Computational Optimization and Applications (2023) 86:967–988
https://doi.org/10.1007/s10589-022-00437-0

Robust and continuous metric subregularity for linear inequality systems

J. Camacho[1] · M. J. Cánovas[1] · M. A. López[2,3] · J. Parra[1]

Received: 4 June 2022 / Accepted: 15 November 2022 / Published online: 16 December 2022

Abstract
This paper introduces two new variational properties, robust and continuous metric subregularity, for finite linear inequality systems under data perturbations. The motivation of this study goes back to the seminal work by Dontchev, Lewis, and Rockafellar (2003) on the radius of metric regularity. In contrast to the metric regularity, the unstable continuity behavoir of the (always finite) metric subregularity modulus leads us to consider the aforementioned properties. After characterizing both of them, the radius of robust metric subregularity is computed and some insights on the radius of continuous metric subregularity are provided.

Keywords Radius of metric subregularity · Linear inequality systems · Calmness · Feasible set mapping

Mathematics Subject Classification 90C31 · 49J53 · 15A39 · 90C05

To the memory of Asen L. Dontchev.

This research has been partially supported by Grant PGC2018-097960-B-C2(1,2) from MICINN, Spain, and ERDF, "A way to make Europe", European Union, and Grant PROMETEO/2021/063 from Generalitat Valenciana, Spain.

✉ J. Parra
parra@umh.es

J. Camacho
j.camacho@umh.es

M. J. Cánovas
canovas@umh.es

M. A. López
marco.antonio@ua.es

[1] Center of Operations Research, Miguel Hernández University of Elche, 03202 Elche, Alicante, Spain

[2] Department of Mathematics, University of Alicante, 03071 Alicante, Spain

[3] CIAO, Federation University, Ballarat, Australia

1 Introduction

In this paper we firstly analyze continuity properties of the *modulus of metric subregularity* for linear inequality systems. This analysis motivates the introduction of new properties named as *robust* and *continuous metric subregularity*. Hereafter we frequently omit the word 'metric' for simplicity. We are particularly concerned with the radius (a sort of distance to ill-posedness) with respect to both properties, as well as with the connection with the *modulus of robust subregularity*. This topic is framed in the broader paradigm

$$\text{radius of } \mathcal{P} = \frac{1}{\text{modulus of } \mathcal{P}} \tag{1}$$

for some stability property, $\mathcal{P}$, which has been widely studied in different contexts (cf. [11, 12, 20]). We also draw the reader's attention to paper [4], devoted to the metric regularity of the inverse feasible set mapping for linear semi-infinite inequality systems (see [14]), where equality (1) holds. We advance that relation (1) does not always hold for the properties analyzed in the present paper.

We deal with (finite) linear inequality systems in $\mathbb{R}^n$ of the form

$$\sigma = \{a_t' x \leq b_t,\ t \in T = \{1, \ldots, m\}\}, \tag{2}$$

where $x \in \mathbb{R}^n$ is the decision variable, regarded as a column-vector, and the prime represents transposition. System σ will be identified with the pair of coefficient functions (a, b), where $a = \left(a_t\right)_{t\in T} \in (\mathbb{R}^n)^T$ and $b = \left(b_t\right)_{t\in T} \in \mathbb{R}^T \equiv \mathbb{R}^m$. For the sake of simplicity in the notation we will identify $(\mathbb{R}^n)^T$ with $\mathbb{R}^{n\times m}$, so that function $a : t \mapsto a_t$ will be regarded as a matrix whose t-th column is a_t . In this way system σ may be abbreviated as $a'x \leq b$. The space $\mathbb{R}^n$ is equipped with an arbitrary norm $\|\cdot\|$, while $\|\cdot\|_*$ stands for its dual norm, given by $\|u\|_* := \max_{\|x\|=1} |u'x|$, whose associated distance is denoted by d_*, and $\mathbb{R}^T$ is endowed with the supremum norm, $\|b\|_\infty := \max_{t\in T} |b_t|$.

In this framework, system σ may be rewritten as the generalized equation

$$\mathcal{G}_a(x) := a'x + \mathbb{R}^m_+ \ni b, \tag{3}$$

where $\mathcal{G}_a : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $\mathbb{R}^m_+$ stands for the subset of elements of $\mathbb{R}^m$ with nonnegative coordinates. For each $a \in \mathbb{R}^{n\times m}$, the inverse multifunction

$$\mathcal{F}_a := \mathcal{G}_a^{-1} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n,$$

given by $x \in \mathcal{F}_a(b) \Leftrightarrow b \in \mathcal{G}_a(x)$, is nothing else but the *feasible set mapping* of system σ under right-hand side perturbations.

Throughout the paper we work with a fixed consistent system denoted by $\overline{\sigma} \equiv \left(\overline{a}, \overline{b}\right)$ and a fixed $\overline{x} \in \mathcal{F}_{\overline{a}}\left(\overline{b}\right)$. We refer to $\overline{a}$, $\overline{b}$ and $\overline{x}$ as the nominal data. Given any property $\mathcal{P}$ of $\mathcal{G}_{\overline{a}}$ fulfilled at the nominal $\left(\overline{x}, \overline{b}\right) \in \mathrm{gph}\mathcal{G}_{\overline{a}}$ (where gph stands for graph), the radius of $\mathcal{P}$-stability at that point is defined as

$$\operatorname{rad}_{\mathcal{P}}\mathcal{G}_{\overline{a}}(\overline{x},\overline{b}) := \inf_{g\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)}\left\{\|g\| \;\middle|\; \mathcal{G}_{\overline{a}}+g \text{ does not have } \mathcal{P} \text{ at } \left(\overline{x},\overline{b}+g\left(\overline{x}\right)\right)\right\}, \tag{4}$$

where $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)$ stands for the space of linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ endowed with the norm subordinated to the norms under consideration in these spaces. This definition of radius is inspired by the one given in [12, Definition 1.4] for the metric regularity property in more general contexts; see also [11] for the property of metric subregularity. In order to adapt this concept to our current notation, let us identify a linear function g with the matrix $g \in \mathbb{R}^{n\times m}$ such that $g(x)$ reads as $g'x$. In this way, denoting by g_t the t-th column of g, we have

$$\|g\| = \max_{\|x\|=1}\|g'x\|_\infty = \max_{\|x\|=1}\max_{t\in T}\left|g_t'x\right| = \max_{t\in T}\left\|g_t\right\|_*.$$

Remark 1 Observe that $\left(\mathcal{G}_{\overline{a}}+g\right)(x) = (\overline{a}+g)'x+\mathbb{R}^m_+$ for all $x\in\mathbb{R}^n$. In other words,

$$\mathcal{G}_{\overline{a}}+g = \mathcal{G}_{\overline{a}+g}. \tag{5}$$

In this way, linear perturbations of $\mathcal{G}_{\overline{a}}$ translate into left-hand side (LHS, in brief) perturbations of the linear inequality system (2). Hence, assuming that $\overline{\sigma}\equiv\left(\overline{a},\overline{b}\right)$ satisfies a certain stability property $\mathcal{P}$, roughly speaking, $\operatorname{rad}_{\mathcal{P}}\mathcal{G}_{\overline{a}}(\overline{x},\overline{b})$ provides the infimum size of LHS perturbations of $\overline{\sigma}$ which cause failure of property $\mathcal{P}$ at the same point $\overline{x}$ with parameter $\overline{b}+g'\overline{x}$.

As already commented in [11, Example 1.1], when $\mathcal{P}$ is the metric subregularity property, then $\operatorname{rad}_{\mathcal{P}}\mathcal{G}_{\overline{a}}(\overline{x},\overline{b})$ is $+\infty$ as, for any $a\in\mathbb{R}^{n\times m}$, $\mathcal{G}_a$ is always metrically subregular at any $(x,b)\in \operatorname{gph}\mathcal{G}_a$ (this fact follows from the classical work of Robinson [24]). Therefore, the associated modulus is always finite, but not necessarily zero (in which case (1) fails). Indeed, the *subregularity modulus* of $\mathcal{G}_{\overline{a}}$ at $(\overline{x},\overline{b})$ is known to coincide with the *calmness modulus* of $\mathcal{F}_{\overline{a}}$ at $(\overline{b},\overline{x})$ which is computed through an implementable formula in Theorem 2; see Sect. 2 for further details. This comment motivates the fact of considering a different (more restrictive) property $\mathcal{P}$ which is not satisfied at all $((a,b),x)$ with $(x,b)\in\operatorname{gph}\mathcal{G}_a$. In this way, the continuous/robust subregularity come into play.

It is important to emphasize the practical repercussions of the metric subregularity property (and its counterparts in terms of *calmness* and *local error bounds*), for instance with respect to the convergence of algorithms. Just observe that finding a solution of our generalized equation $\mathcal{G}_a(x)\ni b$, with b sufficiently close to the nominal $\overline{b}$, might be considerably difficult, whereas the residual (in our case, $\max_{t\in T}\left[a_t'x-\overline{b}_t\right]_+$, where $[\alpha]_+$ represents the positive part of $\alpha\in\mathbb{R}$) is much easier to compute or estimate. Hence, the metric subregularity of $\mathcal{G}_a$ at $(\overline{x},\overline{b})$ with constant κ (see Sect. 2 for the definition) ensures the existence of such a solution whose distance to $\overline{x}$ is no longer than κ times the residual. In particular, if we know an estimate for the rate of convergence of the residual to zero, then we can evaluate the rate of convergence of a sequence of approximate solutions to an exact solution. Two specific applications of calmness modulus are given in [6, Section 5] to the computation of some constants related to the convergence of certain optimization methods. The first one is focused on a particular procedure described in [19, Section 3.1] for a descent method, and the second deals with a concrete regularization scheme for mathematical programs with complementarity constraints introduced in [17]. In [23] we can find several references on the algorithmic repercussions of Hoffman constants (of a global nature, in contrast with the local character of calmness) as well as other related error bounds in establishing convergence properties of a variety of modern convex optimization algorithms.

