

Proof of a conjecture of Kløve on permutation codes under the Chebychev distance

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Abstract Let d be a positive integer and x a real number. Let $A_{d,x}$ be a $d \times 2d$ matrix with its entries

$$a_{i,j} = \begin{cases} x & \text{for } 1 \leq j \leq d+1-i, \\ 1 & \text{for } d+2-i \leq j \leq d+i, \\ 0 & \text{for } d+1+i \leq j \leq 2d. \end{cases}$$

Further, let R_d be a set of sequences of integers as follows:

$$R_d = \{(\rho_1, \rho_2, \dots, \rho_d) | 1 \leq \rho_i \leq d+i, 1 \leq i \leq d, \text{ and } \rho_r \neq \rho_s \text{ for } r \neq s\}.$$

and define

$$\Omega_d(x) = \sum_{\rho \in R_d} a_{1,\rho_1} a_{2,\rho_2} \dots a_{d,\rho_d}.$$

In order to give a better bound on the size of spheres of permutation codes under the Chebychev distance, Kløve introduced the above function and conjectured that

$$\Omega_d(x) = \sum_{m=0}^d \binom{d}{m} (m+1)^d (x-1)^{d-m}.$$

In this paper, we settle down this conjecture positively.

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1 Introduction

A permutation code is a subset of the symmetric group S_n , equipped with a distance metric. Permutation codes are of potential use in various applications such as power-line communications and coding for flash memories used with rank modulation [6, 7]. Permutation codes were extensively studied over the last decade. Hamming metric is naturally the first to be considered. Later, Ulam metric [4] and Kendall τ -metric [2] were introduced and are now the two most investigated metrics. However in [9], a new metric named the Chebyshev metric was proposed by Kløve et al., when they were studying the multi-level flash memory model. A combinatorial survey on metrics related to permutations was given in [3].

The two main questions in coding theory are fundamental limits on the parameters of the code (information rate versus minimum distance) and constructions of codes that attain these limits. It turns out that both topics are difficult for permutation codes. Few explicit constructions were obtained for various metrics and no general bounds better than the GV-bound and Sphere packing bounds were found in [1, 2, 4, 6, 9] except for the Hamming metric [5]. Both the GV-bound and the Sphere packing bound depends on the volume ($V(d, n)$) of a typical “ball” which consists of permutations in S_n at distance at most d from the identity permutation. The calculation of the volume of that ball becomes a crucial problem.

The Chebychev distance $d(p, q)$ between two permutations $p = (p_1, p_2, \dots, p_n)$ and $q = (q_1, q_2, \dots, q_n)$ is defined by

$$d(p, q) = \max_j |p_j - q_j|.$$

Let

$$T_{d,n} = \{p \in S_n \mid |p_i - i| \leq d \text{ for } 1 \leq i \leq n\}.$$

It is clear that $V(d, n) = |T_{d,n}|$. The permanent of an $n \times n$ matrix A is defined by

$$\text{per} A = \sum_{p \in S_n} a_{1,p_1} \dots a_{n,p_n}.$$

Let $A^{(d,n)}$ be the $n \times n$ matrix with $a_{i,j}^{d,n} = 1$ if $|i - j| \leq d$ and $a_{i,j}^{d,n} = 0$ otherwise. Clearly, $V(d, n) = \text{per} A^{(d,n)}$. Although the permanent looks similar to the determinant of a matrix, it is a difficult problem to compute the permanent for general matrices. The celebrated van der Waerden theorem gives a lower bound for the permanent of the so called doubly stochastic $n \times n$ matrix. Here doubly stochastic means that all the elements are non-negative and that the sum of the elements in any row or column is 1. Thus, if A is an $n \times n$ matrix where the sum of the elements in any row or column is a constant k , then van der Waerden theorem gives a lower bound on the permanent of A .

