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Orthogonal one-factorizations of complete multipartite graphs

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Abstract The paper provides a complete solution to the existence problem of two orthogonal one-factorizations of a complete balanced multipartite graph $K_{p\times q}$. In particular, new classes of Howell designs are constructed.

Keywords One-factorization · Orthogonality · Latin square · Room square · Howell design

Mathematics Subject Classification 05C70 · 05B15

1 Introduction

We use standard notation $K_{p\times q}$ for a complete balanced p-partite graph with each part of cardinality q. Let $V(K_{p\times q}) = V_1 \cup V_2 \cup \ldots \cup V_p$, where $V_i \cap V_j = \emptyset$ whenever $i \neq j$. Moreover, we also use the standard symbol $K_{q,q}$ to denote $K_{2\times q}$, a complete balanced bipartite graph on $2q$ vertices.

A *one-factor* in a graph G is a regular spanning subgraph of degree one. A *one-factorization* of G is a set $\mathcal{F} = \{F_1, F_2, \ldots, F_r\}$ of edge-disjoint one-factors such that $E(G) = \bigcup_{i=1}^{r} E(F_i)$. Two one-factorizations $\mathcal{F} = \{F_1, F_2, \ldots, F_r\}$ and $\mathcal{F}' = \{F'_1, F'_2, \ldots, F'_r\}$ are *orthogonal* if $|F_i \cap F'_j| \leq 1$ for all $1 \leq i, j \leq r$.

Orthogonal one-factorizations of complete graphs are well-studied, mostly in terms of Rooms squares, cf. [7,12]. Let m be an odd integer and let S be a set of $m+1$ elements (*symbols*). A *Room square* R of side m is an $m \times m$ array which satisfies the following properties:

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(1) every cell of R is either empty or contains an unordered pair of symbols from S,
(2) every symbol of S occurs exactly once in each row and exactly once in each column of R,
(3) every unordered pair of symbols occurs in precisely one cell in R.
Thus each row and each column of R contain $\frac{m-1}{2}$ empty cells.

The existence of two orthogonal one-factorizations, $\mathcal{F}$ and $\mathcal{F}'$, of a complete graph K_{2n} is equivalent to the existence of a Room square of side $2n-1$: each row corresponds to a one-factor in $\mathcal{F}$ whilst each column represents a one-factor in $\mathcal{F}'$.

The existence problem for Room squares is completely settled.

Theorem 1 [14] *A Room square of side m exists if and only if m is odd and $m \neq 3$ and $m \neq 5$.*

Two orthogonal one-factorizations of a complete bipartite graph $K_{n,n}$ are equivalent to two orthogonal latin squares of side n. A *latin square* of *side* n is an $n \times n$ array in which each cell contains a single symbol from an n-element set S, such that each symbol occurs exactly once in each row and exactly once in each column. Two latin squares, L and L', of side n are *orthogonal* if the n^2 ordered pairs $(L(i, j), L'(i, j))$ are all distinct. Bose, Shrikhande and Parker [3] completely solved the famous Euler's conjecture.

Theorem 2 [3] *A pair of orthogonal latin squares of side n exists whenever $n \neq 2$ and $n \neq 6$.*

The above equivalences can be extended to other classes of regular graphs. Namely, a pair of orthogonal one-factorizations of an s-regular graph G on $2n$ vertices corresponds to the existence of a Howell design of type $(s, 2n)$, for which a graph G is called an *underlying graph*, cf. [15]. Let S be a set of $2n$ symbols. A *Howell design* $H(s, 2n)$ on the symbol set S is an $s \times s$ array that satisfies the following conditions:
(1) every cell is either empty or contains an unordered pair of symbols from S,
(2) every symbol of S occurs exactly once in each row and exactly once in each column of H,
(3) every unordered pair of symbols occurs in at most one cell of H.

