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A structured modified Newton approach for solving systems of nonlinear equations arising in interior-point methods for quadratic programming

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Abstract

The focus in this work is on interior-point methods for inequality-constrained quadratic programs, and particularly on the system of nonlinear equations to be solved for each value of the barrier parameter. Newton iterations give high quality solutions, but we are interested in modified Newton systems that are computationally less expensive at the expense of lower quality solutions. We propose a structured modified Newton approach where each modified Jacobian is composed of a previous Jacobian, plus one low-rank update matrix per succeeding iteration. Each update matrix is, for a given rank, chosen such that the distance to the Jacobian at the current iterate is minimized, in both 2-norm and Frobenius norm. The approach is structured in the sense that it preserves the nonzero pattern of the Jacobian. The choice of update matrix is supported by results in an ideal theoretical setting. We also produce numerical results with a basic interior-point implementation to investigate the practical performance within and beyond the theoretical framework. In order to improve performance beyond the theoretical framework, we also motivate and construct two heuristics to be added to the method.

Keywords Interior-point methods · Modified Newton methods · Quasi-Newton methods · Low-rank updates

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1 Introduction

This work is intended for inequality-constrained quadratic programs on the form

$$\begin{aligned} &\text{minimize } \frac{1}{2}x^T Hx + c^T x \\ &\text{subject to } Ax \geq b, \end{aligned} \tag{IQP}$$

where $H \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. We consider primal–dual interior-point methods which means solving or approximately solving a sequence of systems of nonlinear equations. Application of Newton iterations on each system of nonlinear equations gives an *unreduced* unsymmetric block 3-by-3 system of linear equations, with dimension $n + 2m$, to be solved at each iteration. This system can be put on an equivalent form which contains a *reduced* symmetric block 2-by-2 system of dimension $n + m$, or a *condensed* system of dimension n, see Sect. 2. Both of these systems typically become increasingly ill-conditioned as the iterates converge, whereas the unreduced system may stay well conditioned throughout, see e.g., [1, 2] for analysis of the spectral properties of systems arising in interior-point methods for convex quadratic programs on standard form. The sparsity structure of the unreduced system is maintained in the reduced system. However, the condensed system is typically dense if A contains dense rows. The most computationally expensive part of an interior-point iteration is the solution of the Newton systems that arise, see e.g., [3, 4] for details on the solution of the systems and [5] for a comparison of the solution of unreduced and reduced systems.

We are particularly interested in the convergence towards a solution when the Jacobian of each Newton system is modified so that the system becomes computationally less expensive to solve. In general, there is a trade-off between solving many modified Newton systems, which are computationally less expensive, but typically give lower quality solutions and solving Newton systems which give high quality solutions. We propose an approach where each modified Jacobian is composed of a Jacobian at a previous iteration, whose factorization is assumed to be known, plus one low-rank update matrix per succeeding iteration. A similar strategy have been studied by Gondzio and Sobral [6] in the context of quasi-Newton approaches, where Broyden's rank-1 updates are performed on the Jacobian approximation. If the proposed quasi-Newton approach is started with an exact Jacobian, then the sparsity pattern of the first two block rows is maintained, however the sparsity pattern of third block row is typically lost. In contrast, we consider low-rank update matrices of variable rank which capture the sparsity pattern of all block rows of the Jacobian. Each modified Jacobian may hence be viewed as a Jacobian at a different point. Consequently, the modified Newton approach may also be interpreted in the framework of previous work on e.g., effects of finite-precision arithmetic, stability, convergence and solution techniques for interior-point methods [1–3, 5, 7–13]. The idea of low-rank update matrices in the context of a primal barrier method for linear programming has been considered by Gonzaga [14].

The updates and the theory are given for the unreduced Jacobian but we also discuss how analogous updates can be made on both reduced and condensed systems. The modified Newton approach is also compatible with certain regularization strategies, see e.g., [15–17], although it is outside the scope of this first study.

The work is meant to be an initial study of the structured modified Newton approach. We derive theoretical results in ideal settings to support the choice of update matrix. In addition, we produce numerical results with a basic interior-point algorithm to investigate the practical performance within and beyond the theoretical framework. The numerical simulations were performed on benchmark problems from the repository of convex quadratic programming problems by Maros and Mészáros [18]. We envisage the use of the modified Newton approach as an accelerator to a Newton approach. E.g., when these can be run in parallel for a specific value of the barrier parameter, and the modified Newton approach may utilize factorizations from the Newton approach when it is appropriate.

The manuscript is organized as follows; Sect. 2 contains a brief background to primal–dual interior-point methods and an introduction to the theoretical framework; in Sect. 3 we propose a modified Newton approach and discuss how it relates to some previous work on interior-point methods; Sect. 4 contains a description of the implementation along with two heuristics and a refactorization strategy; in Sect. 5, we give numerical results on convex quadratic programs; finally in Sect. 6 we give some concluding remarks.

1.1 Notation

Throughout, $\rho(M)$ denotes the spectral radius of a matrix M and $|\mathcal{S}|$ denotes the cardinality of a set $\mathcal{S}$. The notion "$\cdot$" is defined as the component-wise multiplication operator, "$\succ 0$" as positive definite, and "$\wedge$" as the logical and. Quantities associated with Newton iterations will throughout be labeled with "$\hat{\ }$". Vector subscript and superscript denote component index and iteration index respectively. The only exception is e_i which denotes the ith unit vector of appropriate dimension. All norms considered are of type 2-norm unless otherwise stated.

2 Background

The theoretical setting is analogous to the setting in a previous work of ours on bound-constrained nonlinear problems [19]. For completeness, we review the background adapted to inequality-constrained quadratic programs in this section. Our interest is focused on the situation as primal–dual interior-point methods converge to a local minimizer $x^* \in \mathbb{R}^n$ with its corresponding multipliers $\lambda^* \in \mathbb{R}^m$ and slack variables $s^* \in \mathbb{R}^m$. Specifically we assume that the iterates of the method converge to a vector $\left(x^{*T}, \lambda^{*T}, s^{*T}\right)^T \triangleq (x^*, \lambda^*, s^*)$ that satisfies

$$Hx^* + c - A^T\lambda^* = 0, \quad \text{(stationarity)} \tag{1a}$$

$$Ax^* - s^* - b = 0, \quad \text{(feasibility)} \tag{1b}$$

$$s^* \geq 0, \quad \text{(non-negativity of slack variables)} \tag{1c}$$

$$\lambda^* \geq 0, \quad \text{(non-negativity of multipliers)} \tag{1d}$$
$$s^* \cdot \lambda^* = 0, \quad \text{(complementarity)} \tag{1e}$$
$$Z(x^*)^T H Z(x^*) \succ 0, \tag{1f}$$
$$s^* + \lambda^* > 0, \quad \text{(strict complementarity)}, \tag{1g}$$
$$A_{\mathcal{A}}(x^*) \text{ of full column rank}, \quad \text{(regularity)}, \tag{1h}$$

where $A_{\mathcal{A}}(x^\star)$ denotes the Jacobian corresponding to the active constraints and $Z(x^*)$ denotes a matrix whose columns span the nullspace of $A_{\mathcal{A}}(x^*)$. First-order necessary conditions for a local minimizer of (IQP) are given by (1a)–(1e). The first-order conditions together with (1f) and (1g) constitute second-order sufficient conditions for a local minimizer of (IQP) [20]. In the theoretical results, we also assume that (x^*, λ^*, s^*) satisfies (1h).

To simplify the notation, we let z denote the triplet (x, λ, s). For a given barrier parameter $\mu \in \mathbb{R}$, we are interested in the function $F_\mu : \mathbb{R}^{n+2m} \to \mathbb{R}^{n+2m}$ given by

$$F_\mu(z) = \begin{pmatrix} Hx + c - A^T\lambda \\ Ax - s - b \\ \Lambda S e - \mu e \end{pmatrix}, \text{ with } z = (x, \lambda, s), \tag{2}$$

where $S = \text{diag}(s)$, $\Lambda = \text{diag}(\lambda)$ and e is a vector of ones of appropriate size. First-order necessary conditions for a local minimizer of (IQP), (1a)–(1e), are satisfied by a vector z, with $s \geq 0$ and $\lambda \geq 0$, that fulfills $F_\mu(z) = 0$ for $\mu = 0$. In interior-point methods $F_\mu(z) = 0$ is solved or approximately solved for a sequence of $\mu > 0$, that approaches zero, while preserving $s > 0$ and $\lambda > 0$. Application of Newton iterations means solving a sequence of systems of linear equations on the form

$$F'(z)\Delta\hat{z} = -F_\mu(z), \tag{3}$$

where $\Delta\hat{z} = (\Delta\hat{x}, \Delta\hat{\lambda}, \Delta\hat{s})$ and $F' : \mathbb{R}^{n+2m} \to \mathbb{R}^{(n+2m)\times(n+2m)}$ is the Jacobian of $F_\mu(z)$, defined by

$$F'(z) = \begin{pmatrix} H & -A^T & \\ A & & -I \\ & S & \Lambda \end{pmatrix}. \tag{4}$$

In consequence it follows that the Jacobian at $z + \Delta z$ can be written as

$$F'(z + \Delta z) = F'(z) + \Delta F'(\Delta z),$$

where

$$\Delta F'(\Delta z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Delta S & \Delta\Lambda \end{pmatrix}, \tag{5}$$

with $\Delta\Lambda = \mathrm{diag}(\Delta\lambda)$ and $\Delta S = \mathrm{diag}(\Delta s)$. The subscript μ has been omitted since F' is independent of the barrier parameter. Under the assumption that Λ is nonsingular the *unreduced* block 3-by-3 system (3) can be reformulated as the *reduced* system

$$\begin{pmatrix} H & A^T \\ A & -\Lambda^{-1}S \end{pmatrix} \begin{pmatrix} \Delta\hat{x} \\ -\Delta\hat{\lambda} \end{pmatrix} = -\begin{pmatrix} Hx + c - A^T\lambda \\ Ax - b - \mu\Lambda^{-1}e \end{pmatrix}, \tag{6}$$

together with $\Delta\hat{s} = -(s - \mu\Lambda^{-1}e) - \Lambda^{-1}S\Delta\hat{\lambda}$. If in addition S is nonsingular, then a Schur complement reduction of $\Lambda^{-1}S$ in (6) gives the *condensed* system

$$(H + A^T S^{-1}\Lambda A)\Delta\hat{x} = -(Hx + c - A^T\lambda) - A^T S^{-1}(\Lambda(Ax - b) - \mu e), \tag{7}$$

together with $\Delta\hat{\lambda} = -S^{-1}(\Lambda(Ax-b)-\mu e)-S^{-1}\Lambda A\Delta\hat{x}$. The focus in the manuscript will mainly be on the unreduced block 3-by-3 system (3). However, analogous reductions of the modified Newton system similar to those of (6) and (7) will also be discussed.

To improve efficiency, many methods seek approximate solutions of $F_\mu(z) = 0$, for each μ. There are different strategies to update μ, e.g., dynamically every iteration or to keep μ fixed until sufficient decrease of a merit function is achieved. Herein, our model method uses the latter. In particular, our model method is similar to the basic interior-point method of Algorithm 19.1 in Nocedal and Wright [21, Ch. 19, p. 567]. However, termination and reduction of μ are based on the merit function $\phi_\mu(x) = \|F_\mu(z)\|$. Similarly, in the theoretical framework we consider the basic condition $\|F_\mu(z)\| \le C\mu$, for some constant $C > 0$, see, e.g., [21, Ch. 17, p. 572]. The additional assumption that all vectors z satisfy $s > 0$ and $\lambda > 0$ is made throughout.

In the remaining part of this section we give some definitions and provide the details for the theoretical framework.

Definition 1 (*Order-notation*) Let $\alpha, \gamma \in \mathbb{R}$ be two positive related quantities. If there exists a constant $C_1 > 0$ such that $\gamma \ge C_1\alpha$ for sufficiently small α, then $\gamma = \Omega(\alpha)$. Similarly, if there exists a constant $C_2 > 0$ such that $\gamma \le C_2\alpha$ for sufficiently small α, then $\gamma = \mathcal{O}(\alpha)$. If there exist constants $C_1, C_2 > 0$ such that $C_1\alpha \le \gamma \le C_2\alpha$ for sufficiently small α, then $\gamma = \Theta(\alpha)$.

Definition 2 (*Neighborhood*) For a given $\delta > 0$, let the neighborhood around z^* be defined by $\mathcal{B}(z^*, \delta) = \{z : \|z - z^*\| < \delta\}$.

Assumption 1 (*Strict local minimizer*) The vector z^* satisfies (1), i.e., second-order sufficient optimality conditions, strict complementarity and regularity hold.

The first of the following two lemmas provides the existence of a neighborhood where the Jacobian is nonsingular. The second lemma gives the existence of a Lipschitz continuous barrier trajectory z^μ in the neighborhood where the Jacobian is nonsingular. The results are well known and can be found in e.g., Ortega and Rheinboldt [22]. See also Byrd, Liu and Nocedal [23] for the corresponding results in a setting similar to the one considered here.

Lemma 1 *Under Assumption 1 there exists $\delta > 0$ such that $F'(z)$ is continuous and nonsingular for $z \in \mathcal{B}(z^*, \delta)$ and*

$$\|F'(z)^{-1}\| \leq M,$$

for some constant $M > 0$.

Proof See [22, p. 46]. □

Lemma 2 *Let Assumption 1 hold and let $\mathcal{B}(z^*, \delta)$ be defined by Lemma 1. Then there exists $\hat{\mu} > 0$ and a Lipschitz continuous function $z^\mu : (0,\ \hat{\mu}] \rightarrow \mathcal{B}(z^*, \delta)$ that satisfies $F_\mu(z^\mu) = 0$ and*

$$\left\| z^\mu - z^* \right\| \leq C_3 \mu,$$

where $C_3 = \sup_{z \in \mathcal{B}(z^*, \delta)} \|F'(z)^{-1} \frac{\partial F_\mu(z)}{\partial \mu}\|$.

Proof The result follows from the implicit function theorem, see e.g., [22, p. 128]. □

The following lemma provides a relation between the distance of vectors z to the barrier trajectory and the quantity $\|F_\mu(z)\|$, when the distance is sufficiently small. A corresponding result is also given by Byrd et al. [23].