By noticing that most rows and columns of $A^{(d,n)}$ have the sum $2d + 1$, Kløve defined a closely related matrix $B^{(d,n)}$ with row sum and column sum $2d + 1$ so that van der Waerden's theorem can be applied. The matrix $B^{(d,n)}$ is defined as follows:

$$b_{i,j} = \begin{cases} 0 & \text{if } i > j + d \text{ or } j > i + d, \\ 2 & \text{if } i + j \leq d + 1 \text{ or } i + j \geq 2n + 1 - d, \\ 1 & \text{otherwise.} \end{cases}$$

With this new defined matrix $B^{(d,n)}$, Kløve [10] gave a lower bound on $V(d, n)$ as follows:

$$V(d, n) > \frac{\sqrt{2\pi n}}{2^{2d}} \left(\frac{2d+1}{e} \right)^n. \quad (1.1)$$

Let $A_{d,2} = (a_{i,j})$ be the upper left corner of $B^{(d,n)}$ which is a $d \times 2d$ matrix defined by

$$a_{i,j} = \begin{cases} 2, & \text{if } 1 \leq j \leq d + 1 - i, \\ 1, & \text{if } d + 2 - i \leq j \leq d + i, \\ 0, & \text{if } d + i + 1 \leq j \leq 2d. \end{cases}$$

For example,

$$A_{1,2} = (2 \ 1), \quad A_{2,2} = \begin{pmatrix} 2 & 2 & 1 & 0 \\ 2 & 1 & 1 & 1 \end{pmatrix}, \quad A_{3,2} = \begin{pmatrix} 2 & 2 & 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Let R_d be a set of sequences of integers as follows:

$$R_d = \{(\rho_1, \rho_2, \dots, \rho_d) \mid 1 \leq \rho_i \leq d + i, 1 \leq i \leq d, \text{ and } \rho_r \neq \rho_s\}.$$

Define

$$\Omega_d = \sum_{\rho \in R_d} a_{1,\rho_1} a_{2,\rho_2} \dots a_{d,\rho_d}.$$

Let

$$\omega_d = \frac{\Omega_d e^d}{(2d+1)^d}.$$

Kløve [9] also gave the following lower bound on $V(d, n)$:

$$V(d, n) > \frac{\sqrt{2\pi(n+2d)}}{\omega_d^2} \left(\frac{2d+1}{e} \right)^n. \quad (1.2)$$

Thus whether (1.2) is an improvement compared with (1.1) depends on the value Ω_d . Kløve [10] gave the first 9 values of Ω_d as follows:

$$3, 18, 170, 2200, 36232, 725200, 17095248, 463936896, 14246942336,$$

which coincide the sequence A074932 in [12], and made the following conjecture.

Conjecture 1 [10, Conjecture 1] *For any positive integer d ,*

$$\Omega_d = \sum_{m=0}^d \binom{d}{m} (m+1)^d.$$

Kløve showed that the equation (1.2) improves on (1.1) if Conjecture 1 is true. Furthermore, let $A_{d,x} = (a_{i,j})$ be the $d \times 2d$ matrix defined by

$$a_{i,j} = \begin{cases} x, & \text{if } 1 \leq j \leq d + 1 - i, \\ 1, & \text{if } d + 2 - i \leq j \leq d + i, \\ 0, & \text{if } d + i + 1 \leq j \leq 2d. \end{cases}$$

and let

$$\Omega_d(x) = \sum_{\rho \in R_d} a_{1,\rho_1} a_{2,\rho_2} \cdots a_{d,\rho_d}.$$

In particular, $\Omega_d(2) = \Omega_d$. Kløve gave the following generalized conjecture and verified it for $d \leq 9$.

Conjecture 2 [10, Conjecture 1] *For any positive integer d ,*

$$\Omega_d(x) = \sum_{m=0}^d \binom{d}{m} (m+1)^d (x-1)^{d-m}. \quad (1.3)$$

In this paper, we shall prove that Conjecture 2 is true.

2 Proof of Kløve's Conjecture

Theorem 3 *For any positive integer d , the identity (1.3) holds.*

Actually, for any $m \times n$ matrix $A = (a_{i,j})$ with $m \leq n$, the permanent function of A is already defined as follows (see, for example, [11]):

$$\text{per}(A) = \sum_{\sigma \in P(n,m)} a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{m,\sigma_m},$$

where $P(n, m)$ denotes the set of all m -permutations of the n -set $\{1, 2, \dots, n\}$.

In fact, by the definition of R_d , we know that R_d is exactly the subset of all d -permutations of the $2d$ -set $\{1, 2, \dots, 2d\}$ such that $\sigma \in R_d$ if and only if $a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{d,\sigma_d} \neq 0$. Hence we have $\Omega_d(x) = \text{per}(A_{d,x})$.

In order to prove Theorem 3, we first give a related combinatorial identity.