Necessary condition for the existence of Howell designs $H(s, 2n)$ is $n \leq s \leq 2n-1$. The existence of an $H(n, 2n)$ comes from two orthogonal one-factorizations of a complete bipartite graph $K_{n,n}$ if $n \neq 2, 6$ and some 6-regular graph if $n = 6$ [13]. There in no $H(2, 4)$. In the other extreme case, an $H(2n-1, 2n)$ is a Room square of side $2n-1$. The existence of Howell designs has been completely determined for all remaining values of s.

Theorem 3 [17] *If s is odd and $n < s < 2n-1$ then there exists an $H(s, 2n)$, except that $H(5, 8)$ does not exist.*

Theorem 4 [2] *If s is even and $n < s < 2n-1$ then there exists an $H(s, 2n)$.*

An important question related to Howell designs concerns properties of graphs which are underlying graphs of Howell designs. While for $s = 2n-1$ and $s = 2n-2$ these graphs are unique (the complete graph K_{2n} and the cocktail party graph $K_{2n} \setminus F$, respectively, where F is a one-factor), determining these graphs in general seems to be hopeless [15,16]. We have to notice that some known constructions may provide Howell designs for certain classes of underlying graphs; in particular, in the case of a powerful recursive "PBD-construction" (cf.

[2,17]), the structure of an underlying graph strongly depends on the choice of parameters, parallel classes in a PBD as well as Howell subdesigns used in the recursion.

It is known that a necessary and sufficient condition for the existence of a one-factorization of a complete balanced multipartite graph $K_{p\times q}$ is that pq is even [11]. The goal of this paper is to show that balanced complete multipartite graphs are underlying graphs of Howell designs; the main result provides a complete solution to the existence problem of two orthogonal one-factorizations of $K_{p\times q}$.

2 Constructions

We first discuss a general recursive construction which in fact is an application of a standard "expansion by latin squares" method.

Lemma 5 *Let p, q and m be integers such that $p \geq 2$, $q \geq 1$, $m \geq 3$ and $m \neq 6$. Suppose there exist two orthogonal one-factorizations of the complete multipartite graph $K_{p\times q}$ and moreover two orthogonal one-factorizations of the complete bipartite graph $K_{m,m}$. Then there exists a pair of orthogonal one-factorizations of the complete multipartite graph $K_{p\times qm}$.*

Proof Let X be the vertex set of $K_{p\times q}$ and let (Y, Y) be the vertex set of $K_{m,m}$. Let $\mathcal{F}^1$, $\mathcal{F}^2$ be two orthogonal one-factorizations of $K_{p\times q}$ on the set X such that $\mathcal{F}^z = \{F_1^z, F_2^z, \ldots, F_{q(p-1)}^z\}$, $z = 1, 2$. Moreover, let $\mathcal{E}^1$, $\mathcal{E}^2$ be a pair of orthogonal one-factorizations of $K_{m,m}$ on (Y, Y) and $\mathcal{E}^z = \{E_1^z, E_2^z, \ldots, E_m^z\}$, $z = 1, 2$.

For each $z = 1, 2$ we construct a one-factorization $\mathcal{D}^z = \{D_{s,t}^z : s = 1, 2, \ldots, q(p-1), t = 1, 2, \ldots, m\}$ of $K_{p\times qm}$ on vertex set $X \times Y$. We replace each edge of $K_{p\times q}$ with one-factorization $\mathcal{E}^z$ as follows: the edge $\{(i, j), (k, l)\}$ belongs to one-factor $D_{s,t}^z$ if $\{i, k\}$ is an edge of F_s^z and $\{j, l\}$ is an edge of E_t^z.