Lemma 3 *Under Assumption 1, let $\mathcal{B}(z^*, \delta)$ and $\hat{\mu}$ be defined by Lemmas 1 and 2 respectively. For $0 < \mu \leq \hat{\mu}$ and z sufficiently close to $z^\mu \in \mathcal{B}(z^*, \delta)$ there exist constants $C_4, C_5 > 0$ such that*

$$C_4 \left\| z - z^\mu \right\| \leq \|F_\mu(z)\| \leq C_5 \left\| z - z^\mu \right\|.$$

Proof See [23, p. 43]. □

The next lemma provides a bound on the Newton direction, $\Delta\hat{z}$, for z sufficiently close to the barrier trajectory.

Lemma 4 *Under Assumption 1, let $\mathcal{B}(z^*, \delta)$ and $\hat{\mu}$ be defined by Lemmas 1 and 2 respectively. For $0 < \mu \leq \hat{\mu}$ and $z \in \mathcal{B}(z^*, \delta)$, let $\Delta\hat{z}$ be the solution of (3) with $\mu^+ = \sigma\mu$, where $0 < \sigma < 1$. If z is sufficiently close to $z^\mu \in \mathcal{B}(z^*, \delta)$ such that $\|F_\mu(z)\| = \mathcal{O}(\mu)$ then*

$$\left\| \Delta\hat{z} \right\| = \mathcal{O}(\mu).$$

Proof Analogous to [19, Lemma 5]. □

3 A structured modified Newton approach

In order to describe the approach and its ideas, we first consider a simple setting with one iteration. For a given $\mu > 0$, consider the interior-point iterate $z^+ \in \mathcal{B}(z^*, \delta)$

defined by $z^+ = z + \Delta\hat{z}$, where $z \in \mathcal{B}(z^*, \delta)$ and $\Delta\hat{z}$ satisfies (3) with $\mu^+ = \sigma\mu$, $0 < \sigma < 1$. Since $\Delta\hat{z}$ has been computed with (3) we assume that a factorization of $F'(z)$ is known. Instead of performing another Newton step $\Delta\hat{z}^+$ at z^+ for some $\mu^{++} = \sigma^+\mu^+$, $0 < \sigma^+ \leq 1$, which requires the solution of (3) with μ^{++} and z^+, we would like to compute an approximate solution, which is computationally less expensive, from

$$B^+\Delta z^+ = -F_{\mu^{++}}(z^+), \quad \text{where } B^+ = F'(z) + U, \tag{8}$$

and U is some low-rank update matrix. A natural question is then how to choose the update matrix U. Gondzio and Sobral [6] consider rank-1 update matrices such that the distance, in Frobenius norm, between B^+ and the previous Jacobian approximation is minimized, when B^+ in addition satisfies the secant condition. They show that the sparsity pattern of the first two block rows is maintained, however the sparsity pattern of the third row block row is typically lost. In contrast, our strategy is, for a given rank restriction r on U, to choose U such that the distance, in both 2-norm and Frobenius norm, between B^+ and the actual Jacobian $F'(z^+)$ is minimized. The sparsity of the Jacobian is maintained, however there is no requirement for B^+ to fulfill the secant condition.

To further support the choice of update matrix we give some additional theoretical results. First, we show that there is a region where the modified Newton approach produces small errors with respect to the Newton direction. In particular, a region that depends on μ where the modified Newton direction approaches the Newton direction as $\mu \to 0$. Later, we also discuss general errors, descent directions with respect to our merit function $\phi_\mu(z)$ and conditions for local convergence.

The error of using the modified Jacobian B^+ of (8) is

$$E^+ = F'(z^+) - B^+ = \Delta F'(\Delta\hat{z}) - U. \tag{9}$$

Given a rank restriction r, $0 \leq r \leq m$, on U, the Eckart–Young–Mirsky theorem gives the update matrix U that minimizes the Jacobian error E^+, in terms of the measure $\|.\|_2$ and $\|.\|_F$. In Proposition 1 below, we give an expression for U and show that the resulting modified Jacobian may be viewed as a Jacobian evaluated at a point $\bar{z}^+ = (\bar{x}^+, \bar{\lambda}^+, \bar{s}^+)$.

Proposition 1 *For $z = (x, \lambda, s)$ and $\Delta z = (\Delta x, \Delta\lambda, \Delta s)$, let $F'(z)$ and $\Delta F'(\Delta z)$ be defined by (4) and (5) respectively, and let $z^+ = z + \Delta z$. For a given rank r, $0 \leq r \leq m$, let $\mathcal{U}_r$ be the set of indices corresponding to the r largest quantities of $\sqrt{(\Delta\lambda_i)^2 + (\Delta s_i)^2}$, $i = 1, \ldots, m$. The optimal solution of*

$$\begin{aligned} \underset{U \in \mathbb{R}^{(2m+n)\times(2m+n)}}{\text{minimize}} \quad & \|F'(z^+) - B^+\| \\ \text{subject to} \quad & B^+ = F'(z) + U, \\ & \text{rank}\,(U) \leq r, \end{aligned}$$

where $\|.\|$ is either of type $\|.\|_2$ or $\|.\|_F$, is

$$U_* = \sum_{i \in \mathcal{U}_r} e_{n+m+i} \left((s_i^+ - s_i) e_{n+i} + (\lambda_i^+ - \lambda_i) e_{m+n+i} \right)^T .$$

In consequence, it holds that

$$B^+ = F'(\bar{z}^+), \text{ with } (\bar{x}_i^+, \bar{\lambda}_i^+, \bar{s}_i^+) = \begin{cases} (x_i^+, \lambda_i^+, s_i^+) & i \in \mathcal{U}_r, \\ (x_i^+, \lambda_i, s_i) & i \in \{1, \dots, m\} \setminus \mathcal{U}_r. \end{cases}$$

Proof Note that $\|F'(z^+) - B^+\| = \|\Delta F'(\Delta z) - U\| = \|E^+\|$ by (9). The result then follows from the Eckart–Young–Mirsky theorem, stated in Theorem 5, together with Lemma 6. The last part of the proposition follows directly from performing the update. □

Proposition 1 shows that each rank-1 term of the sum in U_* added to $F'(z)$ is equivalent to the update of one component-pair, (λ, s), in the Λ and S blocks of the Jacobian. The essence is that adding the rank-r update matrix U_* to $F'(z)$ is equivalent to updating pairs (λ_i, s_i) to $\left(\lambda_i^+, s_i^+\right)$, $i \in \mathcal{U}_r$, and that the modified Jacobian at z^+ may be viewed as a Jacobian evaluated at $\bar{z}^+$. In particular, $r = m$ gives $\bar{z}^+ = z^+$ and $B^+ = F'(z^+)$.

Before we give the analogous result of Proposition 1 in a more general framework, we show that there exists a region where the modified Newton approach may be started without causing large errors in the search direction. In particular, we give a bound on the search direction error $\|\Delta \hat{z}^+ - \Delta z^+\|$, where Δz^+ satisfies (8) with update matrix U_* of rank r as given in Proposition 1. In the derivation, the inverse of B^+ is expressed as a Neumann series which requires $\rho(F'(z^+)^{-1} E^+) < 1$. We first show that among U such that $\text{rank}(U) \le r$, U_* is sound in regard to the reduction of an upper bound of $\rho(F'(z^+)^{-1} E^+)$. Thereafter, in Lemma 5 we show that, for iterates z sufficiently close to the barrier trajectory, the quantity $\|F'(z^+)^{-1} E^+\|$, and consequently also $\rho(F'(z^+)^{-1} E^+)$, is bounded above by a constant times μ. This gives the existence of a region, that depends on the barrier parameter, where $\rho(F'(z^+)^{-1} E^+) < 1$.

By assumption $z^+ \in \mathcal{B}(z^*, \delta)$, hence by Lemma 1 there exists a constant $M > 0$ such that $\|F'(z^+)^{-1}\| \le M$, and it holds that

$$\rho(F'(z^+)^{-1} E^+) \le \|F'(z^+)^{-1} E^+\| \le M \sigma_{max}(\Delta F'(\Delta \hat{z}) - U). \tag{10}$$

Lemma 6 shows that the singular values of $\Delta F'(\Delta \hat{z})$ are given by $\sqrt{(\Delta \hat{\lambda}_i)^2 + (\Delta \hat{s}_i)^2}$, $i = 1, \dots, m$. The largest reduction of the upper bound in (10), among U such that $\text{rank}(U) \le r$, is achieved with the rank-r update matrix U_* of Proposition 1, which gives

$$\begin{aligned} \rho(F'(z^+)^{-1} E^+) \le \|F'(z^+)^{-1} E^+\| &\le M \max_{i=r+1,\dots,m} \sqrt{(\Delta \hat{\lambda}_i)^2 + (\Delta \hat{s}_i)^2} \\ &= M \sqrt{(\Delta \hat{\lambda}_{r+1})^2 + (\Delta \hat{s}_{r+1})^2}, \end{aligned} \tag{11}$$

where the indices $i = 1, \dots, m$ are ordered such that $\sqrt{(\Delta\hat{\lambda}_i)^2 + (\Delta\hat{s}_i)^2}$ are in descending order. Thus supporting the choice of update matrix in regard to the reduction of the upper bound of the spectral radius, and 2-norm, of $F'(z^+)^{-1}E^+$.

Lemma 5 *Under Assumption 1, let $\mathcal{B}(z^*, \delta)$ and $\hat{\mu}$ be defined by Lemmas 1 and 2 respectively. For $0 < \mu \le \hat{\mu}$ and $z \in \mathcal{B}(z^*, \delta)$, define $z^+ = z + \Delta\hat{z}$, where $\Delta\hat{z}$ is the solution of (3) with $\mu^+ = \sigma\mu$, $0 < \sigma < 1$. Moreover, let $E^+ = \Delta F'(\Delta\hat{z}) - U_*$ with U_* defined as the rank-r, $0 \le r < m$, update matrix U_* of Proposition 1. If z is sufficiently close to $z^\mu \in \mathcal{B}(z^*, \delta)$ such that $\|F_\mu(z)\| = \mathcal{O}(\mu)$ and $z^+ \in \mathcal{B}(z^*, \delta)$, then*

$$\|F'(z^+)^{-1}E^+\| \le MC^{(r+1)}\mu, \tag{12}$$

where M is defined by Lemma 1 and $C^{(r+1)} > 0$ is a constant such that $\sqrt{(\Delta\hat{\lambda}_{r+1})^2 + (\Delta\hat{s}_{r+1})^2} \le C^{(r+1)}\mu$ with $\sqrt{(\Delta\hat{\lambda}_i)^2 + (\Delta\hat{s}_i)^2}$, $i = 1, \dots, m$, ordered in descending order. In addition, $C^{(r+1)}$ decreases as r increases.

Proof The point z and direction $\Delta\hat{z}$ satisfy the conditions of Lemma 4, hence there exists a constant $C > 0$ such that $\|\Delta\hat{z}\| \le C\mu$. In consequence, there exist constants $C^{(i)} > 0$, $i = 1, \dots, m$, such that

$$\sqrt{(\Delta\hat{\lambda}_i)^2 + (\Delta\hat{s}_i)^2} \le C^{(i)}\mu, \quad i = 1, \dots, m. \tag{13}$$

If in addition, $\sqrt{(\Delta\hat{\lambda}_i)^2 + (\Delta\hat{s}_i)^2}$, $i = 1, \dots, m$, are ordered in descending order, then $C^{(i)}$, $i = 1, \dots, m$, may be chosen such that $C^{(1)} \ge \cdots \ge C^{(m)}$. A combination of (11) and (13) gives the result. □

The bound in (12) of Lemma 5 shows that $\|F'(z^+)^{-1}E^+\|$, and by (10) also $\rho(F'(z)^{-1}E)$, will be less than unity for sufficiently small μ. Indeed, this is also true when U is a zero matrix, i.e., for a simplified Newton strategy. The derivation of the result of Lemma 5 utilizes that, for a given rank restriction r, U_* of Proposition 1 is the update matrix that gives the tightest bound in (11). In consequence, U_* is also the rank-r update matrix that gives the tightest upper bound in the result of the lemma, with our analysis. Moreover, $C^{(1)} \ge \cdots \ge C^{(m)}$, and consequently $C^{(r+1)}$ decreases with increasing r. In addition, (12) provides an explicit sufficient condition on μ, depending on M, or $\|F'(z^+)^{-1}\|$, and $C^{(r+1)}$, for $\rho(F'(z^+)^{-1}E^+) < 1$.

Next we give a bound on the search direction error at z^+ with the modified Newton equation (8) relative to the Newton equation (3) with μ^{++}. It is shown that the error is bounded by a constant times μ^3 when $\mu^{++} = \mu^+$ and a constant times μ^2 when $\mu^{++} < \mu^+$. As may be anticipated, the bound is tighter when μ is not decreased in the corresponding iteration.

Theorem 2 *Under Assumption 1, let $\mathcal{B}(z^*, \delta)$, M, C_3, $\hat{\mu}$, be defined by Lemmas 1 and 2. For $0 < \mu \le \hat{\mu}$, assume that $z \in \mathcal{B}(z^*, \delta)$ is sufficiently close to $z^\mu \in \mathcal{B}(z^*, \delta)$ such that $\|F_\mu(z)\| = \mathcal{O}(\mu)$. Define $z^+ = z + \Delta\hat{z}$ where $\Delta\hat{z}$ is the solution of (3) with*

$\mu^+ = \sigma\mu$, $0 < \sigma < 1$. Moreover, let Δz^+ be defined by (8) with $\mu^{++} = \sigma^+\mu^+$, $0 < \sigma^+ \leq 1$, and U as the rank-r, $0 \leq r < m$, update matrix U_* of Proposition 1. If $z^+ \in \mathcal{B}(z^*, \delta)$, then there exists $\bar{\mu}$, $0 < \bar{\mu} \leq \hat{\mu}$, such that for $0 < \mu \leq \bar{\mu}$

$$\|\Delta\hat{z}^+ - \Delta z^+\| \leq \frac{MC^{(r+1)}}{1 - MC^{(r+1)}\mu}\big(C_3(1-\sigma^+)\sigma\mu^2 + \mathcal{O}(\mu^3)\big), \tag{14}$$

where $\Delta\hat{z}^+$ is the Newton step at z^+, given by $F'(z^+)\Delta\hat{z}^+ = -F_{\mu^{++}}(z^+)$, and $C^{(r+1)} > 0$ is such that $\sqrt{(\Delta\hat{\lambda}_{r+1})^2 + (\Delta\hat{s}_{r+1})^2} \leq C^{(r+1)}\mu$ with $\sqrt{(\Delta\hat{\lambda}_i)^2 + (\Delta\hat{s}_i)^2}$, $i = 1, \ldots, m$, ordered in descending order. In addition, $C^{(r+1)}$ decreases as r increases.

