



A unified immersed finite element error analysis for one-dimensional interface problems

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Abstract

It has been known that the traditional scaling argument cannot be directly applied to the error analysis of immersed finite elements (IFE) because, in general, the spaces on the reference element associated with the IFE spaces on different interface elements via the standard affine mapping are not the same. By analyzing a mapping from the involved Sobolev space to the IFE space, this article is able to extend the scaling argument framework to the error estimation for the approximation capability of a class of IFE spaces in one spatial dimension. As demonstrations of the versatility of this unified error analysis framework, the manuscript applies the proposed scaling argument to obtain optimal IFE error estimates for a typical first-order linear hyperbolic interface problem, a second-order elliptic interface problem, and the fourth-order Euler-Bernoulli beam interface problem, respectively.

Keywords Immersed finite elements · Error analysis · Bramble Hilbert lemma · Scaling argument

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1 Introduction

Partial differential equations with discontinuous coefficients arise in many areas of sciences and engineering such as heat transfer, acoustics, structural mechanics, and electromagnetism. The discontinuity of the coefficients results in multiple challenges in the design and the analysis of numerical methods and it is an active area of research in the communities of finite element, finite volume, as well as finite difference method.

The immersed finite element (IFE) methods can use an interface independent mesh to solve an interface problem. Many publications were about IFE methods using either linear [22–24], bilinear [19, 26], or trilinear polynomials [18, 34]. IFE methods have been applied to a variety of problems, such as parabolic interface problems [20, 28, 30], hyperbolic interface problems [7], the acoustic interface problem [8, 33] and Stokes and Navier–Stokes interface problems [3, 14, 21, 35]. IFE methods with higher degree polynomials have also been explored [2, 4, 5, 14, 17, 29]. In particular, Adjerid and Lin [5] constructed IFE spaces of arbitrary degree and analyzed their approximation capabilities.

In [8, 33], Adjerid and Moon discussed IFE methods for the following acoustic interface problem

$$\begin{cases} p_t(x, t) = \rho(x)c(x)^2 v_x(x, t), & x \in (a, \alpha) \cup (\alpha, b), \\ \rho(x)v_t(x, t) = p_x(x, t), & x \in (a, \alpha) \cup (\alpha, b), \\ [v]_{x=\alpha} = [p]_{x=\alpha} = 0. \end{cases} \quad (1.1)$$

where ρ, c are equal to ρ_+, c_+ on interval (α, b) and to ρ_-, c_- on (a, α) . Assuming that the exact solution (v, p) has sufficient regularity in (a, α) and (α, b) , respectively, we can follow the idea in [32] to show that the exact solution satisfies the following so-called extended jump conditions:

$$\frac{\partial^k}{\partial x^k} p(\alpha^+, t) = r_k^p \frac{\partial^k}{\partial x^k} p(\alpha^-, t), \quad \frac{\partial^k}{\partial x^k} v(\alpha^+, t) = r_k^v \frac{\partial^k}{\partial x^k} v(\alpha^-, t), \quad k = 0, 1, \dots, \quad (1.2)$$

for certain positive constants r_k^p and r_k^v . In [8, 33], IFE spaces based on polynomials of degree up to 4 were developed with these extended jump conditions, and these IFE spaces were used with a discontinuous Galerkin (DG) method to solve the above acoustic interface problem with pertinent initial and boundary conditions. Numerical examples presented in [8, 33] demonstrated the optimal convergence of this DG IFE method, but we have not seen any error analysis about it in the related literature.

Extended jump conditions have also been used in the development of higher degree IFE spaces for solving other interface problems [2, 4, 5, 14, 17, 29]. This motivates us to look for a unified framework for the error analysis for methods based on IFE spaces constructed with the extended jump conditions such as those in (1.2). As an initial effort, our focus here is on one-dimensional interface problems.

One challenge in error estimation for IFE methods is that the scaling argument commonly used in error estimation for traditional finite element methods cannot be directly

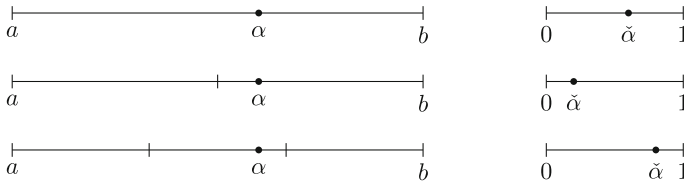


Fig. 1 The relative position of the interface (on the right) changes as the we refine the mesh (on the left)

applied. In the standard scaling argument, local finite element spaces on elements in a sequence of meshes are mapped to the same finite element space on the reference element via an affine transformation. However, the same affine transformation will map the local IFE spaces on interface elements in a sequence of meshes to different IFE spaces on the reference element because of the variation of interface location in the reference element, see the illustration in Fig. 1. A straightforward application of the scaling argument to the analysis of the approximation capability of an IFE space will result in error bounds of a form $C(\check{\alpha})h^l$, i.e., the constant factor $C(\check{\alpha})$ in the derived error bounds depend on the location of the interface in the reference element, and this kind of error bounds cannot be used to show the convergence of the related IFE method unless one can show that the constant factor $C(\check{\alpha})$ is bounded for all $\check{\alpha}$ in the reference element, which, to our best knowledge, is difficult to establish. Alternative analysis techniques such as multi-point Taylor expansions are used [5] which becomes awkward for higher degree IFE spaces, particularly so for higher degree IFE spaces in higher dimension. To circumvent this predicament of the classical scaling argument, we introduce a mapping between the related Sobolev space and the IFE space by using weighted averages of the derivatives in terms of the coefficients in the jump conditions. We show that the Sobolev norm of the error of this mapping can be bounded by the related Sobolev semi-norm. This essential property enables us to establish a Bramble–Hilbert type lemma for the IFE spaces, and, to our best knowledge, this is the first result that makes the scaling argument applicable in the error analysis of a class of IFE methods. For demonstrating the versatility of this unified error analysis framework, we apply it to establish, for the first time, the optimal approximation capability of the IFE space designed for the acoustic interface problem (1.1). Similarly, we apply this immersed scaling argument to the IFE space designed for an elliptic interface problem considered in [5] as well as the IFE space for the Euler-Bernoulli Beam interface problem considered in [25, 27, 36] leading to much simpler and elegant proofs. Moreover, a novel approach was recently introduced in [6] for constructing immersed finite element functions to address two-dimensional interface problems. This method constructs IFE functions in Frenet coordinates, where the interface is a line segment along the vertical axis, transformed from the interface curve using the Frenet apparatus based on the differential geometry of the interface. The IFE functions in Frenet coordinates satisfy jump conditions along this simplified interface with respect to just one variable in the horizontal axis. Consequently, we envision that our research might serve as a precursor for the analysis of this new class of IFE functions.

The paper is organized as follows. In Sect. 2, we introduce the notation and spaces used in the rest of the paper. In Sect. 3, we restrict ourselves to study of the IFE functions

on the interval $[0, 1]$, we show that they have similar properties to polynomials, for example, they both have the same maximum number of roots, they both admit a Lagrange basis and they both satisfy an inverse and a trace inequality. In Sect. 4, we define the notion of uniformly bounded reference IFE (RIFE) operators and how the scaling argument is applicable using an immersed Bramble–Hilbert lemma. In Sect. 5, we study the convergence of the DG-IFE method for the acoustic interface problem (1.1). In Sect. 6, we give shorter and simpler proofs for the optimal convergence of IFE methods for the second-order elliptic interface problem as well as the fourth-order Euler-Bernoulli beam interface problem.

2 Preliminaries

Throughout the article, we will consider a bounded open interval $I = (a, b)$ with $|a|, |b| < \infty$, and let $\alpha \in I$ be the interface point dividing I into two open intervals $I^- = (a, \alpha)$, $I^+ = (\alpha, b)$. This convention extends to any other open interval $B \subseteq \mathbb{R}$ with $B^- = B \cap (-\infty, \alpha)$ and $B^+ = B \cap (\alpha, \infty)$. For every bounded open interval B not containing α , let $W^{m,p}(B)$ be the usual Sobolev space on B equipped with the norm $\|\cdot\|_{m,p,B}$ and the seminorm $|\cdot|_{m,p,B}$. We are particularly interested in the case of $p = 2$ corresponding to the Hilbert space $H^m(B) = W^{m,2}(B)$, and we will use $\|\cdot\|_{m,B}$ and $|\cdot|_{m,B}$ to denote $\|\cdot\|_{m,2,B}$ and $|\cdot|_{m,2,B}$, respectively, for convenience. We will use $(\cdot, \cdot)_B$ and $(\cdot, \cdot)_{w,B}$ to denote the classical and the weighted L^2 inner product defined as

$$(f, g)_B = \int_B f(x)g(x) dx, \quad (f, g)_{w,B} = \int_B w(x)f(x)g(x) dx, \quad w(x) > 0, \quad \forall x \in B.$$

Given a positive finite sequence $r = (r_i)_{i=0}^{\tilde{m}}$, $\tilde{m} \geq 0$ and an open interval B containing α , we introduce the following piecewise Sobolev space:

$$\begin{aligned} \mathcal{H}_{\alpha,r}^{m+1}(B) = \left\{ u \mid u|_{B^\pm} \in H^{m+1}(B^\pm), u^{(k)}(\alpha^+) = r_k u^{(k)}(\alpha^-), \right. \\ \left. \forall k = 0, 1, \dots, m \right\}, \quad 0 \leq m \leq \tilde{m}. \end{aligned} \quad (2.1)$$

The norms, semi-norms and the inner product that we will use on $\mathcal{H}_{\alpha,r}^{m+1}(B)$ are

$$\begin{aligned} \|\cdot\|_{m+1,B} = \sqrt{\|\cdot\|_{m+1,B^-}^2 + \|\cdot\|_{m+1,B^+}^2}, \quad |\cdot|_{m,B} = \sqrt{|\cdot|_{m,B^-}^2 + |\cdot|_{m,B^+}^2}, \\ \text{and } (f, g)_{w,B} = (f, g)_{w,B^-} + (f, g)_{w,B^+}. \end{aligned}$$

We note, by the Sobolev embedding theory, that $\mathcal{H}_{\alpha,r}^{m+1}(B)$ is a subspace of

$$\mathcal{C}_{\alpha,r}^m(\overline{B}) = \left\{ u \mid u|_{B^\pm} \in C^m(\overline{B^\pm}), u^{(k)}(\alpha^+) = r_k u^{(k)}(\alpha^-), \forall k = 0, 1, \dots, m \right\}, \quad 0 \leq m \leq \tilde{m}. \quad (2.2)$$

By dividing I into N sub-intervals, we obtain the following partition of I :

$$I_k = (x_{k-1}, x_k), \quad \mathcal{T}_h = \{I_k\}_{k=1}^N, \quad a = x_0 < x_1 < \dots < x_N = b, \\ h = \max_{1 \leq k \leq N} (x_k - x_{k-1}).$$

We will assume that there is $k_0 \in \{1, 2, \dots, N\}$ such that $x_{k_0-1} < \alpha < x_{k_0}$, which is equivalent to $\alpha \in \overset{\circ}{I}_{k_0}$. We define the discontinuous immersed finite element space $W_{\alpha,r}^m$ on the interval I as

$$W_{\alpha,r}^m(\mathcal{T}_h) = \left\{ \varphi \mid \varphi|_{I_k} \in \mathcal{P}^m(I_k) \text{ for } k \in \{1, \dots, N\} \setminus \{k_0\} \text{ and } \varphi|_{I_{k_0}} \in \mathcal{V}_{\alpha,r}^m(I_{k_0}) \right\}, \quad (2.3)$$

where $\mathcal{P}^m(I_k)$ is the space of polynomials of degree at most m on I_k and $\mathcal{V}_{\alpha,r}^m(I_{k_0})$ is the local immersed finite element (LIFE) space defined as:

$$\mathcal{V}_{\alpha,r}^m(I_{k_0}) = \left\{ \varphi \in \mathcal{C}_{\alpha,r}^m(I_{k_0}) \mid \varphi|_{I_{k_0}^s} \in \mathcal{P}^m(I_{k_0}^s), \quad s = +, - \right\}, \quad 0 \leq m \leq \tilde{m}. \quad (2.4)$$

In discussions from now on, given a function v in $\mathcal{H}_{\alpha,r}^{m+1}$ (or $\mathcal{C}_{\alpha,r}^m$ or $\mathcal{V}_{\alpha,r}^m$), its derivative is understood in the piecewise sense unless specified otherwise. By definition, we can readily verify that $\mathcal{V}_{\alpha,r}^{m-1}(I_{k_0}) \subset \mathcal{V}_{\alpha,r}^m(I_{k_0})$ for a given finite sequence $r = (r_i)_{i=0}^{\tilde{m}}$, $\tilde{m} \geq 1$. In order to study the LIFE space $\mathcal{V}_{\alpha,r}^m(I_{k_0})$, we will investigate the properties of the corresponding reference IFE (RIFE) space $\mathcal{V}_{\check{\alpha},r}^m(\check{I})$ on the reference interval $\check{I} = [0, 1]$ with an interface point $\check{\alpha} \in (0, 1)$. Our goal is to extend the scaling argument to such IFE spaces and use the IFE scaling argument to show IFE spaces such as $\mathcal{V}_{\alpha,r}^m(I_{k_0})$ have the optimal approximation capability, i.e., every function in $\mathcal{H}_{\alpha,r}^{m+1}(I_{k_0})$ can be approximated by functions from the IFE space $\mathcal{V}_{\alpha,r}^m(I_{k_0})$ at the optimal convergence rate.

