



The extended Lissajous–Levi-Civita transformation

Slawomir Breiter¹ · Krzysztof Langner¹

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Abstract

Action-angle variables for the Levi-Civita regularized planar Kepler problem were introduced independently first by Chenciner and then by Deprit and Williams. The latter used explicitly the so-called Lissajous variables. When applied to the transformed Keplerian Hamiltonian, the Lissajous transformation encounters the difficulty of being defined in terms of the constant frequency parameter, whereas the Kepler problem transformed into a harmonic oscillator involves the frequency as a function of an energy-related canonical variable. A simple canonical transformation is proposed as a remedy for this inconvenience. The problem is circumvented by adding to the physical time a correcting term, which occurs to be a generalized Kepler's equation. Unlike previous versions, the transformation is symplectic in the extended phase space and allows the treatment of time-dependent perturbations. The relation of the extended Lissajous–Levi-Civita variables to the classical Delaunay angles and actions is given, and it turns out to be a straightforward generalization of the results published by Deprit and Williams.

Keywords Perturbed Kepler problem · Regularization · Levi-Civita variables · Lissajous transformation

1 Introduction

The combination of Sundman time regularization and parabolic coordinates, first studied by Goursat (1889) and then by Levi-Civita (1906), converts the planar Kepler problem into an isotropic harmonic oscillator with two degrees of freedom. However elegant, this Levi-Civita (LC) transformation involves a subtle point: it has to be performed on a fixed energy level—the value defining the oscillator's frequency. When executed as a canonical transformation, the LC transformation must be performed in the extended phase space, and then, the frequency becomes an explicit function of the momentum conjugate to physical time. Sometimes the problem is discarded by replacing the momentum with its numerical value, but such kind of

✉ Slawomir Breiter
breiter@amu.edu.pl

Krzysztof Langner
krzysztof.langner@amu.edu.pl

¹ Astronomical Observatory Institute, Faculty of Physics, Adam Mickiewicz University, Słoneczna 36, 61-286 Poznań, Poland

sweeping under carpet cannot withstand the addition of time-dependent perturbation terms to the Keplerian Hamiltonian.

The set of action-angle variables for the LC-transformed Keplerian problem should preferably account for the degeneracy, so that the transformed Hamiltonian depends on a single-action variable. This goal was first achieved by Chenciner (1986), but his results, published in a lecture notes preprint of the Paris Observatory, remained practically unknown, until they were picked up by Jacques Féjóz—first in his 1999 Ph.D. dissertation, and then in an article (Féjóz 2001). Yet, Féjóz has not noticed that meanwhile, Deprit and Williams (1991) obtained a similar set of the action-angle variables, as the crowning of the sequence of papers introducing the so-called Lissajous transformation (Deprit 1991). The latter approach will serve as the landmark in the present work, so we will refer to the variables under the name of the Lissajous–Levi-Civita (LLC) set.

For the completeness of the picture, let us add that the LLC variables appear to be closely related (if not identical) to the action-angle set of the Kepler problem derived already by Levi-Civita himself (Levi-Civita 1913), yet without resorting to regularization. The variables attracted some attention for a while (Andoyer 1913; de Sitter 1913), but then apparently fell into oblivion, with few exceptions like Ferrer and Lara (2009). Indeed, without the Sundman time as an independent variable, the ‘isoenergetic variables’ of Levi-Civita lose much of their flavour.

The last section of Deprit and Williams (1991) signals the problem arising from the fact that the Lissajous transformation is defined for an oscillator with a fixed frequency—a parameter independent on time and variables. But the Levi-Civita transformation is performed in the extended phase space, so the frequency depends on one of the variables—the energy-related momentum. In the present work, we show how to adapt the Lissajous transformation to the regularized perturbed (possibly time-dependent) Kepler problem in the extended phase space. In Sect. 2, we briefly recall the canonical LC regularization and the Lissajous transformation for an oscillator with constant frequency parameter. As a minor novelty, the canonicity of the latter is shown using a differential one form. Then, the extended LLC transformation is presented in Sect. 3. Against the common habit, we do not restrict the discussion of the LC and LLC transformations to the purely Keplerian Hamiltonian, where some difficulties disappear, although a separate subsection is devoted to this case. Finally, the relations between the LLC and the Delaunay variables are derived in Sect. 4.

2 Basic tools

2.1 The Levi-Civita transformation

Let us consider a time-dependent, planar problem in the extended phase space of Cartesian coordinates

$$\mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2, \quad (1)$$

and momenta

$$\mathbf{X} = X_1 \mathbf{e}_1 + X_2 \mathbf{e}_2, \quad (2)$$

with the time–energy pair x_0, X_0 appended, so that the Hamiltonian function

$$\mathcal{H}(x_0, \mathbf{x}, X_0, \mathbf{X}) = \mathcal{H}_0(\mathbf{x}, \mathbf{X}) + \mathcal{R}(x_0, \mathbf{x}, \mathbf{X}) + X_0, \quad (3)$$

is the sum of the Keplerian part with the gravitational parameter μ

$$\mathcal{H}_0 = \frac{\mathbf{X} \cdot \mathbf{X}}{2} - \frac{\mu}{\sqrt{\mathbf{x} \cdot \mathbf{x}}}, \tag{4}$$

of the perturbation \mathcal{R} and of the momentum X_0 . As it is seen from the canonical equations of motion

$$\frac{d\mathbf{x}}{dt} = \frac{\partial \mathcal{H}}{\partial \mathbf{X}} = \mathbf{X} + \frac{\partial \mathcal{R}}{\partial \mathbf{X}}, \tag{5}$$

