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Liquidity in Financial Networks

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Abstract
This study investigates the amount of liquidity that is necessary to settle a given network of financial obligations. In our analysis, we assume *sequential* settlement, which is standard in real-world interbank settlement systems, instead of *simultaneous* settlement, which is typically assumed in relevant research. We develop a graph-theoretic model and apply a flow network technique to investigate how the interconnected feature could affect the required liquidity. Our main contribution is to show that the effect of the interconnected feature is characterized with our original concepts regarding "twist" properties—*arc-twisted* and *vertex-twisted*—that are defined on the basis of the concept of "cycles." Each of the "twist" properties refers to certain inconsistency in the dynamics of settlements. The characterization provides a consistent and fundamental perspective of how a *hub* or other network structures affect the required liquidity. We further investigate the quantitative implications of the "twist" properties for real-world networks of obligations by examining networks with *clustered* structures and *small-world* structures. We show that "twist" properties have non-linear effects on the required liquidity against an increase in the amount of obligations.

Keywords Financial crisis · Bank · Modeling

1 Introduction

The financial crisis of 2008 raised the concern that default of a bank may cause a "domino" of default through interconnected financial obligations among banks. To ensure the resilience of financial system, the Basel Committee introduced a liquidity regulation under the framework of Basel lll; banks are required to hold sufficient liquidity against short-term liquidity shocks. For the purpose of liquidity regulation in general, it is crucial to assess the required liquidity appropriately. The assessment hinges on, among other things, the assumption of an underlying settlement system. In

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this respect, seminal theoretical studies in the literature on financial contagion[1] effectively assume "simultaneous" settlement, by which financial obligations are canceled out whenever possible, although "sequential" settlement is the standard in the modern interbank settlement systems. The assumption of simultaneous settlement leaves the relevant analyses highly tractable by allowing fixed-point arguments, although it could considerably underestimate the amount of funds to prevent a "domino" effect of default.

Purpose and Framework
This study offers a framework to evaluate the amount of liquidity—funds available for settlements—that is necessary to prevent default under "sequential" settlement. Our aim is to clarify how the interconnected feature of financial obligations could affect the required amount of funds. For this purpose, we assume an exogenous network of obligations and abstract their formation. We focus on the total amount of funds necessary to settle all obligations, and investigate how the amount depends on network topologies. We perform our investigation along with two polar liquidity scenarios. One is assumed as a situation in times of financial distress, by which funds circulate least efficiently. The other is assumed as a benchmark situation in times of non-distress, by which funds circulate most efficiently. In our formulation as graph problems, the pair of scenarios amounts to deriving the upper bound and lower bound of the required amount of funds against possible "orders" of settlements.

Approach and Technique
We characterize each lower bound and upper bound, applying a basic technique to analyze "flow" networks. Specifically, we decompose a network of obligations on the basis of cycles. This helps to separate mathematically the relevant "ordering" factor from the relevant "flow" factor. We refer to the former as the *synchronization* factor, and the latter as the *domain* factor, for reasons that become clearer soon. In terms of the interconnection of network, the *synchronization* factor captures the interconnected feature of network, while the *domain* factor—cycles and the relevant amount of obligations—serves as the reference basis for the interconnection.

Main Results: Qualitative Aspects
The main contribution of this study is to reveal the qualitative aspect of the *synchronization* factor with two of our original concepts, *arc-twisted* and *vertex-twisted*, which are formally defined on a directed graph. Specifically, with regard to the lower bound, the *domain* factor refers to the efficient recycling of funds within each *domain*—a cycle of obligations—without considering the interrelation between *domains*, while the synchronization factor, which is characterized by *arc-twisted*, refers to the additional amount of funds required that is sourced from the interrelation between *domains*. With regard to the upper bound, the domain factor refers to the least efficient recycling of funds within a *domain* without considering the interre-

[1] The pioneering work of Eisenberg and Noe (2001) provides an analytical framework, which is adopted for further investigation by Cifuentes et al. (2005), Gourieroux et al. (2013), Elliott et al. (2014), and Acemoglu et al. (2015). Glasserman and Young (2016) provide a survey on financial contagion.

lation between *domains*, while the synchronization factor, which is characterized by *vertex-twisted*, refers to the amount of subtraction that is sourced from the interrelation between *domains*.

Thus, the interconnected feature of network is essentially captured by *arc-twisted* for the case of the lower bound, and by *vertex-twisted* for the case of the upper bound. A natural concern is the relation between the two concepts. In this respect, we show a simple relation, such that *vertex-twisted* indicates *arc-twisted* but not vice-versa. This result consolidates our findings on the qualitative implications of the interconnected feature. We illustrate that our characterization with the "twist" concepts provides a consistent and fundamental perspective on the underlying mechanism for a *hub* or other network structures to affect each lower/upper bound.

Results: Quantitative Aspects
We further analyze the quantitative implications of the "twist" concepts for real-world payment networks. Our approach is to focus on classes of network that capture key properties of real-world payment networks. The key properties observed in real-world payment networks have been discussed as a combination of high degree of *clustering* and relatively short *average path lengths*, as highlighted by Soramäki et al. (2007) with regard to Fedwire, by Rordam and Bech (2009) with regard to Danish interbank money flows, and by Inaoka et al. (2004) and Imakubo and Soejima (2010) with regard to BOJ-NET. This pair of properties is typically referred to as a *small-world* structure.[2] We show that for each lower and upper bound, an increase of obligations has a non-linear effect on the required amount of funds through the *synchronization* factor. This result suggests that appropriate consideration of the *synchronization* factor is indispensable in evaluating the funds required for real-world payment networks.

Interbank Settlement Systems
Here, we elaborate on the background of our assumption of "sequential" settlement. A traditional interbank settlement system used to be a simultaneous-type system called a designated-time net settlement system, whereby settlements are conducted on a designated-time basis and the obligations are offset against each other as much as possible. Against the backdrop of the expanding volume of obligations and technological advances for real-time transactions, many interbank settlement systems have changed and now adopt a sequential-type system called real-time gross settlement (RTGS), which settles obligations on a real-time basis.[3] In the RTGS system, obligations are no longer offset against each other but instead are settled on an individual gross basis. Although several interbank settlement systems further adopt systems that partially combine the offsetting service to the RTGS system, they still operate on the basis of RTGS system.[4] Our assumption of "sequential" settlement simulates the RTGS sys-

[2] Small-world structures are examined by Watts and Strogatz (1998), Watts (1999), and broadly in subsequent literature for various types of networks beyond payment networks.

[3] Fedwire changed to RTGS in 1982, the Clearing House Automated Payments System began using it in 1996, and BOJ-NET adopted it in 2001. The World Bank (2013) documents that 116 of 139 surveyed countries had adopted RTGS up to 2010.

[4] The additional offsetting service to the RTGS system is called a liquidity-saving mechanism. For its functioning and relevant discussion, see Martin and McAndrews (2008, 2010).

tem without any offsetting service, and is intended to serve as a benchmark analysis for applications that incorporate the offsetting service.

Literature and Contribution
In the literature of financial network, a distinctive feature of our two concepts, *arc-twisted* and *vertex-twisted*, is their reference to dynamics. Such existing concepts as connectivity and concentration are largely static,[5] because the main focus of such research is examination of the consequence of balance-sheet linkage, effectively assuming "simultaneous" settlement. Few theoretical studies explicitly assume certain "sequential" settlement to discuss the relevance of network topologies. An exception is Rotemberg (2011), whose analysis is most relevant to the present study. Rotemberg (2011) theoretically examines the lower bound of the required amount of funds in a different setting. The author assumes "sequential" settlement, which allows an obligation to be settled in arbitrary installments.[6] However, in reality, the unit of settlements is not flexibly adjusted in many interbank settlements.[7] In our study, we do not allow arbitrary installments "ex-post," that is, the unit of settlement is fixed when a network of obligations is given; when the given obligations are settled, each obligation needs to be settled at once in each given unit.[8] Rotemberg (2011) shows that multiplicity of cycles could be a source of inefficient circulation of funds in deriving the lower bound. From the perspective of our characterization, multiplicity of cycles corresponds to the "domain" factor. The assumption of flexible installments fails to capture the "synchronization" factor.[9] Indeed, the key contribution of our study is to reveal that the static concept of "domain" factor is insufficient; however, a dynamic concept regarding the "synchronization" factor is indispensable for examining RTGS systems. The implications of the dynamic aspect of RTGS systems have also been examined in simulation-based studies, such as Beck and Soramäki (2001) and Gal-

[5] With regard to the analysis of financial contagion, the implications of the "connectedness" or "connectivity" of networks are extensively examined by Allen and Gale (2000), Freixas et al. (2000), Lagunoff and Schreft (2001),Cifuentes et al. (2005), Nier et al. (2007), Castiglionesi and Navarro (2008), Caballero and Simsek (2009), Allen et al. (2010), Gai and Kapadia (2010), Battiston et al. (2012a), and Zawadowski (2013). Simulation studies for examining the contribution of connectivity and concentration are conducted by Nier et al. (2007) and Battiston et al. (2012b). Statistical concepts that refer to concentration are proposed by Elsinger et al. (2006) and Cont et al. (2013).

[6] Rotemberg (2011) examines a class of a Euler graph. Our analysis deals with a general class that encompasses that class. Appendix A.7 presents the graph and shows the relevance of his study to ours.

[7] One reason implied by our analysis is that paying in multiple installments sometimes does not benefit the payer (i.e. reduce the cost of financing liquidity), but instead benefits some other banks. This externality is not endogenized under settlement systems in which the cost of finance is essentially burdened by the financiers. In major interbank settlement systems, intraday liquidity is served by the central bank with an explicit fee or on the basis of collateral served in advance, where the financier is burdened with the cost of finance.

[8] Note that "ex-ante" installments can be captured in our framework such that a debt of 10 is given as two debts of 3 and 7. Then, at the time of settlement, each debt with 3 and with 7 needs be settled at once in the unit of 3 and 7 for each, respectively.

[9] Note that Rotemberg (2011) does not deal with the upper bound. For the case of the upper bound, we find that even under the assumption of flexible installments, the "synchronization" factor—characterized by *vertex-twisted*—is relevant.

biati and Soramäki (2011).[10] Our theoretical analysis complements these works by clarifying the source of complications relevant to the dynamics.

Applications of the Framework
Hayakawa (2018) applies our framework to discuss the liquidity issues of introducing central clearing counterparties (CCPs), and discusses how a CCP's offsetting service could affect (i.e. increase or decrease) overall liquidity needs. By understanding CCPs' offsetting service as the elimination of arcs in the network model, Hayakawa (2018) reveals potential negative effects of introducing a CCP, and discusses the effects quantitatively in relation to network topologies. The study illustrates the usefulness of the framework in discussing the liquidity issues of settlements from the perspective of network topologies.

The rest of the paper is structured as follows. Section 2 introduces our model and mathematical framework. Section 3 provides motivational observations, especially regarding several example "networks" that are particularly relevant to our analysis. Section 4 overviews our main results in reference to the relevant theorems, and provides preliminary analysis. Section 5 shows our main results, formally introducing the "twist" properties—*arc-twist* and *vertex-twist*. Section 6 presents additional results regarding the quantitative implications of the "twist" properties. Section 7 provides concluding remarks. The appendix includes proofs of the relevant results.

2 Model and Framework

There are a finite number of banks. We consider two stages for the formation of interbank obligations and their settlements. In Stage 1, debts among banks are exogenously given. The contracts specify that all relevant obligations be settled in Stage 2. We discuss the required amount of funds in Stage 2 along with two polar liquidity scenarios that are intended to capture settlements in each "good time" and "bad time," as we elaborate in Sect. 2.3.

We express interbank obligations formed in Stage 1 utilizing a flow network representation, which we introduce in Sect. 2.1. Then, the settlements of interbank obligations in Stage 2 are expressed as a network with additional elements that specify "orders" and "liquidity inputs," which we introduce in Sect. 2.2. Section 2.3 introduces our liquidity scenarios and presents our formulation as a pair of graph problems.

2.1 Interbank Obligations: *f-network* $\langle V, A, f \rangle$

We express the obligations formed in Stage 1 utilizing a flow network representation.[11] The obligations are expressed with a *flow network* (***f-network***) $\mathcal{N}^f = \langle V, A, f \rangle$. V specifies a set of *vertices*, which corresponds to banks. $A =$

[10] Beck and Soramäki (2001)'s simulation study considers the seriousness of gridlock in the context of gridlock resolution mechanisms. In the perspective of network topology, their study indicates that a cycle in the network of payments could be a source of ill-functioning of a settlement system.

[11] For basic terminologies of flow networks, we follow the textbook usage. See, for example, Ahuja et al. (1993).

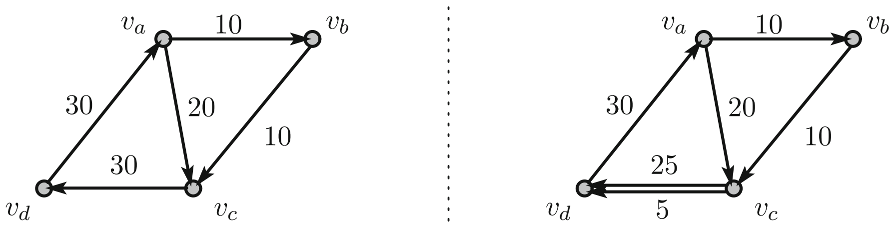

Fig. 1 Examples of closed f-networks: graphical representation. Notes: On the left side of the figure, $V = \{v_a, v_b, v_c, v_d\}$, $A = \{(v_a, v_b), (v_a, v_c), (v_b, v_c), (v_c, v_d), (v_d, v_a)\}$, $f((v_a, v_b)) = f((v_b, v_c)) = 10$, $f((v_a, v_c)) = 20$, $f((v_c, v_d)) = f((v_d, v_a)) = 30$. The right side of the figure shows a situation with multiple obligations between v_c and v_d. Each is distinguished as $f((v_c, v_d, 1)) = 25$, $f((v_c, v_d, 2)) = 5$ while the other obligations are the same as on the left

$\{(v, w, n)|v, w \in V, n = 1, 2, \ldots\}$ specifies a set of *arcs*, where each arc (v, v', k) shows that v has some amount of obligation to v', and k is used as an index that distinguishes among multiple arcs for (v, v'). The indices are not usually mentioned in order to avoid being notationally cumbersome; however, the multiplicity of the arcs is not trivial for our analysis, as discussed in the analysis section. Then, *flow* $f : A \to \mathcal{R}_+$ expresses the amount of the relevant obligations.

Throughout the study, we confine ourselves to a class of f-networks that are *closed*. In words, an f-network $\langle V, A, f\rangle$ is closed when for each $v \in V$, the total amount of obligations owed by v to the others equals the total amount of claims held by v. We show the formal definition below. Figure 1 provides examples of closed f-networks. The left f-network is later used to illustrate our analytical method. The right f-network is presented simply as a reminder that our framework allows multiple arcs between the same two vertices in the same direction.

Definition 1 (*Closed f-network*)
f-network $\langle V, A, f\rangle$ is *closed* if $f_v^I = f_v^O$ for every $v \in V$, where $f_v^I \equiv \sum_{v' \in V} \sum_{l=1,2,\ldots} f((v', v, l))$, $f_v^O \equiv \sum_{v' \in V} \sum_{m=1,2,\ldots} f((v, v', m))$.

In the above Definition 1, f_v^I denotes the total amount of claims held by v—the total amount of funds that should flow into v. Meanwhile, f_v^O denotes the total amount of obligations v owes to the others—the total amount of funds that should flow out of v. To clarify the definition, we could restate it as follows: for each $v \in V$, the weighted node input degree (i.e. f_v^I) equals the weighted node output degree (i.e. f_v^O).

Remark
The main reason we confine our focus to the class of *closed* f-networks is their mathematical tractability in our analysis. We consider our findings to be essentially applicable also to f-networks that are not closed, as we discuss in the analysis section. In addition, the class of *closed* f-networks serves as a benchmark to examine the implications of "sequential" settlement, particularly in comparison to "simultaneous" settlement, since no funds would be required under "simultaneous" settlement for arbitrary *closed* f-networks.

2.2 Settlement of Obligations: *fsp-network* $\langle V, A, f, s, p\rangle$

In Stage 2, the interbank obligations are settled sequentially on a gross basis. Given interbank obligations $\langle V, A, f\rangle$, we let one-to-one mapping (*sequence*) $s : A \rightarrow \{1, 2, \ldots, |A|\}$ show the relative order of settlements. It states that the obligations specified by A are settled individually in this order such that obligation $a \in A$ with $s(a) = k$ is settled in k-th order.

In order to evaluate the required liquidity, we assume that funds are input to each bank only initially in Stage 2, and that lending/borrowing funds between banks is not allowed. Thus, funds that can be used by a bank for its payments are either those initially put to the bank, or those the bank has received in the course of settlements. Along with this assumption, let (*potential*) $p : V \rightarrow \mathcal{R}_{0+}$ indicate the amount of funds *put* to each bank initially in Stage 2 (e.g. $p(v) = 10$ indicates $v \in V$ initially holds 10 units of funds.)

We form a different concept of network called the ***fsp-network*** $\mathcal{N}^{fsp} = \langle V, A, f, s, p\rangle$. We define the *e-covered* (exact covered) property for fsp-networks, such that for e-covered $\langle V, A, f, s, p\rangle$, $\sum_{v\in V} p(v)$ indicates the total amount of required funds for $\langle V, A, f, s\rangle$. The *e-covered* property consists of two requirements: first, there is *no shortage* of funds at any point of settlement, and second, there are no *redundant* funds that are not used for any of the settlements. Before proceeding to our formal definition, Fig. 2 shows examples of e-covered fsp-networks. For each f-network $\langle V, A, f\rangle$ in the figure, the order specified by $s : A \rightarrow \{1, 2, \ldots, |A|\}$ is shown on the upper right of the amount of corresponding obligation, and each initial liquidity holding specified by $p : V \rightarrow \mathcal{R}_{0+}$ is shown in boldface alongside each vertex. It is easily confirmed for the fsp-network shown on the left that 30 units of funds put to the vertex on the bottom-left is sufficient to settle all the obligations given the indicated orders of settlements, and any amount smaller than this falls short of the required amount for the corresponding settlements. For the right-hand fsp-network, it is similarly confirmed that the indicated funds are sufficient to settle all obligations, and any smaller amount of funds for any vertex causes a shortage for the corresponding settlement.