Following the convention in the error analysis literature for finite element methods, we will often use a generic constant C in estimates whose value varies depending on the context, but this generic constant is independent of h and the interface $\alpha \in I_{k_0}$ or $\check{\alpha} \in \check{I}$ unless otherwise declared.

3 Properties of the RIFE space $\mathcal{V}_{\alpha,r}^m(\check{I})$

For a given function $\check{\varphi} \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$, we will write $\check{\varphi} = (\check{\varphi}_-, \check{\varphi}_+)$ where $\check{\varphi}_s = \check{\varphi}|_{\check{I}^s} \in \mathcal{P}^m(\check{I}^s)$ for $s = +, -$. Additionally, we will use $\check{\varphi}_s^{(k)}(\check{\alpha})$ to denote $\lim_{x \rightarrow \check{\alpha}^s} \check{\varphi}_s^{(k)}(x)$, $s = \pm$ for a given integer $k \geq 0$. For clarity, we will use s' to denote the dual of s , i.e., if $s = \pm$, then $s' = \mp$.

Lemma 3.1 *Let $\tilde{m} \geq m \geq 0$, $\{r_k\}_{k=0}^{\tilde{m}} \subset \mathbb{R}_+$, $\check{\alpha} \in (0, 1)$ and $s \in \{+, -\}$. The following statements hold*

1. For every $\check{\varphi}_s \in \mathcal{P}^m(\check{I}^s)$ there is a unique $\check{\varphi}_{s'} \in \mathcal{P}^m(\check{I}^{s'})$ such that $\check{\varphi} = (\check{\varphi}_-, \check{\varphi}_+) \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$.
2. The dimension of $\mathcal{V}_{\check{\alpha},r}^m(\check{I})$ is $m + 1$.
3. The set $\{\mathcal{N}_{\check{\alpha},r}^k\}_{k=0}^m$, where

$$\mathcal{N}_{\check{\alpha},r}^k(x) = \begin{cases} (x - \check{\alpha})^k, & x \in \check{I}^-, \\ r_k(x - \check{\alpha})^k, & x \in \check{I}^+, \end{cases} \tag{3.1}$$

forms a basis of $\mathcal{V}_{\check{\alpha},r}^m(\check{I})$ and will be referred to as the canonical basis.

Proof We will prove the statements in order:

1. Let $\check{\varphi}_{\pm} \in \mathcal{P}^m(\check{I}^{\pm})$, then $(\check{\varphi}_-, \check{\varphi}_+) \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$ if and only if

$$\check{\varphi}_{\mp}^{(k)}(\check{\alpha}) = (r_k)^{\mp 1} \check{\varphi}_{\pm}^{(k)}(\check{\alpha}), \quad k = 0, 1, \dots, m,$$

which uniquely defines a polynomial $\check{\varphi}_{\mp} \in \mathcal{P}^m(\check{I}^{\mp})$:

$$\check{\varphi}_{\mp}(x) = \sum_{k=0}^m \frac{(r_k)^{\mp 1} \check{\varphi}_{\pm}^{(k)}(\check{\alpha})}{k!} (x - \check{\alpha})^k.$$

2. We have shown that the maps $\check{\varphi}_- \mapsto \check{\varphi}$ is well defined and injective which implies that the map $\check{\varphi} \mapsto \check{\varphi}_-$ is surjective since every $\check{\varphi}_- \in \mathcal{P}^m(\check{I}^-)$ can be extended to $\check{\varphi} \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$. Hence, $\mathcal{V}_{\check{\alpha},r}^m(\check{I})$ is isomorphic to $\mathcal{P}^m(\check{I}^-)$ implying that the dimension of $\mathcal{V}_{\check{\alpha},r}^m(\check{I})$ is $m + 1$.
3. We only need to show that $\{\mathcal{N}_{\check{\alpha},r}^k\}_{k=0}^m$ is linearly independent: Assume that $\check{\varphi} = \sum_{k=0}^m c_k \mathcal{N}_{\check{\alpha},r}^k \equiv 0$, then $\check{\varphi}_- \equiv 0$ which implies that $c_k = 0$ for all $k = 0, 1, \dots, m$. □

The results in Lemma 3.1 allows us to introduce an extension operator that maps $\check{\varphi}_s$ to $\check{\varphi}_{s'}$.

Definition 3.1 Let $\tilde{m} \geq m \geq 0$, $\{r_k\}_{k=0}^{\tilde{m}} \subset \mathbb{R}_+$, $\check{\alpha} \in (0, 1)$ and $s \in \{+, -\}$. We define the extension operator $\mathcal{E}_{\check{\alpha},r}^{m,s'} : \mathcal{P}^m(\check{I}^s) \rightarrow \mathcal{P}^m(\check{I}^{s'})$ that maps every $\check{\varphi}_s \in \mathcal{P}^m(\check{I}^s)$ to $\mathcal{E}_{\check{\alpha},r}^{m,s'}(\check{\varphi}_s) = \check{\varphi}_{s'}$ such that $(\check{\varphi}_-, \check{\varphi}_+) \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$.

By Lemma 3.1, the extension operator $\mathcal{E}_{\check{\alpha},r}^{m,s'}$ is well-defined and is linear. Furthermore, by Lemma 3.1 again, this extension operator is also invertible. Consequently, the dimension of the RIFE space is the same as the dimension of the traditional polynomial space of the same degree.

Next, we will estimate the operator norm of $\mathcal{E}_{\check{\alpha},r}^{m,s'}$. Let $\check{h}_- = \check{\alpha}$, $\check{h}_+ = 1 - \check{\alpha}$ be the lengths of the sub-intervals \check{I}^{\pm} formed by $\check{\alpha}$. First, let us consider the following example $\check{\varphi} = (\check{\varphi}_-, \check{\varphi}_+) = \mathcal{N}_{\check{\alpha},r}^m$ defined by (3.1), we have

$$\begin{aligned} \left\| \mathcal{E}_{\check{\alpha},r}^{m,s'}(\check{\varphi}_-) \right\|_{0,\check{I}^+} &= \|\check{\varphi}_+\|_{0,\check{I}^+} = |r_m| \left(\frac{\check{h}_+}{\check{h}_-} \right)^m \sqrt{\frac{\check{h}_+}{\check{h}_-}} \|\check{\varphi}_-\|_{0,\check{I}^-}, \\ \text{where } \check{h}_- &= \check{\alpha}, \check{h}_+ = 1 - \check{\alpha}. \end{aligned}$$

Hence, if $h_- > h_+$, we get $\|\check{\varphi}_+\|_{0,\check{I}^+} \leq |r_m| \|\check{\varphi}_-\|_{0,\check{I}^-}$. In the following lemma, we will show that a similar result holds for all $\check{\varphi} \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$. Consequently, for every interface position $\check{\alpha} \in I$, one of the two extension operators $\mathcal{E}_{\check{\alpha},r}^{m,s'}$, $s' = -, +$ will be bounded independently of $\check{\alpha}$.

Lemma 3.2 *There exists a constant $C > 0$ that depends on m such that for every $(\check{\varphi}_-, \check{\varphi}_+) \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$, we have*

$$\begin{aligned} \left\| \mathcal{E}_{\check{\alpha},r}^{m,s'} \check{\varphi}_s \right\|_{0,\check{I}^{s'}} &= \|\check{\varphi}_{s'}\|_{0,\check{I}^{s'}} \leq C \sqrt{\frac{\check{h}_{s'}}{\check{h}_s}} \left(\max_{0 \leq i \leq m} r_i^s \right) \max \left(1, \left(\frac{\check{h}_{s'}}{\check{h}_s} \right)^m \right) \|\check{\varphi}_s\|_{0,\check{I}^s}, \\ s = +, - \end{aligned} \tag{3.2}$$

where $s' = \mp$ for $s = \pm$. In particular, if $\check{h}_s \geq \check{h}_{s'}$, we have

$$\|\check{\varphi}_{s'}\|_{0,\check{I}^{s'}} \leq C \sqrt{\check{h}_{s'}} \left(\max_{0 \leq i \leq m} r_i^{s'} \right) \|\check{\varphi}_s\|_{0,\check{I}^s}. \tag{3.3}$$

Proof First, we note that (3.3) is a straightforward consequence of (3.2). Here, we only need to prove (3.2) for $s = -$ since the case $s = +$ can be proven similarly. For every $(\check{\varphi}_-, \check{\varphi}_+) \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$, we first define $\hat{\varphi}_- \in \mathcal{P}^m([0, 1])$ as $\hat{\varphi}_-(\xi) = \check{\varphi}_-(\check{h}_-\xi)$ which yields

$$\hat{\varphi}_-^{(i)}(1) = \check{h}_-^i \check{\varphi}_-^{(i)}(\check{\alpha}), \quad i = 0, 1, \dots, m. \tag{3.4}$$

Now, let us write $\check{\varphi}_+$ as a finite Taylor sum around $\check{\alpha}$ and use $\check{\varphi}_+^{(i)}(\check{\alpha}) = r_i \check{\varphi}_-^{(i)}(\check{\alpha})$ to obtain:

$$\check{\varphi}_+(x) = \sum_{i=0}^m \check{\varphi}_+^{(i)}(\check{\alpha}) \frac{(x - \check{\alpha})^i}{i!} = \sum_{i=0}^m r_i \check{\varphi}_-^{(i)}(\check{\alpha}) \frac{(x - \check{\alpha})^i}{i!}.$$

Using (3.4), we can replace $\check{\varphi}_-^{(i)}(\check{\alpha})$ by $\check{h}_-^{-i} \hat{\varphi}_-^{(i)}(1)$:

$$\check{\varphi}_+(x) = \sum_{i=0}^m r_i \hat{\varphi}_-^{(i)}(1) \check{h}_-^{-i} \frac{(x - \check{\alpha})^i}{i!}. \tag{3.5}$$

We square and integrate (3.5), then we apply the change of variables $z = x - \check{\alpha}$ to get

$$\|\check{\varphi}_+\|_{0,\check{I}^+}^2 = \int_{\check{\alpha}}^1 \left(\sum_{i=0}^m r_i \hat{\varphi}_-^{(i)}(1) \check{h}_-^{-i} \frac{(x - \check{\alpha})^i}{i!} \right)^2 dx = \int_0^{\check{h}_+} \left(\sum_{i=0}^m r_i \hat{\varphi}_-^{(i)}(1) \left(\frac{z}{\check{h}_-} \right)^i \frac{1}{i!} \right)^2 dz.$$

We can bound r_i and $|\hat{\varphi}_-^{(i)}(1)|$ by their maximum values for $0 \leq i \leq m$ and we can bound $\left(\frac{z}{\check{h}_-}\right)^i$ by

$$\left(\frac{z}{\check{h}_-} \right)^i \leq \max \left(1, \left(\frac{\check{h}_+}{\check{h}_-} \right)^m \right).$$

We also have $\sum_{i=0}^m \frac{1}{i!} \leq e$. Using these bounds, we get

$$\begin{aligned} \|\check{\varphi}_+\|_{0,\check{I}^+}^2 &\leq \left(\max_{0 \leq i \leq m} r_i \right)^2 \left(\max_{0 \leq i \leq m} |\hat{\varphi}_-^{(i)}(1)| \right)^2 \max \left(1, \left(\frac{\check{h}_+}{\check{h}_-} \right)^m \right)^2 \int_0^{\check{h}_+} e^2 dz \\ &= \left(\max_{0 \leq i \leq m} r_i \right)^2 \left(\max_{0 \leq i \leq m} |\hat{\varphi}_-^{(i)}(1)| \right)^2 \max \left(1, \left(\frac{\check{h}_+}{\check{h}_-} \right)^m \right)^2 \check{h}_+ e^2. \end{aligned} \tag{3.6}$$

Since $\mathcal{P}^m([0, 1])$ is a finite dimensional space, all norms are equivalent. In particular, there is a constant $C(m)$ such that

$$\left(\max_{0 \leq i \leq m} |p^{(i)}(1)| \right) \leq C(m) \|p\|_{0,[0,1]}, \quad \forall p \in \mathcal{P}^m([0, 1]),$$

which leads to

$$\left(\max_{0 \leq i \leq m} |\hat{\varphi}_-^{(i)}(1)| \right)^2 \leq C(m) \|\hat{\varphi}_-\|_{0,[0,1]}^2. \tag{3.7}$$

By using a change of variables, we can show that

$$\|\hat{\varphi}_-\|_{0,[0,1]}^2 = \frac{1}{\check{h}_-} \|\check{\varphi}_-\|_{0,\check{I}^-}^2. \tag{3.8}$$

Finally, we combine (3.6), (3.7) and (3.8) to get

$$\|\mathcal{E}_{\check{\alpha},r}^{m,+} \check{\varphi}_-\|_{0,\check{I}^+}^2 = \|\check{\varphi}_+\|_{0,\check{I}^+}^2 \leq C(m) \left(\max_{0 \leq i \leq m} r_i \right)^2 \left(\frac{\check{h}_+}{\check{h}_-} \right)^2 \max \left(1, \left(\frac{\check{h}_+}{\check{h}_-} \right)^m \right)^2 \|\check{\varphi}_-\|_{0,\check{I}^-}^2$$

which is (3.2) for $s = -$. □

Next, we will use the bounds on the extension operator $\mathcal{E}_{\check{\alpha},r}^{m,s}$ to establish inverse inequalities which are independent of $\check{\alpha}$ for the RIFE space.