$$\frac{d\mathbf{X}}{dt} = -\frac{\partial \mathcal{H}}{\partial \mathbf{x}} = -\frac{\mu \mathbf{x}}{(\mathbf{x} \cdot \mathbf{x})^{\frac{3}{2}}} - \frac{\partial \mathcal{R}}{\partial \mathbf{x}}, \tag{6}$$

$$\frac{dx_0}{dt} = \frac{\partial \mathcal{H}}{\partial X_0} = 1, \tag{7}$$

$$\frac{dX_0}{dt} = -\frac{\partial \mathcal{H}}{\partial x_0} = -\frac{\partial \mathcal{R}}{\partial x_0}, \tag{8}$$

the variable x_0 , formally distinct from time t , is actually equal to it up to an additive constant. Let us simply set $x_0 = t$ as the solution of Eq. (7), because then we can identify the time-dependent perturbation $\mathcal{R}(x_0, \mathbf{x}, \mathbf{X}) = \mathcal{R}(t, \mathbf{x}, \mathbf{X})$ as the function obtained by a direct replacement of time t by its formal counterpart x_0 , the latter being a dependent variable. Accordingly, Eq. (8) implies that the variations of $\mathcal{H}_0 + \mathcal{R}$, induced by the time dependence of \mathcal{R} , are compensated by X_0 , so that $\mathcal{H} = \text{const.}$ along the solution. The value of \mathcal{H} depends on the choice of X_0 as one of the initial conditions, and it can be arbitrary. But if the change of independent variable is to be performed using the ‘Poincaré trick’ (in the words of Meyer et al. 2009), it is necessary to choose $X_0 = -\mathcal{H}_0 - \mathcal{R}$, so that the motion takes place on the manifold $\mathcal{H} = 0$. In addition, we assume that only the motion leading to strictly positive values of X_0 is to be considered in the present work. The collision singularity of \mathcal{H}_0 imposes another constraint of $\|\mathbf{x}\| \neq 0$, so the initial problem is considered in the phase space Σ_x defined as

$$\Sigma_x = \{(x_0, \mathbf{x}, X_0, \mathbf{X}) : x_0 \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^2/\{\mathbf{0}\}, X_0 \in \mathbb{R}_+, \mathbf{X} \in \mathbb{R}^2\}. \tag{9}$$

The LC transformation combines two ingredients: the Sundman time transformation that regularizes the motion, and the parabolic coordinates linearizing the system. Following Deprit and Williams (1991), we consider the transformation ϕ depending on a constant parameter $\alpha > 0$ having the dimension of length, which helps to conserve the units of time and length in the new variables. Then, in the extended phase space

$$\begin{aligned} \Sigma_y &= \{(y_0, \mathbf{y}, Y_0, \mathbf{Y}) : y_0 \in \mathbb{R}, \mathbf{y} \in \mathbb{R}^2, Y_0 \in \mathbb{R}_+, \mathbf{Y} \in \mathbb{R}^2\}, \\ \Sigma'_y &= \Sigma_y / \{\mathbf{y} = \mathbf{0}\}, \end{aligned} \tag{10}$$

the LC transformation is

$$\phi : (\Sigma'_y; \tau) \rightarrow (\Sigma_x; t), \tag{11}$$

where

$$\mathbf{x} = \frac{y_1^2 - y_2^2}{\alpha} \mathbf{e}_1 + \frac{2y_1y_2}{\alpha} \mathbf{e}_2, \tag{12}$$

$$\mathbf{X} = \frac{1}{2r} [(y_1Y_1 - y_2Y_2) \mathbf{e}_1 + (y_1Y_2 + y_2Y_1) \mathbf{e}_2], \tag{13}$$

$$x_0 = y_0, \tag{14}$$

$$X_0 = Y_0, \tag{15}$$

and

$$r = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \frac{\mathbf{y} \cdot \mathbf{y}}{\alpha}. \tag{16}$$

The new independent variable τ is the Sundman time, related to the physical time t through the differential equation

$$\frac{d\tau}{dt} = \frac{\alpha}{4r} = \frac{\alpha^2}{4(y_1^2 + y_2^2)}, \tag{17}$$

which mimics the relation between the mean and the eccentric anomalies in Kepler problem.

A straightforward substitution, or the construction of canonical extension of the point transformation (12), shows that ϕ is a Mathieu transformation conserving the Pfaffian 1-form

$$-\mathcal{H} dt + X_0 dx_0 + \mathbf{X} \cdot d\mathbf{x} = -\mathcal{H} d\tau + Y_0 dy_0 + \mathbf{Y} \cdot d\mathbf{y}, \tag{18}$$

where the transformed Hamiltonian is¹

$$\mathcal{H} = \left(\frac{dt}{d\tau}\right) \phi^\# \mathcal{H} = \frac{1}{2} (Y_1^2 + Y_2^2) + \frac{\omega^2}{2} (y_1^2 + y_2^2) - \frac{4\mu}{\alpha} + \mathcal{P} = 0. \tag{19}$$

In the above expression, \mathcal{P} is a modified perturbing Hamiltonian

$$\mathcal{P}(y_0, \mathbf{y}, \mathbf{Y}) = \frac{4(y_1^2 + y_2^2)}{\alpha^2} \phi^\# \mathcal{R}, \tag{20}$$

and ω is a function of energy

$$\omega = \frac{2\sqrt{2Y_0}}{\alpha}. \tag{21}$$

The transformation ϕ is weakly canonical (equality (18) holds only on the manifold $\mathcal{H} = 0$) and 2:1 homomorphic. It could be made bijective if the phase space Σ'_y is replaced by a quotient space with the equivalence class $(\mathbf{y}, \mathbf{Y}) \sim (-\mathbf{y}, -\mathbf{Y})$, but we are not going to follow this path in the present work. Let us also add that restricting ϕ to Σ'_y does not mean that we cannot consider trajectories passing through $\mathbf{y} = \mathbf{0}$ while studying the flow generated by \mathcal{H} ; it only means that such a point cannot be mapped to Σ_x because of singular expressions (13).