Proof See Appendix A. □

Similarly as for Lemma 5, the result of Theorem 2 is also valid for U as a zero matrix. The essence is again that, among update matrices U such that $\text{rank}(U) \leq r$, the rank-r update matrix U_* of Proposition 1 is the matrix that provides the tightest bound on (14) with our analysis. As mentioned, U_* is also the matrix that gives the upper bound in (12), and in addition, $C^{(r+1)}$ decreases with increasing r. Consequently, the bound in (14) decreases with increasing r, and larger values of r may thus also increase the region where the result of Theorem 2 is valid, i.e., the region where the proposed modified Newton approach may be a viable alternative.

3.1 At a general iteration *k*

In this section we give a result analogous to Proposition 1, at iteration k, $k \geq 1$, in a damped modified Newton setting. Consider the sequence $\{z^i\}_{i=0}^k$ generated by $z^{i+1} = z^i + \alpha^i\Delta z^i$, $i = 0, \ldots, k-1$, where α^i is the step size. Suppose that each Δz^i satisfies

$$B^i\Delta z^i = -F_{\mu^i}(z^i), \quad \text{with } B^i = \begin{cases} F'(z^0) & i = 0, \\ B^{i-1} + U^i & i = 1, \ldots, k-1, \end{cases} \tag{15}$$

for some $\mu^i > 0$ and update matrix U^i of rank r^i. If at $k = 1$, for a given rank r^k, the update matrix is chosen as the optimal solution of the optimization problem in Proposition 1, then $B^1 = F'(\bar{z}^1)$, for some $\bar{z}^1$. Inductively, at an iteration k, $k \geq 1$, for a given $\bar{z}^{k-1}$ and rank r^k, $0 \leq r^k \leq m$, we wish to choose U^k as the optimal solution of

$$\begin{aligned} \underset{U \in \mathbb{R}^{(2m+n)\times(2m+n)}}{\text{minimize}} \quad & \|F'(z^k) - B^k\| \\ \text{subject to} \quad & B^k = B^{k-1} + U, \quad B^{k-1} = F'(\bar{z}^{k-1}), \\ & \text{rank}(U) \leq r^k, \end{aligned} \tag{16}$$

where $\|.\|$ is either of type $\|.\|_2$ or $\|.\|_F$. The optimal solution of (16), the update of $\bar{z}^k$ from $\bar{z}^{k-1}$, and the resulting optimal B^k are shown in Proposition 3. This is analogous to the update from z^0 to $\bar{z}^1$ given in Proposition 1. The essence is that the rank-r^k update matrix, defined by the solution of (16), is equivalent to updating information corresponding to the r^k largest quantities $\sqrt{(\lambda_i^k - \bar{\lambda}_i^{k-1})^2 + (s_i^k - \bar{s}_i^{k-1})^2}$. In essence, the r^k largest deviations from the Newton step are corrected, and $r^k = m$ gives $B^k = F'(z^k)$.

Proposition 3 *At iteration k, $k \geq 1$, for given vectors z^k, $\bar{z}^{k-1}$ and rank r^k, $0 \leq r^k \leq m$, consider optimization problem (16). The optimal solution of (16) is*

$$U_*^k = \sum_{i \in \mathcal{U}_{r^k}} e_{n+m+i} \left((s_i^k - \bar{s}_i^{k-1}) e_{n+i} + (\lambda_i^k - \bar{\lambda}_i^{k-1}) e_{m+n+i} \right)^T,$$

where $\mathcal{U}_{r^k}$ is the set of indices corresponding to the r^k largest quantities of $\sqrt{(\lambda_i^k - \bar{\lambda}_i^{k-1})^2 + (s_i^k - \bar{s}_i^{k-1})^2}$, $i = 1, \ldots, m$. In consequence, it holds that

$$B^k = F'(\bar{z}^k), \text{ with } \bar{z}^k = (\bar{x}^k, \bar{\lambda}^k, \bar{s}^k) = \begin{cases} (x_i^k, \lambda_i^k, s_i^k) & i \in \mathcal{U}_{r^k}, \\ (x_i^k, \bar{\lambda}_i^{k-1}, \bar{s}_i^{k-1}) & i \in \{1, \ldots, m\} \setminus \mathcal{U}_{r^k}. \end{cases}$$

Proof The proof is analogous to that of Proposition 1 with $z = \bar{z}^{k-1}$, $z^+ = z^k$ and $\Delta z = z^k - \bar{z}^{k-1}$. □

In the described approach, U_*^k of Proposition 3 gives $B^k = F'(\bar{z}^k)$ for some $\bar{z}^k$. A direct consequence is that iterates which become primal–dual feasible, i.e., satisfies the first two block equations of (3), will remain so. Moreover, at iteration k, the error of using the modified Jacobian is

$$E^k = F'(z^k) - B^k = F'(z^k) - (B^{k-1} + U_*^k) = F'(z^k) - F'(\bar{z}^k), \tag{17}$$

and it holds that

$$\|E^k\|_2 \leq \|E^k\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |E_{ij}^k|^2} = \sqrt{\sum_{i=1}^{m} \left((s_i^k - \bar{s}_i^k)^2 + (\lambda_i^k - \bar{\lambda}_i^k)^2 \right)} = \|z^k - \bar{z}^k\|_2,$$

since $x^k = \bar{x}^k$. Hence it follows that

$$\|F'(z^k) - F'(\bar{z}^k)\|_2 \leq \|F'(z^k) - F'(\bar{z}^k)\|_F = \|z^k - \bar{z}^k\|_2. \tag{18}$$

In fact, for any z and $\tilde{z}$ it holds that

$$\|F'(z) - F'(\tilde{z})\|_2 \leq \|F'(z) - F'(\tilde{z})\|_F \leq \|z - \tilde{z}\|_2,$$

which implies that the Lipschitz constant of F' may be chosen as one. Recall that, among the update matrices U^k such that $\text{rank}(U^k) \leq r^k$, U_*^k of Proposition 3 is the update matrix that minimizes $\|F'(z^k) - F'(\bar{z}^k)\|_F$. Consequently, by (18), U_*^k is also the update matrix that minimizes $\|z^k - \bar{z}^k\|_2$.

Next we show that the update matrix U_*^k of Proposition 3 is sound with respect to reducing an upper bound of the relative error of the search direction Δz^k. At iteration k, $k \geq 1$, Δz^k satisfies (15) which equivalently can be written as

$$(F'(z^k) - E^k)(\Delta \hat{z}^k - \epsilon^k) = -F_{\mu^k}(z^k), \quad \epsilon^k = \Delta \hat{z}^k - \Delta z^k,$$

where $\Delta \hat{z}^k$ satisfies $F'(z^k)\Delta \hat{z}^k = -F_{\mu^k}(z^k)$. Standard perturbation analysis gives

$$\frac{\|\Delta \hat{z}^k - \Delta z^k\|}{\|\Delta z^k\|} \leq \|F'(z^k)^{-1}\|\|E^k\|. \tag{19}$$

Given the restrictions on the update, U_*^k of Proposition 3 is the update that minimizes $\|E^k\|$, and in consequence gives the largest reduction in the upper bound on the relative error (19).

3.2 Convergence

In this section we discuss convergence towards the barrier trajectory, i.e., convergence of the inner loop of Algorithm 1. We first give a condition for descent direction with respect to our merit function, as our setting is compatible with linesearch strategies. Thereafter, results are given in a setting where unit steps are assumed to be tractable. We give conditions on B^k, and thus r^k, so that the modified Newton approach converges locally. The theoretical setting is here widened slightly from the previous sections, in that we wish to quantify the effects of the modified Newton approach also for larger values of μ. For a given $\mu > 0$, iterate z and vector Δz, consider the merit function and the univariate function

$$\phi_\mu(z) = \|F_\mu(z)\|, \text{ and } \varphi_\mu(\alpha) = \|F_\mu(z + \alpha \Delta z)\|, \text{ respectively.}$$

The directional derivative is then

$$\nabla \phi_\mu(z)^T \Delta z = \varphi'_\mu(\alpha)|_{\alpha=0} = \frac{d}{d\alpha}\|F_\mu(z + \alpha \Delta z)\||_{\alpha=0} = \frac{\Delta z^T F'(z)^T F_\mu(z)}{\|F_\mu(z)\|}.$$

At iteration k, $k \geq 0$, z^k and Δz^k in the modified Newton approach satisfies $F'(\bar{z}^k)\Delta z^k = -F_\mu(z^k)$, for some $\bar{z}^k$. If $F'(\bar{z}^k)$ is nonsingular then the directional derivative is

$$\begin{aligned}\nabla \phi_\mu(z^k)^T \Delta z^k &= -\frac{1}{\|F_\mu(z^k)\|} F_\mu(z^k)^T F'(z^k) F'(\bar{z}^k)^{-1} F_\mu(z^k) \\ &= -\|F_\mu(z^k)\| - \frac{1}{\|F_\mu(z^k)\|} F_\mu(z^k)^T E^k F'(\bar{z}^k)^{-1} F_\mu(z^k),\end{aligned} \tag{20}$$

with E^k as in (17). From (20) it follows that Δz^k is a descent direction with respect to ϕ_μ if

$$\|F_\mu(z^k)\|^2 > -F_\mu(z^k)^T E^k F'(\bar{z}^k)^{-1} F_\mu(z^k). \tag{21}$$

Under the restrictions of the update, the rank-r^k matrix U_*^k of Proposition 3 is chosen such that $\|E^k\|$ is minimized. In addition, $\|E^k\| = 0$ for $r^k = m$. A descent direction can hence always be ensured for r^k sufficiently large. Moreover, in our theoretical setting, Lemma 5 gives the existence of a region, that depends on μ, where the modified Newton approach may be initiated so that $\|E^k\|$ is sufficiently small for (21) to hold, even when $r^k = 0$. However, the essence is again that U_*^k gives the largest reduction of $\|E^k\|$, among U^k such that $\text{rank}(U^k) \le r^k$.

Next we give a result on local convergence and discuss the modified Newton approach in the framework of inexact Newton methods. Note that the local convergence result in the proposition shows balance between quadratic rate of convergence, which would follow if $B^k = F'(z^k)$, i.e., $C_6 = 0$, and $\|F'(z^k)^{-1}\| \le C_7$ for all k, and linear rate of convergence, which would follow when B^k differs from $F'(z^k)$.

Proposition 4 *For a given $\mu > 0$, assume that z^μ exists. At an iteration k_0, consider the sequence of iterates generated by $z^{k+1} = z^k + \Delta z^k$, $k = k_0, k_0 + 1, \dots,$ where each Δz^k satisfies (15), with $\mu^k = \mu$ and update matrix of rank-r^k, $0 \le r^k \le m$, given by U_*^k of Proposition 3. Assume that at each iteration k, $(B^k)^{-1}$ exists and that r^k is chosen such that for all k, $\|(B^k)^{-1}\|\|F'(z^k) - B^k\| \le C_6$ for some $C_6 < 1$. If in addition, there is a C_7 such that for all k, $\|(B^k)^{-1}\| \le C_7$, then it holds that*

$$\|z^{k+1} - z^\mu\| \le \frac{C_7}{2}\|z^k - z^\mu\|^2 + C_6\|z^k - z^\mu\|, \tag{22}$$

so that z^k converges to z^μ if

$$\|z^{k_0} - z^\mu\| \le \frac{1 - C_6}{C_7}. \tag{23}$$

Proof Under the conditions of the proposition, Δz^k and z^k satisfy $B^k \Delta z^k = -F_\mu(z^k)$. At iteration $k+1$ the error may be written as

$$\begin{aligned} z^{k+1} - z^\mu &= z^k - (B^k)^{-1} F_\mu(z^k) - z^\mu \\ &= (B^k)^{-1}\big(F_\mu(z^\mu) - F_\mu(z^k) - F'(z^k)(z^\mu - z^k)\big) \\ &\quad - (B^k)^{-1}\big(B^k - F'(z^k)\big)(z^\mu - z^k), \end{aligned}$$

where $(B^k)^{-1}F'(z^k)(z^\mu - z^k)$ has been added and subtracted in the second equality. Taking 2-norm while considering Lipschitz continuity of F', see end of Sect. 3.1, and norm inequalities give

$$\|z^{k+1} - z^\mu\| \le \frac{\|(B^k)^{-1}\|}{2}\|z^k - z^\mu\|^2 + \|(B^k)^{-1}\|\|B^k - F'(z^k)\|\|z^k - z^\mu\|. \quad (24)$$

Insertion of the assumed C_6 and C_7 into (24) gives (22). Finally, if $\|z^k - z^\mu\| \le (1 - C_6)/C_7$, then (22) gives

$$\begin{aligned}\|z^{k+1} - z^\mu\| &\le \frac{C_7}{2}\|z^k - z^\mu\|^2 + C_6\|z^k - z^\mu\| \le \left(\frac{1 - C_6}{2} + C_6\right)\|z^k - z^\mu\| \\ &\le \frac{1 + C_6}{2}\|z^k - z^\mu\|,\end{aligned}$$

so that z^{k_0} satisfying (23) converges to z^μ, as $C_6 < 1$. □

Conditions for local convergence may also be obtained when the modified Newton approach is interpreted in the context of inexact Newton methods [24]. For a given $\mu > 0$, such steps may be viewed on the form

$$F'(z^k)\Delta z^k = -F_\mu(z^k) + q^k, \text{ where } \|q^k\|/\|F_\mu(z^k)\| \le \eta^k. \quad (25)$$

The sequence of iterates $z^k + \Delta z^k$ converges to z^μ, with at least linear rate, for z^0 sufficiently close to z^μ if $\eta^k < 1$ uniformly. Given that the iterates converge, the convergence is superlinear if and only if $\|q^k\| = o(\|F_\mu(z^k)\|)$, as $k \to \infty$.