Lemma 3.3 *Let $\tilde{m} \geq m \geq 0$, $\{r_k\}_{k=0}^{\tilde{m}} \subset \mathbb{R}_+$, $\check{\alpha} \in (0, 1)$. Then there exists $C(m, r) > 0$ independent of $\check{\alpha}$ such that for every $\check{\varphi} \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$ we have*

$$|\check{\varphi}|_{i,\check{I}} \leq C(m, r) \|\check{\varphi}\|_{0,\check{I}}, \quad 0 \leq i \leq m + 1. \tag{3.9}$$

Proof The estimate given in (3.9) obviously holds for $i = 0$ and $i = m + 1$. Without loss of generality, assume that $\check{h}_- \geq \check{h}_+$, this implies that $\check{h}_- \geq \frac{1}{2}$. Then, using the classical inverse inequality [12] we have

$$\|\check{\varphi}'\|_{0,\check{I}^-} \leq C\check{h}_-^{-1} \|\check{\varphi}\|_{0,\check{I}^-} \leq 2C \|\check{\varphi}\|_{0,\check{I}^-}. \tag{3.10}$$

By the Taylor expansion of $\check{\varphi}'_+(x)$ at $x = \check{\alpha}$, we have

$$\check{\varphi}'_+ = \mathcal{E}_{\check{\alpha},\tau(r)}^{m-1,+} \left(\frac{d}{dx} \check{\varphi}_- \right), \tag{3.11}$$

where $\tau : (r_0, r_1, \dots, r_m) \mapsto (r_1, \dots, r_m)$ is the shift operator. By (3.3) and the inverse inequality given in (3.10), we have

$$\|\check{\varphi}'_+\|_{0,\check{I}^+} \leq C(m) \left(\max_{1 \leq i \leq m} r_i \right) \|\check{\varphi}'\|_{0,\check{I}^-} \leq C(m, r) \|\check{\varphi}\|_{0,\check{I}^-}. \tag{3.12}$$

Therefore, we have

$$|\check{\varphi}|_{1,\check{I}} = \|\check{\varphi}'\|_{0,\check{I}} \leq C(m, r) \|\check{\varphi}\|_{0,\check{I}^-} \leq C(m, r) \|\check{\varphi}\|_{0,\check{I}}$$

which proves (3.9) for $i = 1$. Applying similar arguments, we can prove (3.9) for other values of i . □

Since $\check{\varphi}_+ = \mathcal{E}_{\check{\alpha},r}^{m,+}(\check{\varphi}_-)$, the formula in (3.11) leads to the following identity about the permutation of the classical differential operator and the extension operator:

$$\frac{d}{dx} \mathcal{E}_{\check{\alpha},r}^{m,+}(\check{\varphi}_-) = \check{\varphi}'_+ = \mathcal{E}_{\check{\alpha},\tau(r)}^{m-1,+} \left(\frac{d}{dx} \check{\varphi}_- \right), \quad \forall m \geq 1. \tag{3.13}$$

As a piecewise function $\check{\varphi} = (\check{\varphi}_-, \check{\varphi}_+) \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$, the value of $\check{\varphi}$ at $\check{\alpha}$ is not defined in general since the two sided limits $\check{\varphi}_-(\check{\alpha})$ and $\check{\varphi}_+(\check{\alpha})$ could be different if $r_0 \neq 1$. However, if $\check{\varphi}_s(\check{\alpha}) = 0$ then $\check{\varphi}_{s'}(\check{\alpha}) = 0$ for $s = +, -$. Furthermore, the multiplicity of $\check{\alpha}$ as a root of $\check{\varphi}_-$ is the same as its multiplicity as a root of $\check{\varphi}_+$. This observation motivates to define $\check{\alpha}$ as a root of $\check{\varphi}$ of multiplicity d if $\check{\alpha}$ is a root of $\check{\varphi}_-$ of multiplicity d . The following theorem shows that the number of roots of a non-zero function $\check{\varphi} \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$ counting multiplicities cannot exceed m (similar to a polynomial

of degree m), this theorem will be crucial to establish the existence of a Lagrange-type basis in $\mathcal{V}_{\check{\alpha},r}^m(\check{I})$ and constructing an immersed Radau projection later in Sect. 5.

In the discussions below, we will omit the phrase “counting multiplicities” for the sake of conciseness. For example, we say that $(x - 2)^2$ has two roots in \mathbb{R} .

Theorem 3.1 *For $\check{m} \geq m \geq 0$, $\{r_k\}_{k=0}^{\check{m}} \subset \mathbb{R}_+$, and $\check{\alpha} \in (0, 1)$, every non-zero $\check{\varphi} \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$ has at most m roots.*

Proof We start from the base case $m = 0$ and then proceed by induction. Let $\check{\varphi} \in \mathcal{V}_{\check{\alpha},r}^0(\check{I})$. if $\check{\varphi} \not\equiv 0$ then $\check{\varphi} = c \mathcal{N}_{\check{\alpha},r}^0$ for some $c \neq 0$. In this case, $\check{\varphi}$ has no roots since $\mathcal{N}_{\check{\alpha},r}^0$ has no roots. Now, assume that for every positive sequence q , the number of roots of any non-zero $\check{\varphi} \in \mathcal{V}_{\check{\alpha},q}^{m-1}(\check{I})$ is at most $m - 1$. Next, we will show that for a given positive sequence r , every function $\check{\varphi} \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$ has at most m roots by contradiction:

Assume that $\check{\varphi} = (\check{\varphi}_-, \check{\varphi}_+) \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$ is a non-zero function that has j distinct roots $\{\xi_i\}_{i=1}^j$ of multiplicities $\{d_i\}_{i=1}^j$ such that $D = d_1 + d_2 + \dots + d_j > m$. Therefore, $\xi_j > \check{\alpha}$ and $\xi_1 \leq \check{\alpha}$ because $\check{\varphi}_{\pm} \in \mathcal{P}^m(\check{I}^s)$. let ξ_{i_0} be the largest root that is not larger than $\check{\alpha}$, i.e.,

$$0 \leq \xi_1 < \xi_2 < \dots < \xi_{i_0} \leq \check{\alpha} < \xi_{i_0+1} < \dots < \xi_j \leq 1,$$

By the definition of $\check{\varphi} = (\check{\varphi}_-, \check{\varphi}_+)$, $\check{\varphi}_-$ has $D_1 = d_1 + d_2 + \dots + d_{i_0}$ roots in $[\xi_1, \xi_{i_0}]$ and $\check{\varphi}_+$ has $D_2 = d_{i_0+1} + d_{i_0+2} + \dots + d_j$ roots in $[\xi_{i_0+1}, \xi_j]$. Therefore, $\check{\varphi}'_-$ has $D_1 - 1$ roots in $[\xi_1, \xi_{i_0}]$ and $\check{\varphi}'_+$ has $D_2 - 1$ roots in $[\xi_{i_0+1}, \xi_j]$. It remains to show that $\check{\varphi}'$ has an additional root in (ξ_{i_0}, ξ_{i_0+1}) . To show that, we consider two cases:

- $\xi_{i_0} = \check{\alpha}$: In this case, $\check{\varphi}$ is continuous and $\check{\varphi}'_+(\check{\alpha}) = \check{\varphi}'_+(\xi_{i_0+1}) = 0$. By the mean value theorem, we conclude that $\check{\varphi}'_+$ has a root in (ξ_{i_0}, ξ_{i_0+1}) .
- $\xi_{i_0} < \check{\alpha}$: Assume that $\check{\varphi}'(x) > 0$ for all $x \in (\xi_{i_0}, \xi_{i_0+1}) \setminus \{\check{\alpha}\}$, then $\check{\varphi}'_-(\check{\alpha}) > 0$. Since $r_0 > 0$, we have $\check{\varphi}'_+(\check{\alpha}) > 0$. By integrating $\check{\varphi}'_+$ from $\check{\alpha}$ to ξ_{i_0+1} , we get $0 = \check{\varphi}_+(\xi_{i_0+1}) > 0$, a contradiction. A similar conclusion follows if we assume that $\check{\varphi}'(x) < 0$ for all $x \in (\xi_{i_0}, \xi_{i_0+1}) \setminus \{\check{\alpha}\}$. Therefore, $\check{\varphi}'$ changes sign at some $x_0 \in (\xi_{i_0}, \xi_{i_0+1})$. Since, $\check{\varphi}'$ does not change sign at $\check{\alpha}$, then $x_0 \neq \check{\alpha}$ and $\check{\varphi}'(x_0) = 0$ (because $\check{\varphi}'$ is continuous at x_0).

In either case above, $\check{\varphi}'$ has $(D_1 - 1) + (D_2 - 1) + 1 = D - 1 > m - 1$ roots in \check{I} which contradicts the induction hypothesis since $\check{\varphi}' \in \mathcal{V}_{\check{\alpha},\tau(r)}^{m-1}(\check{I})$. Therefore, $\check{\varphi}$ has at most m roots. □

The previous theorem allows us to establish the existence of a Lagrange basis on $\mathcal{V}_{\check{\alpha},r}^m(\check{I})$ for every choice of nodes and for every degree m which was proved by Moon in [33] for a few specific cases $m = 1, 2, 3, 4$.

Theorem 3.2 *Let $\check{m} \geq m \geq 0$, $\{r_k\}_{k=0}^{\check{m}} \subset \mathbb{R}_+$, and $\check{\alpha} \in (0, 1)$. Assume $\xi_0, \xi_1, \dots, \xi_m$ are $m + 1$ distinct points in \check{I} , then there is a Lagrange basis $\{L_i\}_{i=0}^m$ of $\mathcal{V}_{\check{\alpha},r}^m(\check{I})$ that satisfies*

$$L_i(\xi_j) = \delta_{i,j}, \quad 0 \leq i, j \leq m. \tag{3.14}$$

Proof For each $0 \leq i \leq m$, we construct $\tilde{L}_i \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$ such that $\tilde{L}_i(\xi_j) = 0$ for all $j \neq i$ by writing \tilde{L}_i as

$$\tilde{L}_i = \sum_{i=0}^m a_i \mathcal{N}_{\check{\alpha},r}^i,$$

for some $\{a_i\}_{i=0}^m$ chosen such that

$$\tilde{L}_i(\xi_j) = \sum_{i=0}^m a_i \mathcal{N}_{\check{\alpha},r}^i(\xi_j) = 0, \quad \forall j \in \{0, 1, \dots, i-1, i+1, \dots, m\}. \quad (3.15)$$

The Eq. (3.15) form a homogeneous system of m equations with $m + 1$ unknowns. Therefore, it has a non-zero solution. From Theorem 3.1, we know that $\tilde{L}_i(\xi_i) \neq 0$; otherwise, \tilde{L}_i would have $m + 1$ roots. This allows us to define L_i as

$$L_i(x) = \frac{1}{\tilde{L}_i(\xi_i)} \tilde{L}_i(x).$$

By (3.14), $L_i, 0 \leq i \leq m$ are linearly independent. Consequently, $\{L_i\}_{i=0}^m$ is a basis for $\mathcal{V}_{\check{\alpha},r}^m(\check{I})$ since its dimension is $m + 1$ from Lemma 3.1. □

In addition to having a Lagrange basis, the RIFE space has an orthogonal basis with respect to $(\cdot, \cdot)_{w,\check{I}}$ as stated in the following theorem in which we also show that if a function $\check{\varphi} \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$ is orthogonal to $\mathcal{V}_{\check{\alpha},r}^{m-1}(\check{I})$ with respect to $(\cdot, \cdot)_{w,\check{I}}$, then $\check{\varphi}$ has exactly m distinct interior roots similar to the classical orthogonal polynomials. Although the theorem holds for a general weight w , we will restrict our attention to a piecewise constant function w :

$$w(x) = \begin{cases} w_-, & x \in \check{I}^-, \\ w_+, & x \in \check{I}^+, \end{cases} \quad (3.16)$$

where w_{\pm} are positive constants. The result of this theorem can also be considered as a generalization for the theorem about the orthogonal IFE basis described in [13] for elliptic interface problems.