Concerning interior-point methods, in [5, Section 4] the well-known *central path* construction associated with a linear programming problem is considered. If $\{(x(\mu), y(\mu), z(\mu))$, for $\mu > 0\}$ denotes such a path and Λ is the primal-dual solution set for the original problem (corresponding to $\mu = 0$), then, under appropriate hypotheses, [5, Theorem 4.1] shows that

$$d((x(\mu), y(\mu), z(\mu)), \Lambda) \leq \kappa\mu$$

for μ small enough, where κ is directly related with the calmness modulus of a suitable feasible set mapping defined in terms of the nominal problem's data, so that constant κ can be computed through an implementable procedure as it involves only fixed elements. A closely related problem is tackled in [1, Corollary 3], where an application to the convergence of a certain path-following algorithmic scheme, also in terms of calmness constants, is developed.

Aside the importance of the regularity concepts themselves, the study of related radii is also relevant. As already mentioned in [11, Section 5], the radius of nonsingularity of matrices is ultimately related to their condition number, and preconditioning is a highly efficient tool for enhancing computations in numerical linear algebra. In that paper the authors also suggest that different radius expressions could be utilized in procedures for conditioning of problems of feasibility and optimization. For a wider insight on conditioning, see [2].

The present paper is structured as follows: Sect. 2 sets up the necessary notation and preliminary results. Section 3 deals with the continuity behavior of the subregularity modulus of linear inequality systems under LHS perturbations, which is analyzed in two steps. First, Theorem 3 sheds light on the stability of the end set of polyhedra. As a consequence of this result, the continuity of the subregularity modulus is characterized in Theorem 4. In Sect. 4 we introduce the properties of robust and continuous subregularity and characterize them in Theorem 5 and Corollary 2 , respectively. Section 5 computes the radius of robust subregularity (Theorem 6) and gives some insights on the radius of continuous subregularity (see Example 4). The paper finishes with a section of conclusions and future research.

2 Preliminaries

Firstly, let us give some definitions and notations used along the paper. Given $S \subset \mathbb{R}^{\ell}$, $\ell \in \mathbb{N}$, conv$S$ denotes the convex hull of S. From the topological side, int$S$, cl$S$, and bd$S$ stand for the interior, the closure, and the boundary of S, respectively. Additionally, if S is convex, its *end set* (introduced in [15]) is defined as

$$\text{end}\, S := \left\{ u \in \text{cl}\, S \mid \nexists \mu > 1 \text{ such that } \mu u \in \text{cl}\, S \right\}.$$

Here we recall the lower/inner and upper/outer limit of sets in the Painlevé-Kuratowski sense (cf. [22, p. 13], see also [25, p. 152]). Given two metric spaces X and A and a family of subsets of X, $\left\{X_a\right\}_{a\in A}$, we say $x \in \text{Lim}\inf_{a\to\overline{a}} X_a$ if for each sequence $\{a^r\}_{r\in\mathbb{N}}$ converging to $\overline{a}$ there exist $r_0 \in \mathbb{N}$ and $\{x^r\}_{r\geq r_0}$ verifying $x^r \in X_{a^r}$ for all $r \geq r_0$ and $\lim_{r\to\infty} x^r = x$. Regarding the outer limit, $x \in \text{Lim}\sup_{a\to\overline{a}} X_a$ if $x = \lim_{r\to\infty} x^r$ with $x^r \in X_{a^r}$ for some sequence $\{a^r\}_{r\in\mathbb{N}}$ converging to $\overline{a}$.

A set-valued mapping $\mathcal{M} : X \rightrightarrows Y$ between metric spaces (with both distances denoted by d) is said to be (metrically) *subregular* at $(\overline{x}, \overline{y}) \in \text{gph}\, \mathcal{M}$ if there exist a constant $\kappa \geq 0$ together with a neighborhood U of $\overline{x}$ such that

$$d(x, \mathcal{M}^{-1}(\overline{y})) \leq \kappa d(\overline{y}, \mathcal{M}(x)) \quad \text{for all } x \in U. \tag{6}$$

Here $d(x, C) := \inf_{y\in C} d(x, y)$ denotes the point-to-set distance, with $d(x, \emptyset) = +\infty$. Throughout the paper we assume $1/0 = +\infty$ and $1/(+\infty) = 0$. The infimum of constants κ in (6), over the set of all possible (κ, U) is called the *subregularity modulus* of $\mathcal{M}$ at $(\overline{x}, \overline{y})$ and it is denoted by subreg $\mathcal{M}(\overline{x}, \overline{y})$.

The subregularity property of $\mathcal{M}$ at $(\overline{x}, \overline{y}) \in \text{gph}\, \mathcal{M}$ is known to be equivalent to the calmness of its inverse $\mathcal{M}^{-1}$ at $(\overline{y}, \overline{x})$ and it is also known that subreg $\mathcal{M}(\overline{x}, \overline{y})$ coincides with the calmness modulus of $\mathcal{M}^{-1}$ at $(\overline{y}, \overline{x})$ (cf. [13, 16, 18, 22, 25]).

Our focus is on mapping $\mathcal{G}_a$, with $a \in \mathbb{R}^{n\times m}$, given in (3), where a point-based formula (in terms of the given data) for its subregularity modulus is known (see Theorem 2). More specifically, given our nominal data $\overline{\sigma} \equiv \left(\overline{a}, \overline{b}\right)$ and $\overline{x} \in \mathcal{F}_{\overline{a}}\left(\overline{b}\right)$, such expression of subreg $\mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right)$ appeals to the set of active indices at $\overline{x}$,

$$T_{\overline{\sigma}}\left(\overline{x}\right) := \left\{ t \in T \mid \overline{a}_t' \overline{x} = \overline{b}_t \right\},$$

and involves the family $\mathcal{D}_{\overline{a}}$ (introduced in [9, Section 4] under the name $\mathcal{D}\left(\overline{x}\right)$) of subsets $D \subset T_{\overline{\sigma}}\left(\overline{x}\right)$ such that system

$$\left\{ \begin{array}{l} \overline{a}_t' d = 1,\ t \in D, \\ \overline{a}_t' d < 1,\ t \in T_{\overline{\sigma}}\left(\overline{x}\right) \setminus D, \end{array} \right\} \tag{7}$$

is consistent in the variable $d \in \mathbb{R}^n$. Observe that if $D \in \mathcal{D}_{\overline{a}}$ and d is such a solution, then $\left\{\overline{a}_t,\ t \in D\right\}$ is contained in the hyperplane $\{z \in \mathbb{R}^n \mid d'z = 1\}$, which leaves $\{0_n\} \cup \{\overline{a}_t,\ t \in T_{\overline{\sigma}}\left(\overline{x}\right) \setminus D\}$ on one of its two associated open half-spaces.

Another key tool in the present paper is the family of sets $\mathcal{D}_{\overline{a}}^0$ (see [7, Section 3.2]) formed by all $D \subset T_{\overline{\sigma}}\left(\overline{x}\right)$ such that system

$$\left\{ \begin{array}{l} \overline{a}_t' d = 0,\ t \in D, \\ \overline{a}_t' d < 0,\ t \in T_{\overline{\sigma}}(\overline{x}) \setminus D, \end{array} \right\} \tag{8}$$

has nonzero solutions in the variable $d \in \mathbb{R}^n$. Now, if $D \in \mathcal{D}_{\overline{a}}^0$ and $d \in \mathbb{R}^n \backslash \{0_n\}$ satisfies (8), then the hyperplane $\{z \in \mathbb{R}^n \mid d'z = 0\}$ contains $\{0_n\} \cup \{\overline{a}_t,\ t \in D\}$ and leaves $\{\overline{a}_t,\ t \in T_{\overline{\sigma}}(\overline{x}) \setminus D\}$ on one of its two associated open half-spaces.

Theorem 1 *Let* $\overline{\sigma} \equiv \left(\overline{a}, \overline{b}\right)$ *and* $\overline{x} \in \mathcal{F}_{\overline{a}}\left(\overline{b}\right)$. *Then*

$$\operatorname{end}\operatorname{conv}\left\{\overline{a}_t,\ t \in T_{\overline{\sigma}}(\overline{x})\right\} = \bigcup_{D \in \mathcal{D}_{\overline{a}}} \operatorname{conv}\left\{\overline{a}_t,\ t \in D\right\}. \tag{9}$$

Proof It is a direct consequence of [21, Corollary 2.1 and Remark 2.3]. □

Theorem 2 *Let* $\overline{\sigma} \equiv \left(\overline{a}, \overline{b}\right)$ *and* $\overline{x} \in \mathcal{F}_{\overline{a}}\left(\overline{b}\right)$. *Then*

$$\begin{aligned} \operatorname{subreg} \mathcal{G}_{\overline{a}}(\overline{x}, \overline{b}) &= d_*\left(0_n, \operatorname{end}\operatorname{conv}\left\{\overline{a}_t,\ t \in T_{\overline{\sigma}}(\overline{x})\right\}\right)^{-1} \\ &= \max_{D \in \mathcal{D}_{\overline{a}}} d_*\left(0_n,\ \operatorname{conv}\left\{\overline{a}_t,\ t \in D\right\}\right)^{-1}. \end{aligned}$$

Proof For $\overline{x} \in \operatorname{bd} \mathcal{F}_{\overline{a}}\left(\overline{b}\right)$ the result follows from [9, Theorem 4] together with Theorem 1. If $\overline{x} \in \operatorname{int} \mathcal{F}_{\overline{a}}\left(\overline{b}\right)$, then $\mathcal{D}_{\overline{a}} = \{\emptyset\}$ and

$$0 = \operatorname{subreg} \mathcal{G}_{\overline{a}}(\overline{x}, \overline{b}) = d_*\left(0_n, \emptyset\right)^{-1}.$$