The canonical equations of motion

$$\frac{dy_j}{d\tau} = Y_j + \frac{\partial \mathcal{P}}{\partial Y_j}, \tag{22}$$

$$\frac{dY_j}{d\tau} = -\omega^2 y_j - \frac{\partial \mathcal{P}}{\partial y_j}, \tag{23}$$

$$\frac{dy_0}{d\tau} = \frac{\partial \mathcal{H}}{\partial Y_0} = \frac{4}{\omega \alpha^2} \frac{\partial \mathcal{H}}{\partial \omega} = \frac{4(y_1^2 + y_2^2)}{\alpha^2}, \tag{24}$$

$$\frac{dY_0}{d\tau} = -\frac{\partial \mathcal{P}}{\partial y_0}, \tag{25}$$

with $j \in \{1, 2\}$, represent a perturbed isotropic harmonic oscillator (22, 23), whose frequency varies according to Eq. (25), since

$$\frac{d\omega}{d\tau} = \frac{4}{\omega \alpha^2} \frac{dY_0}{d\tau}. \tag{26}$$

¹ We use the pull-back symbol $\phi^\# \mathcal{H}$ as the equivalent of the composition $\mathcal{H} \circ \phi$.

Equation (24) simply repeats the Sundman transformation (17).

In the original formulation (Goursat 1889; Levi-Civita 1906), the parameter α was absent (i.e. equal to 1 and dimensionless). Moreover, only the time-independent problems (with Hamiltonians constant in the phase space (\mathbf{x}, \mathbf{X})) were discussed. This allowed to detach the time transformation (17) from the canonical framework and to speak about the motion on a given, fixed energy level $\mathcal{H}_0 + \mathcal{R} = K$, instead of the manifold $\mathcal{H} = 0$. In other words, our Y_0 was replaced by a fixed parameter $K = -Y_0$, not being a canonical variable. This approach is widespread; even if some scaling is applied to the parabolic coordinates, it is either using a constant α (Deprit and Williams 1991), or some function of the constant parameter K (Chenciner 1986). Obviously, problems are to be expected with this approach in the time-dependent problems, because then K is neither a constant, nor even an explicit function of time alone.

2.2 Lissajous transformation

Although Deprit (1991) proposed three distinct variants of what he named the Lissajous transformation, we focus on only one of them, which results from a simple combination of two canonical polar transformations, and in such form had been found by Chenciner around 1986 or by Vorobyev and Zaslavsky (1987). In one step, it could be defined as a homomorphic map

$$\lambda : \Sigma_L \rightarrow \Sigma_Y, \tag{27}$$

from

$$\Sigma_L = \{(l, g, L, G) : (l, g) \in \mathbb{T}^2, L \geq 0, -L \leq G \leq L\}, \tag{28}$$

to

$$\Sigma_Y = \{(y_1, y_2, Y_1, Y_2) \in \mathbb{R}^4\}, \tag{29}$$

depending on a fixed parameter $\omega > 0$:

$$\begin{aligned} y_1 &= \sqrt{\frac{L+G}{2\omega}} \cos(l+g) - \sqrt{\frac{L-G}{2\omega}} \cos(l-g), \\ y_2 &= \sqrt{\frac{L+G}{2\omega}} \sin(l+g) + \sqrt{\frac{L-G}{2\omega}} \sin(l-g), \\ Y_1 &= -\sqrt{\frac{\omega(L+G)}{2}} \sin(l+g) + \sqrt{\frac{\omega(L-G)}{2}} \sin(l-g), \\ Y_2 &= \sqrt{\frac{\omega(L+G)}{2}} \cos(l+g) + \sqrt{\frac{\omega(L-G)}{2}} \cos(l-g). \end{aligned} \tag{30}$$

The variables admit a transparent interpretation, describing the motion on the coordinate plane (y_1, y_2) , where the orbit is an ellipse with the centre at $y_1 = y_2 = 0$. In terms of the ellipse major and minor semi-axes a and b , the momenta are $L = \omega(a^2 + b^2)/2$, $G = \omega ab$, the angle g is the polar angle of the minor semi-axis², and $l + g$ is related to the polar angle of the moving point (Deprit 1991, Fig. 1).

Asking if the Lissajous transformation is canonical, one finds that the Pfaffian 1-forms of the two variable sets differ by a total differential (Deprit 1991)

$$Y_1 dy_1 + Y_2 dy_2 - L dl - G dg = dJ_2, \tag{31}$$

² Actually, it is better to define the polar angle of the major axis as $g + \frac{\pi}{2}$, to avoid problems with $G = 0$, when the minor axis vanishes.

where J_2 , the primitive function of the transformation (Arnold et al. 1997), is

$$J_2 = \frac{y_1 Y_1 + y_2 Y_2}{2} = \frac{\sqrt{L^2 - G^2} \sin 2l}{2}. \tag{32}$$

But the differential form (31) makes sense only if \mathbf{y}, \mathbf{Y} defined through relations (30) are differentiable with respect to the Lissajous variables. It means that we have to exclude $|G| = L$, and (as a consequence of new bounds $|G| < L$) also $L = 0$, where singularities appear. Thus, the canonical Lissajous transformation

$$\lambda : \Sigma'_L \rightarrow \Sigma'_Y, \tag{33}$$

requires a restricted domain

$$\Sigma'_L = \{(l, g, L, G) : (l, g) \in \mathbb{T}^2, L > 0, -L < G < L\}, \tag{34}$$

and, accordingly, a reduced image

$$\Sigma'_Y = \{(y_1, y_2, Y_1, Y_2) \in \mathbb{R}^4 / \mathcal{E}\}, \tag{35}$$

where

$$\mathcal{E} = \{(y_1, y_2, Y_1, Y_2) : 4\omega^2 (y_1 Y_2 - y_2 Y_1)^2 = (Y_1^2 + Y_2^2 + \omega^2 (y_1^2 + y_2^2))^2\}. \tag{36}$$

Even if restricted to Σ'_L , the Lissajous transformation remains homomorphic with respect to the angles. Indeed, given L and G , the pairs (l, g) and $(l + \pi, g + \pi)$ map onto the same point in Σ'_Y .

