $j = 3, \ldots, n$, meets $\langle \pi_1, \pi_2 \rangle$ in the same $(k - t + 1)$-subspace. But this contradicts $s \geq 2$.
Case (ii): We can suppose, without loss of generality, that

$$\dim(\pi_1 \cap V') = k - t \quad \text{and} \quad \dim(\pi_2 \cap V') = k - t + 1.$$

Clearly, $\pi_1 \cap V' \not\subseteq \pi_2 \cap V'$, otherwise

$$\pi_1 \cap W_h = \pi_1 \cap V' \subseteq \pi_2 \cap V'.$$

This implies that $\pi_1 \cap \pi_2 \subseteq W_h$, for $h = 1, \ldots, s$, and then $\mathcal{S}$ is a $(k - t)$-junta with center $\pi_1 \cap \pi_2$. Since $W = \pi_1 \cap \pi_2 \cap V'$ is a $(k - t - 1)$-space, there exists a t-space X_1 contained in π_1 and a $(t - 1)$-space X_2 contained in π_2 both disjoint from V', for $i = 1, 2$, and such that $\langle X_1, X_2 \rangle = 2t - 2$. Then

$$\pi_1 = \langle \pi_1 \cap V', X_1 \rangle \text{ and } \pi_2 = \langle \pi_2 \cap V', X_2 \rangle.$$

We note explicitly that

$$\pi_1 \cap V' = \pi_1 \cap W_h \subseteq W_h, \tag{12}$$

for $h \in \{1, \ldots, s\}$, since $\dim(\pi_1 \cap V') = k - t$.
Case (iii): Now, we suppose that $\pi_1 \cap V'$ and $\pi_2 \cap V'$ are hyperplanes of V', say V_0 and W_0, respectively. Then, there exists X_i, $i = 1, 2$, a $(t - 1)$-space in π_i disjoint from V', such that

$$\pi_1 = \langle V_0, X_1 \rangle \text{ and } \pi_2 = \langle W_0, X_2 \rangle.$$

Again, by Grassmann's Formula, we obtain that X_1, X_2, V' are linearly independent and $\dim\langle X_1, X_2 \rangle = 2t - 2$.

So, the discussion in Case (ii) provides us with an example described in Class III, while Case (iii) gives an example in Class IV. □

Remark 3.3 Let $\mathcal{W} = \{W_1, \ldots, W_s, \pi_2 \cap V'\}$ be the set of $(k - t + 1)$-spaces in V' with $2 \leq s \leq n - 3$.
In Case (ii), if $s \geq 3$ by Formula (12), the first s subspaces in $\mathcal{W}$ form a sunflower with center $\pi_1 \cap V'$, and $\pi_2 \cap V'$ not through $\pi_1 \cap V'$.
In Case (iii), considering $\pi_1 \cap V'$ and $\pi_2 \cap V'$, one of them or both could be in $\{W_1, \ldots, W_s\}$. If $s = 2$, at most one of $\pi_1 \cap V'$ and $\pi_2 \cap V'$ can coincide with W_1 or W_2. Otherwise,

$W_1 \cap W_2 = \pi_1 \cap \pi_2$ and it is contained in all elements of $\mathcal{S}$.
In particular, if $s = 2$ and $n = 4$, it is straightforward to see that exactly one of $\pi_1 \cap V'$ and $\pi_2 \cap V'$ must necessarily be equal to W_1 or W_2.

We are now in the position to prove the main result of this section.

Proof of Theorem 3.1 We assume that $\mathcal{S}$ is not a $(k-t)$-junta and denote the elements of $\mathcal{S}$ by $\pi_1, \pi_2, \ldots, \pi_n$. We will consider all possible orderings of the spaces in $\mathcal{S}$ such that the parameters $(\delta_2, \ldots, \delta_n)$ are non-increasing. Since $\dim\langle\mathcal{S}\rangle = k + (n-1)(t-1) + 1$, we have the equality in (2) of Theorem 2.1. Hence, if $m \geq 3$, we have

$$(\delta_2, \ldots, \delta_n) = (\underbrace{t, \ldots, t}_{m-1 \text{ times}}, \underbrace{t-1, \ldots, t-1}_{n-m-1 \text{ times}}, t+1-m), \tag{13}$$

otherwise $m = 2$ and we have

$$(\delta_2, \ldots, \delta_n) = (t, t-1, \ldots, t-1). \tag{14}$$

- Suppose that we can find a permutation of the elements in $\mathcal{S}$ such that $\delta(\mathcal{S})$ is as in (13), for $m \geq 3$. Then, by Lemma 2.4, it follows that $\mathcal{S}$ belongs to Class I.
- If otherwise $(\delta_2, \ldots, \delta_n)$ is as in (14), by Proposition 2.3, there is no $(k-t)$-sunflower of maximal dimension with at least three petals. The result then follows by Lemma 3.2.

□

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