To prove orthogonality of $\mathcal{D}^1$ and $\mathcal{D}^2$ we suppose to the contrary that there are two distinct edges, $\{(i, j), (k, l)\}$ and $\{(i', j'), (k', l')\}$ of $K_{p\times qm}$ that belong together to the same two one-factors, $D_{s,t}^1$ and $D_{s',t'}^2$. We consider two cases:

(1) $i = i'$ and $k = k'$. Then $j \neq j'$ and $l \neq l'$. Moreover, $\{j, l\}$ and $\{j', l'\}$ are both in the same two one-factors E_t^1 and $E_{t'}^2$, a contradiction to the orthogonality of $\mathcal{E}^1$ and $\mathcal{E}^2$.
(2) $i \neq i'$ or $k \neq k'$. Then $\{i, k\}$ and $\{i', k'\}$ are two distinct edges of both F_s^1 and $F_{s'}^2$, a contradiction to the orthogonality of $\mathcal{F}^1$ and $\mathcal{F}^2$.

□

When $q = 1$ we immediately get the following.

Corollary 6 *Let p and q be integers such that p is even, $p \geq 8$, $m \geq 3$ and $m \neq 6$. There exists a pair of orthogonal one-factorizations of a complete multipartite graph $K_{p\times m}$.* □

The second construction is based on Room frames. Let $\{S_1, S_2, \ldots, S_k\}$ be a partition of the set S. An $\{S_1, S_2, \ldots, S_k\}$-*Room frame* is an $|S| \times |S|$ array, F, indexed by S, which satisfies the following properties:

(1) every cell of F is either empty or contains an unordered pair of symbols from S,
(2) the subarrays $S_i \times S_i$ are empty, for $1 \leq i \leq k$ (these subarrays are called *holes*),
(3) every symbol $x \notin S_i$ occurs exactly once in each row s and exactly once in each column t, for any $s, t \in S_i$,
(4) pairs occurring in F are those $\{s, t\}$, where $(s, t) \in (S \times S) \setminus \bigcup_{i=1}^{k}(S_i \times S_i)$.

The *type* of a Room frame F is a multiset $\{|S_i| : 1 \leq i \leq k\}$. An "exponential" notation is used to describe types; a Room frame has type $t_1^{u_1} t_2^{u_2} \dots t_l^{u_l}$ if there are u_i subsets of cardinality t_i, $1 \leq i \leq l$. A Room frame of type t^u (one hole size) is called *uniform*. In particular, a Room square of side m is equivalent to a Room frame of type 1^m.

The existence problem for uniform Room frames is completely solved.

Theorem 7 [5,6,8–10] *Suppose t and u are positive integers, $u \geq 4$ and $(t, u) \neq (1, 5)$ and $(2, 4)$. Then there exists a uniform Room frame of type t^u if and only if $t(u-1)$ is even.*

Room frames are key structures in the "filling in holes" construction for Howell designs, cf. [4]. In particular, applying this construction for uniform Room frames yields Howell designs with complete balanced multipartite graphs as underlying graphs.

Lemma 8 *Let t and u be integers such that $t \geq 3$, $t \neq 6$, $u \geq 4$ and $t(u-1)$ is even. Then there exists a Howell design $H(ut, ut+t)$ whose underlying graph is $K_{(u+1)\times t}$.*

Proof By Theorem 7, there exists a Room frame F of type t^u on a set S of cardinality tu. Let $S_1, S_2, \dots, S_u$ be sets corresponding to holes of F, $S_i \subset S$ and $|S_i| = t$ for each $i = 1, 2, \dots u$. Let S_{u+1} be a set containing t elements, none of them in the set S.

For each pair of sets (S_i, S_{u+1}), $i = 1, 2, \dots u$, by Theorem 2, there exists a pair of orthogonal latin squares of side t which correspond to two orthogonal one factorizations of complete bipartite graph $K_{t,t}$ with bipartition (S_i, S_{u+1}), and moreover which are equivalent to a Howell design H_i of type $(t, 2t)$ on the set $S_i \cup S_{u+1}$. It is easy to see that each hole $S_i \times S_i$ of F can be filled with H_i. In this way we obtain a Howell design H on the set $S \cup S_{u+1}$. Notice that none of unordered pairs with both elements in the same S_i, $i = 1, 2, \dots, u+1$, occurs in H. Thus $K_{(u+1)\times t}$ is an underlying graph of H. □

The well-known starter-adder construction, as a basic method to obtain Room squares, can be generalized for Howell designs, cf. [1]. Let G be an abelian group of order s. A *Howell starter* in G, where $s + 1 \leq 2n \leq 2s$, is a set $S_{s,n} = \{\{x_i, y_i\} : 1 \leq i \leq s-n\} \cup \{\{x_i\} : s-n+1 \leq i \leq n\}$ that satisfies:

(1) $\{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq s-n\} = G$,
(2) $(x_i - y_i) \neq \pm(x_j - y_j)$ if $i \neq j$.