Theorem 3.3 *Let $\tilde{m} \geq m \geq 1, \{r_k\}_{k=0}^{\tilde{m}} \subset \mathbb{R}_+, \check{\alpha} \in (0, 1)$, and let $w : \check{I} \rightarrow \mathbb{R}_+$ be defined as in (3.16), then there is a non-zero $\check{\varphi} \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$ such that*

$$(\check{\varphi}, \check{\psi})_{w,\check{I}} = \int_{\check{I}} w(x) \check{\varphi}(x) \check{\psi}(x) dx = 0, \quad \forall \check{\psi} \in \mathcal{V}_{\check{\alpha},r}^{m-1}(\check{I}). \quad (3.17)$$

Furthermore, $\check{\varphi}$ has exactly m distinct roots in the interior of \check{I} .

Proof Existence is a classical result of linear algebra. The proof of the second claim follows the same steps used for orthogonal polynomials: Note that $\check{\varphi}$ has at least one root of odd multiplicity in the interior of \check{I} since $(\check{\varphi}, \mathcal{N}_{\check{\alpha},r}^0)_{w,\check{I}} = 0$.

Assume that $\check{\varphi}$ has $j < m$ distinct roots $\{\xi_i\}_{i=1}^j$ of odd multiplicity in the interior of \check{I} . Following the ideas in the proof of Theorem 3.2, we can show that there is $\check{\psi}_0 \in \mathcal{V}_{\check{\alpha},r}^j(\check{I})$ such that $\check{\psi}_0(\xi_i) = 0$ for $1 \leq i \leq j$.

Furthermore, all roots of $\check{\psi}_0$ are simple according to Theorem 3.1 since the sum of multiplicities cannot exceed j , and $\check{\psi}_0$ changes sign at these roots. This means that $w\check{\varphi}\check{\psi}_0$ does not change sign on \check{I} . As a consequence, $(\check{\varphi}, \check{\psi}_0)_{w,\check{I}} \neq 0$ which contradicts the assumption $(\check{\varphi}, \check{\psi})_{w,\check{I}} = 0$ for all $\check{\psi} \in \mathcal{V}_{\check{\alpha},r}^{m-1}(\check{I})$ since $\mathcal{V}_{\check{\alpha},r}^j(\check{I}) \subseteq \mathcal{V}_{\check{\alpha},r}^{m-1}(\check{I})$. \square

For every integer m with $\tilde{m} \geq m \geq 1$, we use $\mathcal{Q}_{\check{\alpha},w,r}^m(\check{I})$, $m \geq 1$ to denote the orthogonal complement of $\mathcal{V}_{\check{\alpha},r}^{m-1}(\check{I})$ in $\mathcal{V}_{\check{\alpha},r}^m(\check{I})$ with respect to the weight w , that is

$$\mathcal{Q}_{\check{\alpha},w,r}^m(\check{I}) = \left\{ \check{\varphi} \in \mathcal{V}_{\check{\alpha},r}^m(\check{I}) \mid (\check{\varphi}, \check{\psi})_{w,\check{I}} = 0, \forall \check{\psi} \in \mathcal{V}_{\check{\alpha},r}^{m-1}(\check{I}) \right\}.$$

According to Theorem 3.3, one can see that $\check{\varphi} \mapsto \sqrt{\check{\varphi}(0)^2 + \check{\varphi}(1)^2}$ defines a norm on $\mathcal{Q}_{\check{\alpha},w,r}^m(\check{I})$ which is one-dimensional. Thus, it is equivalent to the L^2 norm and the quantity $\frac{\sqrt{\check{\varphi}(0)^2 + \check{\varphi}(1)^2}}{\|\check{\varphi}\|_{0,\check{I}}}$ depends only on $\check{\alpha}$, w and r (and not on the choice of $\check{\varphi} \in \mathcal{Q}_{\check{\alpha},w,r}^m(\check{I})$). Furthermore, The following lemma shows that the equivalence constant is independent of the interface location. This result will be crucial later in the analysis of Radau projections.

Lemma 3.4 *Let $\tilde{m} \geq m \geq 1$, $\{r_k\}_{k=0}^{\tilde{m}} \subset \mathbb{R}_+$, $\check{\alpha} \in (0, 1)$ and w be defined as in (3.16) then, there exist $C(m, w, r)$ and $\tilde{C}(m, w, r) > 0$ independent of $\check{\alpha}$ such that for every $\check{\varphi} \in \mathcal{Q}_{\check{\alpha},w,r}^m(\check{I})$, we have*

$$\sqrt{\check{\varphi}(0)^2 + \check{\varphi}(1)^2} \geq C(m, w, r) \|\check{\varphi}\|_{0,\check{I}} \geq \tilde{C}(m, r) \|\check{\varphi}\|_{1,\check{I}}. \tag{3.18}$$

Proof The inequality on the right follows from the inverse inequality (3.9) for the IFE functions. For a proof of the inequality on the left, see ‘‘Appendix A’’. \square

4 An immersed Bramble–Hilbert lemma and the approximation capabilities of the LIFE space

In this section, we will develop a new version of Bramble–Hilbert lemma that applies to functions in $\mathcal{H}_{\check{\alpha},r}^{m+1}(\check{I})$ and its IFE counterpart. This lemma will serve as a fundamental tool for investigating the approximation capability of IFE spaces. In the discussions below, we will use $\mathbb{1}_B$ for the indicator function of a set $B \subset \mathbb{R}$, i.e.,

$$\mathbb{1}_B(x) = \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{if } x \notin B, \end{cases} \quad \forall x \in \mathbb{R},$$

and we define $w_i = r_i \mathbb{1}_{\check{I}^-} + \mathbb{1}_{\check{I}^+}$ for $i = 0, 1, \dots, m$.

Theorem 4.1 *Let $\tilde{m} \geq m \geq 0$, $\{r_k\}_{k=0}^{\tilde{m}} \subset \mathbb{R}_+$, $\check{\alpha} \in (0, 1)$, and $v \in \mathcal{H}_{\check{\alpha}, r}^{m+1}(\check{I})$. Assume $(w_i, v^{(i)})_{\check{I}} = 0$ for $i = 0, 1, \dots, m$. Then, there exists $C(i, r) > 0$ independent of $\check{\alpha}$ such that*

$$\|v\|_{i, \check{I}} \leq C(i, r) |v|_{i, \check{I}}, \quad i = 0, 1, \dots, m + 1. \tag{4.1}$$

Proof Let $v \in \mathcal{H}_{\check{\alpha}, r}^{m+1}(\check{I})$ and assume $(w_i, v^{(i)})_{\check{I}} = 0$ for $i = 0, 1, \dots, m$. Because w_0 is such that $w_0 v$ is continuous and since $v|_{\check{I}^\pm} \in H^1(\check{I}^\pm)$, we have $w_0 v \in H^1(\check{I})$. Therefore, for any given $x, y \in \check{I}$, we have

$$w_0(x)v(x) - w_0(y)v(y) = \int_y^x w_0(z)v'(z) dz$$

We integrate this identity on \check{I} with respect to x and use $(w_0, v^{(0)})_{\check{I}} = 0$ to get

$$-w_0(y)v(y) = \int_0^1 \int_y^x w_0(z)v'(z) dz dx, \quad \forall y \in \check{I}.$$

Taking the absolute value and applying the Cauchy–Schwarz inequality, we get

$$|v(y)| \leq \frac{1}{\min(1, r_0)} |w_0(y)v(y)| \leq \frac{\max(1, r_0)}{\min(1, r_0)} |v|_{1, \check{I}} = \max(r_0, r_0^{-1}) |v|_{1, \check{I}},$$

which implies $\|v\|_{0, \check{I}} \leq \max(r_0, r_0^{-1}) |v|_{1, \check{I}}$. Since $v' \in \mathcal{H}_{\check{\alpha}, \tau(r)}^m(\check{I})$, where τ is the shift operator described in (3.11), we can use the same reasoning to show that

$$|v|_{1, \check{I}} \leq \max(r_1, r_1^{-1}) |v|_{2, \check{I}}.$$

Repeating the same arguments, we can obtain

$$|v|_{i, \check{I}} \leq \max(r_i, r_i^{-1}) |v|_{i+1, \check{I}}, \quad i = 0, 1, \dots, m \tag{4.2}$$

which leads to (4.1) with

$$C(i, r) = \sqrt{1 + \sum_{k=0}^i \prod_{j=k}^i \max(r_j^2, r_j^{-2})}.$$

□

Lemma 4.1 *Let $\tilde{m} \geq m \geq 0$, $\{r_k\}_{k=0}^{\tilde{m}} \subset \mathbb{R}_+$, $\check{\alpha} \in (0, 1)$ and $v \in \mathcal{H}_{\check{\alpha},r}^{m+1}(\check{I})$. Then, there is a unique $\check{\pi}_{\check{\alpha},r}^m v \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$ that satisfies*

$$\int_{\check{I}} w_i(x) \frac{d^i}{dx^i} \left(v(x) - \check{\pi}_{\check{\alpha},r}^m v(x) \right) dx = 0, \quad \forall i = 0, 1, \dots, m. \tag{4.3}$$

Proof For $v \in \mathcal{H}_{\check{\alpha},r}^{m+1}(\check{I})$, to see that $\check{\pi}_{\check{\alpha},r}^m v$ exists and is unique, we consider the problem of finding $\check{\varphi} \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$ such that

$$(w_i, \check{\varphi}^{(i)})_{\check{I}} = (w_i, v^{(i)})_{\check{I}}, \quad \text{for } i = 0, 1, \dots, m. \tag{4.4}$$

By Lemma 3.1, we can express $\check{\varphi}$ in terms of the canonical basis

$$\check{\varphi} = \sum_{j=0}^m c_j \mathcal{N}_{\check{\alpha},r}^j.$$

Then, by (4.4), the coefficients \mathbf{c} of $\check{\varphi}$ are determined by the linear system $\mathbf{A}\mathbf{c} = \mathbf{b}$, where $A = (A_{i,j})$ is a triangular matrix with $A_{i,j} = (w_i, (\mathcal{N}_{\check{\alpha},r}^j)^{(i)})_{\check{I}}$, $0 \leq i, j \leq m$ and diagonal entries

$$A_{i,i} = (w_i, (\mathcal{N}_{\check{\alpha},r}^i)^{(i)})_{\check{I}} = i!(h_- + r_i h_+) \neq 0, \quad 0 \leq i \leq m.$$

Therefore, A is invertible and $\check{\pi}_{\check{\alpha},r}^m v = \check{\varphi}$ is uniquely determined by (4.3). □

We note that the mapping $\check{\pi}_{\check{\alpha},r}^m : v \in \mathcal{H}_{\check{\alpha},r}^{m+1}(\check{I}) \mapsto \check{\pi}_{\check{\alpha},r}^m v \in \mathcal{V}_{\check{\alpha},r}^m(\check{I})$ is linear because of the linearity of integration.

We now present an immersed version of the Bramble–Hilbert lemma [11] which can be considered a generalization of the one-dimensional Bramble–Hilbert lemma in the sense that if $r \equiv 1$, then, this immersed Bramble–Hilbert lemma recovers the classical Bramble–Hilbert lemma.