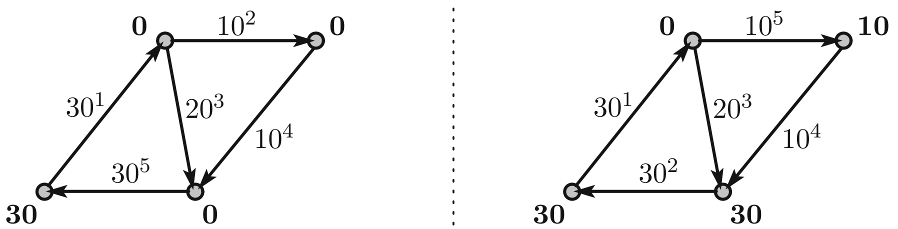

Fig. 2 Examples of e-covered fsp-networks: graphical representation. Notes: These two *fsp-networks* are defined on the same *f-network* $\langle V, A, f\rangle$ shown on the left side of Figure 1. The left-hand *fsp-network* shows $\langle V, A, f, s, p\rangle$ with $s((v_d, v_a)) = 1$, $s((v_a, v_b)) = 2$, $s((v_a, v_c)) = 3$, $s((v_b, v_c)) = 4$, $s((v_c, v_d)) = 5$, $p(v_a) = p(v_b) = p(v_c) = 0$, $p(v_d) = 30$. The right *fsp-network* shows (V, A, f, s', p') with $s'((v_d, v_a)) = 1$, $s'((v_a, v_b)) = 5$, $s'((v_a, v_c)) = 3$, $s'((v_b, v_c)) = 4$, $s'((v_c, v_d)) = 2$, $p'(v_a) = 0$, $p'(v_b) = 10$, $p'(v_c) = p'(v_d) = 30$. The order specified by s is shown on the upper right of the amount of corresponding obligation, and each initial liquidity holding specified by p is shown in boldface alongside each vertex

We prepare to define *covered* property, then proceed to our formal definition of *e-covered* property. Given fsp-network $\langle V, A, f, s, p\rangle$, let $t = 0, 1, \dots, |A|$ be a period in which obligation $a \in A$ with $s(a) = t$ is settled at the beginning of each period. The aggregate amount of funds received by $v \in V$ in period t is denoted as $f^I_{v,t} = \sum_{v' \in V} \sum_{l=1,2,\dots} 1_{\{s((v',v,l))=t\}} f((v', v, l))$, while the aggregate amount paid to other banks is denoted as $f^O_{v,t} = \sum_{v' \in V} \sum_{m=1,2,\dots} 1_{\{s((v,v',m))=t\}} f((v, v', m))$. We denote $p^t(v)$ as the amount held by $v \in V$ at the end of period t, by letting $p^t(v) = p^{t-1}(v) + (f^I_{v,t} - f^O_{v,t})$ for $t = 1, 2, \dots, |A|$ and $p^0(v) = p(v)$. Now, we say $\langle V, A, f, s, p\rangle$ is *covered* if $p^t(v) \geq 0$ for every $v \in V$ and every $t = 0, 1, \dots, |A|$. Based on the *covered* property, the *e-covered* property is defined as follows.

Definition 2 (*E-covered property of fsp-networks*)

fsp-network $\langle V, A, f, s, p\rangle$ is *e-covered* (exact covered) if (no shortage): $\langle V, A, f, s, p\rangle$ is *covered*, and (no redundancy): There is no other $p' : V \to R_{0+}$ such that $\langle V, A, f, s, p'\rangle$ is *covered*, $p'(v) \leq p(v)$ for every $v \in V$, and $p'(v') < p(v')$ for some $v' \in V$.

In words, the e-covered fsp-network $\langle V, A, f, s, p\rangle$ indicates that the amount of funds initially assigned to each vertex v (i.e. $p(v)$) is just sufficient to settle all the obligations owed by v, given the order of the settlements specified by s.

We proceed to clarify the implication of the e-covered property by presenting a minimization problem (Problem A) and associated lemma.

(Problem A: Minimization Among Covered fsp-Networks)

For a given $\langle V, A, f, s\rangle$ where $\langle V, A, f\rangle$ is closed,

$$\min_p \sum_{v \in V} p(v),$$
$$\text{s.t., } \langle V, A, f, s, p\rangle \text{ is covered.}$$

Lemma 1

Given $\langle V, A, f, s\rangle$ where $\langle V, A, f\rangle$ is closed, $\langle V, A, f, s, p\rangle$ that attains the value for Problem A is e-covered and unique.

The result is immediate based on the definition of e-covered. Based on the lemma, we examine the required amount of funds in terms of the uniquely derived *e-covered* fsp-network for a given closed f-network and *sequence*. When $\langle V, A, f, s, p\rangle$ is e-covered, we call $\sum_{v \in V} p(v)$ its *circulation.*

2.3 Liquidity Scenarios and the Relevant Graph Problems

We focus on two polar scenarios for our analysis: one in the good time, and the other in the bad time. The good time refers to the financial state in which settlements are executed under the best coordination, and the total required funds are the minimum possible. By contrast, the bad time refers to the state in which settlements are executed under the worst coordination, and the total required funds are the maximum possible. The bad time is meant to represent a period of financial distress, while the good time is interpreted as a benchmark state for a time of non-distress.

Formally, the total required funds in each scenario are derived by each of the following minimization and maximization problems, called min/max settlement fund circulation problems (MIN- SFC and MAX- SFC).[12] For a given closed f-network, MIN- SFC (MAX- SFC) derives the lower (upper) bound of the total required funds against every possible "order" of settlement. For the formulation, let $S(A)$ denote the set of available *sequences* for given $\langle V, A, f \rangle$.

(MIN- SFC *in Original Form*)

Given a closed f-network $\langle V, A, f \rangle$,

$$\min_{s \in S(A)} \sum_{v \in V} p(v),$$

$$\textit{s.t. fsp-network } \langle V, A, f, s, p \rangle \textit{ is e-covered.}$$

(MAX- SFC *in Original Form*)

Given a closed f-network $N^f = \langle V, A, f \rangle$,

$$\max_{s \in S(A)} \sum_{v \in V} p(v),$$

$$\textit{s.t. fsp-network } \langle V, A, f, s, p \rangle \textit{ is e-covered.}$$

For example, for the closed f-network shown on the left side of Fig. 1, the left side of Fig. 2 shows an fsp-network that gives the value derived by MIN- SFC, while the right side of the figure shows an fsp-network that gives the value derived by MAX- SFC. Throughout the study, we call the value derived by each MIN- SFC and MAX- SFC *min/max-circulation* and denote them as $x^{min}(N^f)$, $x^{max}(N^f)$ for the input closed f-network N^f.

Remark on the Formulation

Our particular focus in setting up the problems of MIN- SFC and MAX- SFC is on analyzing the implications of the sequential feature of settlements given the level of coordination. In this regard, the *good* and *bad* liquidity scenarios are meant to be analytical scenarios, and there are limitations in applying our approach to real-world scenarios. For one thing, our formulation ignores the endogenous nature of the timing decision by banks themselves. In particular, bank-run behavior might spread among financial institutions with regard to their mutual obligations, which could switch the situation suddenly from the *good* to *bad* liquidity scenario. Meanwhile, in *bad* scenarios, banks could simply claim their pending debts, while attempting to postpone their own debts at the same time. Such a "simultaneous" claim situation is not captured in our sequential setting; however, we note that our analysis could still help in discussing the necessary amount of liquidity that would cease or prevent the "simultaneous" claim situation.

3 Motivational Observations

In this section, we motivate the introduction of our "twist" properties—*arc-twisted* and *vertex-twisted*—and illustrate their relevance to min/max-circulation, referring to several example f-networks.

[12] Hayakawa et al. (2019) show NP-hardness of each MIN- SFC and MAX- SFC for closed f-networks.

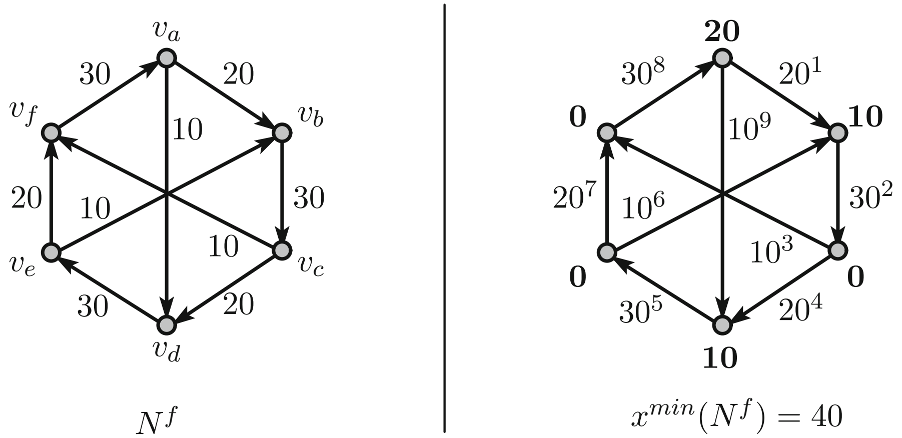

Fig. 3 Notes: $V = \{v_a, v_b, v_c, v_d, v_e, v_f\}$, $A = \{(v_a, v_b), (v_b, v_c), (v_c, v_d), (v_d, v_e), (v_e, v_f), (v_f, v_a), (v_a, v_d), (v_c, v_f), (v_e, v_b)\}$, $f((v_f, v_a)) = f((v_b, v_c)) = f((v_d, v_e)) = 30$, $f((v_a, v_b)) = f((v_c, v_d)) = f((v_e, v_f) = 20$, $f((v_a, v_d) = f((v_c, v_f)) = f((v_e, v_b)) = 10$

3.1 Min-Circulation and *arc-twisted*

For the fsp-network shown on the left side of Fig. 2, we observe that the min-circulation for the underlying f-network equals the largest obligation in the amount (i.e. $max_{a \in A} f(a)$) for the underlying closed f-network. Actually, the following result is immediate.

Observation 1 (*Lower bound of min-circulation*)
For an arbitrary closed f-network $N^f = \langle V, A, f \rangle$, we have $x^{min}(N^f) \geq max_{a \in A} f(a)$.

The inequality arises for other f-networks. For example, see Fig. 3, where $x^{min}(N^f) = 40 > max_{a \in A} f(a) = 30$ for the given $N^f = \langle V, A, f \rangle$. We show that, for this specific example f-network, the additionally required funds $x^{min}(N^f) - max_{a \in A} f(a) = 10$ sources from the "twist" property of *arc-twisted*.[13] To explain the *arc-twisted* property informally, for the f-network in Fig. 3, take two "cycles," as shown in Fig. 4. We say that the two cycles are in *arc-twisted* relation, based on certain inconsistency between the two cycles. The formal definition is shown in our analysis section. The following results illustrate the relevant inconsistency for our example f-network.

Observation 2 (*Arc-twisted cycles*)

i. (Cycle on the left side of Fig. 4) Take a closed f-network $N^f = \langle V, A, f \rangle$ where $\langle V, A \rangle$ is given as the cycle shown on the left side of Fig. 4.

[13] Note that the additionally required funds could also be sourced from another factor, which we could refer to as "multiplicity of cycles." This point becomes clearer in our analysis section. Here, we point out that *arc-twisted* is one of the possible factors.

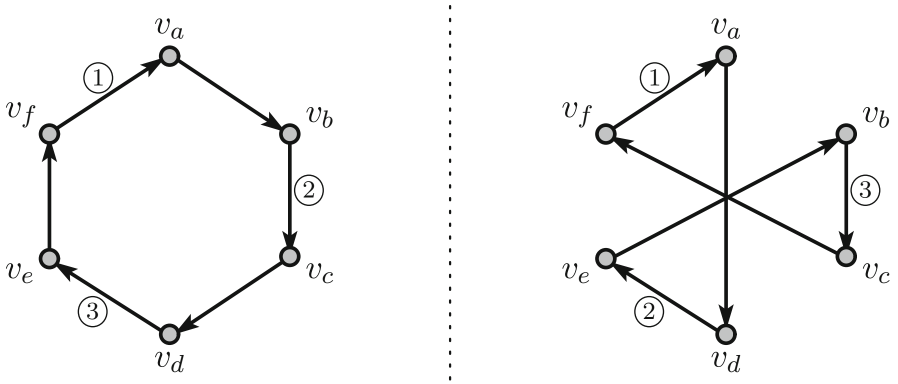

Fig. 4 Notes: This figure shows two cycles included in the f-network shown in Fig. 3

If *sequence* s starting with $s((v_f, v_a)) = 1$ attains the min-circulation of N^f, then we have $s((v_b, v_c)) < s((v_d, v_e))$.

ii. (Cycle on the right of Fig. 4) Take a closed f-network $N^f = \langle V, A, f \rangle$ where $\langle V, A \rangle$ is given as the cycle shown in the right side of Fig. 4.

If *sequence* s starting with $s((v_f, v_a)) = 1$ attains the min-circulation of N^f, then we have $s((v_b, v_c)) > s((v_d, v_e))$.

Remember that in the original f-network shown in Fig. 3, each of the three arcs (v_f, v_a), (v_b, v_c), and (v_d, v_e) needs be settled at once, that is, not in any multiple installments. Thus, Observation 2 implies that an efficient order of funds to circulate for each cycle cannot be realized at the same time for the original f-network, and this inconsistency generates the additionally required funds. In our analysis section, we show how this observation is formally generalized.

Implications of slicing and hub

Here, we further observe that several different changes on the f-network shown in Fig. 3 serve to eliminate the "additionally required" funds. First, see Fig. 5, where an arc (v_b, v_c) for the original f-network is *sliced* into two arcs. A comparison of the two f-networks indicates that the "additionally required" funds source from the restriction such that an arc in the amount of 30 units (e.g. (v_b, v_c)) need be settled at once, whereas when stated from the perspective of the f-network in Fig. 5, the restriction is interpreted as the *sliced* arcs $(v_b, v_c, 1)$ and $(v_b, v_c, 2)$ needing to be *synchronized* to settle.

Figure 6 shows a different way to avoid the additional funds. There, a *hub* vertex v_g is added to the original f-network. The fsp-network shown on the right side of the figure attains the min-circulation, indicating that the added *hub* serves to coordinate the timing of settlements. These observations indicate that the "twist" property of *arc-twisted* is more fundamental in discussing min-circulation than the other concepts are, such as *hub*.

The Non-linearity Brought by arc-twisted Cycles

Now, we proceed to observe a notable consequence of the existence of *arc-twisted* cycles. For the f-network shown in Fig. 3, we consider an increase in the amount of

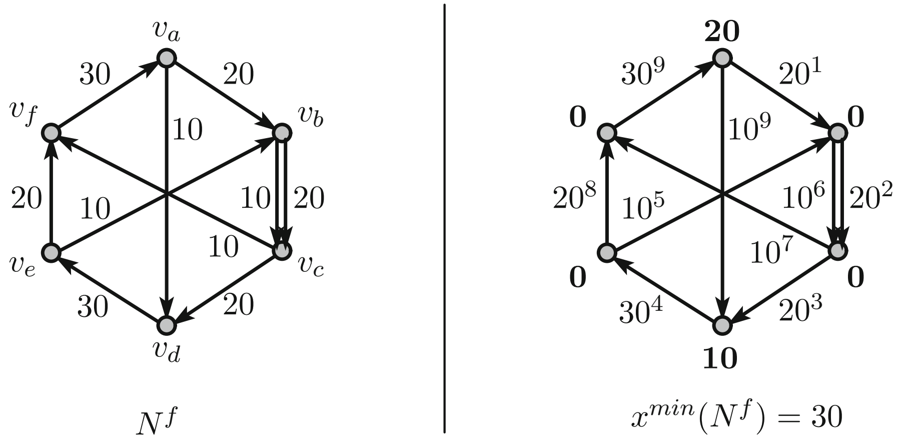

Fig. 5 *Slicing*. Notes: The figure shows an f-network constructed from the f-network shown in Figure 3, by letting an arc (v_b, v_c) be *sliced* into two

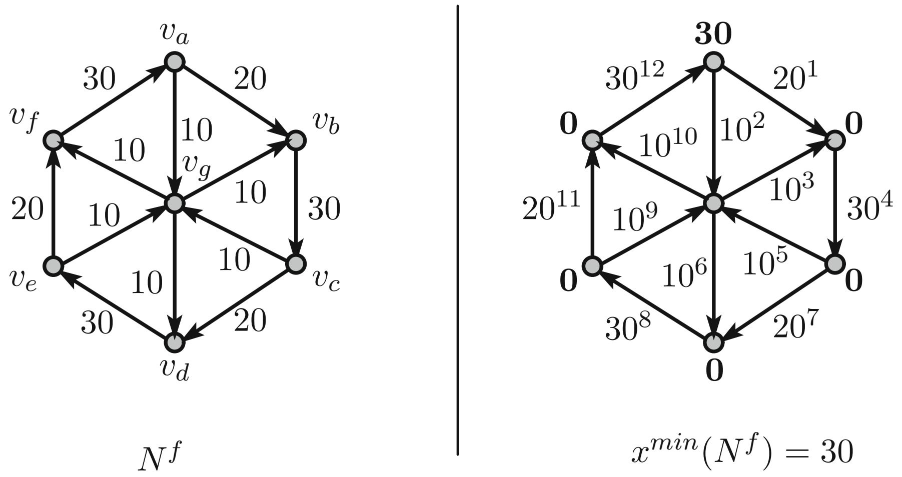

Fig. 6 *Hub*. Notes: The figure shows an f-network that constructed from the f-network shown in Fig. 3, by adding a *hub* vertex v_g such that the vertex is added in the middle of each of (v_a, v_d), (v_c, v_f), and (v_e, v_b) to separate one arc into two (e.g. (v_a, v_d) is separated into (v_a, v_g) and (v_g, v_d))

obligations on the basis of a cycle. Suppose that we increase the amount of obligations on the basis of the cycle shown on the right side of Fig. 4 such that each arc in the cycle has an additional 5 units of funds, which we show on the left side of Fig. 7. Observe that the *sequence* that attains the min-circulation for the original f-network shown in Fig. 3 also attains the min-circulation for that in Fig. 7, where a corresponding fsp-network is shown on the right side. However, when we keep increasing the amount on the basis of the same cycle, at a certain point, the same *sequence* no longer attains min-circulation. For example, see the f-network shown on the left side of Fig. 8, where an amount of 15 units is added instead of 5 units in the previous figure. Now, the

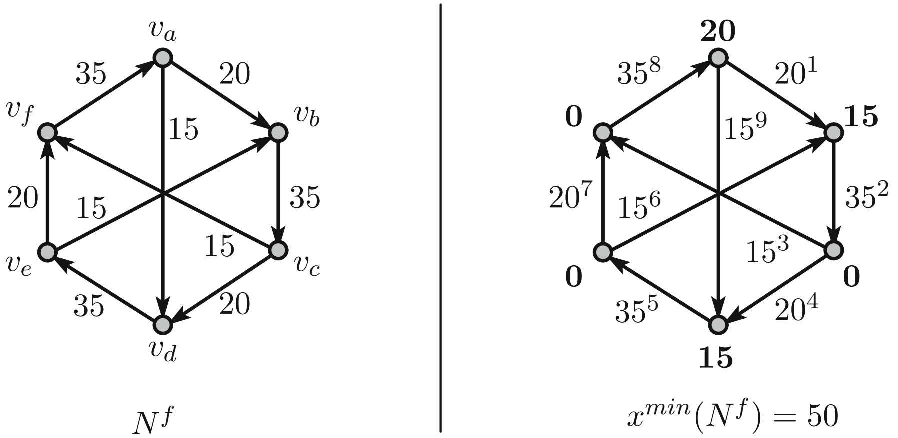

Fig. 7 Notes: Compared to Fig. 3, *flow* is increased in the amount of 5 units with respect to each arc in the cycle $(v_a, v_d, v_e, v_b, v_c, v_f, v_a)$ (e.g. $f((v_a, v_d)) = 10 + 5$ and $f((v_d, v_e)) = 30 + 5$))

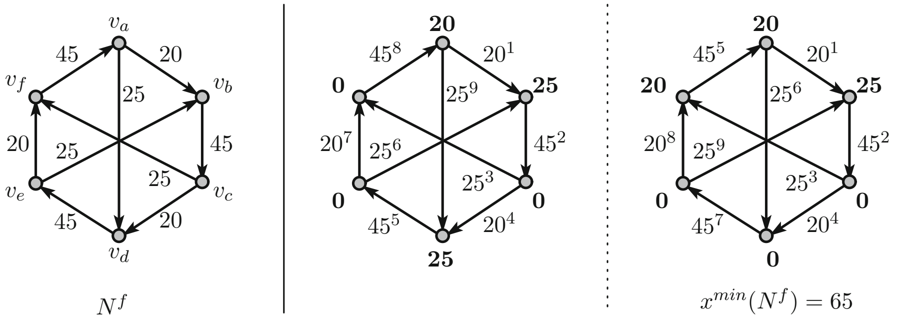

Fig. 8 Notes: Compared to Fig. 3, *flow* is increased in the amount of 15 units with respect to each arc in the cycle $(v_a, v_d, v_e, v_b, v_c, v_f, v_a)$. The fsp-network shown in the middle of the figure implies that the *sequence* that attains the min-circulation for the original f-network shown in Fig. 3 no longer attains the min-circulation of the f-network shown on the left side of this figure, where the circulation is shown as $70(= 20 + 25 + 25)$ units. The min-circulation turns to 65 units, for which a corresponding fsp-network is shown on the right side of the figure

circulation attained for the same *sequence* is 70 units as shown in the middle of the figure; however, the min-circulation is smaller in this case, at 65 units, for which a corresponding fsp-network is shown on the right side of the figure. We summarize these observations below.