Applying the canonical Lissajous transformation to an isotropic oscillator Hamiltonian

$$\mathcal{H}_0 = \frac{1}{2} (Y_1^2 + Y_2^2) + \frac{\omega^2}{2} (y_1^2 + y_2^2), \tag{37}$$

one obtains a simple function

$$\lambda^\# \mathcal{H}_0 = \mathcal{H}_0 = \omega L. \tag{38}$$

This, together with a 2π -periodicity of Eq. (30) in l and g , shows that the Lissajous variables form an action-angle set for the harmonic oscillator with 2 degrees of freedom (37), properly accounting for the degeneracy of the system.

Finally, let us add that the canonicity condition (31) may be violated if ω , instead of being a constant parameter, depends on some variables. Then, the partial derivatives with respect to the frequency come into play, and—if their influence cannot be encapsulated into a total differential—the Hamiltonian (38) becomes meaningless, since the motion is no longer described by the canonical equations of motion.

For example, in the simple case of $\omega = \omega(t)$, the new Hamiltonian \mathcal{H}_0 should be complemented with an appropriate remainder

$$\mathcal{H}_0 = \omega L + \frac{J_2 \dot{\omega}}{\omega}. \tag{39}$$

3 Extended Lissajous–Levi-Civita transformation

3.1 Extended transformations

A naive composition of ϕ and λ transformations, when applied to the Kepler problem (even unperturbed), would result in a Hamiltonian whose derivative with respect to Y_0 does not

reproduce a correct equation for the evolution of y_0 . In these circumstances, let us repeat the Levi-Civita and Lissajous transformations, but this time without fixed parameters. The domains of both the transformations are restricted to secure the differentiability.

In the extended Levi-Civita transformation ϕ_e , we postulate that \mathbf{x} and \mathbf{X} are still defined as in Eqs. (12) and (13), that the Sundman time transformation remains in the form (17) and that the new energy-like momentum remains equal to X_0 , but we allow α to be a function of all variables. This choice implies the modification of the formal time variable x_0 definition, so

$$\phi_e : (\Sigma_v; \tau) \rightarrow (\Sigma_x; t), \tag{40}$$

with

$$\Sigma_v = \{(v_0, \mathbf{y}, Y_0, \mathbf{Y}) : v_0 \in \mathbb{R}, \mathbf{y} \in \mathbb{R}^2/\{\mathbf{0}\}, Y_0 \in \mathbb{R}_+, \mathbf{Y} \in \mathbb{R}^2\}, \tag{41}$$

and the choice of v_0 is the minimum flexibility needed to secure the canonical form of the transformation. This can be achieved if the differential forms satisfy

$$X_0 dx_0 + \mathbf{X} \cdot d\mathbf{x} - Y_0 dv_0 - \mathbf{Y} \cdot d\mathbf{y} = d\Phi_e, \tag{42}$$

with a primitive function Φ_e . Assume that

$$x_0 = v_0 + B(\mathbf{y}, \mathbf{Y}, Y_0), \tag{43}$$

with yet unknown differentiable function B . Substituting Eqs. (12), (13), (15), and (43), we find the constraint

$$Y_0 dB - \frac{\mathbf{y} \cdot \mathbf{Y}}{2} \frac{d\alpha}{\alpha} = d\Phi_e. \tag{44}$$

Admitting the risk of overlooking some potentially interesting variants, we assume that

$$\begin{aligned} \alpha &= \alpha(Y_0), \\ B(\mathbf{y}, \mathbf{Y}, Y_0) &= \mathbf{y} \cdot \mathbf{Y} B_0(Y_0), \\ \Phi_e(\mathbf{y}, \mathbf{Y}, Y_0) &= \mathbf{y} \cdot \mathbf{Y} \Phi_0(Y_0), \end{aligned} \tag{45}$$

because then condition (44) turns into

$$[Y_0 B_0 - \Phi_0] d(\mathbf{y} \cdot \mathbf{Y}) - \frac{\mathbf{y} \cdot \mathbf{Y}}{2\alpha} \left[\frac{d\alpha}{dY_0} + 2\alpha \left(\frac{d\Phi_0}{dY_0} - Y_0 \frac{dB_0}{dY_0} \right) \right] dY_0 = 0, \tag{46}$$

satisfied if

$$B_0 = \frac{-1}{2\alpha} \frac{d\alpha}{dY_0}, \tag{47}$$

$$\Phi_0 = Y_0 B_0 = -\frac{Y_0}{2\alpha} \frac{d\alpha}{dY_0}. \tag{48}$$

Thus, any differentiable $\alpha(Y_0)$ chosen, it gives the specific formal time correction B , guaranteeing that the transformation ϕ_e is canonical, with the new Hamiltonian function

$$\mathcal{H}_e = \frac{Y_1^2 + Y_2^2}{2} + \frac{\omega^2}{2} (y_1^2 + y_2^2) - \frac{4\mu}{\alpha} + \frac{4r}{\alpha} \phi_e^\# \mathcal{R}, \tag{49}$$

where the frequency ω is a function of Y_0 both directly and through α

$$\omega(Y_0) = \frac{\sqrt{8Y_0}}{\alpha(Y_0)}. \tag{50}$$

Note that the choice of α can be done up to an arbitrary constant multiplier, with no consequences for the expression for B , or for the canonicity of the transformation.