Lemma 4.2 *Let $\tilde{m} \geq m \geq i \geq 0$, $\{r_k\}_{k=0}^{\tilde{m}} \subset \mathbb{R}_+$, $\check{\alpha} \in (0, 1)$. Assume $\check{P}_{\check{\alpha},r}^m : \mathcal{H}_{\check{\alpha},r}^{m+1}(\check{I}) \rightarrow \mathcal{V}_{\check{\alpha},r}^m(\check{I})$ is a linear map that satisfies the following two conditions*

1. $\check{P}_{\check{\alpha},r}^m$ is a projection on $\mathcal{V}_{\check{\alpha},r}^m(\check{I})$ in the sense that

$$\check{P}_{\check{\alpha},r}^m \check{\varphi} = \check{\varphi}, \quad \forall \check{\varphi} \in \mathcal{V}_{\check{\alpha},r}^m(\check{I}). \tag{4.5}$$

2. There exists an integer j , $0 \leq j \leq m + 1$ such that $\check{P}_{\check{\alpha},r}^m$ is bounded with respect to the norm $\|\cdot\|_{j,\check{I}}$ as follows:

$$\left\| \check{P}_{\check{\alpha},r}^m v \right\|_{i,\check{I}} \leq C \|v\|_{j,\check{I}}, \quad \forall v \in \mathcal{H}_{\check{\alpha},r}^{m+1}(\check{I}). \tag{4.6}$$

Then, there exists $C(m, r) > 0$ independent of $\check{\alpha}$, such that

$$\|v - \check{P}_{\check{\alpha},r}^m v\|_{i,\check{I}} \leq C(m, r) \left(1 + \|\check{P}_{\check{\alpha},r}^m\|_{i,j,\check{I}}\right) |v|_{m+1,\check{I}}, \quad v \in \mathcal{H}_{\check{\alpha},r}^{m+1}(\check{I}), \quad (4.7)$$

where

$$\|\check{P}_{\check{\alpha},r}^m\|_{i,j,\check{I}} = \sup \left\{ \|\check{P}_{\check{\alpha},r}^m v\|_{i,\check{I}} \mid v \in \mathcal{H}_{\check{\alpha},r}^{m+1}(\check{I}) \text{ and } \|v\|_{j,\check{I}} = 1 \right\}.$$

Proof Since $\check{P}_{\check{\alpha},r}^m$ is a projection in the sense of (4.5), we have $\check{P}_{\check{\alpha},r}^m \check{\pi}_{\check{\alpha},r}^m v = \check{\pi}_{\check{\alpha},r}^m v$. Using the triangle inequality and (4.6), we obtain

$$\begin{aligned} \|v - \check{P}_{\check{\alpha},r}^m v\|_{i,\check{I}} &\leq \|v - \check{\pi}_{\check{\alpha},r}^m v\|_{i,\check{I}} + \|\check{P}_{\check{\alpha},r}^m (v - \check{\pi}_{\check{\alpha},r}^m v)\|_{i,\check{I}}, \\ &\leq \left(1 + \|\check{P}_{\check{\alpha},r}^m\|_{i,j,\check{I}}\right) \|v - \check{\pi}_{\check{\alpha},r}^m v\|_{j,\check{I}} \leq \left(1 + \|\check{P}_{\check{\alpha},r}^m\|_{i,j,\check{I}}\right) \|v - \check{\pi}_{\check{\alpha},r}^m v\|_{m+1,\check{I}}, \\ &\quad \forall v \in \mathcal{H}_{\check{\alpha},r}^{m+1}(\check{I}). \end{aligned}$$

Then, applying Lemma 4.1 and Theorem 4.1 to the right hand side of the above estimate leads to (4.7). □

Next, we extend the results of Lemma 4.2 to the physical interface element $I_{k_0} = [x_{k_0-1}, x_{k_0-1} + h]$. Following the tradition in finite element analysis, for every function φ defined on the interface element I_{k_0} , we can map it to a function $\mathcal{M}\varphi = \check{\varphi}$ defined on the reference interval \check{I} by the standard affine transformation:

$$\mathcal{M}\varphi(\xi) = \check{\varphi}(\xi) = \varphi(x_{k_0-1} + h\xi), \quad \xi \in \check{I} = [0, 1]. \quad (4.8)$$

Furthermore, given a mapping $P_{\alpha,r}^m : \mathcal{H}_{\alpha,r}^{m+1}(I_{k_0}) \rightarrow \mathcal{V}_{\alpha,r}^m(I_{k_0})$, we can use this affine transformation to introduce a mapping $\check{P}_{\check{\alpha},r}^m : \mathcal{H}_{\check{\alpha},r}^{m+1}(\check{I}) \rightarrow \mathcal{V}_{\check{\alpha},r}^m(\check{I})$ such that

$$(\check{P}_{\check{\alpha},r}^m \check{v})(\xi) = (P_{\alpha,r}^m v)(x_{k_0-1} + h\xi) = (P_{\alpha,r}^m v)(x) \text{ with } \xi \in \check{I} \text{ or } x = x_{k_0-1} + h\xi \in I_{k_0}. \quad (4.9)$$

It can be verified that the mappings \mathcal{M} , $P_{\alpha,r}^m$ and $\check{P}_{\check{\alpha},r}^m$ satisfy the following commutative diagram:

$$\begin{array}{ccc} \mathcal{H}_{\alpha,r}^{m+1}(I_{k_0}) & \xrightarrow{P_{\alpha,r}^m} & \mathcal{V}_{\alpha,r}^m(I_{k_0}) \\ \downarrow \mathcal{M} & & \downarrow \mathcal{M} \\ \mathcal{H}_{\check{\alpha},r}^{m+1}(\check{I}) & \xrightarrow{\check{P}_{\check{\alpha},r}^m} & \mathcal{V}_{\check{\alpha},r}^m(\check{I}) \end{array} \quad (4.10)$$

We now use the immersed Bramble–Hilbert lemma in the scaling argument to obtain estimates for the projection error $v - P_{\alpha,r}^m v$.

Theorem 4.2 *Let $\tilde{m} \geq m \geq i \geq 0$, $\{r_k\}_{k=0}^{\tilde{m}} \subset \mathbb{R}_+$, $\check{\alpha} \in (0, 1)$. Assume $P_{\alpha,r}^m : \mathcal{H}_{\alpha,r}^{m+1}(I_{k_0}) \rightarrow \mathcal{V}_{\alpha,r}^m(I_{k_0})$ is a linear operator such that $\check{P}_{\check{\alpha},r}^m$ defined by (4.9) satisfies the assumption of Lemma 4.2 for an integer j with $0 \leq j \leq m + 1$. Then, there exists $C(m, r) > 0$ independent of α such that*

$$|v - P_{\alpha,r}^m v|_{i,I_{k_0}} \leq Ch^{m+1-i} \left(1 + \left\| \check{P}_{\check{\alpha},r}^m \right\|_{i,j,\check{I}} \right) |v|_{m+1,I_{k_0}}, \quad v \in \mathcal{H}_{\alpha,r}^{m+1}(I_{k_0}). \tag{4.11}$$

Proof The proof follows the same argument as for the classical case $r_k = 1$, $k = 0, 1, \dots, m$ [15]. We start by applying the change of variables $x \mapsto h^{-1}(x - x_{k_0-1})$ to we obtain

$$|v|_{m+1,I_{k_0}} = h^{-1} h^{-m-1} |\check{v}|_{m+1,\check{I}}, \quad |v - P_{\alpha,r}^m v|_{i,I_{k_0}} = h^{-1} h^{-i} |\check{v} - \check{P}_{\check{\alpha},r}^m \check{v}|_{i,\check{I}}. \tag{4.12}$$

Next, we combine Lemma 4.2 and (4.12) to obtain

$$\begin{aligned} |v - P_{\alpha,r}^m v|_{i,I_{k_0}} &= h^{-i-1} |\check{v} - \check{P}_{\check{\alpha},r}^m \check{v}|_{i,\check{I}} \leq h^{-i-1} C(m, r) \left(1 + \left\| \check{P}_{\check{\alpha},r}^m \right\|_{i,j,\check{I}} \right) |\check{v}|_{m+1,\check{I}} \\ &= h^{-i-1} C(m, r) \left(1 + \left\| \check{P}_{\check{\alpha},r}^m \right\|_{i,j,\check{I}} \right) h^{m+2} |v|_{m+1,I_{k_0}} \\ &= C(m, r) h^{m+1-i} \left(1 + \left\| \check{P}_{\check{\alpha},r}^m \right\|_{i,j,\check{I}} \right) |v|_{m+1,I_{k_0}}. \end{aligned}$$

□

Nevertheless, the estimate (4.11) does not directly lead to the convergence of $P_{\alpha,r}^m v$ to v as $h \rightarrow 0$ unless we can show that $\left\| \check{P}_{\check{\alpha},r}^m \right\|_{i,j,\check{I}}$ is uniformly bounded with respect to $\check{\alpha} \in \check{I}$, and this can be addressed by the *uniform boundedness* of $P_{\alpha,r}^m$ defined as follows.

Definition 4.1 *Let $\tilde{m} \geq m \geq 0$, $\{r_k\}_{k=0}^{\tilde{m}} \subset \mathbb{R}_+$, $\check{\alpha} \in \check{I}$, and let $\{\check{P}_{\check{\alpha},r}^m\}_{0 < \check{\alpha} < 1}$ be a collection of projections in the sense of (4.5) such that $\check{P}_{\check{\alpha},r}^m : \mathcal{H}_{\check{\alpha},r}^{m+1}(\check{I}) \rightarrow \mathcal{V}_{\check{\alpha},r}^m(\check{I})$. We call $\{\check{P}_{\check{\alpha},r}^m\}_{0 < \check{\alpha} < 1}$ a uniformly bounded collection of RIFE projections provided that there exists a constant $C > 0$ independent of $\check{\alpha}$ and an integer j with $0 \leq j \leq m + 1$ such that*

$$\left\| \check{P}_{\check{\alpha},r}^m v \right\|_{0,\check{I}} < C \|v\|_{j,\check{I}}, \quad \forall v \in \mathcal{H}_{\check{\alpha},r}^{m+1}(\check{I}), \quad \forall \check{\alpha} \in (0, 1), \tag{4.13}$$

and the associated collection of maps $\{P_{\alpha,r}^m\}_{\alpha \in I_{k_0}}$ defined in (4.9) is called a uniformly bounded collection of LIFE projections.

Lemma 4.3 *Let $\tilde{m} \geq m \geq 0, \{r_k\}_{k=0}^{\tilde{m}} \subset \mathbb{R}_+, \check{\alpha} \in \check{I}$. Assume $\{\check{P}_{\check{\alpha},r}^m\}_{0 < \check{\alpha} < 1}$ is a uniformly bounded collection of RIFE projections. Then, there exists a constant C independent of $\check{\alpha}$ such that*

$$\|\check{P}_{\check{\alpha},r}^m v\|_{i,\check{I}} \leq C \|v\|_{m+1,\check{I}}, \quad \forall v \in \mathcal{H}_{\check{\alpha},r}^{m+1}(\check{I}), \quad 0 \leq i \leq m + 1. \tag{4.14}$$

Proof Assume that $\{\check{P}_{\check{\alpha},r}^m\}_{0 < \check{\alpha} < 1}$ is a uniformly bounded collection of RIFE projections, then,

$$\|\check{P}_{\check{\alpha},r}^m v\|_{0,\check{I}} \leq C \|v\|_{j,\check{I}}$$

for an integer j with $0 \leq j \leq m + 1$. By Lemma 3.3, we further have

$$\begin{aligned} \|\check{P}_{\check{\alpha},r}^m v\|_{i,\check{I}} &\leq c(m, r) \|\check{P}_{\check{\alpha},r}^m v\|_{0,\check{I}} \leq c(m, r)C \|v\|_{j,\check{I}} \leq c(m, r)C \|v\|_{m+1,\check{I}}, \\ &0 \leq i \leq m + 1 \end{aligned}$$

which implies the uniform boundedness stated in (4.14). □

Now, we can derive an error bound for a collection of uniformly bounded LIFE projections that implies convergence.

Theorem 4.3 *Let $\tilde{m} \geq m \geq 0, \{r_k\}_{k=0}^{\tilde{m}} \subset \mathbb{R}_+, \check{\alpha} \in \check{I}$. Assume that $\{P_{\alpha,r}^m\}_{\alpha \in I_{k_0}}^\circ$ is a uniformly bounded collection of LIFE projections. Then, there exists a constant $C > 0$ independent of α and h such that*

$$|v - P_{\alpha,r}^m v|_{i,I_{k_0}} \leq Ch^{m+1-i} |v|_{m+1,I_{k_0}}, \quad \forall v \in \mathcal{H}_{\alpha,r}^{m+1}(I_{k_0}), \quad \forall i = 0, 1, \dots, m. \tag{4.15}$$

Proof By Lemma 4.3, we know that $\{P_{\alpha,r}^m\}_{\alpha \in I_{k_0}}^\circ$ satisfies (4.6) for $i = 0, 1, 2, \dots, m$ with $j = m + 1$. Consequently, we have

$$\|\check{P}_{\check{\alpha},r}^m\|_{i,m+1,\check{I}} = \sup \left\{ \|\check{P}_{\check{\alpha},r}^m v\|_{i,\check{I}} \mid v \in \mathcal{H}_{\check{\alpha},r}^{m+1}(\check{I}) \text{ and } \|v\|_{m+1,\check{I}} = 1 \right\} \leq C.$$

Then, applying these to (4.11) established in Theorem 4.2 yields (4.15). □

The simplest example of a uniformly bounded collection of LIFE projections is the L^2 projection $\check{P}r_{\check{\alpha},r}^m : \mathcal{H}_{\check{\alpha},r}^{m+1}(\check{I}) \rightarrow \mathcal{V}_{\check{\alpha},r}^m(\check{I})$ defined by

$$(q, \check{P}r_{\check{\alpha},r}^m v)_{\check{I}} = (q, v)_{\check{I}}, \quad \forall q \in \mathcal{V}_{\check{\alpha},r}^m(\check{I}).$$

Choosing $q = \check{\text{Pr}}_{\check{\alpha},r}^m v$, we get $\|\check{\text{Pr}}_{\check{\alpha},r}^m v\|_{0,\check{I}} = \|v\|_{0,\check{I}} \leq \|v\|_{m+1,\check{I}}$. By Lemma 4.3, $\{\check{\text{Pr}}_{\check{\alpha},r}^m\}_{\alpha \in I_{k_0}}$ is a uniformly bounded collection of projections. Consequently, by Theorem 4.3, we can obtain the following optimal approximation capability of the LIFE space.