Observation 3 (*Non-linearity associated with arc-twisted*)
Given f-network N^f shown on the left side of Fig. 3, suppose that *flow* is increased for the arcs in cycle $(v_a, v_d, v_e, v_b, v_c, v_f, v_a)$ by the amount of $m > 0$.[14] Denote the derived f-network as $\tilde{N}^f$. Then, we have as follows:

i. if $m \leq 10$, then any *sequence* that attains the min-circulation for N^f also attains the min-circulation for $\tilde{N}^f$, and $x^{min}(\tilde{N}^f) = x^{min}(N^f) + 2m$.

[14] The case with $m = 5$ is shown in Fig. 7, and the case with $m = 15$ is shown in Fig. 8.

ii. if $m > 10$, then none of the *sequences* that attain the min-circulation for N^f attains the min-circulation for $\tilde{N}^f$, and $x^{min}(\tilde{N}^f) = x^{min}(N^f) + 20 + (m - 10)$.

According to Observation 3, the marginal effect of increasing *flow* on the cycle is 2 if $m \leq 10$, while it is 1 if $m > 10$. To observe the intuition, note for the fsp-network shown in Fig. 3, the funds are "efficiently circulated" on the basis of the cycle $(v_a, v_b, v_c, v_d, v_e, v_f, v_a)$ shown on the left side of Fig. 4. (i.e. the *sequence* for the cycle is monotonically increasing when we start by $s((v_a, v_b)) = 1$, which is followed by $s((v_b, v_c)) = 2$, $s((v_c, v_d)) = 4$, $s((v_d, v_e)) = 5$, $s((v_e, v_f)) = 7$, $s((v_f, v_a)) = 8$.) By contrast, the funds are not efficiently circulated on the basis of the cycle shown on the right side of Fig. 4 (i.e. the *sequence* for the cycle is never monotonically increasing when we start by any of the vertices.). The reason that the former cycle—$(v_a, v_b, v_c, v_d, v_e, v_f, v_a)$—is "chosen" to attain efficient circulation is that the *flow* endowed to the cycle is basically larger than that endowed to the other cycle (e.g. $f((v_e, v_f)) = 20 > f((v_a, v_d)) = 10$). The former cycle should be chosen as long as $m \leq 10$ in the notation of Observation 3; but once $m > 10$, the other cycle should now be chosen, and thus, *sequences* that attain the min-circulation need to be changed accordingly, as illustrated on the right side of Fig. 8. Note that such non-linearity is not observed for the f-networks shown in Fig. 2, for any pair or set of the cycles included.

Implications of Non-linearity for Interbank Settlements

The observed non-linearity has implications for real-world settlements, specifically regarding the effect of *partial* offsetting. In real-world settlements, it is plausible that offsetting is executed only partially.[15] Using the notation in Observation 3, suppose $m > 10$ at first, and then, m decreases to track the change of the marginal effect of offsetting obligations on the basis of the cycle. The observation states that once $m < 10$, the marginal effect doubles.

To see the implications of the same observation in a different context, compare two ways of *partial* offsetting for the f-network shown in Fig. 3; that is, offsetting on the basis of the cycle shown on the left side of Fig. 4, or the right side of the figure. Related to Observation 3, we observe that the marginal effect of offsetting is twice as large for the right cycle as for the left cycle.

3.2 Max-Circulation and *vertex-twisted*

As observed for the fsp-network shown on the right side of Fig. 2, the max-circulation for the underlying f-network is strictly less than the aggregate amount of obligations (i.e. $\sum_{a \in A} f(a)$). Specifically, for the shown f-network, the max-circulation equals $\sum_{a \in A} f(a) - max_{a \in A} f(a)$. Actually, we obtain the following result.

Observation 4

For an arbitrary closed f-network $N^f = \langle V, A, f \rangle$, we have $x^{max}(N^f) \leq \sum_{a \in A} f(a) - max_{a \in A} f(a)$.

Again, the inequality arises for other f-networks. For example, see Fig. 9, where $x^{max}(N^f) = 110 < \sum_{a \in A} f(a) - max_{a \in A} f(a) = 140 - 20$ for the given $N^f =$

[15] In reality, not all the obligations are settled through CCPs. Liquidity saving mechanisms typically work within a few minutes, indicating that it is possible to offset only a fraction of obligations in a day.

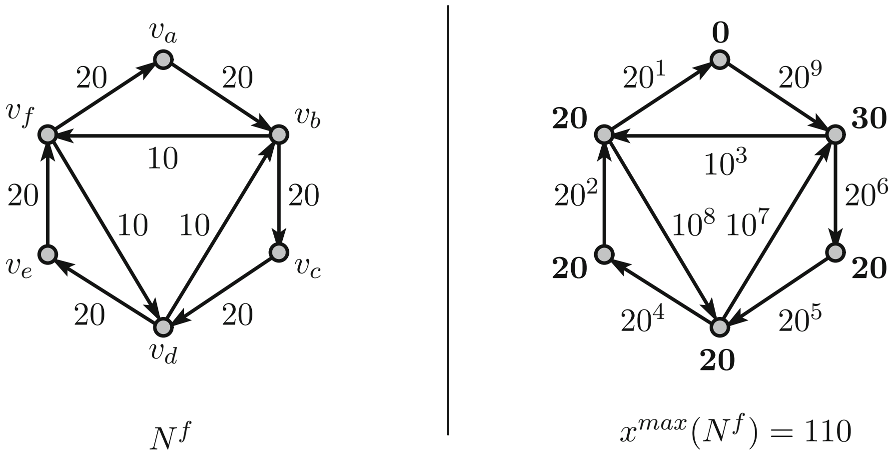

Fig. 9 Notes: $V = \{v_a, v_b, v_c, v_d, v_e, v_f\}$, $A = \{(v_a, v_b), (v_b, v_c), (v_c, v_d), (v_d, v_e), (v_e, v_f), (v_f, v_a), (v_b, v_f), (v_f, v_d), (v_d, v_b)\}$, $f((v_a, v_b)) = f((v_b, v_c)) = f((v_c, v_d)) = f((v_d, v_e)) = f((v_e, v_f)) = f((v_f, v_a) = 20$, $f((v_b, v_f) = f((v_f, v_d)) = f((v_d, v_b)) = 10$

$\langle V, A, f\rangle$. We show that the subtracted amount $10(= 120 - 110)$ is sourced from the "twist" property, which we term *vertex-twisted* for the case of max-circulation. To explain the *vertex-twisted* property informally, for the f-network shown in Fig. 9, take the two "cycles" shown in Fig. 10. We say that the two cycles are in *vertex-twisted* relation, based on a certain inconsistency arising between the two cycles. The following observations are relevant to the inconsistency.

Observation 5

i. (Cycle on the left side of Fig. 10) Take a closed f-network $N^f = \langle V, A, f\rangle$ where $\langle V, A\rangle$ is given as the cycle shown on the left side of Fig. 10.

If *sequence* s starting with $s((v_f, v_a)) = 1$ attains the max-circulation of N^f, then, we have $s((v_d, v_e)) < s((v_b, v_c))$.

ii. (Cycle on the right side of Fig. 10) Take a closed f-network $N^f = \langle V, A, f\rangle$ where $\langle V, A\rangle$ is given as the cycle shown on the right side of Fig. 10.

If *sequence* s starting with $s((v_f, v_a)) = 1$ attains the max-circulation of N^f, then, we have $s((v_d, v_b)) > s((v_b, v_f))$.

For the case of max-circulation, the implied order of vertices is relevant. In order to clarify this point, proceed to Fig. 11, where the direction of the inner cycle is now reversed. The max-circulation becomes 120 units, where there is no longer a subtracted amount. Focus on vertices v_f and v_d; each has multiple outgoing and incoming arcs. For the case of v_f, we observe that all outgoing arcs—(v_f, v_a) and (v_f, b)—are settled before all incoming arcs—(v_d, v_f) and (v_e, v_f). The same observation holds for the case of v_d. In words, we could say that the outgoing arcs for

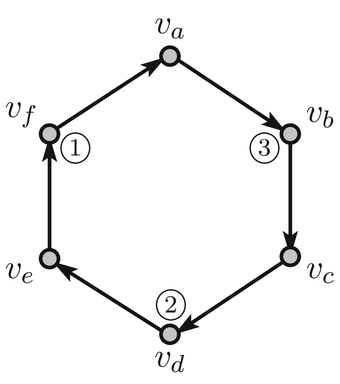

Fig. 10 Notes: The above figure shows two cycles included in the f-network shown in Fig. 9

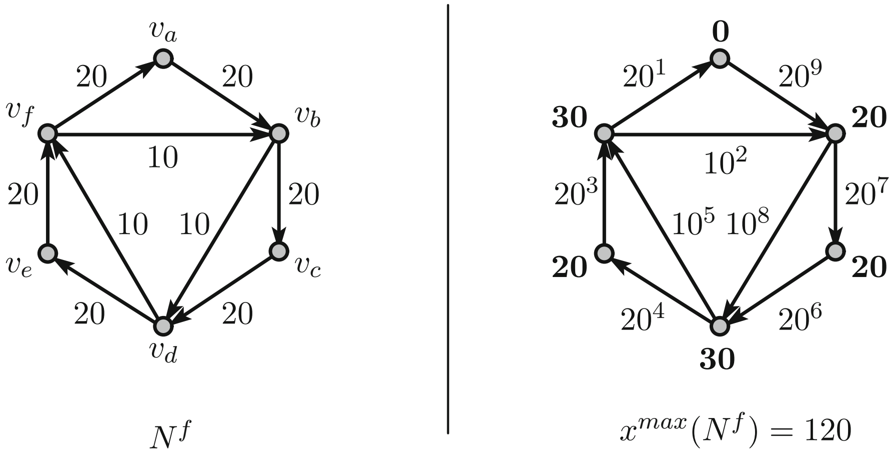

Fig. 11 Notes: Compared to Fig. 9, the direction of the inner cycle is reversed

the vertex are *synchronized* to be settled before any incoming arc. This would be an instruction for the least efficient circulation of funds to be attained; however, Observation 5 indicates that such *synchronization* is not fully executable for the f-network shown in Fig. 9. This is essentially because there is an arc—(v_f, v_d)—between two vertices v_d and v_f. With regard to v_f, when all outgoing arcs are settled before all incoming arcs, (v_f, v_d) needs be settled before (v_f, v_e). Then, the instruction is for (v_d, v_e) to be settled after (v_e, v_f) for the purpose of the least efficient circulation of funds. Combining these, the instruction is that (v_f, v_d) is settled before (v_d, v_e). Now, we observe inconsistency; with regard to v_d, in order for all outgoing arcs to be settled before all incoming arcs, (v_d, v_e) needs be settled before (v_f, v_d). Thus, with regard to v_f, all outgoing arcs cannot be settled before all incoming arcs. This inconsistency that arises between the two cycles is formally referred to as *vertex-twisted* relation.

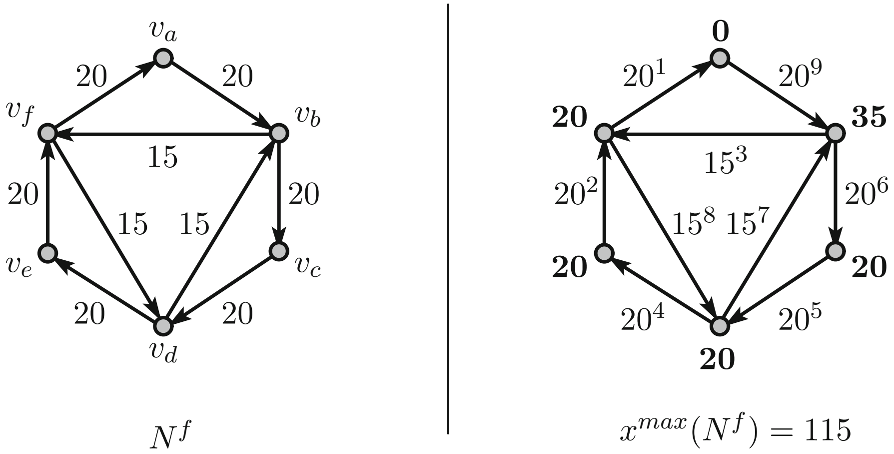

Fig. 12 Notes: Compared to Figure 9, *flow* is increased in the amount of 5 with respect to each arc in cycle (v_b, v_f, v_d, v_b) (e.g. $f((v_b, v_f)) = 10 + 5$)

Non-linearity Brought by Vertex-Twisted Cycles

Now, we proceed to observe the non-linearity for the case of max-circulation, which is brought by *vertex-twisted* cycles. For the f-network shown in Fig. 9, we consider to increase the amount of obligations on the basis of a cycle. Suppose that we increase the amount of obligations on the basis of the cycle shown on the right side of Fig. 10 such that each arc in the cycle is given an additional 5 units, which we show on the left side of Fig. 12. Observe that the *sequence* that attains the max-circulation for the original f-network shown in Fig. 9 also attains the max-circulation for that in Fig. 12, where a corresponding fsp-network is shown on the right side. However, when we keep increasing the amount on the basis of the same cycle, at a certain point, the same *sequence* no longer attains the max-circulation. For example, see the f-network shown on the left side of Fig. 13, where an amount of 15 units is added instead of 5 units in the previous figure. Now, the circulation attained for the same *sequence* is 125 units as shown in the middle of the figure; however, the max-circulation is smaller in this case, at 130 units, for which a corresponding fsp-network is shown on the right side of the figure. We summarize these observations below.

Observation 6 (*Non-linearity associated with vertex-twisted*)

Given f-network N^f shown on the left side of Fig. 9, suppose that *flow* is increased for the arcs in cycle $(v_a, v_d, v_e, v_b, v_c, v_f, v_a)$ by the amount of $m > 0$.[16] Denote the derived f-network as $\tilde{N}^f$. Then, we have:

i. if $m \leq 10$, then any of the *sequences* that attain the max-circulation for N^f also attains the max-circulation for $\tilde{N}^f$, and $x^{max}(\tilde{N}^f) = x^{max}(N^f) + m$.
ii. if $m > 10$, then none of the *sequences* that attain the max-circulation for N^f attains the max-circulation for $\tilde{N}^f$, and $x^{max}(\tilde{N}^f) = x^{max}(N^f) + 10 + 2(m - 10)$.

[16] The case with $m = 5$ is shown in Fig. 7, and the case with $m = 15$ is shown in Fig. 8.

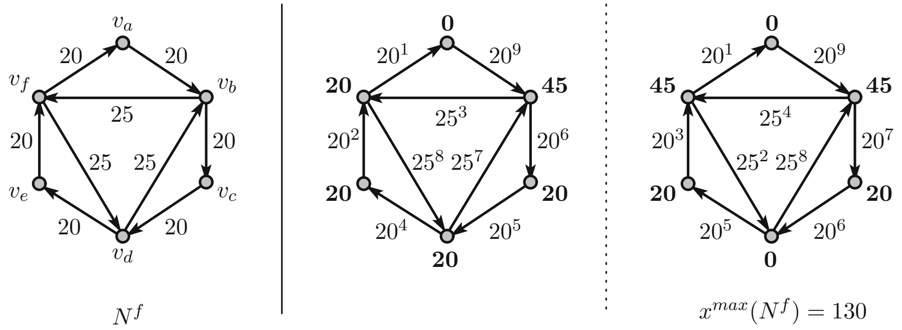

Fig. 13 Notes: Compared to Fig. 9, *flow* is increased in the amount of 15 with respect to each arc in cycle (v_b, v_f, v_d, v_b). The fsp-network shown in the middle of the figure implies that the *sequence* that attains the max-circulation for the original f-network shown in Fig. 9 no longer attains the max-circulation of the f-network shown on the left side of this figure, where the circulation is shown as $125(= 45 + 20 * 4)$ units. The max-circulation becomes 130 units, for which a corresponding fsp-network is shown on the right side of the figure

According to Observation 6, the marginal effect of increasing *flow* on the cycle is 1 if $m \leq 10$, while it is 2 if $m > 10$. To observe the intuition, note for the fsp-network shown in Fig. 9 that the funds are least efficiently circulated on the basis of cycle $(v_a, v_b, v_c, v_d, v_e, v_f, v_a)$, which is shown on the left side of Fig. 10 (i.e. for each arc except one—(v_a, v_b), initially held funds are used at the timing of the settlement.). By contrast, the funds are not least efficiently circulated on the basis of the other cycle (v_b, v_f, v_d, v_b) shown on the right side of Fig. 10 (i.e. for each arc except two—(v_f, v_d) and (v_d, v_b), initially held fund is used at the timing of the settlement. The reason that the former cycle–$(v_a, v_b, v_c, v_d, v_e, v_f, v_a)$—is chosen to attain least efficient circulation is that the *flow* endowed to the cycle is basically larger than that endowed to the other cycle. The former cycle should be chosen as long as $m \leq 10$ in the notation of Observation 6; but once $m > 10$, the other cycle should now be chosen, and thus, the *sequence* that attains the max-circulation needs to be changed accordingly, as illustrated on the right side of Fig. 13. Note that such non-linearity is not observed for the f-networks shown in Fig. 2, regarding any pair or set of the cycles included. In the same manner as the case of min-circulation, the observed non-linearity associated with the max-circulation has implications for *partial* offsetting in the context of interbank settlements.

3.3 *Arc-twisted* and *vertex-twisted*

We later formally state that *arc-twisted* indicates *vertex-twisted*, but not vice versa. Specifically, the two cycles in Fig. 4 are *arc-twisted*, and thus, they are also *vertex-twisted*. The two cycles in Fig. 10 are *vertex-twisted*, but they are not *arc-twisted*. This result consolidates the observations regarding *arc-twisted* and *vertex-twisted*.

4 Overview and Preliminary Analysis

4.1 Overview of the Main Results

Since we need to take several steps to present our main results, we provide an overview in this section. With regard to min/max-circulation, our base results are summarized as follows.

$$x^{min/max}(.) = \underbrace{(\textbf{“flow” relevant part})}_{\textit{domain}\text{ factor}} + \underbrace{(\textbf{“order” relevant part})}_{\textit{synchronization}\text{ factor}}.$$

Based on the preliminary results provided in Theorems 1, 2, and 3, Theorem 4 presents our formal presentation of the above separation into two parts. The theorem also shows our characterization of the first part of the equation with the concept of *domain*, which refers to the extent that each unit of settlement funds can circulate; a cycle is shown as the unit for funds to circulate. With regard to the second part of the equation, which is our main focus, our characterization is presented in Theorem 5 for min-circulation with our *arc-twisted* concept, and in Theorem 6 for max-circulation with our *vertex-twisted* concept. Theorem 7 consolidates our main findings by showing the relation between the "twist" concepts, as mentioned in the last part of Sect. 3. Further results focusing on the quantitative aspects of the "twisted" properties—non-linearity in min/max-circulation—are discussed in Sect. 6.