□

Remark 2 (On semi-infinite systems) For the sake of completeness, let us comment on some facts which may arise when the set T indexing the constraints is infinite. To start with, in the case when T is a compact metric space and $t \mapsto \left(a_t, b_t\right)$ is continuous on T, the set $T_{\overline{\sigma}}(\overline{x})$ is also compact and [21, Corollary 2.1 and Remark 2.3] ensures, denoting $\mathcal{B}(\overline{x}) := \bigcup_{D \in \mathcal{D}_{\overline{a}}} \operatorname{conv}\left\{\overline{a}_t,\ t \in D\right\}$, that

$$\mathcal{B}(\overline{x}) \subset \operatorname{end}\operatorname{conv}\left\{\overline{a}_t,\ t \in T_{\overline{\sigma}}(\overline{x})\right\} \subset \operatorname{cl}\mathcal{B}(\overline{x}),$$

hence,

$$d_*\left(0_n, \operatorname{end}\operatorname{conv}\left\{\overline{a}_t,\ t \in T_{\overline{\sigma}}(\overline{x})\right\}\right) = \inf_{D \in \mathcal{D}_{\overline{a}}} d_*\left(0_n,\ \operatorname{conv}\left\{\overline{a}_t,\ t \in D\right\}\right),$$

which generalizes to this continuous semi-infinite case the second equality in Theorem 2. The first equality in Theorem 2 holds under the following regularity condition (see [21, Corollary 2.1, Remark 2.3 and Corollary 3.2]): "There exists a neighborhood W of $\overline{x}$ such that

$$\mathcal{F}_{\overline{a}}(\overline{b}) \cap W = \left(\overline{x} + (\operatorname{cone}\left\{\overline{a}_t,\ t \in T_{\overline{\sigma}}(\overline{x})\right\})^{\circ}\right) \cap W\text{''}, \tag{10}$$

where X° denotes the (negative) polar of X. Observe that this condition is held at all points of polyhedral sets and, for instance, at the vertex of the ice-cream cone. Indeed, the fulfilment of the condition (10) at all points of $\mathcal{F}_{\overline{a}}(\overline{b})$ is equivalent the fact that system $\overline{\sigma}$ is locally polyhedral (see [21, Corollary 3.3] and also [14, Section 5.2]). To the authors knowledge, the exact computation of subreg $\mathcal{G}_{\overline{a}}(\overline{x},\overline{b})$ for more general semi-infinite systems via a point-based formula (in terms exclusively of the nominal data $\overline{a},\overline{b},\overline{x}$) remains as an open problem.

3 On the continuity of the subregularity modulus

Given the nominal data $\overline{\sigma} \equiv \left(\overline{a},\overline{b}\right)$ and $\overline{x} \in \mathcal{F}_{\overline{a}}\left(\overline{b}\right)$, we follow the perturbation structure of [12, p. 496]. In other words, we are considering arbitrary LHS perturbations $a-\overline{a}$ of $\overline{\sigma}$ and, in order to preserve feasibility of $\overline{x}$, the corresponding right-hand side perturbations are given by $\overline{b}+\left(a-\overline{a}\right)'\overline{x}$. In this way, (5) with $g=a-\overline{a}$ shifts $\left(\overline{x},\overline{b}\right)$ to $\left(\overline{x},\overline{b}+\left(a-\overline{a}\right)'\overline{x}\right)$. In the sequel, it will be useful to note that, denoting the set of active indices of system (2) at $x\in\mathcal{F}_a(b)$ by $T_{(a,b)}(x) := \left\{t\in T \mid a_t'x=b_t\right\}$, we have

$$T_{\left(a,\overline{b}+\left(a-\overline{a}\right)'\overline{x}\right)}\left(\overline{x}\right) = T_{\overline{\sigma}}\left(\overline{x}\right) \text{ for all } a\in\mathbb{R}^{n\times m}. \tag{11}$$

Now we introduce the function $\mathcal{S}:\mathbb{R}^{n\times m}\to\mathbb{R}$ given by

$$\mathcal{S}(a) := \operatorname{subreg}\mathcal{G}_a(\overline{x},\overline{b}+\left(a-\overline{a}\right)'\overline{x}),\ a\in\mathbb{R}^{n\times m}. \tag{12}$$

In order to simplify the notation, in $\mathcal{S}(a)$ we omit the dependence on the nominal data $\overline{a}$, $\overline{b}$, and $\overline{x}$. Taking Theorem 2 and equality (11) into account, the end set of $\operatorname{conv}\left\{a_t, t\in T_{\overline{\sigma}}\left(\overline{x}\right)\right\}$ constitutes a crucial ingredient in the computation of $\mathcal{S}(a)$ for any $a\in\mathbb{R}^{n\times m}$. The following subsection is devoted to analyzing the stability behavior of this end set under perturbations of the a_t's; this is a subject of independent interest.

3.1 Stability of the end set of polyhedra

This subsection is intended to be self-contained as far as our statements on $\left\{a_t,\ t\in T_{\overline{\sigma}}\left(\overline{x}\right)\right\}$ could be given for any finite family in $\mathbb{R}^n$, not necessarily coming from a linear inequality system. In this way, set $T_{\overline{\sigma}}\left(\overline{x}\right)$ could be replaced by any finite index set. Accordingly, throughout this subsection we consider a finite index set I. For each $a=\left(a_t\right)_{t\in I}\in(\mathbb{R}^n)^I$, we define

$$E(a) := \operatorname{end}\operatorname{conv}\left\{a_t,\ t\in I\right\}, \tag{13}$$

and the families $\mathcal{D}_a$ and $\mathcal{D}_a^0$ coming from replacing in (7) and (8), respectively, $T_{\overline{\sigma}}\left(\overline{x}\right)$ by I and $\overline{a}$ by a. Recall that, from Theorem 1,

$$E(a) = \bigcup_{D \in \mathcal{D}_a} \operatorname{conv}\left\{a_t,\ t \in D\right\},\ \text{for each } a \in (\mathbb{R}^n)^I. \tag{14}$$

The following lemma provides the Painlevé-Kuratowski upper/outer limit of $\mathcal{D}_a \subset 2^I$ (the subsets of I), with a approaching $\overline{a}$; in it, the finite set 2^I is endowed with the discrete topology.

Lemma 1 *Given* $\overline{a} \in (\mathbb{R}^n)^I$, *we have*:

(i) $\operatorname*{Lim\ sup}_{a \to \overline{a}} \bigcup_{D \in \mathcal{D}_a} \operatorname{conv}\left\{a_t,\ t \in D\right\} = \bigcup_{D \in \operatorname*{Lim\ sup}_{a \to \overline{a}} \mathcal{D}_a} \operatorname{conv}\left\{\overline{a}_t,\ t \in D\right\}$;

(ii) $\operatorname*{Lim\ sup}_{a \to \overline{a}} \mathcal{D}_a = \left\{S \subset I \mid \exists D \in \mathcal{D}_{\overline{a}} \cup \mathcal{D}^0_{\overline{a}} \text{ with } S \subset D\right\}$.

Proof (i) Let $u \in \operatorname{Lim\ sup}_{a \to \overline{a}} \bigcup_{D \in \mathcal{D}_a} \operatorname{conv}\left\{a_t,\ t \in D\right\}$ be written as $u = \lim_{r \to \infty} u^r$ with $u^r = \sum_{t \in D_r} \lambda^r_t a^r_t$, $\sum_{t \in D_r} \lambda^r_t = 1$, $\lambda^r_t \geq 0$ for all $t \in D_r$, for certain $D_r \in \mathcal{D}_{a^r}$ associated with some sequence $a^r \to \overline{a}$. Since $D_r \subset I$ (finite) for all r, it is not restrictive to assume (by taking a suitable subsequence) that $\left\{D_r\right\}_{r \in \mathbb{N}}$ is constant, say $D_r = D$, and $\left\{\lambda^r_t\right\}_r$ converges to some $\lambda_t \geq 0$ for each $t \in D$, hence $\sum_{t \in D} \lambda_t = 1$ and $u = \sum_{t \in D} \lambda_t \overline{a}_t$, with

$$D \in \operatorname*{Lim\ sup}_{r \to \infty} \mathcal{D}_{a^r} \subset \operatorname*{Lim\ sup}_{a \to \overline{a}} \mathcal{D}_a.$$

Now, let us prove '$\supset$'. Take $u = \sum_{t \in \widetilde{D}} \lambda_t \overline{a}_t$ with $\sum_{t \in \widetilde{D}} \lambda_t = 1$, $\lambda_t \geq 0$ for all $t \in \widetilde{D}$ and $\widetilde{D} \in \operatorname*{Lim\ sup}_{a \to \overline{a}} \mathcal{D}_a$. Then, there exists $a^r \to \overline{a}$ with $\widetilde{D} \in \mathcal{D}_{a^r}$ for all r, which entails

$$\bigcup_{D \in \mathcal{D}_{a^r}} \operatorname{conv}\left\{a^r_t,\ t \in D\right\} \ni \sum_{t \in \widetilde{D}} \lambda_t a^r_t \to u.$$

Accordingly, $u \in \operatorname{Lim\ sup}_{a \to \overline{a}} \bigcup_{D \in \mathcal{D}_a} \operatorname{conv}\left\{a_t,\ t \in D\right\}$.

(ii) We start by proving the inclusion '$\supset$'. Let $S \subset I$ be such that $S \subset D$ for some $D \in \mathcal{D}_{\overline{a}} \cup \mathcal{D}^0_{\overline{a}}$. If $D \in \mathcal{D}_{\overline{a}}$, take $p = 1$, otherwise ($D \in \mathcal{D}^0_{\overline{a}}$) take $p = 0$. In any case, let $d \in \mathbb{R}^n \setminus \{0_n\}$ be such that

$$\begin{cases} \overline{a}'_t d = p,\ t \in D, \\ \overline{a}'_t d < p,\ t \in I \setminus D. \end{cases}$$

Define the sequence by

$$a^r_t := \begin{cases} \overline{a}_t + \frac{1}{r} d, & t \in S, \\ \overline{a}_t, & t \in I \setminus S, \end{cases}$$

so that, denoting by $\|\cdot\|_2$ the Euclidean norm,

$$\left(a^r_t\right)' \left(p + \frac{1}{r} \|d\|^2_2\right)^{-1} d \begin{cases} = 1,\ t \in S, \\ < 1,\ t \in I \setminus S, \end{cases}$$

for all $r \in \mathbb{N}$; i.e., in both cases ($p = 0$ or $p = 1$), $S \in \mathcal{D}_{a^r}$ for all $r \in \mathbb{N}$. Therefore, $S \in \limsup_{a \to \overline{a}} \mathcal{D}_a$.