The modified Newton approach can be put onto the form of (25) under the assumption that $I - E^k F'(z^k)^{-1}$ is nonsingular, where E^k is given by (17). Nonsingularity may be ensured with an update matrix U_*^k of Proposition 3 of sufficiently large rank r^k, $0 \le r^k \le m$, or by starting the modified Newton approach when μ is sufficiently small, as shown by Lemma 5. A straightforward calculation shows that (15) at iteration k, with $\mu^k = \mu$, can be written as

$$F'(z^k)\Delta z^k = -F_\mu(z^k) + \left(I - (I - E^k F'(z^k)^{-1})^{-1}\right)F_\mu(z^k). \quad (26)$$

Identification of terms in (25) and (26) gives

$$q^k = \left(I - (I - E^k F'(z^k)^{-1})^{-1}\right)F_\mu(z^k).$$

If in addition $\|E^k F'(z^k)^{-1}\| < 1$, then $q^k = \sum_{j=1}^{\infty}(E^k F'(z^k)^{-1})^j F_\mu(z^k)$. Norm inequalities and standard geometric series results give

$$\|q^k\| \le \frac{\|E^k F'(z^k)^{-1}\|}{1 - \|E^k F'(z^k)^{-1}\|}\|F_\mu(z^k)\|.$$

Local convergence towards the barrier trajectory follows if, at each iteration k, r^k of Proposition 3 is chosen such that $\|E^k F'(z^k)^{-1}\|/(1 - \|E^k F'(z^k)^{-1}\|) < 1$ uniformly. Moreover, the convergence is superlinear if in addition r^k is chosen such that $\|E^k F'(z^k)^{-1}\|/(1 - \|E^k F'(z^k)^{-1}\|) \to 0$ as $k \to \infty$.

Similarly, the modified Newton approach may also be viewed in the framework of inexact interior-point methods in order to study conditions for global convergence, see e.g., [25–27]. However, our analysis have only given further technical conditions which are outside the scope of this initial work. Instead, we have chosen to limit our focus to basic supporting results and to the study of practical performance with update matrices of low-rank at larger values of μ.

3.3 Reduced systems

The ideas presented so far have been on the unreduced unsymmetric block 3-by-3 system (3). In this section we describe the corresponding reduced and condensed system which are similar to those of (6) and (7).

In essence, the system of linear equations to be solved for each iterate z in the modified Newton approach takes the form

$$F'(\bar{z})\Delta z = -F_\mu(z), \tag{27}$$

for some $\bar{z}$. System (27) may be reformulated on the reduced form

$$\begin{pmatrix} H & A^T \\ A & -\bar{\Lambda}^{-1}\bar{S} \end{pmatrix} \begin{pmatrix} \Delta x \\ -\Delta\lambda \end{pmatrix} = -\begin{pmatrix} Hx + c - A^T\lambda \\ Ax - b - s + \bar{\Lambda}^{-1}(\Lambda S e - \mu e) \end{pmatrix}, \tag{28}$$

together with $\Delta s = -\bar{\Lambda}^{-1}(\Lambda S e - \mu e) - \bar{\Lambda}^{-1}\bar{S}\Delta\lambda$. Schur complement reduction of $\bar{\Lambda}^{-1}\bar{S}$ in (28) gives the condensed form

$$\begin{aligned}(H + A^T\bar{S}^{-1}\bar{\Lambda}A)\Delta x = &-(Hx + c - A^T\lambda) \\ &- A^T\bar{S}^{-1}(\bar{\Lambda}(Ax - b - s) + \Lambda S e - \mu e),\end{aligned} \tag{29}$$

together with $\Delta\lambda = -\bar{S}^{-1}(\bar{\Lambda}(Ax - b - s) + \Lambda S e - \mu e) - \bar{S}^{-1}\bar{\Lambda}A\Delta x$. As mentioned, the proposed rank-r update matrix of Proposition 3 is equivalent to updating r component pairs $(\bar{\lambda}, \bar{s})$. The change between iterations in the matrices of (28) and (29) is thus of rank r. In consequence, low-rank updates on the factorization of the matrix of (28), or (29), may also be considered.

3.4 Compatibility with previous work on interior-point methods

In order to have simple notation, we have chosen to formulate our problem on the form (IQP), with inequality constraints only. Analogous results hold for quadratic programs on standard form, as considered in [2, 6, 13, 15, 16]. However, when working with reduced systems, then the update will be on the diagonal of the H-matrix in the symmetric block 2-by-2 indefinite system.

Moreover, the proposed approach is also compatible with regularized methods for quadratic programming, e.g., [15–17], as long as the scaling of the regularization is not changed at iterations where the modified Jacobian is updated by a low-rank matrix. The scaling of the regularization may be changed at a refactorization step, e.g., on the form suggested in (34) of Sect. 4.

As each modified Jacobian may be viewed as a Jacobian evaluated at a different point, the modified Newton approach may also be interpreted in the framework of previous work on stability, effects of finite-precision arithmetic and spectral properties of the arising systems, e.g., [1–3, 5, 7–13].

4 Implementation

All numerical simulations were performed in `matlab` on benchmark problems from the repository of convex quadratic programming problems by Maros and Mészáros [18]. Many of these problems contain both linear equality and linear inequality constraints. However, in order not to complicate the description of the implementation with further technical details, we choose to give the description for problems on the same form as in previous sections. Note however that some of the parameters will depend on quantities related to the format of benchmark problems.

4.1 Basic algorithm

The aim is to study the fundamental behavior of the modified Newton approach as primal–dual interior-point methods converge. In particular, when each search direction is generated with a modified Newton equation on the form (15), with update matrix U_*^k of Proposition 3, relative to a Newton equation on the form (3). In order not to risk combining effects of the proposed approach with effects from other features in more advanced methods, we chose to implement the modified Newton approach in a simple interior-point framework. Moreover, all systems of linear equations were solved with `matlab`'s built in direct solver. No low-rank update of factorizations was used or implemented for the numerical tests. In consequence, the results are not dependent on the particular factorization or the procedure used to update the factors. For low-rank updates of matrix factorizations, see e.g., [28–31]. Our basic interior-point algorithm is similar to Algorithm 19.1 in Nocedal and Wright [21, Ch. 19, p. 567], however, termination and the update of μ are based on the merit function $\phi_\mu(z) = \|F_\mu(z)\|$.

Algorithm 1 Basic interior-point method (IQP).

$k \leftarrow 0,\ \mu^k \leftarrow$ inital μ, $\quad (x^k, \lambda^k, s^k) \leftarrow$ Point such that $\lambda^k > 0, s^k > 0$ and $\|F_{\mu^k/\sigma}(x^k, \lambda^k, s^k)\| < \mu^k/\sigma$.
while $\|F_0(z^k)\| > \varepsilon^{tol}$ **do**
  while $\|F_{\mu^k}(z^k)\| > \mu^k$ **do**
    $(\Delta x^k, \Delta\lambda^k, \Delta s^k) \quad \leftarrow$ search direction
    $(\alpha_P^k, \alpha_D^k) \quad \leftarrow \big(\min\{1, 0.98\alpha_P^{max,k}\}, \min\{1, 0.98\alpha_D^{max,k}\}\big)$
    $(x^{k+1}, \lambda^{k+1}, s^{k+1}) \quad \leftarrow (x^k + \alpha_P^k \Delta x^k, \lambda^k + \alpha_D^k \Delta\lambda^k, s^k + \alpha_P^k \Delta s^k)$
    $\mu^{k+1} \leftarrow \mu^k, \quad k \leftarrow k+1$
  end while
  $\mu^{k+1} \leftarrow \sigma\mu^k$
end while

In Algorithm 1 at iteration k, $\alpha_P^{max,k}$ and $\alpha_D^{max,k}$ are the maximum feasible step sizes for s^k along Δs^k and λ^k along $\Delta\lambda^k$ respectively.

Our reference method, which in all experiments is denoted by `Newton`, is defined by Algorithm 1 where the search direction at iteration k, $\Delta z^k = (\Delta x^k, \Delta\lambda^k, \Delta s^k)$, satisfies (3). The method, whose behavior we aim to study, is defined by Algorithm 1 where the search direction satisfies (15), with an update matrix of rank $r^k = r$ given by U_*^k of Proposition 3. This method is in the numerical experiments denoted by `mN-r`(r). Although the rank of the update matrices can be varied between the iterations, this initial study is limited to update matrices of constant rank in order to keep the comparisons clean.

4.2 Benchmark problems

Each problem was pre-processed and put on an equivalent form with n x-variables, m_{in} inequality constraints and m_{eq} equality constraints. The total number of variables in the primal–dual formulation is thus $N = n + m_{eq} + 2m_{in}$ variables, see Appendix A for a description and formulation of the systems that arise. A trivial equality constraint that fixed a variable at any of its bounds was removed from the problem along with the variable. A problem was accepted if $m_{in} \geq 4$ and, in addition, if `Newton` converged from a given initial solution. Due to the simplicity of `Newton` convergence was not achieved for some problems due to reasons as, non-trivial equality constraints fixing variables at its bounds, singular Jacobians caused by linearly dependent equality constraints, etc. Moreover, we were not able to run `CONT-300`, `BOYD1` and `BOYD2` due to memory restrictions. These conditions reduced the benchmark set, $\mathcal{P}$, to 90 problems (out of 138). The problems were divided into the three subsets: small, $\mathcal{S}$, medium, $\mathcal{M}$, and large, $\mathcal{L}$. The sets were defined as follows: $\mathcal{S} = \{p \in \mathcal{P} : N < 500\}$, $\mathcal{M} = \{p \in \mathcal{P} : 500 \leq N < 10000\}$ and $\mathcal{L} = \{p \in \mathcal{P} : N \geq 10000\}$. Consequently, $|\mathcal{S}| = 25$, $|\mathcal{M}| = 37$ and $|\mathcal{L}| = 28$. The specific problems of each group and details on their individual sizes can be found in Appendix A.

4.3 Heuristics

The theoretical results in Sect. 3 concern iterates sufficiently close to the trajectory for sufficiently small μ, or when the rank of each update matrix is sufficiently large. However, we are also interested in studying the behavior of the modified Newton approach beyond this setting. In particular, for larger vales of μ and when each update matrix is of low-rank. In this section, we discuss the behavior in these cases. In order to improve performance we also suggest two heuristics and a refactorization strategy. In essence, the heuristics allow for change of indices in the set $\mathcal{U}_{r^k}$ of Proposition 3, and the refactorization strategy limits the total rank change on an initial Jacobian.

Numerical experiments with mN-r(r), for small r, have shown that convergence may slow down due to small step sizes α_P^k and α_D^k. Small step sizes can be caused by few components in the modified Newton direction which differ considerably from those in the Newton direction. We first show some numerical evidence of this behavior and suggest a partial explanation on which we base two heuristics. The effectiveness of each heuristic is then illustrated, and finally a refactorization strategy is included in the modified Newton approach. Step sizes and convergence, in terms of the measure $\|F_\mu\|$ with $\mu = 0$, for Newton and mN-r(r) with $r = [0,\ 2,\ 4]$ are shown in the left-hand side of Fig. 1. The results are for benchmark problem qafiro with parameters $\mu^0 = 1, \sigma = 0.1$ and $\varepsilon^{tol} = 10^{-6}$. The right-hand side of the figure shows the inverse of the limiting step sizes and the relative error in the search direction at the iteration marked by the red circle of mN-r(2), hence large spikes imply small step sizes. Moreover, the figure only contains negative components of the modified Newton direction. The result for $r = 0$, i.e., simplified Newton, is given to illustrate that low-rank updates can indeed make a difference compared to a simplified Newton approach for which some of our theoretical results are still valid, although in a smaller region.

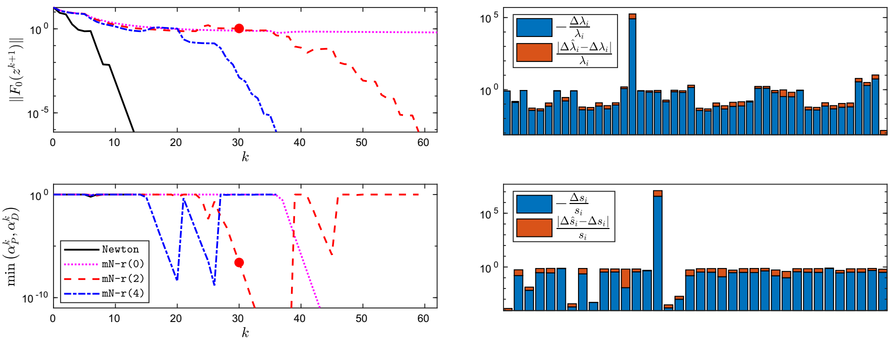

Fig. 1 The left-hand side shows step sizes and convergence on benchmark problem qafiro. The right-hand side shows the inverse of the limiting step sizes and the relative error in the search direction for negative components of the modified Newton direction in mN-r(2), at the iteration marked by the red circle

The results in the left-hand side of Fig. 1 indicate that convergence may slow down with the low-rank modified Newton approach due to small step sizes. The right-hand side of Fig. 1 suggests that small steps may be caused by large relative errors in certain components of the search direction. The results are similar to those shown by Gondzio and Sobral [6] for quasi-Newton approaches, hence indicating that the proposed modified Newton approach suffers from the same phenomenon as quasi-Newton approaches. In theory, zero steps are not harmful for the modified Newton approach, as long as the Newton step makes progress from this point, since after m/r iterations with zero steps the modified Jacobian will indeed be the Jacobian at that point. In practice however, close to zero steps have negative effects on the convergence. In consequence we would like to understand what causes these steps and how to avoid them.