Remark

- Our summary equation implies that a specific change of an underlying network can be decomposed into two effects. In the discussion paper version of this paper (Hayakawa 2014), we define a group of *network transformations*, which includes the operations of *slicing* and *separation*—letting one arc be two by adding a vertex in the middle of the arc—mentioned in Sect. 3, and present the effects in terms of the *domain* and *synchronization* factors. The following statement summarizes the relevant results with the symbolic expression $\Delta_{transform}$, which refers to the extent to which min/max-circulation is changed by each of our *network transformation* operations on the input closed f-network.[17]

$$\Delta_{transform} x^{min/max}(.) = (\textit{domain}\text{ effect}) + (\textit{synchronization}\text{ effect}).$$

4.2 Preliminary Analysis

We say that an f-network is not *connected* when we can separate its vertices into two groups with no arc between them. Throughout this article, without loss of generality we focus on f-networks that are *connected*.

Our approach is based on the observation that a closed f-network can be *decomposed* into several closed f-networks. For the formal statement, we first define *decomposition* of the f-network as follows.

[17] Hayakawa (2018) applies the results to investigate the role of CCP.

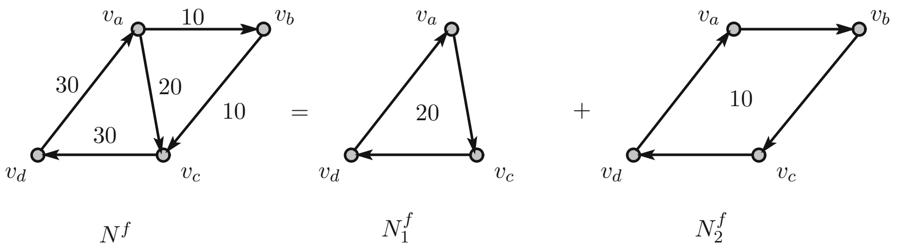

Fig. 14 Example of *decomposition* of an f-network. Notes: $N_1^f = \langle V_1, A_1, f_1 \rangle$, where $V_1 = \{v_a, v_c, v_d\}$, $A_1 = \{(v_a, v_c), (v_c, v_d), (v_d, v_a)\}$, $f_1(a \in A_1) = 20$. $N_2^f = \langle V_2, A_2, f_2 \rangle$, where $V_2 = \{v_a, v_b, v_c, v_d\}$, $A_2 = \{(v_a, v_b), (v_b, v_c)(v_c, v_d), (v_d, v_a)\}$, $f_2(a \in A_2) = 10$. For each decomposed f-network, the constant value of *flow* (i.e. 20 and 10 units respectively) is shown in the center

Definition 3 (*Decomposition of f-network*)
f-network $N^f = \langle V, A, f \rangle$ is *decomposed* into $\left\{ N_k^f = \langle V_k, A_k, f_k \rangle \right\}_{k=1,2,\ldots,K}$ if

i) $V = \cup_{1 \leq k \leq K} V_k$, $A = \cup_{1 \leq k \leq K} A_k$, and
ii) $\forall a \in A$, $f(a) = \sum_{k \in K'} f_k(a)$, where $K' = \left\{ k'' | a \in A_{k''} \right\}$.

We simply denote a decomposition of f-network N^f as $N^f = \sum_{k=1}^{K} N_k^f$. Figure 14 shows an example of *decomposition* of an f-network.

A specific type of *decomposition*, which we term *closed-cycle decomposition*, is the basis for our analyses. For *closed-cycle decomposition*, each decomposed f-network is *closed*, and consists of one *cycle*. The *decomposition* in Fig. 14 is actually a *closed-cycle decomposition*. Below, we prepare to define *cycle* and the relevant concepts, before proceeding to define *closed-cycle decomposition*.

Given a multi-arc directed graph $\langle V, A \rangle$, we denote the set of vertices included in $A' \subseteq A$ as $V_{A'}$ and the set of arcs that includes $v \in V$ as A_v. $A' \subseteq A$ is a *path* from $v \in V_{A'}$ to $v' \in V_{A'}$ if we can order vertices $V_{A'}$ such that $(v, v_1, v_2, \ldots, v_k, v')$ where the consecutive ordered pairs of vertices (v, v_1), (v_1, v_2)..., (v_k, v') constitute A'. The same arc cannot appear more than once in a path, but the same vertex can. $A' \subseteq A$ is a *cycle* if A' is a path that starts and ends at the same vertex. We say that a cycle is *punctured* if the same vertex appears more than once (except the vertex that appears both at the start and the end), and is *non-punctured* otherwise. For a directed graph G, we denote C_G as the set of cycles included in G and refer to it as the *cycle set* of G.

Our formal definition of *closed-cycle decomposition* is as follows.

Definition 4 (*Closed-cycle decomposition*[18])
An f-network $N^f = \langle V, A, f \rangle$ with $G = \langle V, A \rangle$ is *closed-cycle decomposed* into $\left\{ N_k^f = \langle V_k, A_k, f_k \rangle \right\}_{k=1,2,\ldots,K}$ if

i) $N^f = \sum_{k=1}^{K} N_k^f$ is a *decomposition*, and

[18] Note that a *punctured cycle* is allowed to constitute a decomposed f-network in closed-cycle decomposition. Figure 17 shows a closed-cycle decomposition where a *punctured cycle* is included.

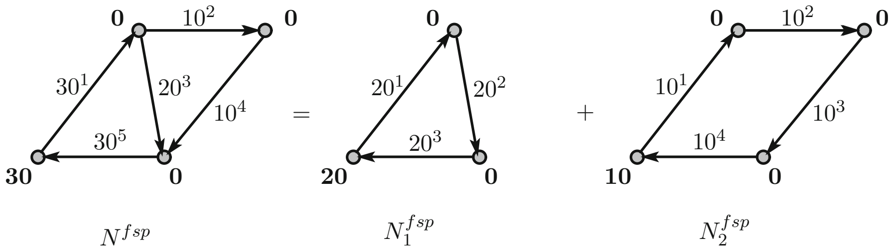

Fig. 15 Example of *decomposition* of fsp-network

ii) $\forall k = 1, 2, \ldots, K$, each N_k^f is *closed* and consists of one *cycle*, where the same cycle does not constitute more than one f-network, that is, for $1 \le k, k' \le K$, $\langle V_k, A_k \rangle \neq \langle V_{k'}, A_{k'} \rangle$ if $k \neq k'$.

With a slight abuse of notation, we let $N^f = \sum_{c \in C} \langle V^c, c, f^c \rangle$ denote a closed-cycle decomposition with $C \subseteq C_G$, where f^c denotes a constant function.

We have the following result for *closed-cycle decomposition*.

Theorem 1 (Ford and Fulkerson 1962)
Any closed f-network can always be closed-cycle decomposed.

Note that Theorem 1 does not ensure *uniqueness*[19] of closed-cycle decomposition, although Fig. 14 shows a case of *uniqueness*.

We now define *decomposition* of fsp-networks.

Definition 5 (*Decomposition of fsp-networks*)
$N^{fsp} = \langle V, A, f, s, p \rangle$ is *decomposed* into $\left\{ N_k^{fsp} = \langle V_k, A_k, f_k, s_k, p_k \rangle \right\}_{k=1,2,\ldots,K}$ if

i. $\langle V, A, f \rangle$ is decomposed into $\{\langle V_k, A_k, f_k \rangle\}_{k=1,2,\ldots,K}$,
ii. each *sequence* s_k is consistent with s in the sense that the ordering is preserved, and
iii. $\sum_k p_k(v) = p(v)$ for every $v \in V$.

When a decomposition of fsp-network $N^{fsp} = \langle V, A, f, s, p \rangle$ is also a closed-cycle decomposition of the corresponding f-network such as $\langle V, A, f \rangle = \sum_{c \in C} \langle V^c, c, f^c, \rangle$, we extend to write $\langle V, A, f, s, p \rangle = \sum_{c \in C} \langle V^c, c, f^c, s^c, p^c \rangle$.

Figures 15 and 16 show examples of *decomposition* of an fsp-network.

Observe that in both Figs. 15 and 16, all decomposed fsp-networks are e-covered. The following Lemma 2 states that it is always possible to perform such decomposition when the given fsp-network is e-covered.

Lemma 2 (E-covered decomposition)
Given a closed f-network $\langle V, A, f \rangle$, for any e-covered fsp-network $N^{fsp} = \langle V, A, f, s, p \rangle$, there exists a decomposition $N^{fsp} = \sum_{c \in C} \langle V^c, c, f^c, s^c, p^c \rangle$ such that $\langle V^c, c, f^c, s^c, p^c \rangle$ is e-covered for every $c \in C$.

[19] Our discussion paper version Hayakawa (2014) investigates the implications of uniqueness of closed-cycle decomposition for the existence of “twist” properties, which is omitted in this version of the paper.

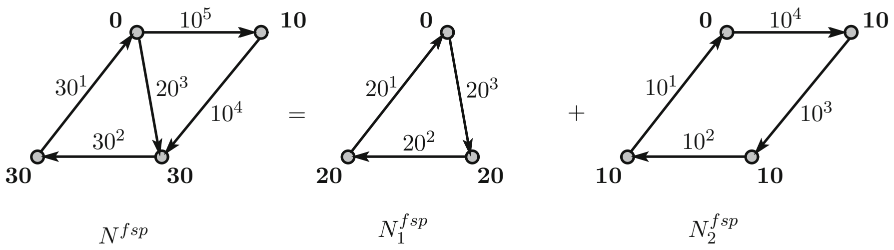

Fig. 16 Example of *decomposition* of fsp-network

Proof See Appendix A.1. □

We say that $N^{fsp} = \sum_{c \in C} \langle V^c, c, f^c, s^c, p^c \rangle$ is an *e-covered decomposition* when N^{fsp} is e-covered and each decomposed $\langle V^c, c, f^c, s^c, p^c \rangle$ is also e-covered.

5 Characterization of Min/Max-Circulation

The purpose of this section is to present our characterization of min/max-circulation. The first step is to rearrange the original minimization/maximization problems utilizing *closed-cycle decomposition*.

Note that in Fig. 15, an e-covered fsp-network that attains the min-circulation for the underlying f-network is expressed with an e-covered decomposition, where the min-circulation is read as $30 = 20 + 10$. Similarly, Fig. 16 shows the case of the max-circulation, where the value is read as $70 = 40 + 30$. This observation implies the following procedure to find the min/max-circulation. First, for a given closed f-network, take a closed-cycle decomposition, and then, let each decomposed f-network be an e-covered fsp-network. We could recover an fsp-network for the given f-network by "adding up"the e-covered fsp-networks—the reverse view of fsp-decomposition. By searching every possible closed-cycle decomposition for the given closed f-network, we can find an fsp-network that attains the min/max-circulation.

The following two Theorems 2 and 3 ensure that such a procedure successfully finds the min/max-circulation. Below, C_{N^f} represents the power set of "the set of cycles" for N^f; the reformulated problems find a set of cycle $C \in C_{N^f}$ that constitutes a closed-cycle decomposition such that $N^f = \sum_{c \in C} \langle V^c, c, f^c \rangle$.

Theorem 2 (MIN-SFC in decomposed form)

Given a closed f-network $N^f = \langle V, A, f \rangle$, the following problem gives the same value as $x^{min}(N^f)$.

$$\min_{s \in S(A), C \in C_{N^f}, \{f^c\}_{c \in C}} \sum_{c \in C} \sum_{v \in V^c} p^c(v).$$

$$\text{s.t. } N^f = \sum_{c \in C} \langle V^c, c, f^c \rangle \text{ is a closed-cycle decomposition,}$$

$$\langle V^c, c, f^c, s^c, p^c \rangle \text{ is e-covered for every } c \in C, \text{ and}$$

$$\langle V, A, f, s, p \rangle = \sum_{c \in C} \langle V^c, c, f^c, s^c, p^c \rangle$$

Proof See Appendix A.2. □

Theorem 3 (MAX-SFC in decomposed form)
Given a closed f-network $N^f = \langle V, A, f \rangle$, the following problem gives the same value as $x^{max}(N^f)$;

$$\max_{s \in S(A), C \in C_{N^f}, \{f^c\}_{c \in C}} \sum_{c \in C} \sum_{v \in V^c} p^c(v).$$

$$s.t.\ N^f = \sum_{c \in C} (V^c, c, f^c) \text{ is a closed-cycle decomposition,}$$

$$\langle V^c, c, f^c, s^c, p^c \rangle \text{ is e-covered for every } c \in C, \text{ and}$$

$$\langle V, A, f, s, p \rangle = \sum_{c \in C} \langle V^c, c, f^c, s^c, p^c \rangle \text{ is e-covered.}$$

Proof See Appendix A.3. □

For each problem in the original form, it is sufficient to search within the e-covered fsp-networks for a given f-network. Lemma 2 ensures that every such e-covered fsp-network is decomposed into e-covered fsp-networks on the basis of closed-cycle decomposition. Thus, essentially, it is sufficient to search within the closed-cycle decomposition for a given f-network.

We could interpret that a redundant procedure that finds a closed-cycle decomposition is inserted for each problem in decomposed form. The decomposed forms need be rearranged once again to proceed to our characterization. There, each of the decomposed form problems is separated into a *decomposition choice part* and a *sequence choice part*. First, we define a sub-problem for each MIN- SFC and MAX- SFC, each of which corresponds to the *sequence choice part* that is characterized with our "twist" properties.

In the sub-problems, as shown formally below, a closed-cycle decomposition $\langle V, A, f \rangle = \sum_{c \in C} \langle V^c, c, f^c \rangle$ is given. Observe that for each decomposed f-network $\langle V^c, c, f^c \rangle$, the min-circulation is f^c. Now, for each decomposed f-network, suppose to take a corresponding e-covered fsp-network $\langle V^c, c, f^c, s^c, p^c \rangle$ such that $\sum_{v \in V^c} p^c(v) = f^c$. If we can "add up" the fsp-networks to construct an e-covered fsp-network $\langle V, A, f, s, c \rangle$, then, the min-circulation is proved as $\sum_{c \in C} f^c$; however, this is not always possible. The sub-problem for the case of minimization is used to derive the minimum value of the discrepancy $(\sum_{c \in C} (\sum_{v \in V^c} p^c(v) - f^c))$, which we call the *residual*. Observe that for the closed-cycle decomposition shown in Fig. 15, the *residual* equals zero, as we easily confirm. The sub-problem for the case of maximization similarly is used to derive the *residual*, and again, an example case of zero residual is shown in Fig. 16.

(Sub-problem for MIN- SFC)

Given a closed f-network N^f *and a closed-cycle decomposition* $\{C \in C_{N^f}, \{f^c\}_{c\in C}\}$,

$$\min_{\{s^c\in S(c)\}_{c\in C}} \sum_{c\in C}\left(\sum_{v\in V^c} p^c(v) - f^c\right)$$
$$\textit{s.t. } \langle V^c, c, f^c, s^c, p^c\rangle \textit{ is e-covered for every } c \in C, \textit{ and}$$
$$\langle V, A, f, s, p\rangle = \sum_{c\in C}\langle V^c, c, f^c, s^c, p^c\rangle.$$

(Sub-problem for MAX- SFC)

Given a closed f-network N^f *and a closed-cycle decomposition* $\{C \in C_{N^f}, \{f^c\}_{c\in C}\}$,

$$\min_{\{s^c\in S(c)\}_{c\in C}} \sum_{c\in C}\left((|c|-1)f^c - \sum_{v\in V^c} p^c(v)\right),$$
$$\textit{s.t. } \langle V^c, c, f^c, s^c, p^c\rangle \textit{ is e-covered for every } c \in C, \textit{ and}$$
$$\langle V, A, f, s, p\rangle = \sum_{c\in C}\langle V^c, c, f^c, s^c, p^c\rangle \textit{ is e-covered.}$$

where $|c|$ denotes the number of arcs that constitute cycle c.

We denote the derived value, *residual*, as $R^{min/max}(N^f, C, \{f^c\}_{c\in C})$.

Now, we rewrite MIN- SFC and MAX- SFC as follows.

(MIN- SFC *in Separated Form*)

Given a closed f-network $N^f = \langle V, A, f\rangle$,

$$\min_{C\in C_{N^f},\{f^c\}_{c\in C}} \sum_{c\in C} f^c + R^{min}\left(N^f, C, \{f^c\}_{c\in C}\right)$$
$$\textit{s.t. } N^f = \sum_{c\in C}\langle V^c, c, f^c\rangle \textit{ is a closed-cycle decomposition.}$$

(MAX- SFC *in Separated Form*)

Given a closed f-network $N^f = \langle V, A, f\rangle$,

$$\max_{C\in C_{N^f},\{f^c\}_{c\in C}} \sum_{c\in C}(|c|-1)f^c - R^{max}\left(N^f, C, \{f^c\}_{c\in C}\right)$$
$$\textit{s.t. } N^f = \sum_{c\in C}\langle V^c, c, f^c\rangle \textit{ is a closed-cycle decomposition },$$

The above separated form is still not well separated in that it does not explicitly refer to which kind of closed-cycle decomposition is to be chosen. We address this issue in the next subsection.

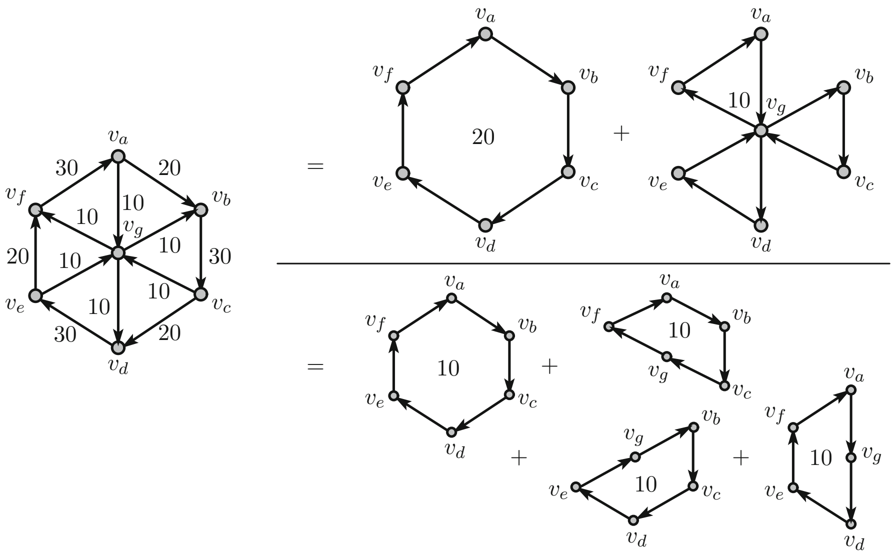

Fig. 17 closed-cycle decomposition and *domination*. Notes: For the f-network shown on the left, two closed-cycle decompositions are shown on the right. The lower-right closed-cycle decomposition is *dominated* by the upper-right closed-cycle decomposition

5.1 Property of *domination* and Min/Max-Circulation

In this subsection, we present our characterization for the *decomposition choice part* of the separated form MIN- SFC and MAX- SFC. We introduce the concept of *domination* that refers to a relation between closed-cycle decompositions for the same f-network. Before proceeding to the formal definition, we see an example f-network shown in Fig. 17 to illustrate the concept. Observe that the f-network shown on the left side of the figure is closed-cycle decomposed into different sets of closed f-networks shown in the right side. The summation of the *flow* on each cycle is $30(= 20 + 10)$ for the upper decomposition, while it is $40(= 10 + 10 + 10 + 10)$ for the lower decomposition. In terms of the smaller value, we say that the lower decomposition is *dominated* by the upper decomposition,[20] which is formally defined below.