Let us prove the converse inclusion, '$\subset$'. Take any $S \in \operatorname{Lim\,sup}_{a \to \overline{a}} \mathcal{D}_a$ and assume the non-trivial case $S \neq \emptyset$. There exists some sequence $a^r \to \overline{a}$ such that $S \in \mathcal{D}_{a^r}$ for all $r \in \mathbb{N}$. Hence, for each r there exists an associated $d^r \in \mathbb{R}^n \backslash \{0_n\}$ such that

$$\begin{cases} \left(a_t^r\right)' d^r = 1, \text{ if } t \in S, \\ \left(a_t^r\right)' d^r < 1, \text{ if } t \in I \setminus S. \end{cases} \tag{15}$$

If $\{d^r\}_{r \in \mathbb{N}}$ is bounded, then we can take a subsequence (denoted as the whole sequence for simplicity) converging to some $d \in \mathbb{R}^n$. Since $\left(a_t^r\right)' d^r = 1$ for $t \in S \neq \emptyset$, we conclude $\overline{a}_t' d = 1$ for those t, which entails $d \neq 0_n$, i.e.,

$$S \subset D := \left\{t \in I \mid \overline{a}_t' d = 1\right\}$$

(the inclusion may be strict) and $D \in \mathcal{D}_{\overline{a}}$.

In the case when $\{d^r\}_{r \in \mathbb{N}}$ is unbounded we may assume (by taking an appropriate subsequence if necessary) that $\|d^r\| \to +\infty$ and $\frac{d^r}{\|d^r\|} \to d \in \mathbb{R}^n$ with $\|d\| = 1$. Then, dividing both sides of (15) by $\|d^r\|$ and letting $r \to +\infty$ we obtain

$$S \subset D := \left\{t \in I \mid \overline{a}_t' d = 0\right\}$$

(the inclusion may be strict again) and $D \in \mathcal{D}_{\overline{a}}^0$. In any case $S \subset D$, with $D \in \mathcal{D}_{\overline{a}} \cup \mathcal{D}_{\overline{a}}^0$, and the proof is complete. □

Theorem 3 *Let $\overline{a} \in (\mathbb{R}^n)^I$. We have*

(i) $\operatorname*{Lim\,inf}_{a \to \overline{a}} E(a) = \bigcup_{D \in \mathcal{D}_{\overline{a}}} \operatorname{conv}\{\overline{a}_t,\ t \in D\} = E(\overline{a})$;

(ii) $\operatorname*{Lim\,sup}_{a \to \overline{a}} E(a) = \bigcup_{D \in \mathcal{D}_{\overline{a}} \cup \mathcal{D}_{\overline{a}}^0} \operatorname{conv}\{\overline{a}_t,\ t \in D\} \supset E(\overline{a})$.

Proof (i) The second equality is established Theorem 1 and it is clear from the definition that $\operatorname{Lim\,inf}_{a \to \overline{a}} E(a) \subset E(\overline{a})$ as E is closed-valued. In order to prove the converse inclusion, take $D \in \mathcal{D}_{\overline{a}}$ and $u = \sum_{t \in D} \lambda_t \overline{a}_t$ for some $\lambda = \left(\lambda_t\right)_{t \in D} \in \mathbb{R}_+^D$ with $\sum_{t \in D} \lambda_t = 1$ and let $d \in \mathbb{R}^n$ with $\overline{a}_t' d = 1$ for all $t \in D$ and $\overline{a}_t' d < 1$ for all $t \in I \backslash D$. Taking any $\{a^r\}_{r \in \mathbb{N}} \subset (\mathbb{R}^n)^I$ converging to $\overline{a}$, define, for each r, $w^r := \sum_{t \in D} \lambda_t a_t^r \in \operatorname{conv}\left\{a_t^r,\ t \in I\right\}$. Then

$$(w^r)' d \to u' d = \sum_{t \in D} \lambda_t \overline{a}_t' d = 1 \text{ as } r \to \infty.$$

On the other hand, for a fixed r and any $v \in \operatorname{conv}\left\{a_t^r,\ t \in I\right\}$, writing $v = \sum_{t \in I} \gamma_t a_t^r$ with $\gamma \in \mathbb{R}_+^I$ and $\sum_{t \in I} \gamma_t = 1$, one has

$$v'd \leq \left(\sum_{t \in I} \gamma_t \overline{a}_t' d \right) + \left(\sum_{t \in I} \gamma_t \left\| a_t^r - \overline{a}_t \right\|_* \|d\| \right) \leq 1 + \left\| a^r - \overline{a} \right\| \|d\|;$$

in particular, $(w^r)'d \leq 1 + \left\| a^r - \overline{a} \right\| \|d\|$. This entails, for each $r \in \mathbb{N}$ such that $(w^r)'d > 0$, the existence of

$$\mu_r \in \left[1, \frac{1 + \left\| a^r - \overline{a} \right\| \|d\|}{(w^r)'d} \right]$$

such that $u^r := \mu_r w^r \in E(a^r)$. Consequently,

$$u = \lim_{r \to \infty} u^r \in \operatorname{Lim\,inf}_{r \to \infty} E(a^r).$$

Since $\{a^r\}_{r \in \mathbb{N}}$ has been arbitrarily chosen, one has $u \in \operatorname{Lim\,inf}_{a \to \overline{a}} E(a)$.

(ii) comes from the Lemma 1 together with (14). □

Remark 3 (i) Theorem 3 (i) establishes the lower semicontinuity in the sense of Berge of mapping E at $\overline{a}$ (equivalently, the inner semicontinuity at $\overline{a}$). Here we do not have an analogous result to Lemma 1 (ii); more specifically, $\operatorname{Lim\,inf}_{a \to \overline{a}} \mathcal{D}_a$ may be strictly contained in $\mathcal{D}_{\overline{a}}$. For instance, for the family

$$\left\{ \overline{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \overline{a}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \overline{a}_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

one has $\operatorname{Lim\,inf}_{a \to \overline{a}} \mathcal{D}_a = \{\emptyset, \{1\}, \{3\}\}$, whereas $\mathcal{D}_{\overline{a}} = \{\emptyset, \{1\}, \{3\}, \{1,2,3\}\}$.

(ii) The union in Theorem 3 (ii) could be confined to those $D \in \mathcal{D}_{\overline{a}} \cup \mathcal{D}_{\overline{a}}^0$ which are maximal with respect to the inclusion order. Moreover, the inclusion '$\supset$' may be strict as Example 1 below shows.

3.2 Lower and upper semicontinuity of the subregularity modulus

Let us consider the nominal data $\overline{\sigma} \equiv \left(\overline{a}, \overline{b}\right)$ and $\overline{x} \in \mathcal{F}_{\overline{a}}\left(\overline{b}\right)$. From now on in the paper, for each $a \in \mathbb{R}^{n \times m}$, we consider the end set defined in (13) in the particular case $I = T_{\overline{\sigma}}\left(\overline{x}\right)$; i.e.,

$$E(a) := \operatorname{end\,conv}\left\{a_t,\ t \in T_{\overline{\sigma}}\left(\overline{x}\right)\right\}. \tag{16}$$

Observe that the index set $T_{\overline{\sigma}}\left(\overline{x}\right)$ does not vary as a varies by virtue of (11). From Theorem 2 we can write

$$\mathcal{S}(a) = d_*\left(0_n, E(a)\right)^{-1}, \text{ for any } a \in \mathbb{R}^{n \times m}. \tag{17}$$

Theorem 4 *Let* $\overline{\sigma} \equiv \left(\overline{a}, \overline{b}\right)$ *and* $\overline{x} \in \mathcal{F}_{\overline{a}}\left(\overline{b}\right)$. *Then*:

(i) *$\mathcal{S}$ is lower semicontinuous at $\overline{a}$; i.e.,*

$$\liminf_{a\to\overline{a}} \mathcal{S}(a) = \left[d_*\left(0_n, \bigcup_{D\in\mathcal{D}_{\overline{a}}} \operatorname{conv}\{\overline{a}_t,\ t\in D\}\right)\right]^{-1} = \mathcal{S}(\overline{a});$$

(ii) *We have*

$$\limsup_{a\to\overline{a}} \mathcal{S}(a) = \left[d_*\left(0_n, \bigcup_{D\in\mathcal{D}_{\overline{a}}\cup\mathcal{D}^0_{\overline{a}}} \operatorname{conv}\{\overline{a}_t,\ t\in D\}\right)\right]^{-1} \geq \mathcal{S}(\overline{a}).$$

Proof (i) Since E is inner semicontinuous at $\overline{a}$, by [25, Proposition 5.11(b)] we have that $d_*(0_n, E(\cdot))$ is upper semicontinuous at $\overline{a}$ and, accordingly, $d_*(0_n, E(\cdot))^{-1}$ is lower semicontinuous at $\overline{a}$.

(ii)

Appealing to (17), we may write

$$\begin{aligned}\limsup_{a\to\overline{a}} \mathcal{S}(a) &= \limsup_{a\to\overline{a}} d_*(0_n, E(a))^{-1} \\ &= \left(\liminf_{a\to\overline{a}} d_*(0_n, E(a))\right)^{-1} \\ &= d_*\left(0_n,\ \operatorname{Lim\ sup}_{a\to\overline{a}} E(a)\right)^{-1} \\ &= d_*\left(0_n, \bigcup\nolimits_{D\in\mathcal{D}_{\overline{a}}\cup\mathcal{D}^0_{\overline{a}}} \operatorname{conv}\{\overline{a}_t,\ t\in D\}\right)^{-1}.\end{aligned}$$

where the third equality follows from [25, Exercise 4.8] and the last one comes from Theorem 3(ii). □

Corollary 1 *If* $\operatorname{Lim\ inf}_{a\to\overline{a}} E(a) = \operatorname{Lim\ sup}_{a\to\overline{a}} E(a) = E(\overline{a})$, *i.e., if E is continuous in the Painlevé-Kuratowski sense, then $\mathcal{S}$ is continuous at $\overline{a}$.*

Proof As in the proof of statement (ii) in Theorem 4, and applying the current assumption, we have

$$\limsup\nolimits_{a\to\overline{a}} \mathcal{S}(a) = d_*\left(0_n,\ \operatorname{Lim\ sup}_{a\to\overline{a}} E(a)\right)^{-1} = d_*\left(0_n, E(\overline{a})\right)^{-1} = \mathcal{S}(\overline{a}).$$

□

Remark 4 Observe that:

(i) $\mathcal{S}$ may fail to be upper semicontinuous at $\overline{a}$, i.e., one can have

$$\limsup_{a\to\overline{a}} \mathcal{S}(a) > \mathcal{S}(\overline{a})$$

and the 'continuity gap' can be finite (Example 1) or infinite (Example 2).