Proceeding to the extended Lissajous transformation λ_e , we essentially conserve the form of λ , but adding a new pair of variables from the extended phase space (u, U) , we will be able to make it canonical in spite of the dependence of ω on Y_0 , indicated in Eq. (50). Thus, in

$$\lambda_e : \Sigma_u \rightarrow \Sigma'_v, \tag{51}$$

with

$$\Sigma_u = \{(u, l, g, U, L, G) : u \in \mathbb{R}, (l, g) \in \mathbb{T}^2, U \in \mathbb{R}_+, L \in \mathbb{R}_+, -L < G < L\}, \tag{52}$$

and

$$\Sigma'_v = \Sigma_v / \mathcal{E}, \tag{53}$$

the transformation $(l, g, L, G) \rightarrow (\mathbf{y}, \mathbf{Y})$ is directly that of λ , i.e. given by Eq. (30), but the canonicity condition in the extended phase space becomes

$$Y_0 d\nu_0 + Y_1 dy_1 + Y_2 dy_2 - U du - L dl - G dg = d\Lambda_e, \tag{54}$$

with yet unknown primitive function Λ_e . Like before, we want to conserve the energy level momentum, assuming $U = Y_0$, so after substitutions, Eq. (54) takes the form

$$(d\nu_0 - du) Y_0 + \frac{d(\mathbf{y} \cdot \mathbf{Y})}{2} - \frac{\mathbf{y} \cdot \mathbf{Y}}{2\omega} d\omega = d\Lambda_e. \tag{55}$$

With similar assumptions as we did for the LC transformation, namely

$$\nu_0 = u + \mathbf{y} \cdot \mathbf{Y} b_0(Y_0), \quad \Lambda = \mathbf{y} \cdot \mathbf{Y} \Lambda_0(Y_0), \tag{56}$$

condition (55) becomes

$$\begin{aligned} & \left(\frac{1}{2} + Y_0 b_0 - \Lambda_0 \right) d(\mathbf{y} \cdot \mathbf{Y}) \\ & + \mathbf{y} \cdot \mathbf{Y} \left(Y_0 \frac{db_0}{dY_0} - \frac{d\Lambda_0}{dY_0} - \frac{1}{2\omega} \frac{d\omega}{dY_0} \right) dY_0 = 0, \end{aligned} \tag{57}$$

and is satisfied by

$$b_0 = \frac{-1}{2\omega} \frac{d\omega}{dY_0} = \frac{-1}{4Y_0} + \frac{1}{2\alpha} \frac{d\alpha}{dY_0}, \tag{58}$$

$$\Lambda_0 = Y_0 b_0 + \frac{1}{2} = \frac{1}{4} + \frac{Y_0}{2\alpha} \frac{d\alpha}{dY_0}. \tag{59}$$

3.2 Final form

Composing the two steps into a single canonical transformation requires the adjustment of the final image set:

$$\phi_e \circ \lambda_e = \psi_e : (\Sigma_u; \tau) \rightarrow (\Sigma'_x; t), \tag{60}$$

where

$$\Sigma'_x = \Sigma_x / \phi_e \mathcal{E}. \tag{61}$$

Then, we finally obtain

$$x_1 = \frac{1}{2\omega\alpha} \left[(L + G) \cos (2l + 2g) + (L - G) \cos (2l - 2g) - 2\sqrt{L^2 - G^2} \cos 2g \right], \tag{62}$$

$$x_2 = \frac{1}{2\omega\alpha} \left[(L + G) \sin (2l + 2g) - (L - G) \sin (2l - 2g) - 2\sqrt{L^2 - G^2} \sin 2g \right], \tag{63}$$

$$X_1 = \frac{-1}{4r} [(L + G) \sin (2l + 2g) + (L - G) \sin (2l - 2g)], \tag{64}$$

$$X_2 = \frac{1}{4r} [(L + G) \cos (2l + 2g) - (L - G) \cos (2l - 2g)], \tag{65}$$

where

$$r = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \frac{L - \sqrt{L^2 - G^2} \cos 2l}{\omega\alpha}, \tag{66}$$

and arbitrary, differentiable functions $\omega(U)$, and $\alpha(U)$ satisfy the constraint

$$\omega\alpha = \sqrt{8U}. \tag{67}$$

The transformation is canonical in the extended phase space thanks to the definition of the last pair of variables

$$X_0 = U, \tag{68}$$

$$x_0 = u - \frac{\sqrt{L^2 - G^2} \sin 2l}{4U} = u - \frac{\mathbf{x} \cdot \mathbf{X}}{2U}. \tag{69}$$

The inverse transformation can be found in Deprit and Williams (1991). Adjusted to the present notation, it gives a unique solution for the momenta

$$L = \frac{(X_1^2 + X_2^2 + 2X_0) r}{\sqrt{2X_0}}, \quad G = 2(x_1 X_2 - x_2 X_1). \tag{70}$$

Knowing L and G , one can find the angles from

$$\sqrt{L^2 - G^2} \cos 2l = \frac{2(X_1^2 + X_2^2 - 2X_0) r}{\sqrt{2X_0}}, \tag{71}$$

$$\sqrt{L^2 - G^2} \sin 2l = x_1 X_1 + x_2 X_2, \tag{72}$$

$$\sqrt{L^2 - G^2} \cos 2g = \frac{x_1 (X_2^2 - X_1^2 - 2X_0) - 2x_2 X_1 X_2}{\sqrt{2X_0}}, \tag{73}$$

$$\sqrt{L^2 - G^2} \sin 2g = \frac{x_2 (X_1^2 - X_2^2 - 2X_0) - 2x_1 X_1 X_2}{\sqrt{2X_0}}. \tag{74}$$

Having combined two 2:1 homomorphisms, we find that the composition remains 2:1 in the angles, but in a different manner. This time, the angle π can be added to either l , or g , or to both of them, whereas in λ and λ_e the addition had to be applied to the two angles simultaneously.