Definition 6 (*dominated, undominated*[21])
Given a directed graph $G = \langle V, A\rangle$ and the sets of cycles C_G, a set of cycles $C \subseteq C_G$ is *dominated* by another set of cycles $C' \subseteq C_G$ (equivalently, C' *dominates* C) if

i) we can take a closed f-network N^f such that $N^f = \sum_{c\in C}\langle V^c, c, f^c\rangle = \sum_{c'\in C'}\langle V^{c'}, c', f^{c'}\rangle$, and

[20] This observation has an implication for discussing how a *hub* can reduce the min-circulation, as we later discuss.

[21] Note that any punctured cycle *dominates* the set of its component non-punctured cycles. For example, see the upper-right of Fig. 17, where a punctured cycle $(v_a, v_g, v_b, v_c, v_g, v_d, v_e, v_g, v_f, v_a)$ *dominates* a set of cycles $\{(v_a, v_g, v_f, v_a), (v_c, v_g, v_b, v_c), (v_e, v_g, v_d, v_e)\}$.

ii) we have $\sum_{c \in C} f^c < \sum_{c' \in C'} f^{c'}$.

Given the power set of "a set of cycles" $\tilde{C}$, we further say that a set of cycles $C \in \tilde{C}$ is *undominated* in $\tilde{C}$ if C is not *dominated* by any $C' \subseteq \tilde{C}$.

For a closed f-network $N^f = \langle V, A, f \rangle$, let C_{N^f} denote the power set of the set of cycles for N^f. Let $C^{und}_{N^f} \subseteq C_{N^f}$ denote the set of *undominated* sets of cycles within C_{N^f}. Then, let $\{C \in C^{und}_{N^f}, \{f^c\}_{c \in C}\}$ denote an *undominated* closed-cycle decomposition for N^f.

The following Theorem 4 presents our characterization of the *decomposition choice part*. Essentially, for each case of min/max-circulation, in choosing *closed-cycle decomposition* along with the MIN- SFC/MAX- SFC in separated form, arbitrary *undominated closed-cycle decomposition* suffices. This implies that min-circulation might not be attained with a *dominated* decomposition. This is actually the case of the decomposition shown in the lower-right of Fig. 17. It is similarly confirmed for the case of max-circulation.

Theorem 4 (Min/max-circulation with undominated closed-cycle decomposition)
Given a closed f-network $N^f = \langle V, A, f \rangle$, regarding the min-circulation, we have:

i) $x^{min}(N^f) = F^{min}(N^f) + min_{C \in C^{und}_{N^f}, \{f^c\}_{c \in C}} R^{min}(N^f, C, \{f^c\}_{c \in C})$, *where* $F^{min}(N^f) = \sum_{c \in C'} f^c$ *for an arbitrary undominated closed-cycle decomposition* $N^f = \sum_{c \in C'} \langle V^c, c, f^c \rangle$.

Regarding max-circulation, we have:

ii) $x^{max}(N^f) = F^{max}(N^f) - min_{C \in C^{und}_{N^f}, \{f^c\}_{c \in C}} R^{max}(N^f, C, \{f^c\}_{c \in C})$, *where* $F^{max}(N^f) = \sum_{c \in C'} (|c| - 1) f^c$ *for an arbitrary undominated closed-cycle decomposition* $N^f = \sum_{c \in C'} \langle V^c, c, f^c \rangle$.

Proof See Appendix A.4. □

Note that $F^{min}(N^f)$ and $F^{max}(N^f)$ indicate that arbitrary *undominated closed-cycle decomposition* has the same value.

5.2 Property of *arc-twisted* and Min-Circulation

As an input for MIN- SFC, take the f-network shown on the left side of Fig. 18, which is a reprint of Fig. 3. We mention in Sect. 3 that the fsp-network shown on the right side of Fig. 18 achieves a min-circulation of 40. Now, from the perspective of closed-cycle decomposition, the min-circulation is derived using the decomposition shown in Fig. 19. We confirm that the *residual* is not zero ($10 > 0$). As the source of the non-zero residual, we mention certain inconsistency between the two cycles, which we formally define as *arc-twisted* relation between cycles. We prepare to define *arc-reverse number*.

Arc-Reverse Number

For a given $G = \langle V, A \rangle$ together with a *sequence* s on A, let cycle $c \in A$ consist of arcs $(a_1, a_2, \ldots a_n, a_{n+1} = a_1)$, where $a_k = (v_k, v_{k+1})$ for $k = 1, 2, \ldots, n$. Then, the

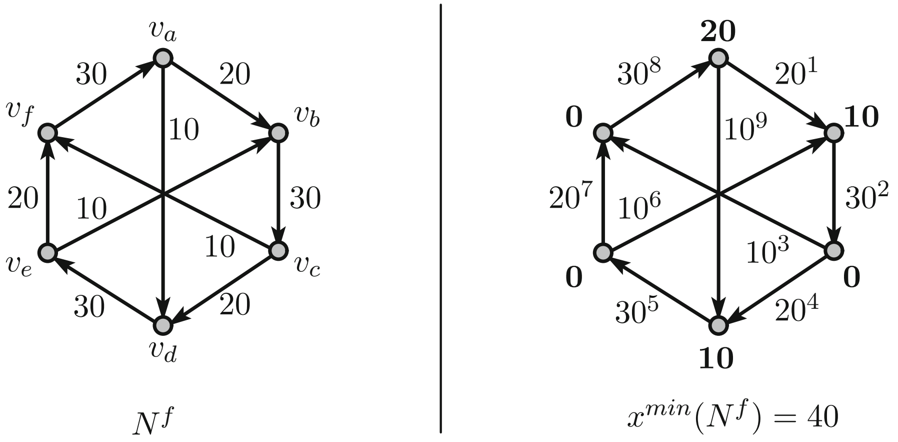

Fig. 18 Example f-network to show the property of *arc-twisted*

arc-reverse number is defined as $r^{atwi}(c, s) = \sum_{k=1}^{n} 1_{\{s(a_k)>s(a_{k+1})\}}$. When there are multiple ways to index arcs for a cycle and accordingly multiple values of the *arc-reverse number* (which is possible when the cycle is *punctured*), we set the arc-reverse number as the minimum among them.

Arc-Twisted
We say that cycles in $C \subseteq C_G$ are in *arc-twisted* relation, or simply that they are *arc-twisted*, if we cannot take any *sequence* s on A such that $r^{atwi}(c, s) = 1$ for every $c \in C$. We say that the cycles included in $C \subseteq C_G$ are *minimum arc-twisted* when there are no *arc-twisted* cycles $C' \subset C$. Returning to Fig. 19, the two decomposed cycles are *arc-twisted* and *minimum arc-twisted*. Note that *minimum arc-twisted* cycles are not always a pair, as confirmed in Fig. 20.

The following theorem is our main result on the relevance of the *arc-twisted* property for the residual part of min-circulation.

Theorem 5 (arc-twisted and $R^{min}(.)$)
Given a closed f-network N^f and a closed-cycle decomposition, which is characterized by $C \in C_{N^f}$ and $\{f^c\}_{c \in C}$, $R^{min}(N^f, C, \{f^c\}_{c \in C}) = 0$ if and only if C is not arc-twisted.

Proof When C is not *arc-twisted*, we can always take a *sequence* such that the arc-reverse number for every $c \in C$ is 1. This lets us take $\sum_{v \in V^c} p^c(v) = f^c$ for every $c \in C$. Conversely, when $R^{min}(.) = 0$, we can always take the arc-reverse number as 1 for all $c \in C$ under any *sequence* that realizes $R^{min}(.) = 0$. □

5.3 Property of *vertex-twisted* and Max-Circulation

Let the f-network shown on the left side of Fig. 21, which is a reprint of Fig. 9, be our input for the MAX- SFC. We mention in Sect. 3 that the fsp-network shown on

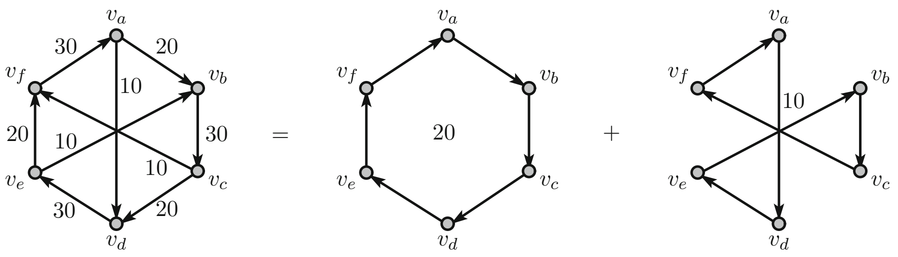

Fig. 19 Closed-cycle decomposition for the f-network shown in Figure 18

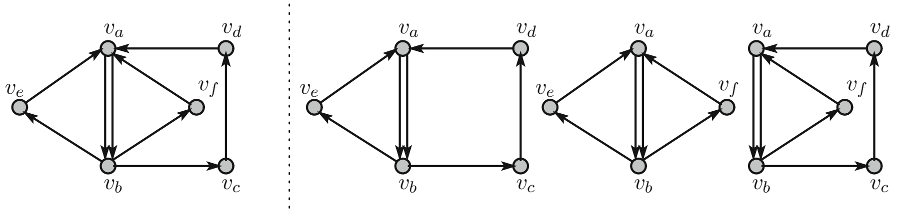

Fig. 20 Examples of three cycles in *arc-twisted* relation. Notes: For the directed graph on the left, the three cycles on the right are *minimum arc-twisted*

the right side of the figure attains the max-circulation as 110. From the perspective of closed-cycle decomposition, the max-circulation is derived using the decomposition shown in Fig. 22. We confirm that the *residual* is not zero ($10 > 0$). As the source of the non-zero residual, we mention certain inconsistency between the two cycles, which we formally define as *vertex-twisted* relation between cycles. We prepare to define the *vertex-reverse number*.

Vertex-Reverse Number
For $\langle V, A\rangle$, define the *vertex-sequence* $s_v : V \rightarrow \{1, 2, \ldots, |V|\}$ as a one-to-one mapping. For a given $\langle V, A\rangle$ together with a *vertex-sequence* s_v on V, let cycle $c \in A$ consist of $(v_1, v_2, \ldots, v_n, v_{n+1})$, where $v_{n+1} = v_1$. Then, the *vertex-reverse number* is defined as $r^{vtwi}(c, s_v) = \sum_{k=1}^{n} 1_{\{s_v(v_k)>s_v(v_{k+1})\}}$. When there are multiple ways to index the vertices for a cycle and accordingly multiple values of the *vertex-reverse number* (which is possible when the cycle is punctured), we set the *vertex-reverse number* for the cycle as the minimum among them.

Vertex-Twisted
Cycles in $C \subseteq C_G$ are in *vertex-twisted* relation, or simply that they are *vertex-twisted* if we cannot take any *vertex-sequence* s_v such that $r^{vtwi}(c, s_v) = 1$ for every $c \in C$. Cycles in $C \subseteq C_G$ are *minimum vertex-twisted* when there are no *vertex-twisted* cycles $C' \subset C$.

Note that although any punctured cycle as in Fig. 23 is *vertex-twisted* by itself, the notion of *vertex-twisted* is not trivial in the sense that cycles that are not punctured can also be *vertex-twisted*, as shown in Figs. 22 and 24.

The following Theorem 6 is our main result on the relevance of the *vertex-twisted* property for the residual part of max-circulation.

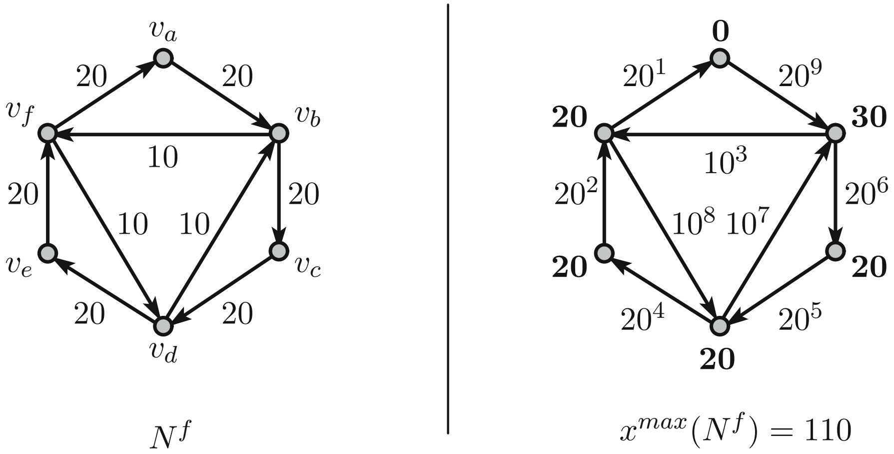

Fig. 21 Example f-network showing the property of *vertex-twisted*

Theorem 6 (Vertex-twisted and $R^{max}(.)$)

Given a closed f-network N^f and its closed-cycle decomposition, which is characterized by $C \in C_{N^f}$ and $\{f^c\}_{c \in C}$, $R^{max}(N^f, C, \{f^c\}_{c \in C}) = 0$ if and only if C is not vertex-twisted.

Proof When C is not *vertex-twisted*, then from its definition, we can always take *vertex-sequence* s_v on vertices in C such that the *vertex-reverse number* is $|c| - 1$ for all $c \in C$. Denote each set of vertices as $V_k = arg_{v \in V} s_v(v) = k$ for $k = 1, 2, \ldots, |V|$. Take a *sequence* on arcs $a_k \in A$, which starts from $v \in V_k$ such that $\sum_1^{k-1} |V_{k-1}| < s(a_k) < \sum_1^k |V_k|$. Since there is no *vertex-twisted* cycle, such a *sequence* s lets us take p^c such that each decomposed fsp-network is e-covered and $\sum_{v \in V^c} p^c(v) = (|c|-1) f^c$. What needs to be shown is that the combined fsp-network with the decomposed fsp-network is e-covered. For each vertex $v \in V$, take any two out-arcs $a' = (v, v'), a'' = (v, v'') \in A$. Then, there is no in-arc $a''' = (v''', v) \in A$ such that $s(a') < s(a''') < s(a'')$. This is true for any two out-arcs. Thus, the combined fsp-network is e-covered.

For the converse direction, take a *sequence s* that realizes $R^{max}(N^f, C, \{f^c\}_{c \in C}) = 0$. Under the *sequence s*, for each cycle $c \in C$ with its set of vertices V^c, take a vertex $v^c \in V^c$ such that $s((v, v^c)) = argmin_{a \in c} s(a)$ and call it the *head-vertex* for c. Then, for every vertex $v' \in V^c \setminus v^c$ with its arc $(v', v) \in c$, there is no arc $a = (v'', v') \in C$ such that $s(a) < s((v', v))$ since otherwise it would immediately lead to $R^{max}(N^f, C, \{f^c\}_{c \in C}) > 0$. This is true for every cycle $c \in C$. Then, we can naturally define partial order $<$ on $v \in V^c$ from *sequence s* such that each *head-vertex* is the largest and becomes smaller in the direction opposite to that indicated by the arcs. We can always take the *vertex-sequence* to be consistent with the order $<$, and under such a *vertex-sequence*, the *vertex-reverse number* is 1 for every cycle $c \in C$. Thus, C is not *vertex-twisted*. □

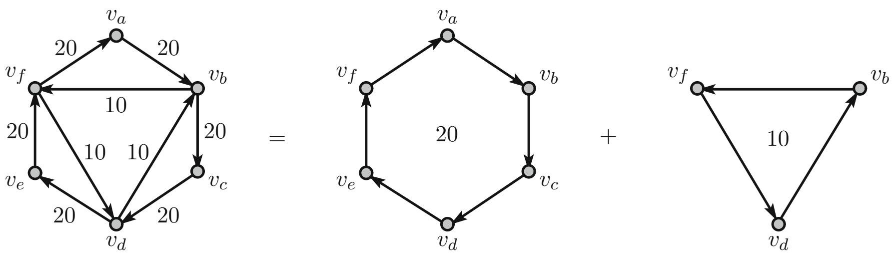

Fig. 22 Closed-cycle decomposition for the f-network shown in Fig. 21

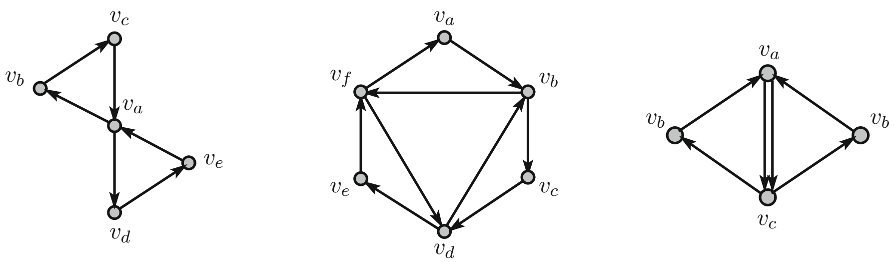

Fig. 23 Examples of *vertex-twisted* cycles. Notes: Each directed graph itself constitutes a punctured cycle

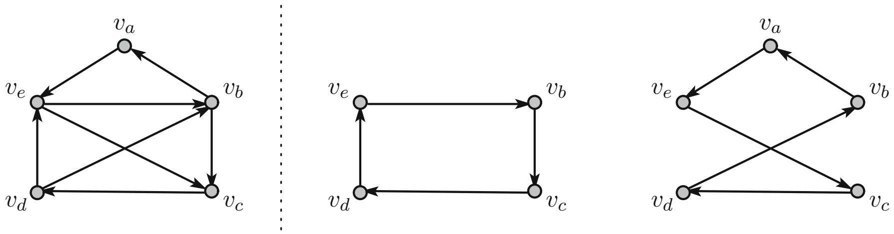

Fig. 24 Example directed graph of two *vertex-twisted* cycles. Notes: For the directed graph shown on the left, we can take the two cycles shown on the right, which are *vertex-twisted*

5.4 *Arc-twisted* and *vertex-twisted*

Note that if the cycles in C are *arc-twisted*, then they are also *vertex-twisted* as stated in the following Theorem 7. The opposite is not always true, as is easily confirmed.

Theorem 7 (arc-twisted and vertex-twisted)
For any set of cycles C, C is vertex-twisted if C is arc-twisted.

Proof Suppose that C is not *vertex-twisted*. Then, by definition, we can take a *vertex-sequence* s_v on the vertices in C such that the *vertex-reverse number* is 1 for all $c \in C$. Take a *sequence* s^c for each $c \in C$ such that $s^c((v, v')) = s_v(v)$. We have $r^{atwi}(c, s^c) = 1$ for every $c \in C$. Since we can always take a *sequence* on the arcs in C such that the *sequence* is consistent with all s^c, C is not *arc-twisted*. □

Policy Implications
In implementing policies that reduce the required liquidity for settlements, the base implication of Theorems 5, 6, and 7 is that the effect of each relevant “twist” property is scenario dependent. For the part of the *arc-twisted* property, which also indicates the *vertex-twisted* property, it would serve to increase the min-circulation but decrease the max-circulation. Thus, even if there is some policy that effectively “adds” the *arc-twisted* property, the implementation of the policy is not always recommended. For the part of the *vertex-twisted* property without the *arc-twisted* property, it would serve to decrease the max-circulation, while would not increase the min-circulation. Thus, it is recommended that a policy that adds such a property should always be implemented.