(ii) The sufficient condition for the continuity of $\mathcal{S}$ given in Corollary 1 is not necessary. Just replace $\overline{a}_3$ in Example 1 below with $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$.

Example 1 Consider the nominal system in $\mathbb{R}^2$, endowed with the Euclidean norm,

$$\overline{\sigma} := \{x_2 \le 0,\ 2x_2 \le 0,\ x_1 + 2x_2 \le 0\},$$

and take $\overline{x} = 0_2$. One easily checks from Theorem 3(*ii*) that $\operatorname*{Lim\,sup}_{a\to\overline{a}} E(a) = \text{conv}\left\{\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\} \bigcup \text{conv}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}\right\}$ while $E(\overline{a}) = \text{conv}\left\{\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$. Regarding function $\mathcal{S}$, from Theorem 4(*ii*), one has

$$\limsup_{a\to\overline{a}} \mathcal{S}(a) = 1 > \mathcal{S}(\overline{a}) = 1/2.$$

Example 2 Consider the nominal system in $\mathbb{R}^2$, endowed with the Euclidean norm,

$$\overline{\sigma} := \{x_2 \le 0,\ -x_2 \le 0\},$$

and take $\overline{x} = 0_2$. Again from Theorem 3(*ii*), we have $\text{Lim sup}_{a\to\overline{a}} E(a) = \text{conv}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}$ while $E(\overline{a}) = \left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}$ and

$$\limsup_{a\to\overline{a}} \mathcal{S}(a) = +\infty,\ \mathcal{S}(\overline{a}) = 1.$$

4 Robust and continuous subregularity

Starting from the fact that

$$\limsup_{a\to\overline{a}} \mathcal{S}(a) \ge \mathcal{S}(\overline{a}) = \liminf_{a\to\overline{a}} \mathcal{S}(a),$$

and taking into account that the inequality above may be strict, this section is firstly devoted to characterizing the finiteness of the continuity gap, i.e., to characterize the condition $\limsup_{a\to\overline{a}} \mathcal{S}(a) < +\infty$, through an alternative (in principle, simpler) condition to the one which can be derived from the explicit formula of Theorem 4(*ii*). In a second stage, we provide a new approach to $\limsup_{a\to\overline{a}} \mathcal{S}(a)$ which allows us to interpret this quantity as a modulus of a robust-type metric subregularity property.

To start with, appealing to the definitions of $\mathcal{D}_{\overline{a}}$ and $\mathcal{D}_{\overline{a}}^0$, recall (7)-(8), one easily checks that

$$\bigcup_{D\in\mathcal{D}_{\overline{a}}\cup\mathcal{D}_{\overline{a}}^0} \text{conv}\,\{\overline{a}_t,\ t\in D\} \subset \text{bd}\,\text{conv}\{\overline{a}_t,\ t\in T_{\overline{\sigma}}(\overline{x})\},$$

and the following proposition is an immediate consequence of this inclusion together with Theorem 4(*ii*).

Proposition 1 *We have*

$$\limsup_{a\to\overline{a}} \mathcal{S}(a) \leq d_*\left(0_n, \mathrm{bd}\,\mathrm{conv}\left\{\overline{a}_t,\ t\in T_{\overline{\sigma}}(\overline{x})\right\}\right)^{-1}. \tag{18}$$

The following example shows that the inequality in Proposition 1 may be strict.

Example 3 [9, Example 4] Let us consider the nominal system, in $\mathbb{R}^2$ endowed with the Euclidean norm,

$$\left\{x_1 \leq 0,\ x_2 \leq 0,\ x_1 + x_2 \leq 0\right\}$$

(associated with indices 1, 2, and 3, respectively) and the nominal solution $\overline{x} = 0_2$; in other words, recalling that each $\overline{a}_t$ is regarded as a column-vector,

$$\overline{a} = \left(\ \overline{a}_1\ \overline{a}_2\ \overline{a}_3\ \right) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \overline{b} = 0_3,$$

which entails $T_{\overline{\sigma}}(\overline{x}) = T = \{1,2,3\}$. Then, after a routine computation, we can show that $\|a - \overline{a}\| \leq \frac{1}{2\sqrt{2}}$ implies

$$\mathrm{end}\,\mathrm{conv}\left\{a_t,\ t\in T_{\overline{\sigma}}(\overline{x})\right\} = \mathrm{conv}\left\{a_1, a_3\right\} \cup \mathrm{conv}\left\{a_2, a_3\right\}. \tag{19}$$

Observe that the condition '$\|a - \overline{a}\| \leq \frac{1}{2\sqrt{2}}$' is not superfluous to ensure (19); indeed, if we take the unitary vector $u = \frac{1}{\sqrt{2}}\binom{1}{1}$ and, for any $\mu > \frac{1}{2\sqrt{2}}$, we consider the perturbed matrix $a^\mu = \left(\ \overline{a}_1 + \mu u\ \overline{a}_2 + \mu u\ \overline{a}_3 - \mu u\ \right)$, then we obtain $\mathrm{end}\,\mathrm{conv}\left\{a_t^\mu,\ t = 1,2,3\right\} = \mathrm{conv}\left\{a_1, a_2\right\}$. Moreover, $\|a - \overline{a}\| \leq \frac{1}{2\sqrt{2}}$ also implies $d_*\left(0_2, \mathrm{bd}\,\mathrm{conv}\left\{a_t,\ t\in T_{\overline{\sigma}}(\overline{x})\right\}\right) = d_*\left(0_2, \mathrm{conv}\left\{a_1, a_2\right\}\right)$. In particular,

$$\begin{aligned}\limsup_{a\to\overline{a}} \mathcal{S}(a) &= d_*\left(0_2, \mathrm{conv}\left\{\overline{a}_1, \overline{a}_3\right\} \cup \mathrm{conv}\left\{\overline{a}_2, \overline{a}_3\right\}\right)^{-1} = 1 \\ &< d_*\left(0_2, \mathrm{bd}\,\mathrm{conv}\left\{\overline{a}_t,\ t\in T_{\overline{\sigma}}(\overline{x})\right\}\right)^{-1} = \sqrt{2}.\end{aligned}$$

In spite of not having equality in (18), $d_*\left(0_n, \mathrm{bd}\,\mathrm{conv}\left\{\overline{a}_t,\ t\in T_{\overline{\sigma}}(\overline{x})\right\}\right)$ can be used to characterize the finiteness of $\limsup_{a\to\overline{a}} \mathcal{S}(a)$, as the following theorem establishes.

Theorem 5 *Given* $\overline{\sigma} \equiv \left(\overline{a}, \overline{b}\right)$ *and* $\overline{x} \in \mathcal{F}_{\overline{a}}\left(\overline{b}\right)$, *the following statements are equivalent*:

(i) $\limsup_{a\to\overline{a}} \mathcal{S}(a)$ *is finite*;

(ii) $0_n \notin \mathrm{bd}\,\mathrm{conv}\left\{\overline{a}_t,\ t\in T_{\overline{\sigma}}(\overline{x})\right\}$;

(iii) *There exist constants* $\kappa \geq 0$ *and* $\varepsilon > 0$ *along with a neighborhood* U *of* $\overline{x}$ *such that*

$$d\left(x, \mathcal{F}_a\left(\overline{b} + (a - \overline{a})'\overline{x}\right)\right) \leq \kappa d\left(\overline{b} + (a - \overline{a})'\overline{x}, \mathcal{G}_a(x)\right) \tag{20}$$

for all $x \in U$ and all $a \in \mathbb{R}^n$ such that $\|a - \overline{a}\| < \varepsilon$.

Moreover, $\limsup_{a \to \overline{a}} \mathcal{S}(a)$ coincides with the infimum of constants κ over the triplets (κ, ε, U) satisfying (20) .

Proof $(i) \Leftrightarrow (ii)$ Implication '$\Leftarrow$' is a direct consequence of Proposition 1. In order to prove the converse implication, assume reasoning by contradiction that $0_n \in \operatorname{bd}\operatorname{conv}\left\{\overline{a}_t,\ t \in T_{\overline{\sigma}}(\overline{x})\right\}$. By separation, consider $d \in \mathbb{R}^n \setminus \{0_n\}$ such that $\overline{a}_t' d \leq 0$ for all $t \in T_{\overline{\sigma}}(\overline{x})$, which necessarily satisfies $D := \left\{t \in T \mid \overline{a}_t' d = 0\right\} \neq \emptyset$. Consider an arbitrary $\varepsilon > 0$ and let $a_t^\varepsilon := \overline{a}_t + \varepsilon d$ for all $t \in T$. For $\widetilde{d} := \left(\varepsilon d' d\right)^{-1} d$ we clearly have $\left(a_t^\varepsilon\right)' \widetilde{d} = 1$ for all $t \in D$ and $\left(a_t^\varepsilon\right)' \widetilde{d} < 1$ for all $t \in T_{\overline{\sigma}}(\overline{x}) \setminus D$. Consequently, taking (11) into account, $D \in \mathcal{D}_{a^\varepsilon}$. Moreover, the fact that $0_n \in \operatorname{bd}\operatorname{conv}\left\{\overline{a}_t,\ t \in T_{\overline{\sigma}}(\overline{x})\right\}$ and the definition of D easily imply $0_n \in \operatorname{conv}\left\{\overline{a}_t,\ t \in D\right\}$ and, then, $\varepsilon d \in \operatorname{conv}\left\{a_t^\varepsilon,\ t \in D\right\}$. Accordingly, recalling Theorem 2, we attain the contradiction

$$\mathcal{S}(a^\varepsilon) \geq d_*\left(0_n, \operatorname{conv}\left\{a_t^\varepsilon,\ t \in D\right\}\right)^{-1} \geq \left(\varepsilon \|d\|_*\right)^{-1} \to +\infty \text{ as } \varepsilon \downarrow 0.$$