The partial solution Δx of (27) satisfies (29). For z sufficiently close to z^*, (29) may be approximated by

$$(H + A_{\mathcal{A}}^T \bar{S}_{\mathcal{A}}^{-1} \bar{\Lambda}_{\mathcal{A}} A_{\mathcal{A}} + A_{\mathcal{I}}^T \bar{S}_{\mathcal{I}}^{-1} \bar{\Lambda}_{\mathcal{I}} A_{\mathcal{I}})\Delta x = -A_{\mathcal{A}}^T \bar{S}_{\mathcal{A}}^{-1}(\Lambda_{\mathcal{A}} S_{\mathcal{A}} e - \mu e) - A_{\mathcal{I}}^T \bar{S}_{\mathcal{I}}^{-1}(\Lambda_{\mathcal{I}} S_{\mathcal{I}} e - \mu e), \qquad (30)$$

where $\mathcal{A}$ and $\mathcal{I}$ are sets of indices corresponding to active and inactive constraints at the solution z^* respectively, i.e., $\mathcal{A} = \{i : s_i^* = 0,\ i = 1, \ldots, m.\}$ and $\mathcal{I} = \{i : \lambda_i^* = 0,\ i = 1, \ldots, m.\}$. If the modified Newton approach is initiated for small μ, or if the rank of each update matrix is sufficiently large, then $\bar{z} \approx z$. If in addition, $A_{\mathcal{A}}\Delta x$ is sufficiently large, i.e, Δx is not in or almost in the null-space of $A_{\mathcal{A}}$ (if it is then the the search direction will not cause limiting steps), then the dominating terms of (30) are

$$A_{\mathcal{A}}^T \bar{S}_{\mathcal{A}}^{-1} \bar{\Lambda}_{\mathcal{A}} A_{\mathcal{A}} \Delta x = -A_{\mathcal{A}}^T \bar{S}_{\mathcal{A}}^{-1}(\Lambda_{\mathcal{A}} S_{\mathcal{A}} e - \mu e). \qquad (31)$$

In essence, (31) is an approximation of (30) in the particular case. By our assumptions $A_{\mathcal{A}}$ has full row rank. Consequently, component-wise (31) gives

$$\frac{\bar{\lambda}_i}{\bar{s}_i} A_i \Delta x = \frac{\mu - \lambda_i s_i}{\bar{s}_i}, \qquad i \in \mathcal{A}, \qquad (32)$$

where A_i denotes the ith row of A. Equation (32) gives an approximate description of how each pair $(\bar{\lambda}_i, \bar{s}_i)$, $i \in \mathcal{A}$ affects $A_i \Delta x$, the inner product of the search direction Δx and the corresponding constraint A_i. This means that each pair $(\bar{\lambda}_i, \bar{s}_i)$, $i \in \mathcal{A}$, affects the angle between Δx and constraint A_i, and/or $\|\Delta x\|$. Both errors in angle and large $\|\Delta x\|$ may cause small step sizes. In the proposed modified Newton approach, depending on the rank of the update matrix, some of the factors in the modified Jacobian, i.e., some components $\bar{\lambda}_i/\bar{s}_i$, $i \in \mathcal{A}$, may contain information from previous iterates. The analysis above suggests that it may be important to update certain components-pairs (λ, s) in order to avoid limiting steps. Such pairs may not be updated with the matrix of Proposition 1 or Proposition 3.

In light of the discussion above and the results of Fig. 1, we construct two heuristics in an attempt to decrease negative effects on convergence caused by small step sizes. Both heuristics have an update matrix U^k analogous to the one given by Proposition 3, with

$$U^k = \sum_{i \in \mathcal{U}^k} e_{n+m+i} \left((s_i^k - \bar{s}_i^{k-1}) e_{n+i} + (\lambda_i^k - \bar{\lambda}_i^{k-1}) e_{m+n+i} \right)^T, \tag{33}$$

where $\mathcal{U}^k$ is an index set of cardinality r. However, not all indices in $\mathcal{U}^k$ are chosen according to the criteria of Proposition 3, so that $\mathcal{U}^k$ of the heuristics may differ from $\mathcal{U}_r$ of Proposition 3. (Note that $r^k = r$, for all k, in the numerical tests.) The first heuristic can have at most two indices that differ between $\mathcal{U}^k$ and $\mathcal{U}_r$, whereas the second is more flexible and can potentially change all r indices. We choose to replace indices instead of adding indices in order to obtain a fair comparison in the study of the heuristics.

Heuristic `H1`

The idea of the first heuristic is to ensure that information corresponding to component-pairs (λ, s) is updated if either limited the step size in the previous iteration. At iteration $k, k \geq 1, \mathcal{U}^k$ is based on $\mathcal{U}_r$ of Proposition 3, but the last one or two indices are replaced by

$$\hat{i}_1 = \underset{i:\Delta\lambda_i^{k-1}<0}{\operatorname{argmin}} \frac{\lambda_i^{k-1}}{-\Delta\lambda_i^{k-1}}, \quad \text{and} \quad \hat{i}_2 = \underset{i:\Delta s_i^{k-1}<0}{\operatorname{argmin}} \frac{s_i^{k-1}}{-\Delta s_i^{k-1}},$$

if $\min_{i:\Delta\lambda_i^{k-1}<0} \frac{\lambda_i^{k-1}}{-\Delta\lambda_i^{k-1}} < 1 \wedge \hat{i}_1 \notin \mathcal{U}_r$ and/or $\min_{i:\Delta s_i^{k-1}<0} \frac{s_i^{k-1}}{-\Delta s_i^{k-1}} < 1 \wedge \hat{i}_2 \notin \mathcal{U}_r$ respectively.

Heuristic `H2`

The principle of the second heuristic is based on the observation in the analysis above. Similarly as in `H1` the idea is to ensure that certain component-pairs (λ, s) are updated. In particular, the components with the largest relative error of the ratio in the left-hand of (32). However, the set of active constraints at the solution is unknown, instead all components which could have limited the step size in the previous iteration are considered in the selection. At iteration k, $k \geq 1, \mathcal{U}^k$ is based on $\mathcal{U}_r$ of Proposition 3, but at most r indices are replaced by the indices corresponding to the, at most, r largest quantities of

$$\frac{|\lambda_i^k / s_i^k - \bar{\lambda}_i^k / \bar{s}_i^k|}{\lambda_i^k / s_i^k}, \quad i \in \mathcal{H}^k$$

where $\mathcal{H}^k = \{i : \Delta\lambda_i^{k-1} < 0 \wedge \frac{\lambda_i^{k-1}}{-\Delta\lambda_i^{k-1}} < 1\} \cup \{i : \Delta s_i^{k-1} < 0 \wedge \frac{s_i^{k-1}}{-\Delta s_i^{k-1}} < 1\}$.

Heuristic test

To demonstrate the impact of heuristic `H1` and `H2` we show results in Fig. 2 which are analogous to those in the left-hand side of Fig. 1. The methods `mN-r(`r`)-H1` and mN-r(r)-H2 denote mN-r(r), $r = [2, 4]$, combined with heuristic H1 and H2 respectively. In addition, Table 1 shows the average of the sum $(\alpha_P^k + \alpha_D^k)/2$ for a subset of the benchmark problems. The subset contains problems from each of the sets $\mathcal{S}$, $\mathcal{M}$ and $\mathcal{L}$.

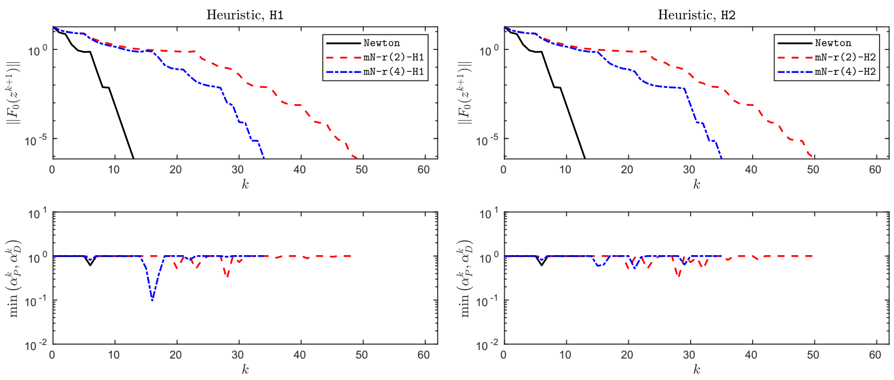

Fig. 2 Step sizes and convergence for mN-r(r), $r = [2, 4]$, combined with heuristic H1 and H2 on benchmark problem qafiro

The results of Fig. 2 show that mN-r(r)-H1 and mN-r(r)-H2, $r = [2, 4]$, use larger step sizes, and converges in fewer iterations, compared to mN-r(r) in Fig. 1. Hence showing that the heuristics H1 and H2 have the intended effect on benchmark problem qafiro.

Table 1 Average of the sum $(\alpha_P^k + \alpha_D^k)/2$ for a subset of the benchmark problems

	Newton	mN-r(2)			mN-r(4)		
			H1	H2		H1	H2
hs118	0.964	0.370	**0.844**	0.814	0.513	0.817	**0.818**
qafiro	0.986	0.762	**0.973**	0.966	0.846	0.957	**0.971**
primal1	0.965	0.677	0.945	**0.956**	0.817	0.936	**0.957**
dualc8	1.000	0.584	0.909	**0.911**	0.845	0.875	**0.916**
laser	0.994	0.072	**0.883**	0.723	0.111	0.865	**0.890**
yao	0.958	0.683	**0.859**	0.444	0.732	**0.753**	0.462
stcqp2	0.999	0.705	**0.971**	0.916	0.699	0.957	**0.976**
ubh1	1.000	0.695	0.995	**0.998**	0.699	0.995	**0.999**
aug2dqp	0.996	0.832	0.952	**0.970**	0.589	**0.997**	0.994

The results in Table 1 indicate that H1 and H2 have the intended effect on more benchmark problems but also that they are not effective on all problems. Part of the reason is that the rank of each update matrix is restricted to 2 and 4 respectively, but for some problems there are many components which limit the step size. For instance, H1 can at most take two of these and H2 can, at most, take as many as the maximum rank of the update. The results for problem yao is an example where H2 does worse than without heuristic. The heuristic replaced indices in $\mathcal{U}_r$ which caused low quality in the search direction, indicating that it may be beneficial to also update the information that is suggested in Proposition 3. Numerical experiments have further shown small step sizes can be avoided by allowing update matrices of varying rank. In particular, update matrices where the rank is determined by the components which potentially limit the step size. However, numerical experiments have also shown that avoiding small step sizes is not sufficient to obtain increased convergence speed. Similarly as in the results of Gondizo and Sobral for quasi-Newton approaches [6], numerical experiments have shown that it is occasionally important to use the Jacobian at z^k instead of $\bar{z}^k$ to improve convergence. In light of this, we will limit the number of allowed steps before the modified Jacobian is refactorized. In consequence the total rank change of an actual Jacobian will be limited. In the following numerical simulations, the modified Newton approaches mN-r(r), mN-r(r)-H1 and mN-r(r)-H2 include a refactorization strategy on the form

$$B^k = \begin{cases} F'(z^k) & k = 0, l+1, 2l+2, 3l+3, \dots, \\ B^{k-1} + U^k & k \neq 0, l+1, 2l+2, 3l+3, \dots, \end{cases} \tag{34}$$

where U^k is given by (33) for index sets $\mathcal{U}^k$ corresponding to H1, H2 for mN-r(r)-H1, mN-r(r)-H2, and $\mathcal{U}_r$ of Propositon 3 for mN-r(r), respectively.

In general, other refactorization strategies or dynamical procedures may be considered, e.g., instead of accepting all steps, refactor or increase the rank of the update matrix if a particular step is deemed bad for some reason.

5 Numerical results

In this section we give results on the form of number of iterations and factorizations. The results are meant to give an initial indication of the performance of the proposed modified Newton approach in a basic interior-point framework. The results are for the methods Newton, mN-r(r), mN-r(r)-H1 and mN-r(r)-H2, with $r = [2, 16]$, described in Sect. 4. In essence the methods differ in how the search direction is computed. The direction at iteration k satisfies (3) in Newton and (15) in the mN-methods. In contrast to Sect. 4, here the mN-methods also include the refactorization strategy described in (34). Due to the large variety in number of inequality constraints and number of variables in each benchmark problem, the parameter l of (34) was defined as the closest integer to $l_{\mathcal{S}}$, $l_{\mathcal{M}}$ and $l_{\mathcal{L}}$ for $p \in \mathcal{S}$, $p \in \mathcal{M}$ and $p \in \mathcal{L}$ respectively, see Table 2 for the specific values. The computational cost of a refactorization of the unreduced, reduced and condensed system all depends on the sparsity structure given by the specific problem. We therefore choose $l_{\mathcal{S}}$, $l_{\mathcal{M}}$ and $l_{\mathcal{L}}$ such that they relate to the full rank change that corresponds to a new factorization. The values of Table 2 were chosen such that a low-rank update is performed as long as the total rank change on an actual Jacobian is not larger than a factor of $1/2$, $1/10$ and $1/100$ for the small, medium and large problems respectively. Moreover, the parameters of Algorithm 1

Table 2 Refactorization parameter for the different problem sizes, m_{in} is the number of inequality constraints

$l_{\mathcal{S}}$	$l_{\mathcal{M}}$	$l_{\mathcal{L}}$
$m_{in}r/2$	$m_{in}r/10$	$m_{in}r/100$

were chosen as follows: $\sigma = 0.1$, termination tolerance $\varepsilon^{tol} = 10^{-6}$ for the small and medium sized problems and $\varepsilon^{tol} = 10^{-5}$ for the large sized problems. In each run, the initial (x^0, λ^0, s^0) was found with `Newton`, with stopping criteria corresponding to the requirement on the initial solution in Algorithm 1.

Results are first shown for problems in the set $\mathcal{S}$, Tables 3 and 4, thereafter for problems in $\mathcal{M}$, Tables 5, 6 and 7, and finally for problems in $\mathcal{L}$, Tables 8, 9 and 10. The results are for three different regions depending on μ^0, namely $\mu^0 = [1,\ 10^{-3},\ 10^{-6}]$. The intention is to illustrate the performance of the modified Newton approach, both close to a solution and in a larger region where the theoretical results are not expected to hold. The results corresponding to $r = 16$ for problems in $\mathcal{S}$ are omitted due to similarity of the performance caused by the refactorization strategy. In all tables the initial factorization of $B^0 = F'(z^0)$ is counted as one factorization. In essence, "1" in the factorization column, `F`, means that no refactorization was performed. Moreover, "-" denotes that the method failed to converge within a maximum number of iterations. For each problem the maximum number of iterations was set to $10N$, where N depends on the number of variables, see Appendix A for the value of N associated with each problem.