Remarkably, the α -dependent terms in Eqs. (43) and (56) cancel in their sum, and Eq. (69) is independent on the choice of function $\alpha(U)$. Nevertheless, this choice matters in the Hamiltonian function

$$\psi_e^\# \left(\frac{4r \mathcal{H}}{\alpha} \right) = \mathcal{M} = \omega L - \frac{4\mu}{\alpha} + \mathcal{F}(u, l, g, U, L, G) = 0, \tag{75}$$

where

$$\mathcal{F} = \frac{4r}{\alpha} \psi_e^\# \mathcal{R}. \tag{76}$$

Although $\psi_e^\# \mathcal{R}$ depends only on U , its multiplier $4r/\alpha$ still depends on α , as well as the frequency in the ωL term of Eq. (75). The term $4\mu/\alpha$ also matters in the extended phase space. On the other hand, \mathcal{M} can be factorized as

$$\mathcal{M} = \alpha^{-1} \mathcal{M}' = 0, \tag{77}$$

where

$$\mathcal{M}' = \sqrt{8U} L - 4\mu + 4r \psi_e^\# \mathcal{R} = 0, \tag{78}$$

depends on U only, regardless of α . In other words, the dynamics depends on $\alpha(U)$ only through the Sundman time definition (17).

Equations of motion generated by \mathcal{M} take the form

$$\begin{aligned} \frac{du}{d\tau} &= \frac{\partial \mathcal{M}}{\partial U} = \frac{4L}{\omega \alpha^2} + \frac{4\mu - L\omega \alpha}{\alpha^2} \frac{d\alpha}{dU} + \frac{\partial \mathcal{F}}{\partial U}, \\ \frac{dU}{d\tau} &= -\frac{\partial \mathcal{M}}{\partial u} = -\frac{\partial \mathcal{F}}{\partial u}, \\ \frac{dl}{d\tau} &= \frac{\partial \mathcal{M}}{\partial L} = \omega + \frac{\partial \mathcal{F}}{\partial L}, \\ \frac{dL}{d\tau} &= -\frac{\partial \mathcal{M}}{\partial l} = -\frac{\partial \mathcal{F}}{\partial l}, \\ \frac{dg}{d\tau} &= \frac{\partial \mathcal{M}}{\partial G} = \frac{\partial \mathcal{F}}{\partial G}, \\ \frac{dG}{d\tau} &= -\frac{\partial \mathcal{M}}{\partial g} = -\frac{\partial \mathcal{F}}{\partial g}. \end{aligned} \tag{79}$$

The first of them can be further simplified using factorization (77)

$$\frac{du}{d\tau} = \frac{1}{\alpha} \frac{\partial \mathcal{M}'}{\partial U} = \frac{1}{\alpha} \left[\frac{\sqrt{2} L}{\sqrt{U}} + 4 \frac{\partial}{\partial U} (r \psi_e^\# \mathcal{R}) \right], \tag{80}$$

but it should not be used for further differentiation, because while $\mathcal{M}' = 0$, its derivatives do not vanish in general.

The solutions of the differential Eq. (79) with the initial conditions in the set Σ_u , parameterized by the independent variable τ , are conjugated by the transformation ψ_e with the solution of Eqs. (5–8) parameterized by t .

3.3 Pure Kepler problem

In the pure Kepler problem, resulting from $\mathcal{F} = 0$, most of the Eqs. (79) have vanishing right-hand sides, except for two:

$$\frac{du}{d\tau} = \frac{4L}{\alpha \sqrt{8U}} = \text{const}, \tag{81}$$

where the simplified form (80) has been taken, and

$$\frac{dl}{d\tau} = \omega = \frac{\sqrt{8U}}{\alpha} = \text{const.} \tag{82}$$

Let us now demonstrate that these equations, combined with the canonical transformation ψ_e , lead to the Kepler equation for the eccentric anomaly E

$$\sqrt{\frac{\mu}{a^3}} t = E - e \sin E, \tag{83}$$

describing the motion on ellipse with semi-major axis a and eccentricity e .

Dividing Eqs. (81) by (82), we find

$$\frac{du}{dl} = \frac{L}{2U}. \tag{84}$$

Assume that both τ and u are measured from the pericentre, when $l = 0$. Then, using Eq. (69) we find

$$x_0 = u - \frac{\sqrt{L^2 - G^2} \sin 2l}{4U} = \frac{L}{4U} \left(2l - \sqrt{1 - \left(\frac{G}{L}\right)^2} \sin 2l \right). \tag{85}$$

Following Deprit and Williams (1991), introduce the ‘pseudo-eccentricity’ ε

$$\varepsilon = \sqrt{1 - \frac{G^2}{L^2}}. \tag{86}$$

In Sect. 4, it is shown that its value becomes equal to e in the pure Kepler problem. On the other hand, the Keplerian energy is a function of the semi-axis a , so $U = \mu/(2a)$, and then $\mathcal{M}' = 0$ implies $L = 4\mu/\sqrt{8U}$, which leads to

$$\frac{L}{4U} = \sqrt{\frac{a^3}{\mu}}. \tag{87}$$

So, with $t = x_0$, we find from (85)

$$\sqrt{\frac{\mu}{a^3}} t = 2l - \varepsilon \sin 2l, \tag{88}$$

which is actually the Kepler equation (83), provided we identify the Lissajous angle l with a half of the eccentric anomaly, and set $e = \varepsilon$, which is possible only in the absence of perturbations, and only as the equality of values.