6 Quantitative Implications of the “twist” Properties

In this section, we show the quantitative implications of the “twist” properties in the context of a real-world payment network by examining certain classes of networks. First, we show the results associated with *clustering*, and then, we proceed to show the results associated with *small-world* feature.

6.1 Networks with *clustered* Structure

We consider two classes of f-networks with clustered structure, each of which highlights the quantitative implications of the “twist” properties.

6.1.1 *Type-1 clustered* Structure

First, as a generalization of the f-network shown on the left side of Fig. 18, we construct *type-1 clustered f-network* $\langle V, A, f\rangle$ as follows.

- Let $V = \{v_1, v_2, \ldots, v_{2n-1}, v_{2n}\}$ for some integer $n \geq 3$. Let $A = A_1 \cup A_2 \cup A_3$.
- Take arcs of (v_{2k-2}, v_{2k-1}) for $k = 1, 2, \ldots, n$ with $v_0 \equiv v_n$, and let this set of arcs be A_1. Take $f(a) = f_1$ for $a \in A_1$.
- Take arcs of (v_{2k}, v_{2k-3}) for $k = 1, 2.., n$, with $v_{-1} \equiv v_{n-1}$, and let this set of arcs be A_2. Take $f(a) = f_2$ for $a \in A_2$.
- Take arcs of (v_{2k-1}, v_{2k}) for $k = 1, 2, \ldots, n$, and let this set of arcs be A_3. Take $f(a) = f_3 = f_1 + f_2$ for $a \in A_3$.

Figure 25 shows the underlying directed graph for a type 1 clustered f-network with $n = 6$. Note that the f-network shown in Fig. 18 is an example of type 1 clustered f-network with; $n = 3$, $A_1 = \{(v_a, v_b), (v_c, v_d), (v_e, v_f)\}$ with $f_1 = f(a \in A_1) = 20$, $A_2 = \{(v_a, v_d), (v_e, v_b), (v_c, v_f)\}$ with $f_2 = f(a \in A_2) = 10$, and $A_3 = \{(v_f, v_a), (v_b, v_c), (v_d, v_e)\}$ with $f_3 = f(a \in A_3) = f(a \in A_1) + f(a \in A_2) = 30$.

Observation 7
Type-1 clustered f-network $\langle V, A, f\rangle$ is closed.

Furthermore, we have the following closed-cycle decomposition. $\langle V, A, f\rangle = \langle V, A_1 \cup A_3, f_1\rangle + \langle V, A_2 \cup A_3, f_2\rangle$, which we simplify as $\langle V, A, f\rangle = \langle V, c_1, f_1\rangle + \langle V, c_2, f_2\rangle$, where c_1 and c_2 are *arc-twisted*.

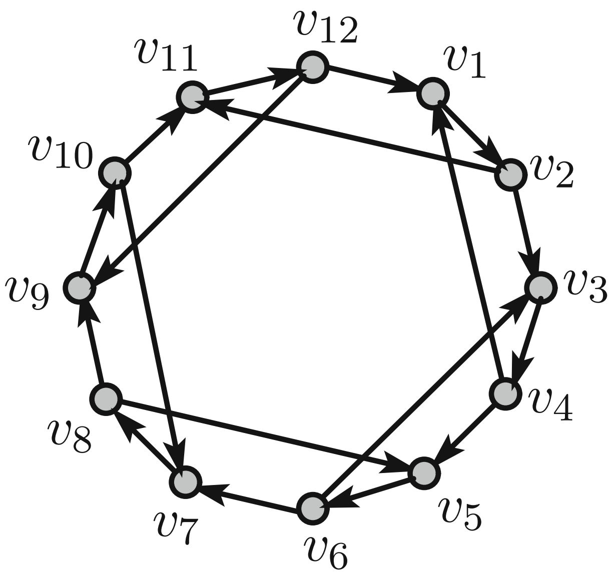

Fig. 25 Type-1 clustered f-network: $n = 6$

The following result explicitly shows the quantitative implications of the property of *arc-twisted* regarding this class of network.

Theorem 8
For a given type-1 clustered f-network $N^f = \langle V, A, f \rangle$ *with* $f_1 \geq f_2$ *in its construction, we have:*

i) $x^{min}(N^f) = f_1 + (n-1)f_2 = (f_1 + f_2) + (n-2)f_2.$
ii) Furthermore, for the case $f_1 > f_2$*, if the above value is realized with sequence* s *on* A*, then* $r^{atwi}(c_1, s) = 1.$

Proof See Appendix A.5 □

The theorem is interpreted as revealing the following two features.[22]

- **Non-proportional feature**
  For a given *type-1 clustered* f-network N^f with $f_1 > f_2$, take another *type-1 clustered* f-network $\tilde{N}^f$ with $f_1 > f_2 + m$. Then, we have

  i) if $f_1 > f_2 + m$, then, $x^{min}(\tilde{N}^f) = x^{min}(N^f) + (n-2)m$, while
  ii) if $f_1 < f_2 + m$, then, $x^{min}(\tilde{N}^f) = x^{min}(N^f) + (n-2)(f_2 - f_1) + (m - f_1)$.

This states that the marginal effect on the min-circulation against an increase in flow regarding the cycle $c_2 (= A_2 \cup A_3)$ is $(n-2)$ until $m < f_1 - f_2$; but once $m > f_1 - f_2$, the marginal effect turns to 1.

[22] These are generalized statements of Observation 3.

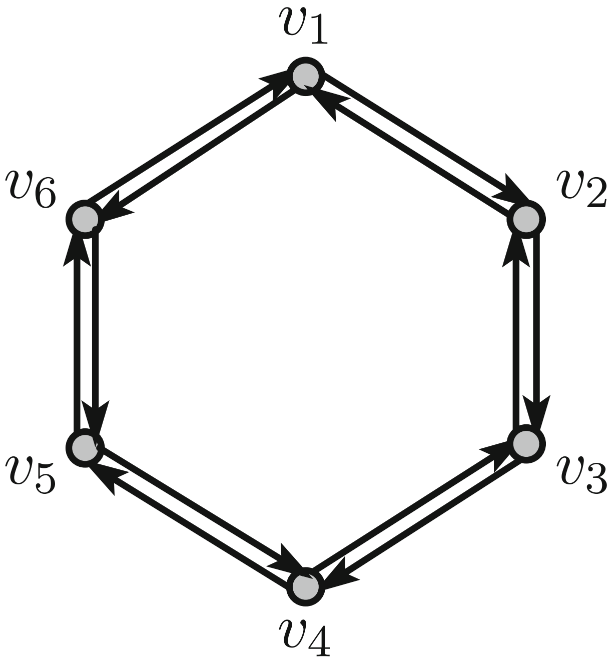

Fig. 26 Type-2 clustered f-network: $n = 6$

- **Regime-change feature**
  For a given *type-1 clustered* f-network $N^f = \langle V, A, f \rangle$ with $f_1 > f_2$, let *sequence* s^* on A attain the min-circulation. Take another *type-1 clustered* f-network $\tilde{N}^f$ with $f_1 > f_2 + m$; then, we have

  i) if $f_1 > f_2 + m$, then, s^* attains the min-circulation also for $\tilde{N}^f$, while
  ii) if $f_1 < f_2 + m$, then, s^* no longer attains the min-circulation for $\tilde{N}^f$.

6.1.2 *Type-2 clustered* Structure

Now, we construct another class of *clustered* f-network. It is intended to capture a key feature of the f-network shown on the left side of Fig. 21, which is introduced as *type-2 clustered f-network* $\langle V, A, f \rangle$ as follows.

– Let $V = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$ for some integer $n \geq 3$. Let $A = A_1 \cup A_2$.
– Take arcs of (v_k, v_{k+1}) for $k = 1, 2, \ldots, n$ with $v_{n+1} \equiv v_1$, and let this set of arcs be A_1. Take $f(a) = f_1$ for $a \in A_1$.
– Take arcs of (v_{k+1}, v_k) for $k = 1, 2, \ldots, n$ with $v_{n+1} \equiv v_1$, and let this set of arcs be A_2. Take $f(a) = f_2$ for $a \in A_2$.

A type-2 clustered f-network consists of two cycles that have the "opposite" direction in the sense of the associated arcs. Figure 26 shows the underlying directed graph for a type-2 clustered f-network with $n = 6$. Although the class might seem fairly special, we could generalize the associated results beyond the class, but this is beyond the scope of this study.[23]

Observation 8

Type-2 clustered f-network $\langle V, A, f\rangle$ is closed.

Furthermore, we have the following closed-cycle decomposition. $\langle V, A, f\rangle = \langle V, A_1, f_1\rangle + \langle V, A_2, f_2\rangle$, where A_1 and A_2 are *vertex-twisted.*

The following result explicitly shows the quantitative implications of the property of *vertex-twisted* regarding this class of network.

Theorem 9

For a given type-2 clustered f-network $N^f = \langle V, A, f\rangle$ *with* $f_1 \geq f_2$ *in its construction, we have:*

i) $x^{max}(N^f) = (n-1)f_1 + f_2 = (n-1)(f_1 + f_2) - (n-2)f_2.$
ii) Furthermore, for the case of $f_1 > f_2$*, if the above value is realized with sequence* s *on* A*, then* $r^{atwi}(A_1, s) = n - 1.$

Proof See Appendix A.6. □

The theorem is interpreted to reveal the following two features.

- **Non-proportional feature**
  For a given *type-2 clustered* f-network N^f with $f_1 > f_2$, take another *type-2 clustered* f-network $\tilde{N}^f$ with $f_1 > f_2 + m$; then, we have

  i) if $f_1 > f_2 + m$, then, $x^{max}(\tilde{N}^f) = x^{min}(N^f) + m$, while
  ii) if $f_1 < f_2 + m$, then, $x^{max}(\tilde{N}^f) = x^{min}(N^f) + (f_2 - f_1) + (n-1)(m - f_1)$.

This states that the marginal effect on the min-circulation against an increase in flow regarding cycle A_2 is 1 until $m < f_1 - f_2$; but once $m > f_1 - f_2$, the marginal effect becomes $n - 2$.

- **Regime-change feature**
  For a given *type-2 clustered* f-network $N^f = \langle V, A, f\rangle$ with $f_1 > f_2$, let *sequence* s^* on A attain the min-circulation. Take another *type-2 clustered* f-network $\tilde{N}^f$ with $f_1 > f_2 + m$; then, we have

  i) if $f_1 > f_2 + m$, then, s^* attains the max-circulation also for $\tilde{N}^f$, while
  ii) if $f_1 < f_2 + m$, then, s^* no longer attains the max-circulation for $\tilde{N}^f$.

6.2 Networks with *small-world* Structure

We construct an f-network with a *small-world* structure by the rewiring procedure, originally proposed by Watts and Strogatz (1998) for the construction of their WS

[23] Observe that we could transform the f-network shown in Fig. 21 to a *type-2 clustered* f-network by merging (v_f, v_a) and (v_a, v_b) into one arc $(v_f, v_b$ with flow 20, and merging the two other pairs of arcs—$\{(v_b, v_c), (v_c, v_d)\}$ and (v_d, v_e) and (v_e, v_f) in the same manner. Note that the transformation reduces the max-circulation proportional to the number of the merge, which is $3 * 20$ in this case. From this perspective, the results regarding the class of *type-2 clustered* f-network can be generalized.

model. In the face of directionality of an f-network, we rewire in a *fully connectivity-preserving* manner, as proposed by Maslov et al. (2003), where both the out-degree and in-degree are preserved. Furthermore, the closed property is to be preserved.

First, we start with *type-1 clustered f-network* $\langle V, A, f \rangle$ where $f(a) \geq f(a')$ for every $a \in A_1, a' \in A_2$ in its construction. Follow the procedure below, for appropriate ratio p $(0 \leq p \leq 1)$.

– Choose $\frac{|V|}{2}p$, an odd number of arcs from A_2, such that there are no two arcs whose vertex constitutes an arc.
– Randomly let the chosen arcs be repaired with each other; then, rewire by exchanging the end vertices while preserving each *flow* (e.g. a pair $(v_a, v_b), (v_c, v_d)$ becomes $(v_a, v_d), (v_c, v_b)$, and the *flow* for each old pair is preserved for each new pair).

Figure 27 shows an example of this rewiring procedure. According to the following Theorem 10, the synchronization factor characterized by the *arc-twisted* property and that by the *vertex-twisted* property are sufficiently preserved by the rewiring.

Theorem 10
For a type-1 clustered f-network N^f, denote its rewired f-network with p as $N^f(p)$. Then, $x^{min}(N^f(p)) \geq f_1 + f_2 + ((n-2) - np) f_2$.

Proof Suppose we have already executed rewiring with k pairs. Observe that rewiring partitions the original cycle formed by c_2 into several cycles; here, we denote the number of cycles as k'. When we further execute rewiring with respect to an additional pair of arcs, the rewiring occurs either among one of the cycles, or between two of them. In the former case, one more cycle is added, and the total number of cycles is $k' + 1$. The latter case always lets the two cycles become one, which makes the total number of cycles $k' - 1$. Focusing on how the residual amount is affected by this, in either case, the min-circulation decreases at most by f_2. □

Next, we similarly rewire *type-2 clustered f-network* $\langle V, A, f \rangle$ where $f(a) \geq f(a')$ for every $a \in A_2, a' \in A_2$ in its construction. Follow the procedure below, for appropriate ratio p $(0 \geq p \geq 1)$.

– Choose $|V|p$, an odd number of arcs from A_2, such that there are no two arcs whose vertex constitutes an arc.
– Randomly let the chosen arcs be repaired with each other; then, rewire by exchanging the end vertices while preserving each *flow* (e.g. a pair $(v_a, v_b), (v_c, v_d)$ becomes $(v_a, v_d), (v_c, v_b)$, and the *flow* for each old pair is preserved for each new pair).

Figure 28 shows an example of this rewiring procedure. According to the following Theorem 11, the synchronization factor characterized by the *vertex-twist* property is sufficiently preserved by the rewiring.

Theorem 11
For a type-2 clustered f-network N^f, denote its rewired f-network with p as $N^f(p)$. Then, $x^{max}(N^f(p)) \leq (n-1)(f_1 + f_2) - ((n-2) - np) f_2$.

Proof The proof is shown similarly to that of Theorem 10. □

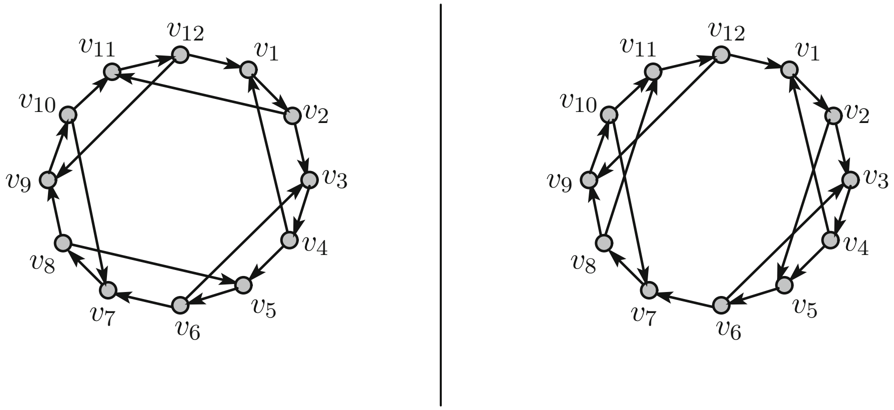

Fig. 27 Example of rewiring for type-1 clustered f-network. Notes: The left network constitutes the type-1 clustered f-network shown in Fig. 25. Rewiring the f-network with $p = \frac{1}{3}$, regarding a pair of arcs (v_2, v_{11}) and (v_8, v_5), we derive the network shown on the right side of the figure, which includes a pair of rewired arcs (v_2, v_5) and (v_8, v_{11})

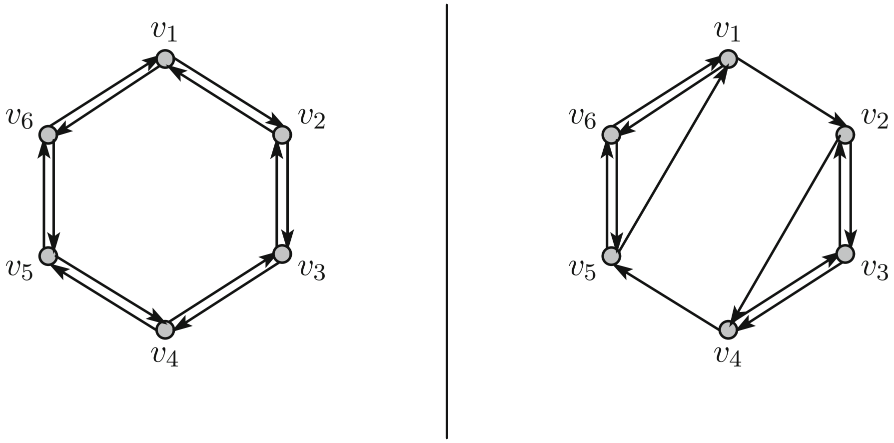

Fig. 28 Example of rewiring for type-2 clustered f-network. Notes: The left network constitutes the type-1 clustered f-network shown in Fig. 26. Rewire the f-network with $p = \frac{1}{3}$, regarding a pair of arcs (v_2, v_1) and (v_5, v_4); then, the network shown on the right side of the figure includes a pair of rewired arcs (v_2, v_4) and (v_5, v_1)

6.3 Discussion

For both the rewiring procedures, a clustered structure appropriately defined[24] would only be locally affected, while the average path length would be affected as much as that in the case of the WS model when n is sufficiently large relative to p. This implies that a small-world structure would emerge with appropriate p. In this sense, Theorems 10 and 11 indicate that the effects of the twist properties can be preserved

[24] There is an debate about how the clustering structure should be formally captured in the face of directionality; see Malliaros and Vazirgiannis (2013), who survey the relevant literature.

under a small-world structure. As it is argued that real-world payment networks have a small-world structure, our twist properties have significance when applied to real-world payment networks.

7 Concluding Remarks

In this study, we introduced a graph-theoretic framework intended to capture interbank settlements in an RTGS system. We analyzed the lower bound and upper bound of the required liquidity for a given payment network, by offering corresponding minimization and maximization graph problems. We showed our characterization for a general class of payment network by introducing two graph properties—*arc-twisted* and *vertex-twisted*. We further showed that these "twist" properties are sources of the non-linear effect on the required liquidity against an increase of the amount of obligation, illustrating the effects for networks with *clustered* and *small-world* structures.

In existing models of financial networks, liquidity is usually discussed by implicitly or explicitly assuming simultaneous settlements. This assumption tends to underevaluate overall liquidity needs, considering that *sequential settlement* is prevalent in reality. Thus, the types of models in the existing literature are not particularly suitable for a discussion of liquidity regulations in which a sufficient level of liquidity requirements is the focus. Our framework enables to discuss liquidity needs explicitly by considering the sequential nature of settlements, and thus, has potential to contribute significantly to the issue. It remains for our future work to enrich the model for the sake of those applications.