Table 3 Number of factorizations and iterations for problems in $\mathcal{S}$ with $\mu^0 = 1$

	Newton	mN-r(2)		mN-r(2)-H1		mN-r(2)-H2	
	F/It	F	It	F	It	F	It
cvxqp1_s	13	2	57	1	43	1	41
cvxqp2_s	14	2	54	2	54	2	54
cvxqp3_s	17	2	74	2	59	2	58
dual1	21	5	188	4	132	3	128
dual2	19	4	150	3	109	3	110
dual3	20	4	168	3	127	3	126
dual4	18	3	91	3	82	3	80
dualc1	10	1	23	1	23	1	23
dualc2	9	1	13	1	13	1	13
dualc5	11	1	15	1	15	1	15
qafiro	14	3	34	3	34	3	33
hs118	15	3	46	3	39	3	42
hs268	14	10	19	10	19	10	19
hs53	8	3	8	3	8	3	8
hs76	11	5	14	5	13	5	14
lotschd	11	4	14	4	15	4	14
primal1	24	5	97	4	83	4	85
primalc1	11	1	24	1	22	1	21
primalc2	10	1	17	1	17	1	17
qadlittl	11	2	39	2	37	2	37
qisrael	20	6	403	3	162	3	163
qpcblend	27	6	150	5	118	5	118
qscagr7	13	2	62	2	51	2	51
qshare2b	21	4	158	3	99	3	99
s268	14	10	19	10	19	10	19

Table 4 Number of factorizations and iterations for problems in $\mathcal{S}$ with $\mu^0 = 10^{-3}$ to the left and $\mu^0 = 10^{-6}$ to the right

	Newton	mN-r(2)		mN-r(2)-H1		mN-r(2)-H2			Newton	mN-r(2)		mN-r(2)-H1		mN-r(2)-H2	
	F/It	F	It	F	It	F	It		F/It	F	It	F	It	F	It
cvxqp1_s	5	1	5	1	5	1	5	cvxqp1_s	3	1	3	1	3	1	3
cvxqp2_s	5	1	5	1	5	1	5	cvxqp2_s	3	1	3	1	3	1	3
cvxqp3_s	6	1	7	1	7	1	7	cvxqp3_s	3	1	3	1	3	1	3
dual1	12	3	86	2	51	2	51	dual1	6	2	43	1	19	1	21
dual2	10	2	50	2	50	2	49	dual2	5	1	8	1	8	1	8
dual3	11	2	58	2	57	2	57	dual3	5	1	13	1	11	1	11
dual4	9	1	37	1	32	1	31	dual4	5	1	6	1	6	1	6
dualc1	5	1	5	1	5	1	5	dualc1	3	1	3	1	3	1	3
dualc2	5	1	5	1	5	1	5	dualc2	3	1	3	1	3	1	3
dualc5	5	1	5	1	5	1	5	dualc5	3	1	3	1	3	1	3
qafiro	4	1	4	1	4	1	4	qafiro	2	1	2	1	2	1	2
hs118	4	1	4	1	4	1	4	hs118	2	1	2	1	2	1	2
hs268	7	5	9	5	9	5	9	hs268	3	2	3	2	3	2	3
hs53	4	2	4	2	4	2	4	hs53	2	1	2	1	2	1	2
hs76	4	2	4	2	4	2	4	hs76	2	1	2	1	2	1	2
lotschd	4	2	4	2	4	2	4	lotschd	2	1	2	1	2	1	2

Table 4 continued

	Newton	mN-r(2)		mN-r(2)-H1		mN-r(2)-H2			Newton	mN-r(2)		mN-r(2)-H1		mN-r(2)-H2	
	F/It	F	It	F	It	F	It		F/It	F	It	F	It	F	It
primal1	12	3	45	2	37	2	37	primal1	5	2	22	1	12	1	12
primalc1	5	1	5	1	5	1	5	primalc1	3	1	3	1	3	1	3
primalc2	5	1	5	1	5	1	5	primalc2	3	1	3	1	3	1	3
qadlittl	5	1	5	1	5	1	5	qadlittl	3	1	3	1	3	1	3
qisrael	7	1	16	1	16	1	16	qisrael	3	1	3	1	3	1	3
qpcblend	15	4	92	3	59	3	60	qpcblend	9	2	45	2	32	2	31
qscagr7	5	1	5	1	5	1	5	qscagr7	3	1	3	1	3	1	3
qshare2b	7	1	16	1	15	1	14	qshare2b	3	1	3	1	3	1	3
s268	7	5	9	5	9	5	9	s268	3	2	3	2	3	2	3

Table 5 Number of factorizations and iterations for problems in $\mathcal{M}$ with $\mu^0 = 1$

	Newton	mN-r(2)		mN-r(2)-H1		mN-r(2)-H2		mN-r(16)		mN-r(16)-H1		mN-r(16)-H2	
	F/It	F	It	F	It	F	It	F	It	F	It	F	It
cvxqp1_m	15	2	140	2	112	2	111	3	29	3	29	3	29
cvxqp2_m	13	2	103	2	102	2	102	2	24	2	24	2	24
cvxqp3_m	16	3	204	2	127	2	121	3	37	3	33	3	34
dualc8	10	2	30	2	30	2	30	3	10	3	10	3	10
gouldqp2	15	4	213	3	168	3	194	4	32	4	31	4	30
gouldqp3	19	4	215	3	206	4	213	5	40	5	40	5	40
ksip	29	6	281	6	268	6	258	11	72	10	68	9	61
laser	18	4	305	2	163	2	178	4	46	4	43	4	42
primal2	23	8	42	8	42	8	42	23	23	23	23	23	23
primal3	24	9	52	8	44	8	44	24	24	24	24	24	24
primal4	21	8	36	8	37	8	37	21	21	21	21	21	21
primalc5	13	3	30	3	31	3	34	8	15	7	13	7	13
primalc8	10	1	17	1	17	1	17	3	10	3	10	3	10
q25fv47	24	10	910	3	215	11	950	13	151	6	67	9	98
qgrow15	16	3	134	3	129	3	129	5	37	5	34	5	33
qgrow22	20	5	402	3	192	3	192	6	69	5	57	5	57
qgrow7	16	4	91	3	65	3	67	6	23	6	21	6	21
qshell	10	1	33	1	25	1	25	2	15	2	15	2	15
qpcstair	25	7	163	5	109	5	109	11	42	10	39	10	39
qcapri	24	10	278	5	123	5	122	12	46	11	42	9	34

Table 5 continued

	Newton	mN-r(2)		mN-r(2)-H1		mN-r(2)-H2		mN-r(16)		mN-r(16)-H1		mN-r(16)-H2	
	F/It	F	It	F	It	F	It	F	It	F	It	F	It
qsctap1	25	10	319	6	181	6	181	11	53	10	47	9	42
qsctap2	24	–	–	4	383	7	759	12	191	7	99	6	85
qsctap3	25	–	–	5	673	6	842	12	232	7	129	7	127
qsc205	23	5	74	5	73	5	73	11	31	11	31	11	31
qscagr25	12	2	64	2	39	2	39	4	19	4	17	4	18
qscsd1	21	6	206	4	124	4	134	9	43	9	41	8	37
qscsd6	26	8	487	4	257	8	478	10	87	9	73	8	71
qscsd8	22	6	701	5	553	4	442	6	102	7	108	5	79
qshare1b	20	7	85	4	48	6	68	14	26	13	24	14	26
values	26	6	108	5	103	5	99	13	36	12	35	12	35
aug3dcqp	26	3	434	3	424	3	430	5	103	5	103	5	102
aug3dqp	27	3	575	4	582	5	837	6	126	6	126	6	129
stadat1	13	2	323	1	191	2	303	2	60	2	49	2	49
stadat2	26	6	1533	4	1114	5	1244	7	231	6	191	6	191
mosarqp1	23	5	651	3	422	4	514	6	110	5	87	5	87
mosarqp2	22	4	229	3	164	3	166	5	46	5	46	5	44
yao	19	3	300	3	204	3	205	5	59	4	47	4	47

Table 6 Number of factorizations and iterations for problems in $\mathcal{M}$ with $\mu^0 = 10^{-3}$

	Newton	mN-r(2)		mN-r(2)-H1		mN-r(2)-H2		mN-r(16)		mN-r(16)-H1		mN-r(16)-H2	
	F/It	F	It	F	It	F	It	F	It	F	It	F	It
cvxqp1_m	8	1	12	1	12	1	12	1	9	1	9	1	9
cvxqp2_m	6	1	6	1	6	1	6	1	6	1	6	1	6
cvxqp3_m	8	1	21	1	20	1	21	2	13	1	12	2	13
dualc8	6	1	6	1	6	1	6	2	6	2	6	2	6
gouldqp2	12	3	151	3	144	3	153	7	54	4	27	4	27
gouldqp3	14	3	145	2	130	2	131	4	28	4	28	4	28
ksip	20	4	185	4	174	5	207	8	53	7	44	7	44
laser	9	1	83	1	32	1	33	2	16	2	15	2	15
primal2	14	5	24	4	21	4	21	12	14	12	14	12	14
primal3	15	5	25	5	25	5	25	12	15	12	15	12	15
primal4	11	4	15	4	15	4	15	9	11	9	11	9	11
primalc5	6	1	6	1	6	1	6	4	6	4	6	4	6
primalc8	6	1	6	1	6	1	6	2	6	2	6	2	6
q25fv47	10	2	111	2	96	7	587	3	25	2	19	3	25
qgrow15	8	1	14	1	13	1	13	2	9	2	9	2	9
qgrow22	11	1	52	1	48	1	48	2	20	2	20	2	20
qgrow7	8	1	12	1	12	1	12	3	8	3	8	3	8
qshell	6	1	6	1	6	1	6	1	6	1	6	1	6
qpcstair	16	4	82	3	63	3	63	6	22	6	22	6	22
qcapri	9	2	37	2	32	2	32	3	10	3	10	3	10
qsctap1	9	2	35	2	34	2	34	3	12	3	12	3	12

Table 6 continued

	Newton	mN-r(2)		mN-r(2)-H1		mN-r(2)-H2		mN-r(16)		mN-r(16)-H1		mN-r(16)-H2	
	F/It	F	It	F	It	F	It	F	It	F	It	F	It
qsctap2	8	1	47	1	17	1	17	2	17	2	16	2	16
qsctap3	9	1	43	1	21	1	22	2	22	2	21	2	21
qsc205	14	3	39	3	39	3	39	6	17	6	17	6	17
qscagr25	6	1	6	1	6	1	6	2	6	2	6	2	6
qscsd1	11	2	56	2	44	2	59	3	14	3	13	3	14
qscsd6	16	5	283	3	138	3	138	6	51	5	37	4	35
qscsd8	11	2	151	1	69	1	74	2	29	2	24	2	24
qshare1b	7	2	15	2	14	2	14	4	8	4	8	4	8
values	17	4	69	4	66	3	54	7	21	7	21	7	21
aug3dcqp	16	2	208	2	196	2	198	2	36	2	36	2	36
aug3dqp	16	2	234	2	238	2	255	3	63	3	63	3	63
stadat1	6	1	6	1	6	1	6	1	6	1	6	1	6
stadat2	20	4	1057	3	697	3	655	6	193	4	131	3	96
mosarqp1	14	3	328	2	186	2	193	3	47	3	45	3	45
mosarqp2	13	2	77	2	77	2	77	2	18	2	17	2	17
yao	9	2	107	1	25	1	25	2	19	2	16	2	16

Table 7 Number of factorizations and iterations for problems in $\mathcal{M}$ with $\mu^0 = 10^{-6}$

	Newton	mN-r(2)		mN-r(2)-H1		mN-r(2)-H2		mN-r(16)		mN-r(16)-H1		mN-r(16)-H2	
	F/It	F	It	F	It	F	It	F	It	F	It	F	It
cvxqp1_m	3	1	3	1	3	1	3	1	3	1	3	1	3
cvxqp2_m	3	1	3	1	3	1	3	1	3	1	3	1	3
cvxqp3_m	3	1	3	1	3	1	3	1	3	1	3	1	3
dualc8	3	1	3	1	3	1	3	1	3	1	3	1	3
gouldqp2	8	2	80	2	73	2	82	3	19	2	17	3	19
gouldqp3	5	1	15	1	14	1	16	2	10	2	9	2	10
ksip	10	2	77	2	55	2	54	3	15	2	13	3	14
laser	3	1	3	1	3	1	3	1	3	1	3	1	3
primal2	4	2	6	2	6	2	6	3	4	3	4	3	4
primal3	5	2	7	2	8	2	8	4	5	4	5	4	5
primal4	3	1	4	1	4	1	4	2	3	2	3	2	3
primalc5	3	1	3	1	3	1	3	2	3	2	3	2	3
primalc8	3	1	3	1	3	1	3	1	3	1	3	1	3
q25fv47	3	1	3	1	3	1	3	1	3	1	3	1	3
qgrow15	3	1	3	1	3	1	3	1	3	1	3	1	3
qgrow22	4	1	8	1	8	1	8	1	8	1	8	1	8
qgrow7	3	1	3	1	3	1	3	1	3	1	3	1	3
qshell	3	1	3	1	3	1	3	1	3	1	3	1	3
qpcstair	7	2	27	2	27	2	27	3	9	3	9	3	9
qcapri	3	1	3	1	3	1	3	1	3	1	3	1	3
qsctap1	3	1	3	1	3	1	3	1	3	1	3	1	3

Table 7 continued

	Newton	mN-r(2)		mN-r(2)-H1		mN-r(2)-H2		mN-r(16)		mN-r(16)-H1		mN-r(16)-H2	
	F/It	F	It	F	It	F	It	F	It	F	It	F	It
qsctap2	3	1	3	1	3	1	3	1	3	1	3	1	3
qsctap3	3	1	3	1	3	1	3	1	3	1	3	1	3
qsc205	6	1	11	1	11	1	11	2	6	2	6	2	6
qscagr25	3	1	3	1	3	1	3	1	3	1	3	1	3
qscsd1	3	1	3	1	3	1	3	1	3	1	3	1	3
qscsd6	5	1	10	1	8	1	9	2	9	1	8	2	9
qscsd8	4	1	4	1	4	1	4	1	4	1	4	1	4
qshare1b	3	1	3	1	3	1	3	2	3	2	3	2	3
values	8	2	24	2	23	2	23	3	9	3	9	3	9
aug3dcqp	7	1	18	1	18	1	18	1	9	1	9	1	9
aug3dqp	7	1	53	1	53	1	53	2	26	2	26	2	26
stadat1	3	1	3	1	3	1	3	1	3	1	3	1	3
stadat2	6	1	32	1	17	1	18	1	15	1	15	1	15
mosarqp1	5	1	8	1	9	1	8	1	7	1	7	1	7
mosarqp2	5	1	10	1	8	1	8	1	5	1	5	1	5
yao	3	1	3	1	3	1	3	1	3	1	3	1	3

Table 8 Number of factorizations and iterations for problems in $\mathcal{L}$ with $\mu^0 = 1$