Corollary 4.1 *Let $\tilde{m} \geq m \geq 0$, $\{r_k\}_{k=0}^{\tilde{m}} \subset \mathbb{R}_+$, $\check{\alpha} \in (0, 1)$. Then, there exists a constant $C > 0$ independent of α and h such that*

$$\min_{q \in \mathcal{Y}_{\alpha,r}^m(I_{k_0})} |q - v|_{0,I_{k_0}} \leq Ch^{m+1} |v|_{m+1,I_{k_0}}, \quad \forall v \in \mathcal{H}_{\alpha,r}^{m+1}(I_{k_0}). \quad (4.16)$$

5 Analysis of an IFE-DG method for a class of hyperbolic systems with discontinuous coefficients

In this section, we employ the results of Sects. 3 and 4 to analyse an IFE-DG method for the acoustic interface problem (1.1). To our knowledge, the analysis of such method for hyperbolic systems has so far not been considered in the literature unlike the IFE methods for elliptic problems. The main challenge that time-dependent problems present is the plethora of possible jump coefficients. For instance, the jump coefficients r_k^p, r_k^v described in (1.2) for the acoustic interface problem are given by:

$$r_{2k}^p = r_{2k}^v = \left(\frac{c_-}{c_+}\right)^{2k}, \quad r_{2k+1}^p = \frac{\rho_+}{\rho_-} \left(\frac{c_-}{c_+}\right)^{2k}, \quad r_{2k+1}^v = \frac{\rho_-}{\rho_+} \left(\frac{c_-}{c_+}\right)^{2k+2}, \quad k = 0, 1, \dots \quad (5.1)$$

The nature of the jump coefficients in (5.1) makes the study of this particular IFE space extremely tedious as observed in [33]. Fortunately, the theory developed in Sects. 3 and 4 is general and applies to any choice of positive jump coefficients.

5.1 Problem statement and preliminary results

Let $I = (a, b)$ be a bounded interval containing α , and let ρ_{\pm}, c_{\pm} be positive constants describing the density and the sound speed in I^{\pm} , respectively. Now, we consider the acoustic interface problem on I

$$\mathbf{u}_t(x, t) + A(x)\mathbf{u}_x(x, t) = 0, \quad x \in I \setminus \{\alpha\}, \quad t > 0, \quad (5.2a)$$

where $\mathbf{u} = (p, u)^T$ is the pressure-velocity couple and

$$A|_{I^{\pm}} = A_{\pm} = \begin{pmatrix} 0 & \rho_{\pm} c_{\pm}^2 \\ \rho_{\pm}^{-1} & 0 \end{pmatrix}. \quad (5.2b)$$

The matrices A_{\pm} can be decomposed as $A_{\pm} = P_{\pm} \Lambda_{\pm} P_{\pm}^{-1}$, where $\Lambda_{\pm} = \text{diag}(-c_{\pm}, c_{\pm})$. Using this eigen-decomposition, we define $A_{\pm}^{\pm} = P_{\pm} \text{diag}(0, c_{\pm}) P_{\pm}^{-1}$,

$A_{\pm}^{-} = P_{\pm} \text{diag}(-c_{\pm}, 0) P_{\pm}^{-1}$, and $|A_{\pm}| = P_{\pm} \text{diag}(c_{\pm}, c_{\pm}) P_{\pm}^{-1}$ to be the positive part, the negative part, and the absolute value of A_{\pm} , respectively. The acoustic interface problem that we are considering here is subject to the following homogeneous inflow boundary conditions

$$A_{-}^{+} \mathbf{u}(a, t) = A_{+}^{-} \mathbf{u}(b, t) = 0, \quad t \geq 0, \tag{5.2c}$$

initial conditions

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad x \in I, \tag{5.2d}$$

and interface condition

$$\mathbf{u}(\alpha^{-}, t) = \mathbf{u}(\alpha^{+}, t), \quad t \geq 0. \tag{5.2e}$$

In the remainder of this section, let $S_{\pm} = \text{diag}(\rho_{\pm}^{-1} c_{\pm}^{-2}, \rho_{\pm})$ and $S(x) = S_{\pm}$ if $x \in I^{\pm}$, then

$$S_{\pm} A_{\pm} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tilde{A}. \tag{5.3}$$

Now, we can multiply (5.2a) by S and write the acoustic interface problem as

$$S(x) \mathbf{u}_t(x, t) + \tilde{A} \mathbf{u}_x(x, t) = 0, \quad x \in I \setminus \{\alpha\}, \quad t > 0. \tag{5.4}$$

Lombard [31], and Moon [33] have shown that by successively differentiating $\llbracket \mathbf{u}(\cdot, t) \rrbracket_{\alpha} = 0$, where $\llbracket \cdot \rrbracket$ is the jump, we obtain

$$\begin{aligned} \llbracket A(\cdot)^k \mathbf{u}(\cdot, t) \rrbracket_{\alpha} = 0 &\iff \frac{\partial^k}{\partial x^k} \mathbf{u}(\alpha^{+}, t) = R_k \frac{\partial^k}{\partial x^k} \mathbf{u}(\alpha^{-}, t), \quad R_k = A_{+}^{-k} A_{-}^k, \\ k &= 0, 1, \dots, m. \end{aligned}$$

Since R_k is diagonal (see part (a) of Lemma 5.1), the condition $\mathbf{u}^{(k)}(\alpha^{+}, t) = R_k \mathbf{u}^{(k)}(\alpha^{-}, t)$ is equivalent to

$$\frac{\partial^k}{\partial x^k} p(\alpha^{+}, t) = r_k^p \frac{\partial^k}{\partial x^k} p(\alpha^{-}, t), \quad \frac{\partial^k}{\partial x^k} u(\alpha^{+}, t) = r_k^u \frac{\partial^k}{\partial x^k} u(\alpha^{-}, t), \tag{5.5}$$

where r_k^p and r_k^u are defined in (5.1). These decoupled interface conditions make the results obtained previously about the approximation capabilities of the LIFE space directly applicable to vector functions in the product spaces

$$\begin{aligned} \mathbb{H}_{\alpha, \mathbf{r}}^{m+1}(I_{k_0}) &= \mathcal{H}_{\alpha, r^p}^{m+1}(I_{k_0}) \times \mathcal{H}_{\alpha, r^u}^{m+1}(I_{k_0}), \quad \mathbb{V}_{\alpha, \mathbf{r}}^m(I_{k_0}) = \mathcal{V}_{\alpha, r^p}^m(I_{k_0}) \times \mathcal{V}_{\alpha, r^u}^m(I_{k_0}), \\ \mathbb{W}_{\alpha, \mathbf{r}}^m(\mathcal{T}_h) &= W_{\alpha, r^p}^m(\mathcal{T}_h) \times W_{\alpha, r^u}^m(\mathcal{T}_h), \end{aligned}$$

where $\mathbf{r} = (r^p, r^u)$. Now, we define the following bilinear forms

$$M(\mathbf{w}, \mathbf{v}) = \sum_{k=1}^N (S\mathbf{v}, \mathbf{w})_{I_k} \tag{5.6a}$$

$$B(\mathbf{w}, \mathbf{v}) = \sum_{k=1}^N (\mathbf{v}', \tilde{A}\mathbf{w})_{I_k} + \sum_{k=0}^N \llbracket \mathbf{v} \rrbracket_{x_k}^T S(x_k) \hat{\mathbf{w}}(x_k), \tag{5.6b}$$

where the numerical flux $\hat{\mathbf{w}}(x_k) = A(x_k)^+ \mathbf{w}(x_k^-) + A(x_k)^- \mathbf{w}(x_k^+)$ at the interior nodes. At the boundary, we have $\hat{\mathbf{w}}(x_N) = A(x_N)^+ \mathbf{w}(x_N^-)$, $\hat{\mathbf{w}}(x_0) = A(x_0)^- \mathbf{w}(x_0^+)$, $\llbracket \mathbf{w} \rrbracket_{x_N} = \mathbf{w}(x_N^-)$ and $\llbracket \mathbf{w} \rrbracket_{x_0} = -\mathbf{w}(x_0^+)$.

Now we define the immersed DG formulation as: Find $\mathbf{u}_h \in C^1([0, T], \mathbb{W}_{\alpha, \mathbf{r}}^m(\mathcal{T}_h))$ such that

$$(\mathbf{u}_h(\cdot, 0), \mathbf{v}_h)_I = (\mathbf{u}_0, \mathbf{v}_h)_I, \quad \forall \mathbf{v}_h \in \mathbb{W}_{\alpha, \mathbf{r}}^m(\mathcal{T}_h), \tag{5.7a}$$

$$M(\mathbf{u}_{h,t}(\cdot, t), \mathbf{v}_h) = B(\mathbf{u}_h(\cdot, t), \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbb{W}_{\alpha, \mathbf{r}}^m(\mathcal{T}_h), \tag{5.7b}$$

We note that the discrete weak form (5.7) and the discrete space $\mathbb{W}_{\alpha, \mathbf{r}}^m(\mathcal{T}_h)$ are identical to the ones described in the IDPGFE formulation in [33].

Next, we will go through some basic properties of the matrices S_{\pm} and A_{\pm} . These properties will be used later in the proof the L^2 stability in Lemma 5.2, in the analysis of the immersed Radau projection and in the convergence estimate.

Lemma 5.1 *Let A_{\pm} be the matrices defined in (5.2b) and let $S_{\pm} = \text{diag}(\rho_{\pm}^{-1} c_{\pm}^{-1}, \rho_{\pm})$, then*

- (a) *For any integer $k \geq 0$, the matrix $A_{\pm}^{-k} A_{\pm}^k$ is diagonal with positive entries.*
- (b) *Let $s \in \{+, -\}$, then there is an invertible matrix P_s such that $A_s = P_s \text{diag}(-c_s, c_s) P_s^{-1}$ and $S_s = P_s^{-T} P_s^{-1}$.*
- (c) *Let $s \in \{+, -\}$, then the matrices $S_s A_s^+$, $S_s A_s^-$ and $S_s |A_s|$ are symmetric. Furthermore, $S_s A_s^+$ is positive semi-definite, $S_s A_s^-$ is negative semi-definite and $S_s |A_s|$ is positive definite.*
- (d) *Let $s, \tilde{s} \in \{+, -\}$, and let $\mathbf{w} \in \mathbb{R}^2$, then*

$$\left(\left\| A_s^{\tilde{s}} \mathbf{w} \right\|^2 + \left| \mathbf{w}^T S_s A_s^{\tilde{s}'} \mathbf{w} \right| = 0 \right) \implies \mathbf{w} = 0. \tag{5.8}$$

where \tilde{s}' is dual of \tilde{s} defined at the beginning of Sect. 3, and $\|\cdot\|$ is Euclidean norm.

- (e) *Let $s \in \{+, -\}$. Then, there is a constant $C(\rho_s, c_s) > 0$ such that*

$$\mathbf{w}^T S_s A_s^+ \mathbf{w} - \mathbf{w}^T S_s A_s^- \mathbf{w} \geq C(\rho_s, c_s) \|\mathbf{w}\|^2, \quad \forall \mathbf{w} \in \mathbb{R}^2. \tag{5.9}$$

Proof (a) We have $A_{\pm}^{2k} = c_{\pm}^2 \text{Id}_2$, where Id_2 is the the 2×2 identity matrix. Therefore,

$$A_{\pm}^{2k} = c_{\pm}^{2k} \text{Id}_2, \quad A_{\pm}^{2k+1} = c_{\pm}^{2k} A_{\pm}, \quad k = 0, 1, \dots \tag{5.10}$$

Using (5.10), we immediately obtain $A_+^{-2k} A_-^{2k} = \left(\frac{c_-}{c_+}\right)^{2k} \text{Id}_2$ and $A_+^{-2k-1} A_-^{2k+1} = \left(\frac{c_-}{c_+}\right)^{2k} A_+^{-1} A_-$. Finally, by direct computation, we have $A_+^{-1} A_- = \text{diag}\left(\frac{\rho_+}{\rho_-}, \frac{\rho_- c_-^2}{\rho_+ c_+^2}\right)$. Hence, $A_+^{-k} A_-^k = \text{diag}(r_k^p, r_k^u)$, where r_k^p and r_k^u are defined in (5.1).

(b) Let

$$P_s = \frac{1}{\sqrt{2\rho_s}} \begin{pmatrix} -c_s \rho_s & c_s \rho_s \\ 1 & 1 \end{pmatrix}, \tag{5.11}$$

then, by a simple computation, we can show that $S_s = P_s^{-T} P_s^{-1}$ and $A_s = P_s \text{diag}(-c_s, c_s) P_s^{-1}$.

(c) We have $S_s A_s^+ = P_s^{-T} \text{diag}(0, c_s) P_s^{-1}$, where P_s is defined in (5.11). Therefore, $S_s A_s^+$ is a symmetric semi-positive definite matrix. The other two claims can be proven similarly.