Lemma 4 *Let m and n be positive integers. Then*

$$\sum_{1 \leq k_1 \leq k_2 \leq \dots \leq k_m \leq n} \prod_{i=0}^m k_i (n+m-i)^{k_{i+1}-k_i} = \binom{n+m-1}{m} n^{n+m-1}, \quad (2.1)$$

where $k_0 = 1$ and $k_{m+1} = n$.

For example, we have

$$\sum_{1 \leq k_1 \leq k_2 \leq k_3 \leq n} k_1 k_2 k_3 (n+3)^{k_1-1} (n+2)^{k_2-k_1} (n+1)^{k_3-k_2} n^{n-k_3} = \binom{n+2}{3} n^{n+2}.$$

Proof of Lemma 4 We compute the multiple sum in the order from k_m to k_1 . It can be proved by induction on $k_{m-1}, k_{m-2}, \dots, k_{m-i-1}$ respectively that

$$\begin{aligned} \sum_{k_m=k_{m-1}}^n k_m(n+1)^{k_m-k_{m-1}} n^{n-k_m} &= (n-k_{m-1}+1)n^{n-k_{m-1}+1}, \\ \sum_{k_{m-1}=k_{m-2}}^n k_{m-1}(n-k_{m-1}+1)(n+2)^{k_{m-1}-k_{m-2}} n^{n-k_{m-1}+1} &= \binom{n-k_{m-2}+2}{2} n^{n-k_{m-2}+2}, \\ &\dots, \\ \sum_{k_{m-i}=k_{m-i-1}}^n k_{m-i} \binom{n-k_{m-i}+i}{i} (n+i+1)^{k_{m-i}-k_{m-i-1}} n^{n-k_{m-i}+i} \\ &= \binom{n-k_{m-i-1}+i+1}{i+1} n^{n-k_{m-i-1}+i+1}. \end{aligned} \quad (2.2)$$

By choosing $i = m-1$ in (2.2), we complete the proof of (2.1). \square

Proof of Theorem 3 It is clear that (1.3) is equivalent to

$$\Omega_d(x+1) = \sum_{m=0}^d \binom{d}{m} (d-m+1)^d x^m. \quad (2.3)$$

Therefore, it suffices to show that the coefficient b_m of x^m in $\Omega_d(x+1)$ is equal to $\binom{d}{m} (d-m+1)^d$. By the definition of $\Omega_d(x+1)$, we know that each x comes from the first term in $x+1$.

To compute b_m , we first choose m x 's from m $(x+1)$'s which are not in the same row nor in the same column of the matrix $A_{d,x+1}$, and then choose $(d-m)$ 1's in the other $d-m$ rows so that no 1's are in the same column. Suppose that the m x 's are chosen from the rows indexed by $d+1-i_1, d+1-i_2, \dots, d+1-i_m$ with $i_1 < i_2 < \dots < i_m$, respectively. By noticing that the $(d+1-i)$ -th row has i $(x+1)$'s and all the x 's we choose must be in different columns, we have $i_1(i_2-1)(i_3-2) \cdots (i_m-m+1)$ ways to do this. As for the number of ways to choose 1's in the remaining rows, we notice that the i -th row has $d+i$ 1's including those 1's in $(x+1)$'s and all these 1's form several right trapezoids in the matrix $A_{d,x+1}$. Therefore, there are $(d+1)^{i_1-1} d^{i_2-i_1-1} (d-1)^{i_3-i_2-1} \cdots (d-m+1)^{d-i_m}$ ways to choose the remaining 1's. It follows that

$$\begin{aligned} b_m &= \sum_{1 \leq i_1 < i_2 < \dots < i_m \leq d} i_1(i_2-1)(i_3-2) \cdots (i_m-m+1) \\ &\quad \times (d+1)^{i_1-1} d^{i_2-i_1-1} (d-1)^{i_3-i_2-1} \cdots (d-m+1)^{d-i_m} \\ &= \sum_{1 \leq k_1 \leq k_2 \leq \dots \leq k_m \leq d-m+1} \prod_{i=0}^m k_i (d+1-i)^{k_{i+1}-k_i}, \end{aligned}$$

where $k_s = i_s - s + 1$ ($s = 1, \dots, m$), $k_0 = 1$, and $k_{m+1} = d-m+1$. By replacing n by $d-m+1$ in (2.1), we obtain $b_m = \binom{d}{m} (d-m+1)^d$. This completes the proof. \square

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