$(i) \Leftrightarrow (iii)$ Implication '$\Leftarrow$' comes from the fact that any $\kappa \geq 0$ as in the statement is a subregularity constant for $\mathcal{G}_a$ at $\left(\overline{x}, \overline{b} + (a - \overline{a})'\overline{x}\right)$; in other words, taking κ and ε as in (iii), we have

$$\mathcal{S}(a) \leq \kappa, \text{ whenever } \|a - \overline{a}\| < \varepsilon,$$

entailing $\limsup_{a \to \overline{a}} \mathcal{S}(a) \leq \kappa$. In order to prove the converse implication assume that $\limsup_{a \to \overline{a}} \mathcal{S}(a)$ is finite and take any $\kappa > \limsup_{a \to \overline{a}} \mathcal{S}(a)$. Let us prove that there exists a neighborhood U of $\overline{x}$ along with $\varepsilon > 0$ such that (20) holds for all $x \in U$ and all $a \in \mathbb{R}^n$ with $\|a - \overline{a}\| < \varepsilon$. To do this we appeal to [10, Theorem 3], which shows –adapted to our current notation– that each $\mathcal{S}(a)$ is indeed a subregularity constant itself with an associated neighborhood U_a, which –see formula (8) in that paper–, taking into account (11) and the 'slack relationship' $\left[\overline{b}_t + \left(a_t - \overline{a}_t\right)'\overline{x}\right] - a_t'\overline{x} = \overline{b}_t - \overline{a}_t'\overline{x}$, is given by

$$U_a := \left\{ x \in \mathbb{R}^n \;\middle|\; \|x - \overline{x}\| < \delta_a := \inf_{t \notin T_{\overline{\sigma}}(\overline{x}),\ a_t \neq 0_n} \frac{\overline{b}_t - \overline{a}_t'\overline{x}}{2\|a_t\|_*} \right\}, \tag{21}$$

with the convention $\inf \emptyset := +\infty$. First, we analyze the case $\{t \in T \setminus T_{\overline{\sigma}}(\overline{x}) \mid \overline{a}_t \neq 0_n\} \neq \emptyset$. Now define $\rho := \min_{t \notin T_{\overline{\sigma}}(\overline{x})} \left(\overline{b}_t - \overline{a}_t'\overline{x}\right) > 0$ (recalling the finiteness of T) and take any

$$0 < \delta_1 < \min\left\{ \frac{\rho}{2\delta_{\overline{a}}}, \min\left\{ \left\| \overline{a}_t \right\|_* \mid t \notin T_{\overline{\sigma}}(\overline{x}),\ \overline{a}_t \neq 0_n \right\} \right\}.$$

Assume $\|a - \overline{a}\| < \delta_1$; then, $a_t \neq 0_n$ whenever $t \notin T_{\overline{\sigma}}(\overline{x})$ and $\overline{a}_t \neq 0_n$. If, for some $t \notin T_{\overline{\sigma}}(\overline{x})$, we have$a_t \neq 0_n$ and $\overline{a}_t = 0_n$, then,

$$\frac{\overline{b}_t - \overline{a}_t'\overline{x}}{2\left\|a_t\right\|_*} \geq \frac{\overline{b}_t - \overline{a}_t'\overline{x}}{2\delta_1} \geq \frac{\rho}{2\delta_1} \geq \delta_{\overline{a}}.$$

Therefore, $\|a - \overline{a}\| < \delta_1$ implies

$$\delta_a \geq \min_{t \notin T_{\overline{\sigma}}(\overline{x}),\ \overline{a}_t \neq 0_n} \frac{\overline{b}_t - \overline{a}_t'\overline{x}}{2\left(\left\|\overline{a}_t\right\|_* + \delta_1\right)} =: \delta_2.$$

Finally, take any $\delta_3 > 0$ satisfying $\|a - \overline{a}\| < \delta_3 \Rightarrow \mathcal{S}(a) \leq \kappa$. Then,

$$\left. \begin{array}{c} \|a - \overline{a}\| < \varepsilon := \min\left\{\delta_1, \delta_3\right\} \\ \|x - \overline{x}\| < \delta_2 \end{array} \right\} \Rightarrow (20) \text{ holds},$$

which establishes (*iii*) in this case.

Consider now the case when $T_{\overline{\sigma}}(\overline{x}) = T$. In this case, $\delta_a = +\infty$ for all $a \in \mathbb{R}^{n \times m}$. Hence, (20) holds whenever $\|a - \overline{a}\| < \delta_3$ (defined as above) and $x \in \mathbb{R}^n$. Finally, assume $\overline{a}_t = 0$ for all $t \in T \backslash T_{\overline{\sigma}}(\overline{x}) \neq \emptyset$. This entails $\overline{b}_t > 0$ for all $t \in T \backslash T_{\overline{\sigma}}(\overline{x})$. Define $\rho := \min\left\{ \overline{b}_t,\ t \in T \backslash T_{\overline{\sigma}}(\overline{x}) \right\}$. Then $\delta_a \geq \frac{\rho}{2\|a - \overline{a}\|}$ for all $a \in \mathbb{R}^{n \times m}$. Accordingly, (20) holds whenever $\|a - \overline{a}\| < \delta_3$ and $\|x - \overline{x}\| < \frac{\rho}{2\delta_3}$.

Moreover, reviewing the previous argument and observing that the proof is valid for any $\kappa > \limsup_{a \to \overline{a}} \mathcal{S}(a)$, we conclude that $\limsup_{a \to \overline{a}} \mathcal{S}(a)$ coincides with the infimum of constants κ over the triplets (κ, ε, U) satisfying (20). □

Definition 1 Given system $\overline{\sigma} \equiv (\overline{a}, \overline{b})$ and $\overline{x} \in \mathcal{F}_{\overline{a}}\left(\overline{b}\right)$, we say that

(i) $\mathcal{G}_{\overline{a}}$ is *robustly subregular* at $\left(\overline{x}, \overline{b}\right)$ if any of the three equivalent conditions of Theorem 5 holds. Regarding Theorem 5(*iii*), the infimum of constants κ over the triplets (κ, ε, U) satisfying (20) is called the *robust subregularity modulus* of $\mathcal{G}_{\overline{a}}$ at $\left(\overline{x}, \overline{b}\right)$ and will be denoted by $\operatorname{rob} \mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right)$. As stated there,

$$\operatorname{rob} \mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right) = \limsup_{a \to \overline{a}} \mathcal{S}(a). \tag{22}$$

(ii) $\mathcal{G}_{\overline{a}}$ is *continuously subregular* at $\left(\overline{x}, \overline{b}\right)$ if $\mathcal{S}$ is continuous at $\overline{a}$.

Remark 5 Condition (*iii*) in Theorem 5 looks like a kind of uniform regularity property with respect to a. Since the term uniform calmness has been already introduced in [3, Definition 1] (to be applied to the feasible set mapping $\mathcal{F}_{\overline{a}}$) with another meaning –uniformly with respect to x in $\mathcal{F}_{\overline{a}}\left(\overline{b}\right)$–, we have preferred here the term robust.

See the comment preceding Corollary 3 below. Also observe that [4, Theorem 2.1], adapted to our current setting, entails that the metric regularity property of $\mathcal{G}_{\overline{a}}$ at $\left(\overline{x}, \overline{b}\right)$ is characterized as $0_n \notin \operatorname{conv}\left\{\overline{a}_t,\ t \in T_{\overline{\sigma}}\left(\overline{x}\right)\right\}$. Indeed, [4, Corollary 3.2] provides the following expression for the the modulus of metric regularity:

$$\operatorname{reg}\mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right) = 1/d_*\left(0_n, \operatorname{conv}\left\{\overline{a}_t,\ t \in T_{\overline{\sigma}}\left(\overline{x}\right)\right\}\right).$$

Accordingly, if $0_n \in \operatorname{int}\operatorname{conv}\left\{\overline{a}_t,\ t \in T_{\overline{\sigma}}\left(\overline{x}\right)\right\}$, then $\mathcal{G}_{\overline{a}}$ is robustly regular but not metrically regular at $\left(\overline{x}, \overline{b}\right)$.

Corollary 2 *For the nominal data* $\overline{\sigma} \equiv (\overline{a}, \overline{b})$*and* $\overline{x} \in \mathcal{F}_{\overline{a}}\left(\overline{b}\right)$, *the following statements are equivalent*:

(i) $\mathcal{G}_{\overline{a}}$ *is continuously subregular at* $\left(\overline{x}, \overline{b}\right)$;
(ii) $\operatorname{rob}\mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right) = \mathcal{S}\left(\overline{a}\right)$;
(iii) *It holds*

$$0 \neq d_*\left(0_n, \bigcup_{D \in \mathcal{D}_{\overline{a}}} \operatorname{conv}\left\{\overline{a}_t,\ t \in D\right\}\right) = d_*\left(0_n, \bigcup_{D \in \mathcal{D}_{\overline{a}} \cup \mathcal{D}^0_{\overline{a}}} \operatorname{conv}\left\{\overline{a}_t,\ t \in D\right\}\right).$$

Proof The proof comes straightforwardly from (22) and Theorem 4. □

5 Radii

In this section we formally introduce the radii announced in the introduction and succeed to compute one of them and give some hints on the other.

Following the notation introduced in (4), let us denote by $\operatorname{rad}_{\text{rob}}\mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right)$ and $\operatorname{rad}_{\text{cont}}\mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right)$ the radius of robust subregularity and continuous subregularity, respectively, of $\mathcal{G}_{\overline{a}}$ at $\left(\overline{x}, \overline{b}\right)$. As a direct consequence of the definitions, continuous subregularity implies robust subregularity, and, hence,

$$\operatorname{rad}_{\text{rob}}\mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right) \geq \operatorname{rad}_{\text{cont}}\mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right). \tag{23}$$

The next technical lemma provides a quite standard result that could be given with more generality. We state it as we need it, in $\mathbb{R}^n$ endowed with the dual norm $\|\cdot\|_*$.