	Newton	mN-r(2)		mN-r(2)-H1		mN-r(2)-H2		mN-r(16)		mN-r(16)-H1		mN-r(16)-H2	
	F/It	F	It	F	It	F	It	F	It	F	It	F	It
aug2dcqp	20	3	205	2	163	2	162	4	46	4	46	4	46
aug2dqp	25	3	305	3	244	3	244	6	69	6	69	6	68
cont-050	17	4	89	4	85	4	85	8	28	7	24	7	25
cont-100	19	4	322	4	312	4	316	7	85	7	81	7	81
cont-101	20	5	439	4	312	4	312	7	79	5	62	6	68
cont-200	20	4	1291	3	815	3	819	5	208	4	158	4	160
cont-201	20	5	1625	4	1216	3	1205	5	208	4	166	4	168
stcqp2	18	3	85	3	85	3	87	5	27	5	27	5	27
stadat3	26	7	400	6	306	6	306	11	82	10	77	11	82
cvxqp1_l	15	2	104	2	104	2	104	3	28	3	29	3	28
cvxqp2_l	18	2	153	2	114	2	114	3	31	3	31	3	31
cvxqp3_l	18	2	141	2	129	2	131	4	49	4	49	4	49
exdata	25	10	344	7	252	7	241	11	51	11	51	11	51
hues-mod	25	3	132	3	132	3	132	7	44	7	44	7	44
huestis	21	2	78	2	78	2	78	4	26	4	26	4	26
liswet1	25	3	146	3	146	3	146	7	44	7	44	7	44
liswet2	33	6	256	5	219	5	220	11	72	9	61	11	72

Table 8 continued

	Newton	mN-r(2)		mN-r(2)-H1		mN-r(2)-H2		mN-r(16)		mN-r(16)-H1		mN-r(16)-H2	
	F/It	F	It	F	It	F	It	F	It	F	It	F	It
liswet3	32	5	230	5	207	5	206	11	72	9	58	9	58
liswet4	33	7	306	5	233	6	255	11	76	9	59	9	59
liswet5	32	6	255	6	257	5	250	11	73	9	58	9	59
liswet6	32	5	234	5	205	5	205	11	72	9	58	9	58
liswet7	20	3	105	3	105	3	105	6	35	6	35	6	35
liswet8	33	9	408	10	479	8	359	15	102	13	84	12	79
liswet9	35	11	513	8	403	8	359	15	101	14	93	10	66
liswet10	33	8	362	6	278	8	402	13	89	14	96	13	84
liswet11	33	9	414	9	421	9	428	15	100	16	106	13	87
liswet12	30	8	361	7	319	9	415	15	100	11	74	10	68
ubh1	9	2	66	2	66	2	66	2	14	2	14	2	14

Table 9 Number of factorizations and iterations for problems in $\mathcal{L}$ with $\mu^0 = 10^{-3}$

	Newton	mN-r(2)		mN-r(2)-H1		mN-r(2)-H2		mN-r(16)		mN-r(16)-H1		mN-r(16)-H2	
	F/It	F	It	F	It	F	It	F	It	F	It	F	It
aug2dcqp	11	1	65	1	66	1	65	2	19	2	19	2	19
aug2dqp	16	2	142	2	142	2	142	4	39	4	39	4	39
cont-050	13	3	61	3	58	3	58	6	20	5	17	5	17
cont-100	15	3	219	3	209	3	213	5	59	5	55	4	42
cont-101	16	4	336	3	209	3	210	5	63	4	39	3	37
cont-200	17	4	1219	3	815	3	815	4	157	3	107	3	108
cont-201	16	4	1220	3	811	3	811	4	157	4	154	3	115
stcqp2	9	1	29	1	28	1	29	2	11	2	11	2	11
stadat3	21	7	399	5	253	4	184	10	74	7	53	9	66
cvxqp1_l	7	1	12	1	12	1	12	1	9	1	9	1	9
cvxqp2_l	10	1	22	1	22	1	22	2	14	2	14	2	14
cvxqp3_l	10	1	22	1	22	1	22	2	17	2	17	2	17
exdata	14	5	154	3	84	3	84	5	21	5	21	5	22
hues-mod	16	2	60	2	60	2	60	4	21	4	21	4	21
huestis	12	1	40	1	40	1	40	2	13	2	13	2	13
liswet1	13	2	56	2	56	2	56	4	22	4	22	4	22
liswet2	21	5	205	3	118	4	154	8	51	6	37	6	37
liswet3	20	5	206	3	121	4	159	7	44	6	37	6	37

Table 9 continued

	Newton	mN-r(2)		mN-r(2)-H1		mN-r(2)-H2		mN-r(16)		mN-r(16)-H1		mN-r(16)-H2	
	F/It	F	It	F	It	F	It	F	It	F	It	F	It
liswet4	21	5	210	5	210	6	268	8	55	6	38	6	38
liswet5	20	5	206	3	118	5	219	7	45	6	37	6	38
liswet6	20	4	189	4	170	5	205	7	44	6	37	6	37
liswet7	9	1	23	1	23	1	23	3	14	3	14	3	14
liswet8	21	7	336	6	271	6	255	9	58	10	63	8	49
liswet9	22	9	411	4	200	4	155	13	87	7	46	7	44
liswet10	21	8	360	5	210	5	213	11	70	13	85	7	46
liswet11	21	8	359	7	315	8	357	11	74	10	63	9	61
liswet12	18	9	413	4	176	4	169	9	61	6	40	5	33
ubh1	6	1	6	1	6	1	6	1	6	1	6	1	6

Table 10 Number of factorizations and iterations for problems in $\mathcal{L}$ with $\mu^0 = 10^{-6}$

	Newton	mN-r(2)		mN-r(2)-H1		mN-r(2)-H2		mN-r(16)		mN-r(16)-H1		mN-r(16)-H2	
	F/It	F	It	F	It	F	It	F	It	F	It	F	It
aug2dcqp	5	1	5	1	5	1	5	1	5	1	5	1	5
aug2dqp	7	1	35	1	35	1	35	2	15	2	15	2	15
cont-050	3	1	3	1	3	1	3	1	3	1	3	1	3
cont-100	6	1	87	1	87	1	87	2	14	2	14	2	14
cont-101	6	1	10	1	10	1	9	1	10	1	10	1	9
cont-200	8	2	407	2	406	2	406	2	53	2	52	2	52
cont-201	7	2	407	2	406	2	406	2	53	2	52	1	44
stcqp2	3	1	3	1	3	1	3	1	3	1	3	1	3
stadat3	8	2	63	1	59	1	56	2	11	2	10	2	10
cvxqp1_l	3	1	3	1	3	1	3	1	3	1	3	1	3
cvxqp2_l	4	1	4	1	4	1	4	1	4	1	4	1	4
cvxqp3_l	5	1	8	1	8	1	8	1	4	1	4	1	4
exdata	4	1	5	1	5	1	5	2	5	2	5	2	5
hues-mod	7	1	12	1	12	1	12	2	8	2	8	2	8
huestis	5	1	6	1	6	1	6	1	5	1	5	1	5
liswet1	4	1	5	1	5	1	5	1	5	1	5	1	5
liswet2	9	2	52	2	52	2	52	3	15	3	15	3	15

Table 10 continued

	Newton	mN-r(2)		mN-r(2)-H1		mN-r(2)-H2		mN-r(16)		mN-r(16)-H1		mN-r(16)-H2	
	F/It	F	It	F	It	F	It	F	It	F	It	F	It
lisweet3	8	2	53	2	53	2	53	3	16	3	16	3	16
lisweet4	9	2	56	2	55	2	67	3	19	3	17	3	17
lisweet5	8	2	53	2	53	1	43	3	16	3	16	3	16
lisweet6	8	2	52	2	52	2	52	3	15	3	15	3	15
lisweet7	3	1	3	1	3	1	3	1	3	1	3	1	3
lisweet8	11	3	125	2	91	2	96	4	27	5	28	5	28
lisweet9	10	3	102	1	37	1	36	4	24	3	14	3	14
lisweet10	11	4	158	2	57	2	57	5	31	3	19	3	19
lisweet11	11	4	186	3	133	4	153	5	28	5	29	5	28
lisweet12	6	2	53	1	19	1	19	2	9	2	9	2	8
ubh1	3	1	3	1	3	1	3	1	3	1	3	1	3

The results in Tables 3, 4, 5, 6, 7, 8, 9 and 10 indicate that the number of factorizations compared to those done by `Newton` may be reduced by instead performing low-rank updates, as with `mN-r`(r), $r = [2,\ 16]$. The reduced number of factorizations is however often at the expense of performing additional iterations with low-rank updates. The total number of iterations and/or factorizations are for many problems, but not for all, further reduced with heuristics `H1` and `H2`, as shown by the results corresponding to `mN-r`(r)-`H1` and `mN-r`(r)-`H2`, $r = [2,\ 16]$. This behavior is most significant in the simulations with larger values of μ, as shown in Tables 3, 5 and 8. Comparing the number of iterations in the `mN`-methods gives an indication of whether the heuristics have been active on a specific problem. The results in Tables 3, 4, 5, 6, 7, 8, 9 and 10 show that the performance of the heuristics varies with each problem, and in addition with μ^0. In particular, the results show that the heuristics are most effective, and hence more likely to be active, at larger values of μ. For smaller values of μ, the `mN`-methods show similar performance, see Tables 7, 10 and the right-hand side of Table 4. This indicates that the heuristics are less likely to have been active for smaller values of μ. Consequently, the `mN`-methods are less likely to produce limiting steps at small values of μ on the benchmark problems. The observation is in line with the results of the theoretical sections. Overall `mN-r(2)` fails to converge for two problems within a maximum number of iterations due to small step sizes. This is overcome with both `H1` and `H2`, as shown by the corresponding results in Table 5.

Numerical experiments have further shown that decreasing the refactorization parameters of Table 2 decreases the number of iterations, and increases the number of factorizations done by the `mN`-methods. In general, an increased rank of each update matrix reduces the number of iterations overall, but due to the refactorization strategy the methods are required to refactorize more often.

Tables 7, 10 and the right-hand side of Table 4 show that low-rank updates give convergence for small values of μ in many of the benchmark problems, even for update matrices of rank-two on large scale problems.

Finally, we want to mention simplified Newton, i.e., `mN-r`(0) without a refactorization strategy. Our simulations showed that this approach was significantly less robust. It is not clear to us how to deduce a refactorization strategy for `mN-r`(0) that gives a fair comparison to the results of Tables 3, 4, 5, 6, 7, 8, 9 and 10. Therefore, we have omitted specific results.

6 Conclusion

In this work we have proposed and motivated a structured modified Newton approach for solving systems of nonlinear equations that arise in interior-point methods for quadratic programming. In essence, the Jacobian of each Newton system is modified to a previous Jacobian plus one low-rank update matrix per succeeding iteration. The modified Jacobian maintains the sparsity pattern of the Jacobian and may thus be viewed as a Jacobian evaluated at a different point. The approach may in consequence be interpreted in the framework of previous works on primal–dual interior-point methods, e.g., effects of finite-precision arithmetic, stability, convergence and solution techniques.

Numerical simulations have shown that small step sizes can have negative effects on convergence with the modified Newton approach, especially at larger values of μ. In order to decrease these negative effects, we have constructed and motivated two heuristics. Further numerical simulations have shown that the two heuristics often increase the step sizes but also that this is not always sufficient to improve convergence. We have therefore also suggested a refactorization strategy. The heuristics and refactorization strategy that we have proposed are merely options, however, the framework allows for both different versions of these as well as other heuristics and/or strategies.

In addition, we have performed numerical simulations on a set of convex quadratic benchmark problems. The results indicated that the number of factorizations compared to those of the Newton based method can be reduced, often at the expense of performing more iterations with low-rank updates. The total number of iterations and/or factorizations were for many problems, but not for all, further reduced with the two heuristics. Although the theoretical results are in the asymptotic region as $\mu \to 0$, or when the rank of each update matrix is sufficiently large, we still obtain interesting numerical results for larger values of μ and update matrices of low-rank.

Our work is meant to contribute to the theoretical and numerical understanding of modified Newton approaches for solving systems of nonlinear equations that arise in interior-point methods. We have laid a foundation that may be adapted and included in more sophisticated interior-point solvers as well as contribute to the development of preconditioners. We have limited ourselves to a numerical study of the accuracy of the approaches in a high-level language. To get a full understanding of the practical performance, precise ways of solving the updated modified Newton systems would have to be investigated further.

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Data availibility The numerical simulations were performed on benchmark problems from the repository of convex quadratic programming problems by Maros and Mészáros [18]. The problems are available at http://www.doc.ic.ac.uk/~im/ and may be downloaded under *Collection of QP Data Files*. Our pre-processing and numerical solution are described in Sect. 4. Additional details on the pre-processing are given in Appendix A.

Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this manuscript.

Appendix A

For completeness we state the Eckart–Young–Mirsky theorem.

Theorem 5 (Eckart–Young–Mirsky theorem) *Let $A \in \mathbb{R}^{m\times n}$ be of rank r and denote its singular value decomposition by $U\Sigma V^T$, where $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ are orthogonal matrices, and $\Sigma = \text{diag}(\sigma) \in \mathbb{R}^{m\times n}$, for $\sigma_1 \geq \cdots \geq \sigma_p \geq 0$, where $p = \min\{m, n\}$. For a given q, $0 < q \leq r$, the optimal solution of*

$$\begin{aligned} \underset{\tilde{A}\in\mathbb{R}^{m\times n}}{\text{minimize}} \quad & \|A - \tilde{A}\| \\ \text{subject to} \quad & \text{rank}\,(\tilde{A}) \leq q, \end{aligned}$$

where $\|.\|$ is either 2-norm or Frobenius norm, is

$$A_* = \sum_{i=1}^{q} \sigma_i u_i v_i^T,$$

with u_i and v_i as the ith column of U and ith column of V respectively.

Proof See [32, Ch. 2]. □

The following lemma contains the singular value decomposition of $\Delta F'(\Delta z)$ given in (5).

Lemma 6 *For $\Delta z = (\Delta x, \Delta\lambda, \Delta s) \in \mathbb{R}^{(n+2m)}$ let $\Delta F'(\Delta z) \in \mathbb{R}^{(n+2m)\times(n+2m)}$ be defined by (5). The singular value decomposition of $\Delta F'(\Delta z)$ can then be written as*

$$\Delta F'(\Delta z) = \sum_{i\in\mathcal{V}} e_{n+m+i} \left(\Delta s_i e_{n+i} + \Delta\lambda_i e_{m+n+i}\right)^T,$$

where $\mathcal{V}$ is a set of indices, $i = 1, \ldots, m$, ordered such that $\sqrt{(\Delta\lambda_i)^2 + (\Delta s_i)^2}$ are in descending order.