(d) We will only consider the case $\tilde{s} = +$ here, the other case can be proven similarly. Consider a vector $\mathbf{w} \in \mathbb{R}^2$ that satisfies

$$\|A_s^+ \mathbf{w}\|^2 + |\mathbf{w}^T S_s A_s^- \mathbf{w}| = 0. \tag{5.12}$$

Now, let $\tilde{\mathbf{w}} = P_s \mathbf{w}$ where P_s is defined in (5.11), then (5.12) can be written as

$$\|P_s \text{diag}(0, c_s) \tilde{\mathbf{w}}\|^2 + \|\text{diag}(-c_s, 0) \tilde{\mathbf{w}}\|^2 = 0,$$

Since both norms are non-negative, we have $\text{diag}(-c_s, 0) \tilde{\mathbf{w}} = 0$ and $P_s \text{diag}(0, c_s) \tilde{\mathbf{w}} = 0$. P_s being invertible, we get $\tilde{\mathbf{w}} = 0$. Consequently, $\mathbf{w} = P_s^{-1} \tilde{\mathbf{w}} = 0$.

(e) We have by direct computation

$$\mathbf{w}^T S_s (A_s^+ - A_s^-) \mathbf{w} = \mathbf{w}^T \begin{pmatrix} \rho_s^{-1} c_s^{-1} & 0 \\ 0 & \rho_s c_s \end{pmatrix} \mathbf{w} \geq \min(\rho_s c_s, \rho_s^{-1} c_s^{-1}) \|\mathbf{w}\|. \tag{5.13}$$

□

Lemma 5.2 *Let \mathbf{u} be a solution to (5.2), and let $\epsilon(t) = \frac{1}{2} (\mathbf{u}(\cdot, t), S(\cdot) \mathbf{u}(\cdot, t))_{0,I}^2$, then*

$$\epsilon'(t) \leq 0, \quad t \geq 0.$$

Proof By multiplying (5.4) by \mathbf{u}^T and integrating on I^\pm , we obtain

$$\int_I \mathbf{u}(x, t)^T S \mathbf{u}_t(x, t) + \mathbf{u}(x, t)^T \tilde{A} \mathbf{u}_x(x, t) dx = 0.$$

The matrices S and \tilde{A} are symmetric. Therefore, we rewrite the previous equation as

$$\epsilon'(t) + \sum_{s=+,-} \int_{I^s} \frac{\partial}{\partial x} \mathbf{u}(x, t)^T \tilde{A} \mathbf{u}(x, t) dx = 0.$$

Since \mathbf{u} is continuous at α (from (5.2e)), we have

$$\epsilon'(t) + \mathbf{u}(b, t)^T \tilde{A} \mathbf{u}(b, t) - \mathbf{u}(a, t)^T \tilde{A} \mathbf{u}(a, t) = 0. \tag{5.13}$$

Now, we can rewrite the term $\mathbf{u}(b, t)^T \tilde{A} \mathbf{u}(b, t)$ as

$$\begin{aligned} \mathbf{u}(b, t)^T \tilde{A} \mathbf{u}(b, t) &= \mathbf{u}(b, t)^T S_+ A_+ \mathbf{u}(b, t) \\ &= \mathbf{u}(b, t)^T S_+ A_+^+ \mathbf{u}(b, t) + \mathbf{u}(b, t)^T S_+ A_+^- \mathbf{u}(b, t) = \mathbf{u}(b, t)^T S_+ A_+^+ \mathbf{u}(b, t), \end{aligned}$$

where the last equality follows from the boundary condition (5.2c). Since SA^+ is symmetric semi-positive definite (see part (c) of Lemma 5.1), we conclude that $\mathbf{u}(b, t)^T \tilde{A} \mathbf{u}(b, t) \geq 0$. Similarly, we have $\mathbf{u}(a, t)^T \tilde{A} \mathbf{u}(a, t) \leq 0$. Therefore,

$$\epsilon'(t) \leq 0.$$

□

The previous lemma shows that $\epsilon(t)$, interpreted as the energy of the system, is decreasing. This is to be expected since the boundary conditions in (5.2c) are dissipative (see [10]). Furthermore, if we let $\epsilon_h(t) = \frac{1}{2} M(\mathbf{u}_h(\cdot, t), \mathbf{u}_h(\cdot, t))$ be the discrete energy, then

$$\epsilon'_h(t) = B(\mathbf{u}_h, \mathbf{u}_h) = -\frac{1}{2} \sum_{k=0}^N \llbracket \mathbf{u}_h \rrbracket_{x_k}^T S(x_k) |A(x_k)| \llbracket \mathbf{u}_h \rrbracket_{x_k} \leq 0. \tag{5.14}$$

The proof of (5.14) follows the same steps as the scalar case described in [16].

5.2 The immersed Radau projection and the convergence analysis

In the remainder of this paper, we use $\mathcal{R}\mathbf{u}$ to denote the global Gauss-Radau projection of \mathbf{u} defined as

$$\mathcal{R}\mathbf{u} \in \mathbb{W}_{\alpha, \mathbf{r}}^m(\mathcal{T}_h) \text{ such that } B(\mathbf{u} - \mathcal{R}\mathbf{u}, \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbb{W}_{\alpha, \mathbf{r}}^m(\mathcal{T}_h). \tag{5.15}$$

Although $\mathcal{R}\mathbf{u}$ is a global projection, it can be constructed on each element independently, the construction of $(\mathcal{R}\mathbf{u})|_{I_k}$ where $k \neq k_0$ can be found in [1, 37] for the scalar case and can be generalized easily to systems. On the interface element, we define the local immersed Radau projection (IRP) operator $\Pi_{\alpha, \mathbf{r}}^m : \mathbb{H}_{\alpha, \mathbf{r}}^{m+1}(I_{k_0}) \rightarrow \mathbb{V}_{\alpha, \mathbf{r}}^m(I_{k_0})$ using 4.10 as

$$\Pi_{\alpha, \mathbf{r}}^m = \mathcal{M}^{-1} \circ \check{\Pi}_{\check{\alpha}, \check{\mathbf{r}}}^m \circ \mathcal{M},$$

where $\check{\Pi}_{\check{\alpha}, \check{\mathbf{r}}}^m : \mathbb{H}_{\check{\alpha}, \check{\mathbf{r}}}^{m+1}(\check{I}) \rightarrow \mathbb{V}_{\check{\alpha}, \check{\mathbf{r}}}^m(\check{I})$ is called the reference IRP operator and it is defined as the solution to the following system:

$$\begin{cases} A_- \check{\Pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}}(0) = A_- \check{\mathbf{u}}(0), & (5.16a) \end{cases}$$

$$\begin{cases} A_+ \check{\Pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}}(1) = A_+ \check{\mathbf{u}}(1), & (5.16b) \end{cases}$$

$$\begin{cases} (\check{A}\mathbf{v}', \check{\Pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}})_{\check{I}} = (\check{A}\mathbf{v}', \check{\mathbf{u}})_{\check{I}}, \quad \forall \mathbf{v} \in \mathbb{V}_{\check{\alpha}, \mathbf{r}}^m(\check{I}). & (5.16c) \end{cases}$$

Next, we will go through some basic properties of the IRP to prove that the IRP is well defined and is uniformly bounded on the RIFE space $\mathbb{V}_{\check{\alpha}, \mathbf{r}}^m(\check{I})$. From there, we can show that the IRP error on the LIFE space $\mathbb{V}_{\check{\alpha}, \mathbf{r}}^m(I_{k_0})$ decays at an optimal rate of $O(h^{m+1})$ under mesh refinement.

Lemma 5.3 *Let A be the matrix function defined in (5.2b) and let $\mathbf{p} \in \mathbb{V}_{\check{\alpha}, \mathbf{r}}^m(\check{I})$, then $A\mathbf{p}' \in \mathbb{V}_{\check{\alpha}, \mathbf{r}}^{m-1}(\check{I})$. Furthermore the map*

$$G : \mathbb{V}_{\check{\alpha}, \mathbf{r}}^m(\check{I}) \rightarrow \mathbb{V}_{\check{\alpha}, \mathbf{r}}^{m-1}(\check{I})$$

$$\mathbf{p} \mapsto A\mathbf{p}'$$

is surjective.

Proof Let $\mathbf{p} \in \mathbb{V}_{\check{\alpha}, \mathbf{r}}^m(\check{I})$ and let $\tilde{\mathbf{p}} = A\mathbf{p}'$, then for a fixed $k \in \{0, 1, \dots, m - 1\}$, we have

$$\begin{aligned} \tilde{\mathbf{p}}^{(k)}(\check{\alpha}^+) &= A_+ \mathbf{p}^{(k+1)}(\check{\alpha}^+) && \text{Using } \tilde{\mathbf{p}} = A\mathbf{p}' \\ &= A_+ A_+^{-k-1} A_-^{k+1} \mathbf{p}^{(k+1)}(\check{\alpha}^-) && \text{By construction of } \mathbb{V}_{\check{\alpha}, \mathbf{r}}^m(\check{I}) \\ &= A_+ A_+^{-k-1} A_-^{k+1} A_-^{-1} \tilde{\mathbf{p}}^{(k)}(\check{\alpha}^-) && \text{Using } \mathbf{p}' = A^{-1} \tilde{\mathbf{p}} \\ &= A_+^{-k} A_-^k \tilde{\mathbf{p}}^{(k)}(\check{\alpha}^-). && (5.17) \end{aligned}$$

Since (4.10) holds for every $k = 0, 1, \dots, m - 1$, we conclude that $\tilde{\mathbf{p}} \in \mathbb{V}_{\check{\alpha}, \mathbf{r}}^{m-1}(\check{I})$.

Now, we show that G is surjective, by the rank-nullity theorem, it suffices to prove that $\dim \ker(G)$ is 2 since $\dim \mathbb{V}_{\check{\alpha}, \mathbf{r}}^m(\check{I}) - \dim \mathbb{V}_{\check{\alpha}, \mathbf{r}}^{m-1}(\check{I}) = 2$. Let $\mathbf{p} \in \ker(G)$, then $A\mathbf{p}' = 0$, since A is invertible, we get $\mathbf{p}' = 0$ which implies that $\mathbf{p} \in \mathbb{V}_{\check{\alpha}, \mathbf{r}}^0(\check{I})$. This shows that $\dim \ker(G) = \dim \mathbb{V}_{\check{\alpha}, \mathbf{r}}^0(\check{I}) = 2$. □

Following the definition of G , we can re-write (5.16c) as

$$(SG(\mathbf{v}), \check{\Pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}})_{\check{I}} = (SG(\mathbf{v}), \check{\mathbf{u}})_{\check{I}}, \quad \forall \mathbf{v} \in \mathbb{V}_{\check{\alpha}, \mathbf{r}}^m(\check{I}).$$

For convenience, we will write $(SG(\mathbf{v}), \check{\Pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}})_{\check{I}} = (G(\mathbf{v}), \check{\Pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}})_{S, \check{I}}$. Now, since G maps $\mathbb{V}_{\check{\alpha}, \mathbf{r}}^m(\check{I})$ onto $\mathbb{V}_{\check{\alpha}, \mathbf{r}}^{m-1}(\check{I})$, we can express the condition (5.16c) as

$$(\mathbf{v}, \check{\Pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}})_{S, \check{I}} = (\mathbf{v}, \check{\mathbf{u}})_{S, \check{I}}, \quad \forall \mathbf{v} \in \mathbb{V}_{\check{\alpha}, \mathbf{r}}^{m-1}(\check{I}). \tag{5.18}$$

Theorem 5.1 *The system (5.16) admits exactly one solution.*

Proof First, we prove that the system admits at most one solution, for that, we only need to show that if $\check{\mathbf{u}} = 0$, then $\check{\Pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}} = 0$. For simplicity, let $\mathbf{q} = (q_1, q_2)^T$ be the solution to (5.16) with $\check{\mathbf{u}} = 0$, and let $r^{(1)} = r^p$ and $r^{(2)} = r^u$, then by (5.18), we have

$$(v, q_i)_{w_i, \check{I}} = 0, \quad \forall v \in \mathcal{V}_{\check{\alpha}, r^{(i)}}^{m-1}(\check{I}), \quad \text{where } w_i = S_{i,i}, \quad i = 1, 2, \quad (5.19)$$

which is equivalent to $q_i \in \mathcal{Q}_{\check{\alpha}, w_i, r^{(i)}}^m(\check{I})$. On the other hand, we have

$$\left(\tilde{A} \mathbf{q}', \mathbf{q} \right)_{\check{I}} = \frac{1}{2} \left(\mathbf{q}(1)^T S_+ A_+ \mathbf{q}(1) - \mathbf{q}(0)^T S_- A_- \mathbf{q}(0) \right) = 0. \quad (5.20)$$

From (5.16a) and (5.16b), we have $A_+^+ \mathbf{q}(1) = A_-^- \mathbf{q}(0) = 0$, then $A_+ \mathbf{q}(1) = A_+^- \mathbf{q}(1)$ and $A_- \mathbf{q}(0) = A_-^+ \mathbf{q}(0)$. Therefore, the Eq. (5.20) becomes

$$\frac{1}{2} \left(\mathbf{q}(1)^T S_+ A_+^- \mathbf{q}(1) - \mathbf{q}(0)^T S_- A_-^+ \mathbf{q}(0) \right) = 0.$$

Now, by Lemma 5.1 part (c), the quantities $\mathbf{q}(1)^T S_+ A_+^- \mathbf{q}(1)$ and $-\mathbf{q}(0)^T S_- A_-^+ \mathbf{q}(0)$ are non-positive, then

$$\mathbf{q}(1)^T S_+ A_+^- \mathbf{q}(1) = \mathbf{q}(0)^T S_- A_-^+ \mathbf{q}(0) = 0,$$

Furthermore, by (5.16a) and (5.16b), we have

$$\|A_+^+ \mathbf{q}(1)\|^2 + |\mathbf{q}(1)^T S_+ A_+^- \mathbf{q}(1)| = \|A_-^- \mathbf{q}(0)\|^2 + |\mathbf{q}(0)^T S_- A_-^+ \mathbf{q}(0)| = 0.$$

At this point, we use Lemma 5.1 part (d) to conclude that $\mathbf{q}(1) = \mathbf{q}(0) = 0$. Therefore, q_i are orthogonal IFE functions (as shown in (5.19)) that vanish on the boundary. By Theorem 3.3, we conclude that $q_i \equiv 0$ for $i = 1, 2$. Equivalently, $\mathbf{q} \equiv 0$.