Lemma 2 *For $i = 1, 2$, let $C_i = \operatorname{conv}\left\{ u_j^i,\ j = 1, ..., m \right\} \subset \mathbb{R}^n$. Assume that for some $u_0 \in \mathbb{R}^n$ we have $d_*\left(u_0, \operatorname{bd} C_1\right) = \delta > 0$ and $\max_{1 \le j \le m} \left\| u_j^1 - u_j^2 \right\|_* \le \varepsilon < \delta$. Then*

$$d_*\left(u_0, \operatorname{bd} C_2\right) \ge \delta - \varepsilon.$$

Proof Firstly we consider the case $u_0 \notin \operatorname{cl} C_1$ and assume, reasoning by contradiction, that $d_*\left(u_0, \operatorname{bd} C_2\right) = \|u_0 - w_2\|_* < \delta - \varepsilon$ for some $w_2 \in C_2$. Let us write $w_2 = \sum_{j=1}^m \lambda_j u_j^2$, with $\lambda_j \ge 0$ for all j and $\sum_{j=1}^m \lambda_j = 1$. Take $w_1 := \sum_{j=1}^m \lambda_j u_j^1 \in C_1$. Then

$$\|u_0 - w_1\|_* \le \|u_0 - w_2\|_* + \sum_{j=1}^m \lambda_j \left\| u_j^2 - u_j^1 \right\|_* < \delta - \varepsilon + \varepsilon = \delta,$$

contradicting the fact that $d_*\left(u_0, \operatorname{bd} C_1\right) = d_*\left(u_0, C_1\right) = \delta$.

Secondly, consider the case $u_0 \in \operatorname{int} C_1$, so that $u_0 + \delta B_* \subset C_1$. Then we will prove that $u_0 + (\delta - \varepsilon) B_* \subset C_2$. The argument here is similar to that of [8, Lemma 6], which we sketch here for completeness. Assume by contradiction that there exists $\widetilde{w}_2 \in \left(u_0 + (\delta - \varepsilon) B_*\right) \backslash C_2$. Then we can strictly separate $\widetilde{w}_2$ and C_2, so that there exists $p \in \mathbb{R}^n$ such that

$$p' \widetilde{w}_2 < p' u_j^2 \text{ for all } j = 1, ..., m. \tag{24}$$

Take $z \in \mathbb{R}^n$ with $\|z\|_* = \varepsilon$ and $p'z = \|p\| \|z\|_*$. Then

$$\left\| \widetilde{w}_2 - z - u_0 \right\|_* \le \left\| \widetilde{w}_2 - u_0 \right\|_* + \|z\|_* \le \delta - \varepsilon + \varepsilon = \delta,$$

entailing $\widetilde{w}_2 - z \in C_1$. Thus write $\widetilde{w}_2 - z = \sum_{j=1}^m \widetilde{\lambda}_j u_j^1$, with $\widetilde{\lambda}_j \ge 0$ for all j and $\sum_{j=1}^m \widetilde{\lambda}_j = 1$. Therefore, recalling (24), we attain the contradiction

$$p' \widetilde{w}_2 - p'z = \sum_{j=1}^m \widetilde{\lambda}_j p' u_j^2 + \sum_{j=1}^m \widetilde{\lambda}_j p' \left(u_j^1 - u_j^2 \right) > p' \widetilde{w}_2 - \|p\| \varepsilon = p' \widetilde{w}_2 - p'z.$$

□

Theorem 6 *Assume that $\mathcal{G}_{\overline{a}}$ is robustly subregular at $\left(\overline{x}, \overline{b}\right)$. Then*

$$\operatorname{rad}_{\text{rob}} \mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right) = d_*\left(0_n,\ \operatorname{bd} \operatorname{conv}\left\{ \overline{a}_t,\ t \in T_{\overline{\sigma}}\left(\overline{x}\right) \right\}\right).$$

Proof Write $\delta := d_*\left(0_n, \operatorname{bd} \operatorname{conv}\left\{ \overline{a}_t,\ t \in T_{\overline{\sigma}}\left(\overline{x}\right) \right\}\right)$ and pick any $a \in \mathbb{R}^n$ with $\|a - \overline{a}\| < \delta$. Then Lemma 2 entails

$$d_*\left(0_n, \operatorname{bd} \operatorname{conv}\left\{ a_t,\ t \in T_{\overline{\sigma}}\left(\overline{x}\right) \right\}\right) \ge \delta - \|a - \overline{a}\|,$$

which in particular implies $0_n \notin \operatorname{bd}\operatorname{conv}\left\{a_t,\ t \in T_{\overline{\sigma}}\left(\overline{x}\right)\right\}$. Therefore, Theorem 5 yields that $\mathcal{G}_a$ is robustly subregular at $\left(\overline{x}, \overline{b} + \left(a - \overline{a}\right)'\overline{x}\right)$. This, together with (5), proves $\operatorname{rad}_{\text{rob}} \mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right) \geq \delta$.

In order to prove the converse inequality, take $u \in \operatorname{bd}\operatorname{conv}\left\{\overline{a}_t,\ t \in T_{\overline{\sigma}}\left(\overline{x}\right)\right\}$ with $\|u\|_* = \delta$. Let us show that $0_n \in \operatorname{bd}\operatorname{conv}\left\{\overline{a}_t - u,\ t \in T_{\overline{\sigma}}\left(\overline{x}\right)\right\}$. It is clear that $0_n \in \operatorname{conv}\left\{\overline{a}_t - u,\ t \in T_{\overline{\sigma}}\left(\overline{x}\right)\right\}$ since, if $u = \sum_{t\in T} \lambda_t \overline{a}_t$, with $\lambda_t \geq 0$ for all $t \in T$ and $\sum_{t\in T} \lambda_t = 1$, then $0_n = \sum_{t\in T} \lambda_t\left(\overline{a}_t - u\right)$. Moreover, a very similar calculation shows that if there existed $\varepsilon > 0$ with $\varepsilon B_* \subset \operatorname{conv}\left\{\overline{a}_t - u,\ t \in T_{\overline{\sigma}}\left(\overline{x}\right)\right\}$, i.e. $0_n \in \operatorname{int}\operatorname{conv}\left\{\overline{a}_t - u,\ t \in T_{\overline{\sigma}}\left(\overline{x}\right)\right\}$, then we would have the contradiction $u + \varepsilon B_* \subset \operatorname{conv}\left\{\overline{a}_t,\ t \in T_{\overline{\sigma}}\left(\overline{x}\right)\right\}$. Now Theorem 5 ensures that $\mathcal{G}_{(\overline{a}_t - u)_{t\in T}}$ is not robustly subregular at $\left(\overline{x}, \left(\overline{b}_t - u'\overline{x}\right)_{t\in T}\right)$. Accordingly, $\operatorname{rad}_{\text{rob}} \mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right) \leq \delta$. □

As an immediate consequence of Theorem 6, together with Proposition 1, (22), and Example 3, we obtain the following result. Observe that this is the opposite inequality to that obtained in general for the radius of metric regularity in [12, Theorem 1.5] (where radius ≥ 1/modulus). The last part of this result, which comes from Theorem 5(ii) together with the definition of robust subregularity, asserts that if $\mathcal{G}_{\overline{a}}$ is robustly subregular at $\left(\overline{x}, \overline{b}\right)$, then the robust subregularity radius is positive. Otherwise, the term 'robust' would sound inappropriate.

Corollary 3 *One has*

$$\operatorname{rad}_{\text{rob}} \mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right) \leq \frac{1}{\operatorname{rob} \mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right)},$$

and the inequality may be strict. Moreover, $\operatorname{rob} \mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right) < +\infty$ *implies* $\operatorname{rad}_{\text{rob}} \mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right) > 0$.

The next example shows that inequality (23) may be strict, as well as provides some hints for the study of the radius of continuous subregularity.

Example 4 Let us consider the nominal system, in $\mathbb{R}^2$ endowed with the Euclidean norm,

$$\overline{\sigma} = \left\{x_1 + 2x_2 \leq 0,\ x_1 + 4x_2 \leq 0,\ 6x_1 + 5x_2 \leq 0\right\}$$

(associated with indices 1, 2 and 3, respectively) and the nominal solution $\overline{x} = 0_2$. Hence, $T_{\overline{\sigma}}\left(\overline{x}\right) = \{1, 2, 3\}$. Let us check that

$$\operatorname{rad}_{\text{cont}} \mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right) = \frac{1}{\sqrt{10}} < \operatorname{rad}_{\text{rob}} \mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right) = \sqrt{5},$$

where the last equality comes from Theorem 6. Indeed, writing $\overline{\sigma} \equiv \left(\overline{a}, \overline{b}\right)$, the minimum perturbation size from $\overline{a}$ making the perturbed a have proportional a_1 and a_2 (i.e., making a_1 and a_2 belong to some straight line in $\mathbb{R}^2$ passing through the origin), obtained by computing, with the well-known Ascoli formula, the distance from $\overline{a}_1$ and $\overline{a}_2$ to such line, is $1/\sqrt{10}$. This minimum perturbation size is attained at the following system $\sigma_\mu \equiv (a^\mu, b^\mu)$ for $\mu = 1/\sqrt{10}$, where the perturbed $a_1^{1/\sqrt{10}}$ and $a_2^{1/\sqrt{10}}$ are the only possible ones with the aimed property and the perturbed $a_3^{1/\sqrt{10}}$ is irrelevant (so that we have kept it as in the nominal system), and where we have followed the criterion $b^\mu = \overline{b} + \left(a^\mu - \overline{a}\right)' \overline{x}$. Define

$$\sigma_\mu = \begin{cases} \left(1 - \frac{3}{\sqrt{10}}\mu\right)x_1 + \left(2 + \frac{1}{\sqrt{10}}\mu\right)x_2 \leq 0, \\ \left(1 + \frac{3}{\sqrt{10}}\mu\right)x_1 + \left(4 - \frac{1}{\sqrt{10}}\mu\right)x_2 \leq 0, \\ 6x_1 + 5x_2 \leq 0. \end{cases}$$

Thus, we have:

$$\begin{aligned} \|a - \overline{a}\| \leq \frac{1}{\sqrt{10}} \Rightarrow \mathcal{S}(a) &= d_*\left(0_2, \operatorname{end}\operatorname{conv}\left\{a_t,\ t \in T_{\overline{\sigma}}(\overline{x})\right\}\right)^{-1} \\ &= d_*\left(0_2, \operatorname{conv}\left\{a_2, a_3\right\}\right)^{-1} = \|a_2\|_*^{-1}. \end{aligned}$$

Hence, $\mathcal{S}$ is continuous in the open ball centered at $\overline{a}$ with radius $1/\sqrt{10}$. On the other hand,

$$\operatorname{end}\operatorname{conv}\left\{a_1^\mu, a_2^\mu, a_3^\mu\right\} = \begin{cases} \operatorname{conv}\left\{a_2^\mu, a_3^\mu\right\}, & \text{if } 0 \leq \mu \leq \frac{1}{\sqrt{10}}, \\ \operatorname{conv}\left\{a_1^\mu, a_2^\mu\right\} \cup \operatorname{conv}\left\{a_2^\mu, a_3^\mu\right\}, & \text{if } \frac{1}{\sqrt{10}} < \mu \leq \frac{5\sqrt{10}}{11}, \end{cases}$$

which entails

$$\mathcal{S}(a^\mu) = \begin{cases} \left\|a_2^\mu\right\|_*^{-1}, \text{ if } 0 \leq \mu \leq 1/\sqrt{10}, \\ \left\|a_1^\mu\right\|_*^{-1}, \text{ if } 1/\sqrt{10} < \mu \leq 5\sqrt{10}/11. \end{cases}$$

Consequently, $\lim_{\mu \to \left(1/\sqrt{10}\right)^+} \mathcal{S}(a^\mu) = \left\|a_1^{1/\sqrt{10}}\right\|_*^{-1} = \frac{\sqrt{10}}{7} > \mathcal{S}\left(a^{1/\sqrt{10}}\right) = \left\|a_2^{1/\sqrt{10}}\right\|_*^{-1} = \frac{\sqrt{10}}{13}$. Putting all together, we have $\operatorname{rad}_{\text{cont}} \mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right) = \frac{1}{\sqrt{10}}$.