Proof The left singular vectors of $\Delta F'(\Delta z)$ are the set of orthonormal eigenvectors of $(\Delta F'(\Delta z))(\Delta F'(\Delta z))^T$, i.e., vectors u such that

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\Delta S)^2 + (\Delta\Lambda)^2 \end{pmatrix} u = \tilde{\lambda} u. \tag{A1}$$

The eigenvectors of $(\Delta F'(\Delta z))(\Delta F'(\Delta z))^T$ are e_i, $i = 1, \ldots, n + 2m$, and the eigenpairs, with nonzero eigenvalues, are $((\Delta\lambda_i)^2 + (\Delta s_i)^2, e_{n+m+i})$, $i = 1, \ldots, m$. Similarly, the right singular vectors are the set of orthonormal eigenvectors of $(\Delta F'(\Delta z))^T(\Delta F'(\Delta z))$, i.e., vectors v such that

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & (\Delta S)^2 & \Delta S\Delta\Lambda \\ 0 & \Delta\Lambda\Delta S & (\Delta\Lambda)^2 \end{pmatrix} v = \tilde{\lambda} v. \tag{A2}$$

The nonzero eigenvalues of (A2) are the same as those in (A1). A straightforward calculation shows that the eigenvector corresponding to the ith nonzero eigenvalue $v_i = \frac{1}{\sqrt{(\Delta\lambda_i)^2+(\Delta s_i)^2}}(\Delta s_i e_{n+i} + \Delta\lambda_i e_{m+n+i})$, $i = 1, \ldots, m$, fulfills (A2) with $\tilde{\lambda}_i = (\Delta\lambda_i)^2 + (\Delta s_i)^2$, $i = 1, \ldots, m$, and in addition that the set of vectors v_i, $i = 1, \ldots, m$, form an orthornormal set. The singular values of $\Delta F'(\Delta z)$ are then given by $\sqrt{\tilde{\lambda}_i}$, $i = 1, \ldots, n + 2m$. □

Proof of Theorem 2

Proof $F'(z^+)$ is nonsingular since $z^+ \in \mathcal{B}(z^*, \delta)$. The modified Jacobian can then be written as

$$B^+ = F'(z^+)(I - F'(z^+)^{-1}E^+).$$

By Lemma 5, $\|F'(z^+)^{-1}E^+\| \leq MC^{(r+1)}\mu$ and hence there exists $\bar{\mu} > 0$, with $\bar{\mu} \leq \hat{\mu}$, such that for $0 < \mu \leq \bar{\mu}$ it holds that $MC^{(r+1)}\mu < 1$. In consequence, for $0 < \mu \leq \bar{\mu}$, $(B^+)^{-1}$ can be expanded as a von Neumann series

$$\begin{aligned}(B^+)^{-1} &= \left(I - F'(z^+)^{-1}E^+\right)^{-1} F'(z^+)^{-1} \\ &= \sum_{j=0}^{\infty} \left(F'(z^+)^{-1}E^+\right)^j F'(z^+)^{-1}.\end{aligned} \tag{A3}$$

The error with respect to the Newton step $\Delta\hat{z}^+$ can with (A3) be written as

$$\begin{aligned}\Delta\hat{z}^+ - \Delta z^+ &= \Delta\hat{z}^+ + (B^+)^{-1}F_{\mu^{++}}(z^+) = \left(I - \left(I - F'(z^+)^{-1}E^+\right)^{-1}\right)\Delta\hat{z}^+ \\ &= \left(I - \sum_{j=0}^{\infty}\left(F'(z^+)^{-1}E^+\right)^j\right)\Delta\hat{z}^+ = -\sum_{j=1}^{\infty}\left(F'(z^+)^{-1}E^+\right)^j\Delta\hat{z}^+.\end{aligned} \tag{A4}$$

Taking 2-norm on both sides of (A4) and making use of norm inequalities give

$$\|\Delta\hat{z}^+ - \Delta z^+\| \leq \sum_{j=1}^{\infty} \|F'(z^+)^{-1}E^+\|^j \|\Delta\hat{z}^+\|. \tag{A5}$$

The sum in (A5) is a geometric series which is convergent since $\|F'(z^+)^{-1}E^+\| < 1$ for $\mu \leq \bar{\mu}$ and hence

$$\|\Delta\hat{z}^+ - \Delta z^+\| \leq \frac{\|F'(z^+)^{-1}E^+\|}{1 - \|F'(z^+)^{-1}E^+\|}\|\Delta\hat{z}^+\|. \tag{A6}$$

Recall that $\Delta\hat{z}^+$ is the solution of $F'(z^+)\Delta\hat{z}^+ = -F_{\mu^{++}}(z^+)$ where $z^+ \in \mathcal{B}(z^*, \delta)$ and hence

$$\begin{aligned}\|\Delta\hat{z}^+\| &= \| - F'(z^+)^{-1} F_{\mu^{++}}(z^+)\| \\ &= \|F'(z^+)^{-1}\big(F_{\mu^{++}}(z^{\mu^{++}}) - F_{\mu^{++}}(z^+) - F'(z^+)(z^{\mu^{++}} - z^+)\big) \\ &\quad + (z^{\mu^{++}} - z^+)\| \\ &\leq \frac{M}{2}\|z^+ - z^{\mu^{++}}\|^2 + \|z^+ - z^{\mu^{++}}\|, \end{aligned} \tag{A7}$$

where $z^{\mu^{++}} : F_{\mu^{++}}(z^{\mu^{++}}) = 0$, and it has been used that the Lipschitz constant of F' equals one, as discussed in the end of Sect. 3.1. Moreover,

$$\begin{aligned}\|z^+ - z^{\mu^{++}}\| &= \|z - F'(z)^{-1} F_{\mu^+}(z) - z^{\mu^{++}}\| \\ &= \|F'(z)^{-1}\big(F_{\mu^+}(z^{\mu^+}) - F_{\mu^+}(z) - F'(z)(z^{\mu^+} - z)\big) + (z^{\mu^+} - z^{\mu^{++}})\| \\ &\leq \frac{M}{2}\|z^{\mu^+} - z\|^2 + \|z^{\mu^+} - z^{\mu^{++}}\| \\ &= \frac{M}{2}\|z^{\mu^+} - z^{\mu} + z^{\mu} - z\|^2 + \|z^{\mu^+} - z^{\mu^{++}}\| \\ &\leq \frac{M}{2}\big(\|z - z^{\mu}\|^2 + 2\|z - z^{\mu}\|\|z^{\mu} - z^{\mu^+}\| + \|z^{\mu} - z^{\mu^+}\|^2\big) \\ &\quad + \|z^{\mu^+} - z^{\mu^{++}}\|, \end{aligned} \tag{A8}$$

where $z^{\mu^+} : F_{\mu^+}(z^{\mu^+}) = 0$. Note that $\|z^{\mu} - z^{\mu^+}\| \leq C_3(\mu - \mu^+) = C_3(1 - \sigma)\mu$, as z^{μ}, $0 < \mu \leq \hat{\mu}$, is Lipschitz continuous by Lemma 2. By assumption there exists a constant $C > 0$ such that $\|F_{\mu}(z)\| \leq C\mu$. Lemma 3 and Lemma 2 applied to (A8) then give

$$\begin{aligned}\|z^+ - z^{\mu^{++}}\| &\leq \frac{M}{2}\left(\frac{C^2}{C_4^2} + 2\frac{C}{C_4}C_3(1 - \sigma) + C_3^2(1 - \sigma)^2\right)\mu^2 + C_3(1 - \sigma^+)\sigma\mu \\ &= C_3(1 - \sigma^+)\sigma\mu + \mathcal{O}(\mu^2). \end{aligned} \tag{A9}$$

A combination of (A7) and (A9) gives

$$\|\Delta\hat{z}^+\| \leq C_3(1 - \sigma^+)\sigma\mu + \mathcal{O}(\mu^2). \tag{A10}$$

Insertion of (A10) into (A6) while making use of Lemma 5 gives the result. □

Pre-processing of benchmark problems

As the benchmark problems in general also contain equality constraints, each problem was pre-processed and put on the form

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}}\ & \tfrac{1}{2}x^T Hx + c^T x \\ \text{subject to}\ & A_{eq}x = b_{eq} \\ & A_{in}x \geq b_{in}, \end{aligned} \tag{A11}$$

where $H \in \mathbb{R}^{n\times n}$, $c \in \mathbb{R}^n$, $A_{eq} \in \mathbb{R}^{m_{eq}\times n}$, $b_{eq} \in \mathbb{R}^{m_{eq}}$, $A_{in} \in \mathbb{R}^{m_{in}\times n}$, and $b_{in} \in \mathbb{R}^{m_{in}}$. First-order necessary conditions for a local minimizer of (A11) can be stated as: i) $Hx + c - A_{eq}^T\lambda_{eq} - A_{in}^T\lambda_{in} = 0$, ii) $A_{eq}x = b_{eq}$, iii) $A_{in}x - s = b_{in}$, iv) $s \cdot \lambda_{in} = 0$, v) $s \geq 0$, vi) $\lambda_{in} \geq 0$, for vectors $\lambda_{eq} \in \mathcal{R}^{m_{eq}}$, $\lambda_{in} \in \mathcal{R}^{m_{in}}$ and $s \in \mathcal{R}^{m_{in}}$. Similarly as in Sect. 2, define $F_\mu : \mathbb{R}^{n+m_{eq}+2m_{in}} \to \mathbb{R}^{n+m_{eq}+2m_{in}}$ by

$$F_\mu(z) = \begin{pmatrix} Hx + c - A_{eq}^T\lambda_{eq} - A_{in}^T\lambda_{in} \\ A_{eq}x - b_{eq} \\ A_{in}x - s - b_{in} \\ \Lambda_{in}Se - \mu e \end{pmatrix}, \text{ with } z = (x, \lambda_{eq}, \lambda_{in}, s). \tag{A12}$$

Primal–dual interior-point methods involve solving or approximately solving $F_\mu(z) = 0$ for a decreasing sequence of $\mu > 0$ while maintaining $\lambda_{in} > 0$ and $s > 0$. Application of Newton iterations gives systems on the form (3) with $F_\mu(z)$ as in (A12), $\Delta\hat{z} = (\Delta\hat{x}, \Delta\hat{\lambda}_{eq}, \Delta\hat{\lambda}_{in}, \Delta\hat{s})$ and $F' : \mathbb{R}^{n+m_{eq}+2m_{in}} \to \mathbb{R}^{(n+m_{eq}+2m_{in})\times(n+m_{eq}+2m_{in})}$ defined by

$$F'(z) = \begin{pmatrix} H & -A_{eq}^T & -A_{in}^T & \\ A_{eq} & & & \\ A_{in} & & & -I \\ & & S & \Lambda_{in} \end{pmatrix}. \tag{A13}$$

Problem data

Number of x-variables, equality constraints, inequality constraints and total number of variables in the primal–dual formulation for problems in the sets $\mathcal{S}$, $\mathcal{M}$ and $\mathcal{L}$ are shown in Tables 11 and 12 respectively.

Table 11 Details on problem size for problems $p \in \mathcal{S}$

	n	m_{eq}	m_{in}	N
cvxqp1_s	100	50	200	350
cvxqp2_s	100	25	200	325
cvxqp3_s	100	75	200	375
dual1	85	1	170	256
dual2	96	1	192	289
dual3	111	1	222	334
dual4	75	1	150	226
dualc1	9	1	232	242
dualc2	7	1	242	250
dualc5	8	1	293	302
qafiro	32	8	51	91
hs118	15	0	59	74
hs268	5	0	5	10
hs53	5	3	10	18
hs76	4	0	7	11
lotschd	12	7	12	31
primal1	325	0	86	411
primalc1	230	0	224	454
primalc2	231	0	236	467
qadlittl	96	14	137	247
qisrael	142	0	316	458
qpcblend	83	43	114	240
qscagr7	140	84	185	409
qshare2b	79	13	162	254
s268	5	0	5	10

Table 12 Details on problem size for problems $p \in \mathcal{M}$ and $p \in \mathcal{L}$

	n	m_{eq}	m_{in}	N		n	m_{eq}	m_{in}	N
cvxqp1_m	1000	500	2000	3500	aug2dcqp	20,200	10,000	20,200	50,400
cvxqp2_m	1000	250	2000	3250	aug2dqp	20,200	10,000	20,200	50,400
cvxqp3_m	1000	750	2000	3750	cont-050	2597	2401	5194	10,192
dualc8	8	1	518	527	cont-100	10,197	9801	20,394	40,392
gouldqp2	699	349	1398	2446	cont-101	10,197	10,098	20,394	40,689
gouldqp3	699	349	1398	2446	cont-200	40,397	39,601	80,794	160,792
ksip	20	0	1001	1021	cont-201	40,397	40,198	80,794	161,389
laser	1002	0	2000	3002	stcqp2	4097	2052	8194	14,343
primal2	649	0	97	746	stadat3	4001	0	11,999	16,000
primal3	745	0	112	857	cvxqp1_l	10,000	5000	20,000	35,000
primal4	1489	0	76	1565	cvxqp2_l	10,000	2500	20,000	32,500
primalc5	287	0	286	573	cvxqp3_l	10,000	7500	20,000	37,500
primalc8	520	0	511	1031	exdata	3000	1	7500	10,501
q25fv47	1571	515	1876	3962	hues-mod	10,000	2	10,000	20,002
qgrow15	645	300	1245	2190	huestis	10,000	2	10,000	20,002
qgrow22	946	440	1826	3212	lisswet1	10,002	0	10,000	20,002
qgrow7	301	140	581	1022	lisswet2	10,002	0	10,000	20,002
qshell	1525	534	1644	3703	lisswet3	10,002	0	10,000	20,002
qpcstair	385	209	532	1126	lisswet4	10,002	0	10,000	20,002
qcapri	337	142	583	1062	lisswet5	10,002	0	10,000	20,002
qsctap1	480	120	660	1260	lisswet6	10,002	0	10,000	20,002
qsctap2	1880	470	2500	4850	lisswet7	10,002	0	10,000	20,002
qsctap3	2480	620	3340	6440	lisswet8	10,002	0	10,000	20,002
qsc205	202	90	315	607	lisswet9	10,002	0	10,000	20,002
qscagr25	500	300	671	1471	lisswet10	10,002	0	10,000	20,002
qscsd1	760	77	760	1597	lisswet11	10,002	0	10,000	20,002
qscsd6	1350	147	1350	2847	lisswet12	10,002	0	10,000	20,002
qscsd8	2750	397	2750	5897	ubh1	17,997	12,000	12,006	42003
qshare1b	225	89	253	567					
values	202	1	404	607					
aug3dcqp	3873	1000	3873	8746					
aug3dqp	3873	1000	3873	8746					
stadat1	2001	0	5999	8000					
stadat2	2001	0	5999	8000					
mosarqp1	2500	0	3200	5700					
mosarqp2	900	0	1500	2400					
yao	2000	0	2001	4001					

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