If $S_{s,n}$ is a Howell starter, then an ordered set $A_{s,n} = \{\{a_i\} : 1 \leq i \leq n\}$ is an *adder* for $S_{s,n}$ if elements in $A_{s,n}$ are distinct and $\{x_i + a_i : 1 \leq i \leq n\} \cup \{y_i + a_i : 1 \leq i \leq s-n\} = G$.

In what follows, we use notation $SA_{s,n} = \{\{x_i, y_i\}^{a_i} : 1 \leq i \leq s-n\} \cup \{\{x_i\}^{a_i} : s-n+1 \leq i \leq n\}$ for a Howell starter $S_{s,n}$ together with an adder $A_{s,n}$. Moreover, we take the cyclic group $\mathbb{Z}_s$ as G.

Lemma 9 *Suppose that there exist a Howell starter $S_{s,n}$ together with an adder $A_{s,n}$ in $\mathbb{Z}_s$ such that $q = 2n - s$ is a divisor of s and moreover none of the pairs in $S_{s,n}$ has the difference of its elements divisible by $p = s/q$. Then a Howell design of type $(s, 2n)$, generated by $S_{s,n}$ and $A_{s,n}$, has an underlying graph $K_{(p+1)\times q}$.*

Proof An $H(s, 2n)$ is constructed on the symbol set V of cardinality $2n$. Let $(V_0, V_1, \dots V_p)$ be a partition of V such that $V_j = \{j, j+p, j+2p, \dots, j+(q-1)p\}$, where $j = 0, 1, \dots, p-1$, and $V_p = \{\infty_1, \infty_2, \dots, \infty_q\}$.

Let us label rows and columns of $H(s, 2n)$ by elements of $\mathbb{Z}_s$. The first row consists of pairs $\{x_i, y_i\}$, for $i = 1, 2, \dots s-n$, and pairs $\{x_i, \infty_{i-s+n}\}$, for $i = s-n+1, s-n+2, \dots n$, each of them in column $-a_i$. It is easy to see that all these pairs form a 1-factor of $K_{(p+1)\times q}$

on V. The remaining cells of the Howell design are filled out by developing the square via the group $\mathbb{Z}_s$; that is, the pair $\{x_i + k, y_i + k\}$ is placed in row k and column $-a_i + k$, and also the pair $\{x_i + k, \infty_{i-s+n}\}$ in a cell in row k and column $-a_i + k$, where all arithmetic is modulo s. Thus, in particular, the first column consists of pairs $\{x_i + a_i, y_i + a_i\}$, for $i = 1, 2, \dots s - n$, and pairs $\{x_i + a_i, \infty_{i-s+n}\}$, for $i = s - n + 1, s - n + 2, \dots n$, which obviously constitute a 1-factor of $K_{(p+1)\times q}$. Due to cyclic rotation of rows and columns we obtain two orthogonal one factorizations of $K_{(p+1)\times q}$. □

An elementary verification shows that the following sets $SA_{s,n}$ are Howell starters and adders for Howell designs of type $(s, 2n)$ whose underlying graphs are $K_{4\times q}$, where $q = n/2$. Notice that none of the pairs in starters has the difference of elements divisible by 3.