6 Conclusions and further research

The following diagram is intended to provide a complete picture of the main results of this work. In it, for each $a \in \mathbb{R}^{n\times m}$, $\mathcal{S}(a)$ and $E(a)$ represent the subregularity modulus and the end set defined in (12) and (16), respectively.

Our starting background is:

- Theorem 1 (from [21, Corollary 2.1 and Remark 2.3]), which establishes

$$E(a) = \bigcup_{D\in\mathcal{D}_{\overline{a}}} \mathrm{conv}\left\{a_t,\ t\in D\right\},\ a\in\mathbb{R}^{n\times m}.$$

- Theorem 2 (derived from [9, Theorem 4]), which yields

$$\mathcal{S}(a) = d_*\left(0_n, E(a)\right)^{-1},\ a\in\mathbb{R}^{n\times m}.$$

Hereafter, (I) and (II) are used as abbreviations as follows:

$$(I) := \left[d_*\left(0_n, \bigcup_{D\in\mathcal{D}_{\overline{a}}} \mathrm{conv}\,\{\overline{a}_t,\ t\in D\}\right)\right]^{-1},$$

$$(II) := \left[d_*\left(0_n, \bigcup_{D\in\mathcal{D}_{\overline{a}}\cup\mathcal{D}^0_{\overline{a}}} \mathrm{conv}\,\{\overline{a}_t,\ t\in D\}\right)\right]^{-1}.$$

The next diagram summarizes the main results of this paper, all of them being new except equality $\mathcal{S}(\overline{a}) = d_*\left(0_n, E(\overline{a})\right)^{-1} = (I)$.

DIAGRAM OF RESULTS ON THE STABLE BEHAVIOR OF $\mathcal{S}$

$$\begin{aligned}
\liminf_{a\to\overline{a}} \mathcal{S}(a) &= \mathcal{S}(\overline{a}) = d_*\left(0_n, E(\overline{a})\right)^{-1} = (I) \\
&\leq \underbrace{\limsup_{a\to\overline{a}} \mathcal{S}(a)}_{\parallel \atop \mathrm{rob}\,\mathcal{G}_{\overline{a}}\left(\overline{x},\overline{b}\right)} = d_*\left(0_n,\ \mathrm{Lim\ sup}_{a\to\overline{a}} E(a)\right)^{-1} = (II) \\
&\leq \underbrace{d_*\left(0_n, \mathrm{bd\,conv}\left\{a_t,\ t\in T_{\overline{\sigma}}(\overline{x})\right\}\right)^{-1}}_{\parallel \atop [\mathrm{rad}_{\ \mathrm{rob}}\,\mathcal{G}_{\overline{a}}\left(\overline{x},\overline{b}\right)]^{-1}} \\
&\leq [\mathrm{rad}_{\ \mathrm{cont}}\,\mathcal{G}_{\overline{a}}\left(\overline{x},\overline{b}\right)]^{-1}.
\end{aligned}$$

Finally, let us point out some remarkable facts:

- Examples 1 and 2 show that the gap with respect to the first inequality of the diagram may be finite or infinite, respectively. Examples 3 and 4 show that the second and the third inequalities may be strict.

- Regarding the second inequality, from Theorem 5, the gap cannot be infinite, as condition $d_*\left(0_n, \operatorname{bd}\operatorname{conv}\left\{a_t,\ t \in T_{\overline{\sigma}}(\overline{x})\right\}\right) = 0$ is equivalent to $\limsup_{a \to \overline{a}} \mathcal{S}(a) = +\infty$.
- The modulus and radius of robust subregularity, $\operatorname{rob} \mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right)$ and $\operatorname{rad}_{\operatorname{rob}} \mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right)$, are computed through point-based formulae (only involving the nominal data $\left(\overline{a}, \overline{b}, \overline{x}\right)$, not appealing to elements in a neighborhood).
- The problem of finding a point-based formula for the radius of continuous subregularity, $\operatorname{rad}_{\operatorname{cont}} \mathcal{G}_{\overline{a}}\left(\overline{x}, \overline{b}\right)$, remains as an open problem; Example 4 provides some hints for future research, as far as it illustrates some of the difficulties which may arise in the computation of this radius.

Acknowledgements The authors are indebted to the anonymous referees and the Coordinating Editor for their valuable critical comments, which have definitely improved the original version of the paper.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Data availability and conflict of interest statement This work is of theoretical nature and has not analyzed or generated any datasets. The authors have no conflicts of interest to declare in relation to this article.

References

1. Argáez, C., Cánovas, M.J., Parra, J.: Calmness of linear constraint systems under structured perturbations with an application to the path-following scheme. Set-Valued Var. Anal. **29**, 839–860 (2021)
2. Bürgisser, P., Cucker, F.: Condition. The geometry of numerical algorithms, grundlehren der mathematischen wissenschaften fundamental principles of mathematical sciences. Springer, New York (2013)
3. Camacho, J., Cánovas, M. J., Parra, J.: From calmness to Hoffman constants for linear inequality systems, SIAM. J. Optim. **32**, 2859–2878 (2022)
4. Cánovas, M.J., Dontchev, A.L., López, M.A., Parra, J.: Metric regularity of semi-infinite constraint systems. Math. Progr. **104B**, 329–346 (2005)
5. Cánovas, M.J., Hall, J.A.J., López, M.A., Parra, J.: Calmness of partially perturbed linear systems with an application to the central path. Optimization **68**, 465–483 (2019)
6. Cánovas, M.J., Hantoute, A., Parra, J., Toledo, F.J.: Calmness modulus of fully perturbed linear programs. Math. Progr. **158A**, 267–290 (2016)
7. Cánovas, M.J., Henrion, R., López, M.A., Parra, J.: Outer limit of subdifferentials and calmness moduli in linear and nonlinear programming. J. Optim. Theory Appl. **169**, 925–952 (2016)
8. Cánovas, M.J., López, M.A., Parra, J., Toledo, F.J.: Distance to ill-posedness and the consistency value of linear semi-infinite inequality systems. Math. Progr. **103A**, 95–126 (2005)

9. Cánovas, M.J., López, M.A., Parra, J., Toledo, F.J.: Calmness of the feasible set mapping for linear inequality systems. Set-Valued Var. Anal. **22**, 375–389 (2014)
10. Cánovas, M.J., Parra, J., Rückmann, J.-J., Toledo, F.J.: Point-based neighborhoods for sharp calmness constants in linear programming. Set-Valued Var. Anal. **25**, 757–772 (2017)
11. Dontchev, A.L., Gfrerer, H., Kruger, A.Y., Outrata, J.V.: The radius of metric subregularity. Set-Valued Var. Anal. **28**, 451–473 (2020)
12. Dontchev, A.L., Lewis, A.S., Rockafellar, R.T.: The radius of metric regularity. Trans. Am. Math. Soc. **355**, 493–517 (2003)
13. Dontchev, A.L., Rockafellar, R.T.: Implicit functions and solution mappings: a view from variational analysis. Springer, New York (2009)
14. Goberna, M.A., López, M.A.: Linear semi-infinite optimization. Wiley, Chichester (UK) (1998)
15. Hu, H.: Characterizations of the strong basic constraint qualifications. Math. Op. Res. **30**, 956–965 (2005)
16. Ioffe, A.D.: Variational analysis of regular mappings. theory and applications. Springer monographs in mathematics. Springer, Cham (2017)
17. Kadrani, A., Dussault, J.-P., Benchakroun, A.: A new regularization scheme for mathematical programs with complementarity constraints. SIAM J. Optim. **20**, 78–103 (2009)
18. Klatte, D., Kummer, B.: Nonsmooth equations in optimization: regularity, calculus, methods and applications, nonconvex optim. appl. 60, Kluwer academic, Dordrecht, The Netherlands, (2002)
19. Klatte, D., Kummer, B.: Optimization methods and stability of inclusions in Banach spaces. Math. Progr. **117B**, 305–330 (2009)
20. Kruger, A.Y., López, M.A., Théra, M.A.: Perturbation of error bounds. Math. Progr. Ser. B **168**, 533–554 (2018)
21. Li, M.H., Meng, K.W., Yang, X.Q.: On error bound moduli for locally Lipschitz and regular functions. Math. Progr. Ser. A **171**, 463–487 (2018)
22. Mordukhovich, B.S.: Variational analysis and generalized differentiation, i: basic theory. Springer, Berlin (2006)
23. Peña, J., Vera, J.C., Zuluaga, L.F.: New characterizations of Hoffman constants for systems of linear constraints. Math. Progr. **187**, 79–109 (2021)
24. Robinson, S.M.: Some continuity properties of polyhedral multifunctions. Math. Progr. Study **14**, 206–214 (1981)
25. Rockafellar, R.T., Wets, R.J.-B.: Variational analysis. Springer, Berlin (1998)