Thus, the Kepler equation has been shown to be an intrinsic fragment of canonical transformation in the extended phase space. Note, however, that relations (84) and (87) are specific to the pure Kepler problem and should not be abused in the perturbed case.

3.4 Two special choices of α

We have demonstrated that the choice of α as a function of U or X_0 influences only the form of the transformed Hamiltonian, but not the complete transformation ψ_e itself. Let us inspect the form of \mathcal{M} if the choice of $\alpha(U)$ is similar to the choice of a fixed parameter α presented in two reference works: Deprit and Williams (1991) and Chenciner (1986).

Deprit and Williams (1991) aimed at respecting the units of time, coordinates, and momenta involved, so they assumed a constant α having the dimension of length. As an appropriate equivalent using the energy variable U , we propose

$$\alpha_1 = \frac{\mu}{U}, \tag{89}$$

which implies the frequency

$$\omega_1 = \frac{\sqrt{8U^3}}{\mu}, \tag{90}$$

and the transformed Hamiltonian

$$\mathcal{M}_1 = \frac{\sqrt{8U^3} L}{\mu} - 4U + \mathcal{F}_1(u, l, g, U, L, G) = 0. \tag{91}$$

In the pure Kepler problem, when $U = \mu/(2a)$, the function α_1 would have a constant value $\alpha_1 = 2a$, and the frequency would be equal to the mean motion n (because $\omega_1 = \sqrt{\mu a^{-3}}$). The angle $l = \omega_1 \tau$ would be equal to $E/2$, because $\dot{\tau} = (a/r)/2$, whereas $\dot{E} = n(a/r)$ in the Keplerian motion.

Chenciner (1986) preferred to use a dimensionless Sundman time and openly stated that his parameter, albeit fixed and treated as numerical, is based upon the value of energy. An analogue in the present framework would be simply

$$\alpha_2 = \sqrt{8U}, \tag{92}$$

with the dimension of velocity (length divided by time), because then $\omega_2 = 1$ (dimensionless), and

$$\mathcal{M}_2 = L - \frac{2\mu}{\sqrt{2U}} + \mathcal{F}_2(u, l, g, U, L, G) = 0. \tag{93}$$

4 Extended LLC and Delaunay variables

The Delaunay variables for the planar two body problem are the action-angle set including the mean anomaly l_D , the longitude of pericentre g_D , and their conjugate momenta $L_D = \sqrt{\mu a}$, $G_D = L_D \sqrt{1 - e^2}$. More precisely, the momenta are defined through their relation to the energy \mathcal{H}_0 of Eq. (4) and to the angular momentum

$$\mathcal{H}_0 = -\frac{\mu^2}{2L_D^2}, \quad G_D = \|\mathbf{x} \times \mathbf{X}\|. \tag{94}$$

Establishing the link between the LLC and the Delaunay variables is not a simple matter because of the essential difference between the definitions of these two sets: the Delaunay L_D is based upon \mathcal{H}_0 regardless of the problem, whereas the LLC momentum L is related to the complete, perturbed energy. This difficulty does not manifest in the angular momentum G_D , because using Eqs. (62–65) in the cross-product of Eq. (94), we find a direct, problem independent relation

$$G_D = \frac{|G|}{2}, \tag{95}$$

but such simplicity is an exception in the entire set of relations.

In order to define L_D , let us use Eq. (94) with \mathcal{H}_0 expressed in term of the LLC variables, which leads to

$$\mathcal{H}_0 = -\frac{\mu^2}{2L_D^2} = \frac{\sqrt{8U}L}{4r} - \frac{\mu}{r} - U, \tag{96}$$

where r depends on four variables: l, L, G , and U , as follows from Eq. (66). In spite of the complicated dependence of the right-hand side on L , the relation can be easily inverted with

$$L = \frac{4\mu}{\sqrt{8X_0}} + \frac{2r}{\sqrt{8X_0}} \left(2X_0 - \frac{\mu^2}{L_D^2} \right), \tag{97}$$

this time with r being a function of the Delaunay variables, in full agreement with Deprit and Williams (1991), provided their constants are replaced by $\omega\alpha = \sqrt{8X_0}$. In the last equation, we have returned to X_0 . Even if it is equal to U , its conjugate x_0 (eventually attached to the Delaunay variables in the extended phase space) is different than u , so it is better to maintain the distinction.

The disparity between the definitions of $G = 2G_D$ and $L = L(l_D, L_D, G_D)$ is responsible for the difference between the osculating eccentricity e and the ‘pseudo-eccentricity’ ε . Comparing the ratios of G/L and G_D/L_D , we find

$$1 - e^2 = \frac{G_D^2}{L_D^2} = (1 - \varepsilon^2) \left(\frac{-\mathcal{H}_0 L^2}{2\mu^2} \right), \tag{98}$$

meaning that e and ε may attain equal values only in the pure Kepler problem.

Linking the angles requires more effort. For the longitude of the pericentre g_D , let us use the Laplace vector

$$\mu \mathbf{e} = \mathbf{X} \times (\mathbf{x} \times \mathbf{X}) - \mu \frac{\mathbf{x}}{r} = \left(2\mathcal{H}_0 + \frac{\mu}{r} \right) \mathbf{x} + (\mathbf{x} \cdot \mathbf{X}) \mathbf{X}. \tag{99}$$

Expressing the latter form in terms of the LLC variables, we obtain

$$\mu e \cos g_D = \frac{\sqrt{8U}}{4L} \varepsilon \cos 2g + (\mathcal{H}_0 + U) x_1, \tag{100}$$

$$\mu e \sin g_D = \frac{\sqrt{8U}}{4L} \varepsilon \sin 2g + (\mathcal{H}_0 + U) x_2. \tag{101}$$

Note that only in the pure Kepler problem one may claim $g = g_D/2$, and $\varepsilon = e$, but even then, the relations refer to the values and cannot be differentiated.