Construction 1 $s \equiv 3 \pmod{24}$, $s \geq 27$
Let $s = 24m + 3$ and $n = 16m + 2$, $m \geq 1$.
$SA_{s,n} = \{\{8m - 2i, 8m + 2 + 4i\}^{16m+2-i}, \{8m - 1 - 2i, 8m + 4 + 4i\}^{4m-i}, \{16m + 1 - 4j, 16m + 2 + 2j\}^{20m+3+j}, \{16m - 1 - 4i, 16m + 3 + 2i\}^{8m+2+i}, \{20m + 3 + 4j\}^{13m+3-j}, \{20m + 4 + 4i\}^{7m+2-i}, \{20m + 5 + 4i\}^{m+1-i}, \{20m + 6 + 4i\}^{19m+3-i} : i = 0, 1, \dots, 2m - 1,\ j = 0, 1, \dots, 2m\}$.

Construction 2 $s \equiv 9 \pmod{24}$, $s \geq 33$
Let $s = 24m + 9$ and $n = 16m + 6$, $m \geq 1$.
$SA_{s,n} = \{\{8m+2-2j, 8m+4+4j\}^{16m+6-j}, \{8m+1-2i, 8m+6+4i\}^{4m+1-i}, \{16m + 5 - 4j, 16m + 6 + 2j\}^{20m+8+j}, \{16m + 3 - 4j, 16m + 7 + 2j\}^{8m+4+j}, \{20m + 8 + 4j\}^{7m+4-j}, \{20m + 9 + 4j\}^{13m+6-j}, \{20m + 10 + 4j\}^{19m+8-j}, \{20m + 11 + 4i\}^{m+1-i} : i = 0, 1, \dots, 2m - 1,\ j = 0, 1, \dots, 2m\}$.

Construction 3 $s \equiv 15 \pmod{24}$, $s \geq 15$
Let $s = 24m + 15$ and $n = 16m + 10$, $m \geq 0$.
$SA_{s,n} = \{\{8m+4-2j, 8m+6+4j\}^{16m+10-j}, \{8m+3-2j, 8m+8+4j\}^{4m+2-j}, \{16m+9 - 4k, 16m + 10 + 2k\}^{20m+13+k}, \{16m + 7 - 4j, 16m + 11 + 2j\}^{8m+6+j}, \{20m + 13 + 4k\}^{7m+5-k}, \{20m+14+4j\}^{m+1-j}, \{20m+15+4j\}^{13m+9-j}, \{20m+16+4j\}^{19m+11-j} : j = 0, 1, \dots, 2m,\ k = 0, 1, \dots, 2m + 1\}$.

Construction 4 $s \equiv 21 \pmod{24}$, $s \geq 21$
Let $s = 24m + 21$ and $n = 16m + 14$, $m \geq 0$.
$SA_{s,n} = \{\{8m+6-2k, 8m+8+4k\}^{16m+14-k}, \{8m+5-2j, 8m+10+4j\}^{4m+3-j}, \{16m+13 - 4k, 16m + 14 + 2k\}^{20m+18+k}, \{16m + 11 - 4k, 16m + 15 + 2k\}^{8m+8+k}, \{20m + 18 + 4k\}^{7m+6-k}, \{20m + 19 + 4k\}^{13m+11-k}, \{20m + 20 + 4k\}^{19m+16-k}, \{20m + 21 + 4j\}^{m-j} : j = 0, 1, \dots, 2m,\ k = 0, 1, \dots, 2m + 1\}$.

Some examples of small order have to be constructed separately.

Example 1 Two orthogonal one-factorizations of $K_{3\times 4}$.
The starter-adder for a Howell design $H(8, 12)$ is $SA_{8,6} = \{\{0, 1\}^1, \{2, 5\}^2, \{3\}^5, \{4\}^7, \{6\}^0, \{7\}^6\}$.

Example 2 Two orthogonal one-factorizations of $K_{4\times 3}$.
The starter-adder for a Howell design $H(9, 12)$ is $SA_{9,6} = \{\{2, 4\}^5, \{3, 7\}^3, \{5, 6\}^7, \{0\}^2, \{1\}^4, \{8\}^0\}$.