A Appendix

A.1 Proof of Lemma 2

For an e-covered $\langle V, A, f, s, p\rangle$ on a closed $\langle V, A, f\rangle$, we derive a set of e-covered fsp-networks for some closed-cycle decomposition of $\langle V, A, f\rangle$ by executing the *unbundling* procedure for each $v \in V$, which proceeds as follows.

Let $A_v^I = \{(v', v)\}_{v' \in V}$ and $A_v^O = \{(v, v')\}_{v' \in V}$ denote the incoming arcs to vertex v and the outgoing arcs to v, respectively. Note that $argmin_{a \in A_v^I} s(a)$ denotes the arc whose sequence is the smallest within the incoming arcs regarding vertex v. We similarly use the notation of $argmax$.

0. Initially, let $A_{v,work}^I = A_v^I$, $A_{v,work}^O = A_v^O$, $f_{work} = f$.

1. We update them until $A^I_{v,work} = A^O_{v,work} = \emptyset$.
Take $a^I_{min}, a^I_{max} \in A^I_{v,work}, a^O_{min} \in A^O_{v,work}$ such that $a^I_{min} = argmin_{a \in A^I_{v,work}} s(a)$, $a^I_{max} = argmax_{a \in A^I_{v,work}} s(a)$, $a^O_{min} = argmin_{a \in A^O_{v,work}} s(a)$.

2a. If $s(a^I_{min}) < s(a^O_{min})$, then let the arcs be a pair with the same amount of *flow* $min\left\{f_{work}(a^I_{min}), f_{work}(a^O_{min})\right\}$.
For the center vertex of the paired arc, endow *potential* 0, and let the *sequence* on the pair of arcs be consistent with the original *sequence* on $\langle V, A, f\rangle$.
Update $f_{work}(a^I_{min}) := f_{work}(a^I_{min}) - min\left\{f_{work}(a^I_{min}), f_{work}(a^O_{min})\right\}$, and $f_{work}(a^O_{min}) : f_{work}(a^O_{min}) - min\left\{f_{work}(a^I_{min}), f_{work}(a^O_{min})\right\}$.
Remove a^I_{min} from $A^I_{v,work}$ if $f_{work}(a^I_{min}) = 0$, and remove a^O_{min} from $A^O_{v,work}$ if $f_{work}(a^O_{min}) = 0$.
Go back to step 1 if either $A^I_{v,work}$ or $A^O_{v,work}$ is not empty; otherwise, end the procedure.

2b. If $s(a^I_{min}) > s(a^O_{min})$, then let the arcs be a pair with the same amount of *flow* $min\left\{f_{work}(a^I_{max}), f_{work}(a^O_{min})\right\}$.
For the center vertex of the paired arc, endow *potential* $min\left\{f_{work}(a^I_{max}), f_{work}(a^O_{min})\right\}$, and let the *sequence* on the pair of arcs be consistent with the original *sequence* on $\langle V, A, f\rangle$.
Update $f_{work}(a^I_{max}) := f_{work}(a^I_{max}) - min\left\{f_{work}(a^I_{max}), f_{work}(a^O_{min})\right\}$, and $f_{work}(a^O_{min}) := f_{work}(a^O_{max}) - min\left\{f_{work}(a^I_{max}), f_{work}(a^O_{min})\right\}$.
Remove a^I_{max} from $A^I_{v,work}$ if $f_work(a^I_{max}) = 0$, and remove a^O_{min} from $A^O_{v,work}$ if $f_work(a^O_{min}) = 0$.
Go back to step 1 if either $A^I_{v,work}$ or $A^O_{v,work}$ is not empty; otherwise, end the procedure.

After the above *unbundling* procedure for each $v \in V$, we let consecutive arcs be arbitrarily connected to each other with each same *flow* and relevant *potential* so as to form a closed-cycle decomposition. Then, we can always connect consecutive pairs by taking an appropriately small unit of *flow* attached when necessary. Furthermore, by attaching an appropriate *sequence* for each derived decomposed f-network that is consistent with the original *sequence*, each derived connected fsp-network is to be e-covered by construction. Figure 30 shows our *unbundling* procedure for the example shown in Fig. 29.

A.2 Proof of Theorem 2

Lemma 2 ensures that our search for fsp-networks on the basis of *closed-cycle decomposed* f-networks always includes fsp-networks that realize the min-circulation. What remains to be shown is that we correctly choose those fsp-networks by minimizing the circulation for *closed-cycle decomposed* f-networks. This is ensured by the following lemma.

Lemma 3
Given a closed f-network $N^f = \langle V, A, f\rangle$, *for any closed-cycle decomposi-*

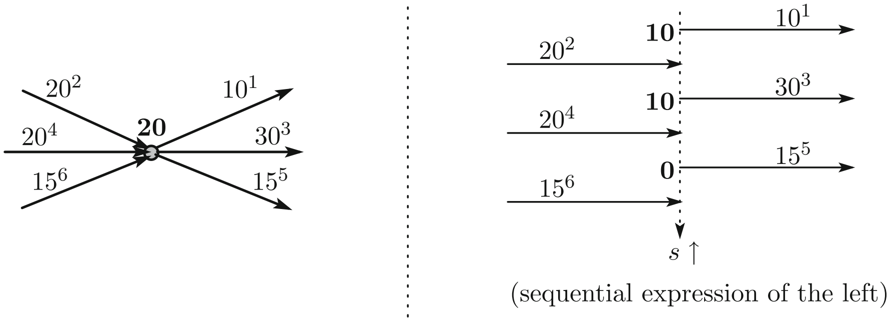

Fig. 29 Before the unbundling procedure

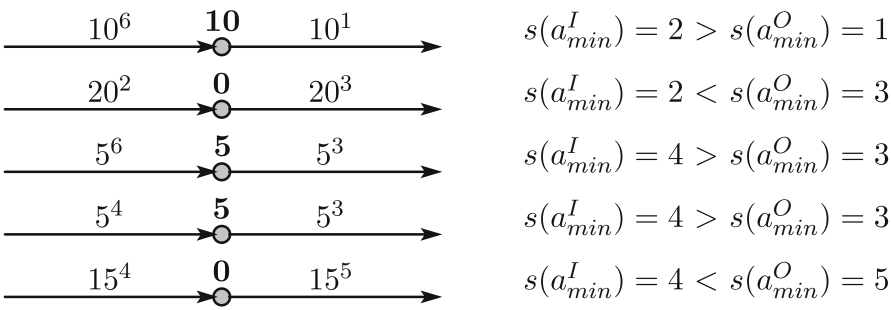

Fig. 30 Unbundling procedure for Fig. 29

tion $N^f = \sum_{c \in C} \langle V^c, c, f^c \rangle$, *if* $\langle V^c, c, f^c, s^c, p^c \rangle$ *is e-covered for every* $c \in C$, *and we can take* $s : V \to \{1, 2, \ldots, |A|\}$ *which is consistent with* $\{s^c\}_{c \in C}$, *then* $\langle V, A, f, s, p \rangle = \sum_{c \in C} \langle V^c, c, f^c, s^c, p^c \rangle$ *is covered.*

Proof As long as $\{s^c\}_{c \in C}$ is consistent with s, it is straightforward that combining covered fsp-networks always gives a covered fsp-network. □

The lemma states that our search on the basis of *closed-cycle decomposed* f-networks never lets us find smaller circulation than the "true" min-circulation. Combining Lemmas 2 and 3, we end our proof.

A.3 Proof of Theorem 3

Our proof is similar to that for Theorem 2. Lemma 2 ensures that our search for fsp-networks on the basis of *closed-cycle decomposed* f-networks always includes fsp-networks that realize the max-circulation. Since we confine our focus to the cases in which combined fsp-networks become e-covered, our search on the basis of *closed-cycle decomposed* f-networks never lets us reach a larger circulation than the "true" max-circulation. This ends our proof.

A.4 Proof of Theorem 4

Lemma 4 provides our proof of the theorem, regarding both min and max-circulation.

Lemma 4 (E-covered decomposition with undominated closed-cycle decomposition) *Given a closed f-network $N^f = \langle V, A, f \rangle$, take an arbitrary e-covered fsp-network $N^{fsp} = \langle V, A, f, s, p \rangle$.*

Then, we always take some e-covered decomposition $N^{fsp} = \sum_{c \in C} \langle V^c, c, f^c, s^c, p^c \rangle$ with undominated closed-cycle decomposition $\{\langle V^c, c, f^c \rangle\}_{c \in C}$.

Proof For the given N^{fsp}, take an e-covered decomposition $\{\langle V^c, c, f^c, s^c, p^c \rangle\}_{c \in C}$ with a closed cycle decomposition $\{\langle V^c, c, f^c \rangle\}_{c \in C}$. We consider to take an undominated closed cycle decomposition by *transforming* the closed cycle decomposition in a manner stated below. We show that, for the derived undominated closed cycle decomposition, we can take an e-covered decomposition of the given N^{fsp}.

For a given closed cycle decomposition, we consider transformation such that for a vertex, taking different pairs of the incoming arcs and the outgoing arcs to form another closed cycle decomposition. For the sake of our formal statement of the transformation, we prepare notations; for a vertex $v \in V$, denote $A^I_v = \left\{(v', v)\right\}_{v' \in V}$, and $A^O_v = \left\{(v, v')\right\}_{v' \in V}$.

First, when either A^I_v or A^O_v is singleton, Lemma 5 ensures that any combination between A^I_v and A^O_v that can generate another closed-cycle decomposition maintains the e-covered property under the same s and p. Now, we proceed to the case in which each of A^I_v and A^O_v is not singleton. Under the supposed closed-cycle decomposition, if any two cycles both commonly include v but have no common arc, then we only need to combine the two cycles into one *punctured* cycle in order to derive an *undominated* closed-cycle decomposition. Lemma 6 ensures that the derived cycles are necessarily e-covered under the same s and p. Then, suppose that there exists an undominated closed-cycle decomposition with some other combination between A^I_v and A^O_v for each $v \in V$. Some repetition of the procedure stated in Lemma 7 ensures that we can always take some undominated closed-cycle decomposition with the original combination between A^I_v and A^O_v. This means that we can take some undominated closed-cycle decomposition under the same s and p that constitutes the same e-covered fsp-network for which the original *undominated* closed-cycle decomposition is another e-covered decomposition. □

Lemma 5 (Invariance of e-covered property: singleton A^I_v or A^O_v)

For a closed f-network $N^f = \langle V, A, f \rangle$, take a closed-cycle decomposition $N^f = \sum_{c \in C} \langle V^c, c, f^c \rangle$ such that it constitutes some e-covered decomposition for some e-covered $N^{fsp} = \langle V, A, f, s, p \rangle$ as $N^{fsp} = \langle V, A, f, s, p \rangle = \sum_{c \in C} \langle V^c, c, f^c, s^c, p^c \rangle$. Then, for any $v \in V$ such that either A^I_v or A^O_v is singleton, take another closed-cycle decomposition $\{\langle V^c, c, f^c \rangle\}_{c \in C'}$ by transforming it such that the combination between A^I_v and A^O_v is different.

Then, we can always take an e-covered decomposition $\langle V, A, f, s, p \rangle = \sum_{c \in C'} \langle V^c, c, f^c, s^c, p^c \rangle$

Proof Under given closed-cycle decomposition $N^f = \sum_{c \in C} \langle V^c, c, f^c \rangle$, suppose that A_v^I or A_v^O is singleton. Then, any *sequence* s on $A_v^I \cup A_v^O$ uniquely determines the relative order between any pair of arcs a, a' where $a \in A_v^I$ and $a' \in A_v^O$. □

Lemma 6 (Invariance property of e-covered decomposition by letting cycles be punctured)

For a closed f-network $N^f = \langle V, A, f \rangle$, take arbitrary e-covered $N^{fsp} = \langle V, A, f, s, p \rangle$ and some e-covered decomposition $N^{fsp} = \sum_{c \in C} \langle V^c, c, f^c, s^c, p^c \rangle$.

Then, if there exist any two cycles $c, c' \in C$ that constitute a punctured cycle $c'' \notin C$, we always take another closed-cycle decomposition that includes the punctured cycle, and it still constitutes an e-covered decomposition for the same $\langle V, A, f, s, p \rangle$.

Proof Take some e-covered decomposition $N^{fsp} = \sum_{c \in C} \langle V^c, c, f^c, s^c, p^c \rangle$. When there exist K cycles $\{c_k\}_{k=1,\ldots,K}$ with *flow* $\{f^{c_k}\}_{k=1,\ldots,K}$ each includes some $v \in V$, denote the relevant arcs as $A_{v,k}^I \equiv c_k \cap A_v^I$, $A_{v,k}^O \equiv c_k \cap A_v^O$.

Take two cycles $c_k, c_{k'} \in C$ that constitute a punctured cycle $c_{K+1} \notin C$. Take another closed-cycle decomposition with c_{K+1} with some *flow* $f^{c_{K+1}} \leq min(f^{c_k}, f^{c_{k'}})$. Let $\langle V^{c_{K+1}}, c_{K+1}, f^{c_{K+1}}, s^{c_{K+1}}, p^{c_{K+1}} \rangle$ be e-covered under the same s on A. We always have $p^{c_{K+1}}(v) \leq p^{c_k}(v) + p^{c_{k'}}(v)$ from Lemma 8. Suppose that $p^{c_{K+1}}(v) < p^{c_k}(v) + p^{c_{k'}}(v)$. Then, for the original decomposition to be e-covered, the redundant amount somehow needs to be absorbed by some other cycles within C_v with respect to v. Again, this contradicts Lemma 8 with respect to other relevant cycles not to be transformed. Thus, $p^{c_{K+1}}(v) = p^{c_k}(v) + p^{c_{k'}}(v)$, which ensures invariance of the e-covered property under our transformation. □

Lemma 7 (Invariance of undominated closed-cycle decomposition)

For a closed $N^f = \langle V, A, f \rangle$, take an undominated closed-cycle decomposition $N^f = \sum_{c \in C} \langle V^c, c, f^c \rangle$. For an arbitrary $v \in V$ such that $|A_v^I| > 1$ and $|A_v^O| > 1$, let $C_v \subset C$ denote the set of cycles that include v. Let $C_v = \{c_1, c_2.., c_K\}$ with $\sum_{k=1}^K f^{c_k} \equiv F_v$, and let $A_{v,k}^I \equiv c_k \cap A_v^I$ and $A_{v,k}^O \equiv c_k \cap A_v^O$.

Take another combination between A_v^I and A_v^O by combining some $a_{v,k}^I \in A_{v,k}^I$ and $a_{v,k'}^O \in A_{v,k'}^O$, and also combine $a_{v,k'}^I \in A_{v,k'}^I$ and $a_{v,k}^O \in A_{v,k}^O$. Further, take an arbitrary amount of flow $f' \leq min(f^{c_k}, f^{c_{k'}})$ for each combination.

Then, we can always take another undominated closed-cycle decomposition with this new combination between A_v^I and A_v^O.

Proof For two cycles $c_k, c_{k'} \in C$ where neither is a subset of the other, if they have some common $v \in V$, we can always take some arc $a \in c_k \cap c_{k'}$ as ensured by Lemma 9. Take a path from a to $a_{v,k}^I$, and $a_{v,k'}^O$ to a, and merge two paths to take a cycle and denote it as c_{K+1}. Furthermore, take its counterpart path from a to $a_{v,k'}^I$, and $a_{v,k}^O$ to a each , and merge them to take a cycle c_{K+2}. Attach *flow* f' on both c_{K+1} and c_{K+2}, and remove the same amount f' from each of the original two cycles $c_k, c_{k'}$ as $f^{c_k} = f^{c_k} - f'$ and $f^{c_{k'}} = f^{c_{k'}} - f'$. The derived set of closed-cycles constitutes a new closed-cycle decomposition. From its construction, the sum of *flow* for the derived cycles is the same as F_v, which shows that the derived closed-cycle decomposition is also undominated. □

Lemma 8 (Invariance of covered property by merge of vertices)

For a closed $\langle V, A, f \rangle$*, take an e-covered* $\langle V, A, f, s, p \rangle$*. Let* $v, v' \in V$ *merge into a single* v'' *that maintains relevant flow, sequence, and potential, to have another fsp-network* $\langle V', A', f', s', p' \rangle$*.*

Then, $\langle V', A', f', s', p' \rangle$ *is always covered.*

Proof From the definition of the e-covered property, for any $v \in V$, at any point of its outflow there remains a sufficient amount of funds. It is straightforward that merging these two vertices preserves the covered property. □

Lemma 9 (Property of undominated closed-cycle decomposition)

For a closed f-network $N^f = \langle V, A, f \rangle$*, take an undominated closed-cycle decomposition* $N^f = \sum_{c \in C} \langle V^c, c, f^c \rangle$*. Then, we have for any two* $c, c' \in C$ *where neither is a subset of the other. Then,* $V^c \cap V^{c'} \neq \emptyset$ *iff* $c \cap c' \neq \emptyset$*.*

Proof It is immediate that $V^c \cap V^{c'} \neq \emptyset$ if $c \cap c' \neq \emptyset$. For the opposite part, suppose that under some undominated closed-cycle decomposition, there exist two cycles c, c' with $\langle V^c, c, f^c \rangle$ and $\langle V^{c'}, c', f^{c'} \rangle$ where $V^c \cap V^{c'} \neq \emptyset$ but $c \cap c' = \emptyset$. Then, we always take another closed-cycle decomposition by letting the cycles combine to take $\langle V^c \cup V^{c'}, c \cup c', min \left\{ f^c, f^{c'} \right\} \rangle$, which dominates the original closed-cycle decomposition. This contradicts our supposition. □

A.5 Proof of Theorem 8

We provide our proof from a rather general perspective, focusing not only on *type-1 clustered* networks but also on clustered networks in which each cluster forms a *tree*, whose definition is stated below. When each cluster is captured by a *tree*, we show that a certain monotonicity enables us to examine min-circulation.

A.5.1 Defluent Partition

First, we prepare terminologies in order to formally describe the relevant monotonicity. The key terminology is *defluent*. Monotonicity implied by *defluent* refers to how *flow* bifurcates, which lies in the monotonicity implied by *tree* with regard to how arcs connect to each other. Figure 31 helps clarify the following terminologies.