In Sect. 3.3, it has been demonstrated that in the pure Kepler problem $l = E/2$, which might be thought to establish a simple link between l and l_D through the Kepler equation

$$l_D = E - e \sin E. \tag{102}$$

But here is the trap, because Eq. (88) involved not the variable l_D itself, but only its time dependence in the pure Kepler problem. In order to link l with E properly, we should consider two functions:

$$e \sin E = \frac{\mathbf{x} \cdot \mathbf{X}}{L_D} = \left(\frac{L \sqrt{-8\mathcal{H}_0}}{4\mu} \right) \varepsilon \sin 2l, \tag{103}$$

$$e \cos E = 1 - \frac{r}{a} = \left(-\frac{2\mathcal{H}_0 L}{\mu \sqrt{8U}} \right) \varepsilon \cos 2l + \left(1 + \frac{2\mathcal{H}_0 L}{\mu \sqrt{8U}} \right). \tag{104}$$

Again, only in the pure Kepler problem (when $\mathcal{H}_0 = -U$ and $\sqrt{8U}L = 4\mu$) it can be stated that numerical values of the variables satisfy $E = 2l$, and $e = \varepsilon$.

5 Conclusions

We have proposed the extension of the standard procedure leading to the action-angle variables of the Lissajous type (or the ‘Levi-Civita–Chenciner–Fejoz’ following Zhao 2015) for a perturbed Keplerian problem subjected to the Levi-Civita regularization. The transformation remains canonical in the extended phase space and does not require artificial fixed parameters related to the energy level. As such, it can be applied even to problems with explicitly time-dependent perturbations. The part of the transformation responsible for the relation of the physical time based x_0 to the new formal variable u turns out to be a generalized Kepler’s equation. After completing the work, we have realized that our approach is a generalization of the transformation introduced by Zhao (2016) for a periodically perturbed rectilinear motion.

Without the extended phase space, the time transformation would be a separate, non-Hamiltonian differential equation, and—as a side effect—the Kepler problem, whose unique frequency depends on energy, would become an isochronous, hence more degenerate, problem of the harmonic oscillator. Both aspects are not without consequences when it comes to studying variational equations for the purpose of some chaos tests.

The transformation has been expressed using two unspecified functions $\alpha(U)$, and $\omega(U)$ of the energy-related momentum U . Their choice is arbitrary and influences only the form of the transformed Hamiltonian function. Previously, these functions had been used as numerical parameters—sometimes to respect the units of time and length, sometimes quite contrarily—to render the dimensionless quantities. The functions α and ω can be selected according to the same requests, as long as they are bound by $\alpha\omega = \sqrt{8U}$, which is necessary for the canonicity in the extended phase space.

Finally, we have linked the LLC and the Delaunay variables. Generally, all the relations between the two sets reduce to those given by Deprit and Williams (1991) if the momentum U is replaced by the fixed parameters α and ω , at the expense of loosing the canonical transformation properties in the extended phase space. There are many delusive short cuts leading to simplified variants of these relations. We were lured into some of them, but ultimately each has been unmasked as either hiding the Keplerian motion postulate, or replacing a variable by its value. Noteworthy, Féjoz (2001) and Zhao (2015) present the simplified relations, equivalent to the statements $L_D = L/2$ etc., but the formulae were derived within the pure Kepler problem and may not be generalized to the perturbed case, even if their modified mass trick is used. For example, in order to annihilate the last term in Eq. (104) by an appropriate modification of μ , one must have a constant \mathcal{H}_0 , whereas in perturbed problems, even if conservative, the energy is being continuously repartitioned into the Keplerian and the perturbation terms—both variable. On the other hand, most of the intricacies disappear if one abandons the ‘isodynamic’ Delaunay variables and works exclusively with the expressions of \mathbf{x} and \mathbf{X} in terms of the ‘isoenergetic’ LLC variables (Levi-Civita 1913).

Quoting the quote from Meyer et al. (2009), ‘no set of coordinates is good enough’. Action-angle sets generally admit the values of actions where the angles are undetermined. The LLC variables share the same weakness as the Delaunay set: circular orbits do not admit meaningful values of l and g . But since the sums $l + g$ or $l - g$ remain properly defined, the solution to this purely geometrical problem is easy to obtain if needed. Yet the LLC variables are immune to a more profound disease of the Delaunay set: by the definition, the action G_D is nonnegative (as the length of angular momentum), whereas the LLC momentum G can be negative.³ This difference is essential when it comes to studying the motion, where the

³ The analogous momentum in the isoenergetic set of Levi-Civita (1913) is also nonnegative by the definition.

perturbation may result in the inversion of the angular momentum direction with the passage through $\mathbf{G} = \mathbf{0}$. In other words, $G = 0$ is an internal point in the domain, so functions of G can be differentiated at this point, whereas $G_D = 0$ is the boundary of the closed interval with only one-sided limit of functions available.

We hope that the extended LLC variables can be useful in the studies of periodically perturbed motion, where radial orbits are of interest, forming the collision manifold (e.g. Boscaggin et al. 2017). In particular, this formulation may be beneficial for planar, elliptic restricted three body problem and its special cases, like the Hill problem.

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Compliance with ethical standards

Conflict of interest The authors S. Breiter and K. Langner declare that they have no conflict of interest.

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