Example 3 Two orthogonal one-factorizations of $K_{4\times 4}$.
The starter-adder for a Howell design $H(12, 16)$ is $SA_{12,8} = \{\{0, 1\}^{11}, \{4, 11\}^2, \{5, 9\}^5, \{6, 8\}^9, \{2\}^7, \{3\}^4, \{7\}^1, \{10\}^6\}$.

3 Main results

Lemma 10 *For every even positive integer q there exist two orthogonal one-factorizations of* $K_{3\times q}$.

Proof We consider separately the following cases. If $q = 2$ then $K_{3\times 2}$ is the cocktail-party graph and the assertion immediately holds by Theorem 4. For $q = 4$ we use two orthogonal one-factorizations of $K_{3\times 4}$ from Example 1. For $q \geq 6$ and $q \neq 12$ we apply the general recursive construction given in Lemma 5 taking as initial graphs $K_{3\times 2}$ and $K_{\frac{q}{2},\frac{q}{2}}$. If $q = 12$ we apply the same construction but we use orthogonal one-factorizations of $K_{3\times 4}$ and $K_{3,3}$. □

Lemma 11 *For every integer* $q \geq 2$ *there exist two orthogonal one-factorizations of* $K_{4\times q}$.

Proof If $q = 2$ then two orthogonal one-factorizations of the cocktail party graph $K_{4\times 2}$ exist by Theorem 4. If $q = 3$ or $q = 4$, two orthogonal one-factorizations of $K_{4\times 3}$ and $K_{4\times 4}$ are given in Examples 2 and 3, respectively. For odd $q \geq 5$ the existence is satisfied by Constructions 1–4 and Lemma 9. For even $q \geq 6$ and $q \neq 12$, the general recursive construction given in Lemma 5 can be used taking as initial graphs $K_{4\times 2}$ and $K_{\frac{q}{2},\frac{q}{2}}$. If $q = 12$ we apply the same construction but we use orthogonal one-factorizations of $K_{4\times 4}$ and $K_{3,3}$. □

Lemma 12 *Let p, q be integers such that p is odd and* $p \geq 5$, *q is even and* $q \geq 2$. *Then there exist two orthogonal one-factorizations of* $K_{p\times q}$.

Proof If $q = 2$ then two orthogonal one-factorizations of $K_{p\times 2}$ exist by Theorem 4. For $q \geq 4$ and $q \neq 6$, the existence of two orthogonal one-factorizations of $K_{p\times q}$ follows directly from Lemma 8. If $q = 6$, a construction in Lemma 5 can be applied for initial graphs $K_{p\times 2}$ and $K_{3,3}$. □

Lemma 13 *Let p, q be integers such that p is even,* $p \geq 6$ *and* $q \geq 2$. *Then there exist two orthogonal one-factorizations of* $K_{p\times q}$.

Proof If $q = 2$ then $K_{p\times 2}$ is the cocktail-party graph and the assertion holds by Theorem 4. For $q \geq 3$ and $q \neq 6$, the existence of two orthogonal one-factorizations of $K_{p\times q}$ follows from Lemma 8. If $q = 6$ then we apply the general recursive construction given in Lemma 5 taking $K_{p\times 2}$ and $K_{3,3}$ as initial graphs. □

Combining Lemmas 10–13 together with Theorems 1 and 2 gives the main result.

Theorem 14 *For any integers p and q such that pq is even,* $p \geq 2$ *and* $q \geq 1$, *a complete balanced multipartite graph* $K_{p\times q}$ *admits a pair of orthogonal one-factorizations, except for* $(p, q) = (2, 2), (2, 6), (4, 1)$ *or* $(6, 1)$. □

Corollary 15 *Let p and q be integers such that pq is even,* $p \geq 2$, $q \geq 1$ *and* (p, q) *is none of the pairs* $(2, 2), (2, 6), (4, 1)$ *and* $(6, 1)$. *Then there exists a Howell design of type* $(pq - q, pq)$ *whose underlying graph is* $K_{p\times q}$. □

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