We start by defining *tree*. First consider a graph that is not directed, which we denote as $\langle V, E \rangle$ where each $e \in E$ is an unordered pair of vertices denoted as $e = \left\{ v, v' \right\}$ with some $v, v' \in V$. We allow multiple edges for the same pair of vertices. We say that $\langle V, E \rangle$ has a *cycle* if there exists a *sequence* of vertices such that every two consecutive vertices are included as a different edge and the same vertex can be shown more than once in the *sequence*. For a directed graph $\langle V, A \rangle$, we can take its underlying undirected graph $\langle V, E \rangle$ by replacing each ordered pair with an unordered pair. Now, we call a directed graph $\langle V, A \rangle$ a *tree* if it is connected and its underlying undirected graph $\langle V, E \rangle$ has no cycle. Note that if $\langle V, A \rangle$ is a *tree*, then it has no *cycle* on the basis of a directed graph, but the opposite is not necessarily true, since our definition

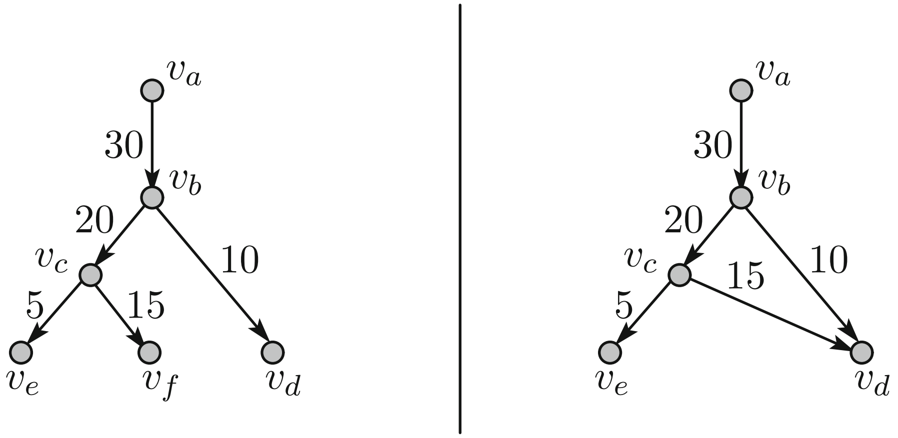

Fig. 31 Examples of *defluent* and relevant notions. Each of the left and right f-networks is *defluent*. The underlying directed graph for the left f-network is a *trunked tree* whose *root* is v_a, *trunk* is (v_a, v_b), and *leaves* are v_d, v_e, and v_f. *Height* is 0 for the *leaves*, 1 for v_c, 2 for v_b, and 3 for v_a. For the right f-network, the underlying directed graph is not itself a *tree*, but the directed graph on the left is its *trunked tree representation* that is *hatcheled* on v_d by adding v_f

of *tree* does not allow even a *cycle* on the basis of an undirected graph to exist. For *tree* $\langle V, A\rangle$, we call each vertex $v \in V$ that has no arc that starts at v a *leaf*.

A *tree* $\langle V, A\rangle$ is called a *rooted tree* if there is one vertex $v \in V$ designated as the *root* such that for any other vertex $v' \in V \setminus v$, there is a *sequence* of vertices from v to v' where every two consecutive vertices $v_k, v_{k'}$ are included as a different arc as $(v_k, v_{k'}) \in A$.

A *tree* $\langle V, A\rangle$ is called a *trunked tree* if (1) it is a *rooted tree*, and (2) there is only one arc (v, v') designated as the *trunk* such that it starts at the root $v \in V$.

For a *rooted tree*, we define the *height* of each vertex within the tree. First we prepare the terminology *length* of a *path*. The *length* of *path* A' is the number of arcs $|A'|$. Now, for a *rooted tree* $\langle V, A\rangle$, the *height* of a vertex $v \in V$ is the maximum *length* of the *path* to the *leaves*.

We define *tree* to obey common usage. Since the definition is too strict for our purpose, we argue whether $\langle V, A\rangle$ has *trunked tree representation* or not. In order to define *trunked tree representation*, we prepare the terminology of *hatchel*. Directed graph $\langle V, A\rangle$ is *hatcheled* into $\langle V', A'\rangle$ when we can take $\langle V', A'\rangle$ through the following procedure. For every $v \in V$ that has no arc that starts at v, when there exist $n > 1$ multiple arcs that end at v, add $n - 1$ vertices. Then, for each of the arcs except arbitrary ones , replace v with each of the added vertices such that each arc ends at each different vertex either at v or at some added vertex. For derived $\langle V', A'\rangle$, any vertex that has no arc that starts at itself has no two arcs that end at itself.

Now, we say that directed graph $\langle V, A\rangle$ has *trunked tree representation* $\langle V', A'\rangle$ when (1) $\langle V, A\rangle$ is *hatcheled* into $\langle V', A'\rangle$, and (2) $\langle V', A'\rangle$ is a *trunked tree*. For f-network $\langle V, A, f\rangle$, when $\langle V, A\rangle$ has *trunked tree representation* $\langle V', A'\rangle$, $\langle V', A', f'\rangle$ is its associated f-network when (1) $f'(a) = f(a)$ for any $a \in A \cap A'$, and (2) for each replacement $a \in A$ with $a' \in A'$ by the procedure of *hatchel*, $f'(a') = f(a)$.

We proceed to define monotonicity regarding *flow* bifurcation. $\langle V, A, f\rangle$ is *defluent* when (1) $\langle V, A\rangle$ has *trunked tree representation* $\langle V', A'\rangle$, and (2) for its associated $\langle V', A', f'\rangle$, for any $v \in V'$ that is neither *root* nor *leaf*, *flow* is balanced as $\sum_{v' \in V'} f(v, v') = \sum_{v' \in V'} f(v', v)$.

We argue partial monotonicity by examining a specific class of *decomposition* called *defluent partition*. For $N^f = \langle V, A, f\rangle$, *decomposition* $\left\{N_k^f = \langle V_k, A_k, f_k\rangle\right\}_{k=1,2,\ldots,K}$ is *partition* when (1) for every $k = 1, 2, \ldots, K$, $\langle V_k, A_k\rangle$ is connected, and (2) for any two N_k^f, $N_{k'}^f$, we have $A_k \cap A_{k'} = \emptyset$.

Finally, a *partition* for $\langle V, A, f\rangle$ is a *defluent partition* when (1) each *partitioned* f-network is *defluent*, and (2) any two partitioned f-networks have no common vertex that is neither *root* nor *leaf* within either of the *defluents*.

For f-network $N^f = \langle V, A, f\rangle$, when we take *defluent partition* $\left\{N_k^f\right\}_{k=1,2,\ldots,K}$, we obtain the set of *edge* vertices denoted as $V^{edge}(\left\{N_k^f\right\}_{k=1,2,\ldots,K}) \subset V$; each vertex is the *root* for the partitioned f-network in the sense that each vertex is the root for each *trunked tree representation*.

For any f-network, there exists some *defluent partition* since a *partition* into each arc is itself a *defluent partition*.

Theorem 12 (Defluent partition)
For a closed f-network $N^f = \langle V, A, f\rangle$*, take an arbitrary defluent partition* $\left\{N_k^f\right\}_{k=1,2,\ldots,K}$ *with the set of edge vertices* $V^{edge}(\left\{N_k^f\right\}_{k=1,2,\ldots,K})$*. Then, the following problem attains the same value with* $x^{min}(N^f)$*.*

$$\begin{aligned} & min_{s,p} \sum_{v \in V} p(v) \\ & \quad s.t., \langle V, A, f, s, p\rangle \text{ is e-covered, and} \\ & \qquad p(v) = 0 \text{ for every } v \in V \setminus V^{edge}\left(\left\{N_k^f\right\}_{k=1,2,\ldots,K}\right). \end{aligned}$$

Proof For our input $N^f = \langle V, A, f\rangle$, take an arbitrary e-covered $\langle V, A, f, s, p\rangle$, which attains the min-circulation $x^{min}(N^f)$.

For a partitioned f-network $N_k^f = \langle V_k, A_k, f_k\rangle$, denote a set of vertices V_k^i with $i = 0, 1, \ldots$ where each V_k^i contains vertices that have *height* i within its *rooted tree*. First find $p(v) > 0$ for $v \in V_k^1$. When we find such v with (v', v), then we always take an arc $a' = (v, v'', n) \in A$ such that $s(a') < s(a)$ with some $n = 0, 1, \ldots$. Now take another *sequence* $s' : A \to \{1, 2, \ldots, A\}$ that lets $s'(a') > s'(a)$ while endowing the same ordering with the *sequence* s for $A \setminus a'$. Take associated e-covered $\langle V, A, f, s', p'\rangle$; then, we have $p'(v) = p(v) - f(a')$ and $p'(v'') \leq p(v'') + f(a')$. The former is true since there is only one arc that ends at v, while the latter is true since there is only one arc that starts at v''. In addition, we have $p'(v''') = p(v''')$ for any $v''' \in V \setminus v, v''$. It leads to $\sum_{v \in V} p'(v) \leq \sum_{v \in V} p(v)$. Furthermore, since $\sum_{v \in V} p(v) = x^{min} p(v)$, we have $\sum_{v \in V} p'(v) = x^{min}(N^f)$.

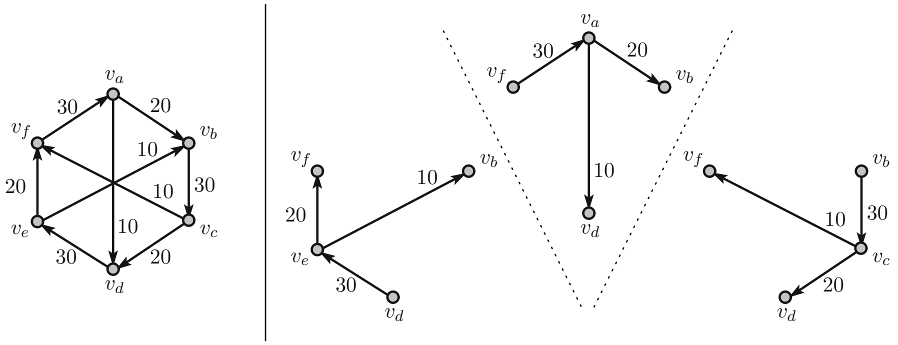

Fig. 32 Example for *defluent partition*. Notes: For the f-network on the left, the three f-networks on the right constitute its *defluent partition* with the set of *edge* vertices $\{v_d, v_f, v_b\}$

For each of the vertices within V_k^1, execute this procedure of replacing the *sequence*. When completed for V_k^1, we proceed to V_k^2. For this case, when we replace the *sequence* regarding vertex $v \in V_k^2$ and arc $(v, v') \in V$, we also maintain the ordering within each *trunked tree* with trunk (v, v') and its leaves so that the potential is different only within the *edge* vertices $V^{edge}(\left\{N_k^f\right\}_{k=1,2,\ldots,K})$. When finished for V_k^2, we proceed to $V_k^3, V_k^4, \ldots$ until we reach each *root*. Such replacement of a *sequence* always realizes the min-circulation with the associated e-covered fsp-network, while letting $p(v'') = 0$ for every vertex except for *root* and *leaves*. By executing the procedure for every *partitioned defluent*, we derive the statement in the theorem (Fig. 32). □

A.5.2 Proof of Theorem 8

We prove the theorem using *defluent partition*.

Proof For our given f-network with some n, take a *defluent partition* $N^f = \sum_{k=1}^n N_k^f$ with $N_k^f = \langle V_k, A_k, f_k \rangle$, which consists of $V_k = \{v_{2k-1}, v_{2k}, v_{2k+1}, v_{2k-3}\}$, $A_k = \{(v_{2k-1}, v_{2k}), (v_{2k}, v_{2k+1}), (v_{2k}, v_{2k-3})\}$, $f_k(a_k) = f(a_k)$ for every $a_k \in A_k$, where we let $v_{-i} = v_{2n-i}$ and $v_{2n+i} = v_i$ for $i = 0, 1, 2$. For our defluent partition, the set of edges is $V^{edge} = \{v_{2k-1}\}_{k=1,2,\ldots,n}$.

From Theorem 12, we only need to examine e-covered $\langle V, A, f, s, p \rangle$ where $p(v_{2k-1}) \geq 0$ while $p(v_{2k}) = 0$ for $k = 1, 2, \ldots n$. Take such a *sequence* s and associated e-covered $\langle V, A, f, s, p \rangle$; then, take another *sequence* s' with its associated e-covered $\langle V, A, f, s', p' \rangle$ where s' is derived from s as follows. Let $s'((v_{2k}, v_{2k+1})) = s'((v_{2k-1}, v_{2k}))+1$, and $s'((v_{2k}, v_{2k-3})) = s'((v_{2k-1}, v_{2k}))+2$ for every $k = 1, 2, \ldots n$, while s' preserves the ordering of s on $\{(v_{2k-1}, v_{2k})\}_{k=1,2,\ldots,n}$. Observe that it satisfies $p'(v_{2k-1}) \geq 0$, $p'(v_{2k}) = 0$ for $k = 1, 2, \ldots n$. Then, we have $\sum_{v \in V} p'(v) \leq \sum_{v \in V} p(v)$. It is immediate that given the ordering among $\{(v_{2k-1}, v_{2k})\}_{k=1,2,\ldots,n}$ and $p'(v_{2k-1}) \geq 0$, $p'(v_{2k}) = 0$ for $k = 1, 2, \ldots n$, the ordering on each (v_{2k}, v_{2k+1}), (v_{2k}, v_{2k-3}) should be set as early as possible for $k = 1, 2, \ldots n$ in order to minimize circulation.

For such *sequence* s', take *sequence* s'' and associated e-covered $\langle V, A, f, s'', p''\rangle$ so that it further satisfies $s((v_{2k-1}, v_{2k})) < s((v_{2k+1}, v_{2k+2}))$ while preserving the ordering within each $\{(v_{2k-1}, v_{2k}), (v_{2k}, v_{2k+1}), (v_{2k}, v_{2k-3})\}$ for $k = 1, 2, \ldots, n$. Note that we have $r^{atwi}(c, s'') = 1$. Now we have $\sum_{v \in V} p''(v) = (f^c + f^{c'}) + (n - 2) * f^{c'} \leq \sum_{v \in V} p(v)$. This is because if there exists one pair of arcs (v_{2m-1}, v_{2m}) and $(v_{2m'-1}, v_{2m'})$ with some $m < m' \in \{1, 2, \ldots, n\}$ such that $s'((v_{2m-1}, v_{2m})) > s((v_{2m'-1}, v_{2m'}))$, then $\sum_{v \in V} p''(v) + (f^c - f^{c'}) \leq \sum_{v \in V} p(v)$. When there exists more than one such pair, the above relation is still satisfied. Note that when $f^c = f^{c'}$, we have $\sum_{v \in V} p''(v) = \sum_{v \in V} p(v)$, which means that the min-circulation can be attained with any $r^{atwi}(c, s') \in \{1, 2, \ldots, n\}$. □

A.6 Proof of Theorem 9

First, we prove the case of $f_1 = f_2$ for *type-2 clustered* f-network $\langle V, A, f\rangle$ with $|V| = n$. We observe that under any e-covered fsp-network $\langle V, A, f, s, p\rangle$, $p(v) \in \{0, f_1, 2f_1\}$ for any $v \in V$. We show that max-circulation is nf_1 in the following steps. Denote $V = \{v_1, v_2, \ldots, v_n\}$ with $v_{n+1} \equiv v_1$ as defined in its construction, such that every two consecutive vertices constitute a cycle by themselves.

Lemma 10
If $\langle V, A, f, s, p\rangle$ attains the max-circulation, then, we do not have $p(v_k) = 2f_1$ and $p(v_{k+1}) = 2f_1$ at the same time for $k = 1, 2, \ldots, n$.

Proof Suppose that $p(v_k) = 2f_1$; then $s((v_k, v_{k+1})) < s((v_{k+1}, v_k))$, which necessarily lets $p(v_{k+1}) \leq f_1$. □

For the next Lemma 11, we prepare the following notation. For type-2 clustered networks, for two $v_k, v_{k'}$, we denote $V_k, V_{k'}$ such that they constitute a partition $V = v_k \cup V_1 \cup v_k \cup V_2$ and each of V_1 and V_2 is connected.

Lemma 11
When $\langle V, A, f, s, p\rangle$ attains the max-circulation, if there exists $v_k, v_{k'}$ such that $p(v_k) = p(v_{k'}) = 2f_1$, then, there exists $v_{k''} \in V_k$ and $v_{k'''} \in V_{k'}$ such that $p(v_{k''}) = p(v_{k'''}) = 0$.

Proof For $p(v_k) = 2f_1$, both out-arcs need be settled before either of the in-arcs such that $s((v_k, v_{k+1})) < s((v_{k+1}, v_k))$, and $s((v_k, v_{k-1})) < s((v_{k-1}, v_k))$. As shown in the previous lemma, $p(v_{k+1}) \leq f_1$, $p(v_{k-1}) \leq f_1$. When we also have $p(v_{k'}) = 2f_1$ with $k' \neq k$, then, proceeding from both vertices to endow a *potential* on each passed vertex as f_1, we need to attain some vertex $v_{k''}$ both of whose neighbors have endowed *potential* f_1. Thus, for $v_{k''}$, both in-arcs are settled before both out-arcs, which necessarily lets $p(v_{k''}) = 0$. This completes our proof. □

Suppose that there are i vertices that are endowed *potential* $2f_1$. Combining Lemmas 10 and 11, we know that $0 \leq i \leq \frac{n}{2}$. When $i = 0$, each vertex can be endowed at most f_1, which lets $sum_{v \in V}\, p(v) \leq nf_1$. Further suppose that each is actually endowed with f_1 *potential*. When we let each one vertex be endowed with $2f_1$ *potential*, there need to be at least two vertices on both sides that need to be endowed with 0 *potential*. We know that this never increases the circulation. This ends our proof for the case of $f_1 = f_2$.

Now, it is almost straightforward to show the case of $f_1 > f_2$. The above proof shows that max-circulation is now attained in the direction with $r^{atwi}(A_1, s) = n - 1$ with $x^{max}(N^f) = (n-1) f_1 + f_2$.

A.7 Results Relevant to Rotemberg (2011)

We follow the notation of Rotemberg (2011) with regard to its target class of network.

The next corollary shows that Rotemberg (2011) treats one of the simplest classes of f-network with no *arc-twisted* cycles.

Corollary 1 (Case of Rotemberg (2011): min-circulation)
For a closed f-network $N^f = \langle V, A, f \rangle$ *that is in a class of* C_N^K *with flow* z [25], *we have* $x_{N^f}^{min} = z$.

Proof Since C_N^K is based on a Euler graph, we can take a cycle c that consists of all the arcs. Furthermore, since the *flow* for each arc equals z, we can take a closed-cycle decomposition with a unique undominated cycle c with *flow* $f(c) = z$. Since a Euler graph has no *arc-twisted* cycles, the min-circulation is realized with c, and the derived value is z. □

The next corollary shows that Rotemberg (2011) treats one of the simplest classes of f-network with no *vertex-twisted* cycles.

Corollary 2 (Case of Rotemberg (2011): max-circulation)
For a closed f-network $N^f = \langle V, A, f \rangle$ *which is in the class of* C_N^K *with flow* z*, suppose that* $N/(K!)$ *is an integer. Then, we have* $x_{N^f}^{max} = z \sum_{k=1}^{K} k * (\frac{N}{k} - 1)$

Proof When $N/(K!)$ is an integer, there exists no *vertex-twisted* cycles within C_G^{np} for any associated graph G, where C_G^{np} denotes the set of non-punctured cycles for G. Observe that we can take a set of cycles $C \subseteq C_G^{np}$ such that C constitutes a closed cycle decomposition with constant flow z for each cycle, and $|C| = \sum_{k=1}^{K} k$, for which there are k number of cycles for each $|c| = \frac{N}{k}$, for $k = 1, 2, \ldots, K$. Since there is no *vertex-twisted* relation between the cycles in C_G^{np}, we take a sequence whereby the circulation for each cycle c with $|c| = \frac{N}{k}$ is $z(\frac{N}{k} - 1)$. Thus, the sequence attains the circulation stated in this corollary. We easily confirm that the circulation equals the max-circulation for the given f-network, observing that any combination of the cycles within C is not dominated by non-punctured cycles. □

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25 We obey the definition of Rotemberg (2011) on C_N^K. N subjects indexed by $i \in [0, 1, \ldots, N-1]$ are arrayed in a circle such that $N-1$ is followed by firm 0. Each subject i has payment z to subjects $i + j$ for each $j = 1, 2, \ldots, K$. $2K \leq N - 1$ is assumed. C_N^K thus defined has a representation as a f-network with constant flow z.

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