To finalize the proof, we only need to show that (5.16) can be written as a square system. Let $A_{\pm} = P_{\pm} \text{diag}(-c_{\pm}, c_{\pm}) P_{\pm}^{-1}$ be an eigen-decomposition of A_{\pm} . Then, (5.16) can be written as

$$\begin{cases} \left(P_-^{-1} \check{\Pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}}(0) \right)_1 &= \left(P_-^{-1} \check{\mathbf{u}}(0) \right)_1, \\ \left(P_+^{-1} \check{\Pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}}(1) \right)_2 &= \left(P_+^{-1} \check{\mathbf{u}}(1) \right)_2, \\ \left(\mathcal{N}_{\check{\alpha}, r^p}^j, (\check{\Pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}})_i \right)_{S_{11}, \check{I}} &= \left(\mathcal{N}_{\check{\alpha}, r^p}^j, \check{p} \right)_{S_{22}, \check{I}}, \quad 1 \leq j \leq m-1, \\ \left(\mathcal{N}_{\check{\alpha}, r^u}^j, (\check{\Pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}})_i \right)_{S_{22}, \check{I}} &= \left(\mathcal{N}_{\check{\alpha}, r^u}^j, \check{u} \right)_{S_{22}, \check{I}}, \quad 1 \leq j \leq m-1, \end{cases} \quad (5.21)$$

which is a system of $2(m + 1)$ equations with $2(m + 1)$ variables. Since the homogeneous system admits at most one solution, we conclude that (5.16) has exactly one solution. \square

Next, we show that $\{\check{\Pi}_{\check{\alpha},\mathbf{r}}^m\}_{0 < \check{\alpha} < 1}$ is uniformly bounded. First, let $\mathbf{p} \in \mathbb{V}_{\check{\alpha},\mathbf{r}}^{m-1}(\check{I})$ be the solution to the following symmetric positive definite system

$$(\mathbf{v}, \mathbf{p})_{S,\check{I}} = (\mathbf{v}, \check{\mathbf{u}})_{\check{I}}, \quad \forall \mathbf{v} \in \mathbb{V}_{\check{\alpha},\mathbf{r}}^{m-1}(\check{I}), \tag{5.22}$$

and let $\mathbf{q} = \check{\Pi}_{\check{\alpha},\mathbf{r}}^m \check{\mathbf{u}} - \mathbf{p}$, then by (5.16c) and (5.18), we have

$$\left(\tilde{A}\mathbf{v}', \mathbf{q}\right)_{\check{I}} = \left(\tilde{A}\mathbf{v}', \check{\Pi}_{\check{\alpha},\mathbf{r}}^m \check{\mathbf{u}} - \mathbf{p}\right)_{\check{I}} = 0, \quad \forall \mathbf{v} \in \mathbb{V}_{\check{\alpha},\mathbf{r}}^m(\check{I}), \tag{5.23}$$

which can be written as

$$(\mathbf{v}, \mathbf{q})_{S,\check{I}} = 0, \quad \forall \mathbf{v} \in \mathbb{V}_{\check{\alpha},\mathbf{r}}^{m-1}(\check{I}).$$

Thus, $\mathbf{q} \in \mathbb{Q}_{\check{\alpha},S,\mathbf{r}}^m(\check{I}) = \mathcal{Q}_{\check{\alpha},S_{11},r^p}^m(\check{I}) \times \mathcal{Q}_{\check{\alpha},S_{22},r^u}^m(\check{I})$. Additionally, by (5.16a) and (5.16b), we have

$$\begin{cases} A_-^-\mathbf{q}(0) = A_-^-(\check{\mathbf{u}}(0) - \mathbf{p}(0)), \\ A_+^+\mathbf{q}(1) = A_+^+(\check{\mathbf{u}}(1) - \mathbf{p}(1)). \end{cases} \tag{5.24}$$

In the next two lemmas, we prove that $\|\mathbf{p}\|_{0,\check{I}}$ and $\|\mathbf{q}\|_{0,\check{I}}$ is bounded by some appropriate norms of $\check{\mathbf{u}}$ independently of $\check{\alpha}$. Both lemmas will be used later in Theorem 5.2 to prove that $\{\check{\Pi}_{\check{\alpha},\mathbf{r}}^m\}_{0 < \check{\alpha} < 1}$ is a uniformly bounded collection of RIFE projections.

Lemma 5.4 *Let $\check{\mathbf{u}} \in \mathbb{H}_{\check{\alpha},\mathbf{r}}^{m+1}(\check{I})$ and $\mathbf{p} \in \mathbb{V}_{\check{\alpha},\mathbf{r}}^{m-1}(\check{I})$ defined by (5.22), then there is $C(\rho, c) > 0$ independent of $\check{\alpha}$ such that $\|\mathbf{p}\|_{0,\check{I}} \leq C(\rho, c) \|\check{\mathbf{u}}\|_{0,\check{I}}$.*

Proof Let $\mathbf{v} = \mathbf{p}$ in (5.22), then $\|\mathbf{p}^T S \mathbf{p}\|_{0,\check{I}} = (\mathbf{p}, \check{\mathbf{u}})_{0,\check{I}} \leq \|\mathbf{p}\|_{0,\check{I}} \|\check{\mathbf{u}}\|_{0,\check{I}}$. On the other hand, by construction of S , we have $\|\mathbf{p}^T S \mathbf{p}\|_{0,\check{I}} \geq C(\rho, c) \|\mathbf{p}\|_{0,\check{I}}$, where $C(\rho, c) = \min(\rho_-^{-1}c_-^{-2}, \rho_+^{-1}c_+^{-2}, \rho_-, \rho_+)$. Therefore,

$$\|\mathbf{p}\|_{0,\check{I}} \leq C(\rho, c) \|\check{\mathbf{u}}\|_{0,\check{I}}.$$

□

Lemma 5.5 *Let $\check{\mathbf{u}} \in \mathbb{H}_{\check{\alpha},\mathbf{r}}^{m+1}(\check{I})$ and let $\mathbf{p} \in \mathbb{V}_{\check{\alpha},\mathbf{r}}^{m-1}(\check{I})$ defined by (5.22). Then, there is $C(\rho, c, m) > 0$ independent of $\check{\alpha}$ such that*

$$\left\| \check{\Pi}_{\check{\alpha},\mathbf{r}}^m \check{\mathbf{u}} - \mathbf{p} \right\|_{0,\check{I}} \leq C(\rho, c, m) \|\check{\mathbf{u}}\|_{1,\check{I}}$$

Proof Let $\mathbf{q} = (q_1, q_2) = \check{\Pi}_{\check{\alpha},\mathbf{r}}^m \check{\mathbf{u}} - \mathbf{p}$, then by (5.23), we have $(\tilde{A}\mathbf{q}', \mathbf{q})_{\check{I}} = 0$ which is equivalent to

$$\mathbf{q}(1)^T \tilde{A}\mathbf{q}(1) - \mathbf{q}(0)^T \tilde{A}\mathbf{q}(0) = 0 \tag{5.25}$$

By decomposing $A_{\pm} = A_{\pm}^+ + A_{\pm}^-$ and using the definition of \tilde{A} in (5.3), we can split (5.25) as

$$\mathbf{q}(1)^T S_+ A_+^+ \mathbf{q}(1) + \mathbf{q}(1)^T S_+ A_+^- \mathbf{q}(1) - \mathbf{q}(0)^T S_- A_-^+ \mathbf{q}(0) - \mathbf{q}(0)^T S_- A_-^- \mathbf{q}(0) = 0. \tag{5.26}$$

Now, from (5.9), we have

$$\mathbf{q}(0)^T S_- A_-^+ \mathbf{q}(0) - \mathbf{q}(0)^T S_- A_-^- \mathbf{q}(0) \geq C_1(\rho, c) \|\mathbf{q}(0)\|^2, \tag{5.27a}$$

$$\mathbf{q}(1)^T S_+ A_+^+ \mathbf{q}(1) - \mathbf{q}(1)^T S_+ A_+^- \mathbf{q}(1) \geq C_1(\rho, c) \|\mathbf{q}(1)\|^2. \tag{5.27b}$$

Next, we sum (5.26), (5.27a) and (5.27b) to obtain

$$\mathbf{q}(1) S_+ A_+^+ \mathbf{q}(1) - \mathbf{q}(0) S_- A_-^- \mathbf{q}(0) \geq \frac{1}{2} C_1(\rho, c) \left(\|\mathbf{q}(0)\|^2 + \|\mathbf{q}(1)\|^2 \right) \tag{5.28}$$

We substitute (5.24) in (5.28) to obtain

$$\mathbf{q}(1) S_+ A_+^+ (\check{\mathbf{u}}(1) - \mathbf{p}(1)) - \mathbf{q}(0) S_- A_-^- (\check{\mathbf{u}}(0) - \mathbf{p}(0)) \geq \frac{1}{2} C_1(\rho, c) \left(\|\mathbf{q}(0)\|^2 + \|\mathbf{q}(1)\|^2 \right) \tag{5.29}$$

Now, we will bound the left hand side from above. First, we have

$$\begin{aligned} & \mathbf{q}(1) S_+ A_+^+ (\check{\mathbf{u}}(1) - \mathbf{p}(1)) - \mathbf{q}(0) S_- A_-^- (\check{\mathbf{u}}(0) - \mathbf{p}(0)) \leq C_2(\rho, c) \\ & \left(\|\mathbf{q}(1)\| \|\check{\mathbf{u}}(1) - \mathbf{p}(1)\| + \|\mathbf{q}(0)\| \|\check{\mathbf{u}}(0) - \mathbf{p}(0)\| \right) \end{aligned} \tag{5.30}$$

Since $\check{\mathbf{u}} - \mathbf{p} \in (H^1(\check{I}))^2$, there is $C_3 > 0$ such that

$$\max \left(\|\check{\mathbf{u}}(0) - \mathbf{p}(0)\|, \|\check{\mathbf{u}}(1) - \mathbf{p}(1)\| \right) \leq C_3 \left(\|\check{\mathbf{u}}\|_{1,\check{I}} + \|\mathbf{p}\|_{1,\check{I}} \right). \tag{5.31}$$

By applying the inverse inequality (3.9) and Lemma 5.4 to $\|\mathbf{p}\|_{1,\check{I}}$, we obtain

$$\begin{aligned} \max \left(\|\check{\mathbf{u}}(0) - \mathbf{p}(0)\|, \|\check{\mathbf{u}}(1) - \mathbf{p}(1)\| \right) & \leq C_4(\rho, c, m) \left(\|\check{\mathbf{u}}\|_{1,\check{I}} + \|\mathbf{p}\|_{0,\check{I}} \right), \\ & \leq C_5(\rho, c, m) \|\check{\mathbf{u}}\|_{1,\check{I}} \end{aligned} \tag{5.32}$$

Now, we substitute (5.32) and (5.30) back into (5.29) and use the inequality $a^2 + b^2 \geq \frac{1}{2}(a + b)^2$ to obtain

$$\left(\|\mathbf{q}(0)\| + \|\mathbf{q}(1)\| \right) \|\check{\mathbf{u}}\|_{1,\check{I}} \geq C_6(\rho, c, m) \left(\|\mathbf{q}(0)\| + \|\mathbf{q}(1)\| \right)^2,$$

which yields

$$\|\mathbf{q}(0)\| + \|\mathbf{q}(1)\| \leq C_7(\rho, c, m) \|\check{\mathbf{u}}\|_{1,\check{I}}. \tag{5.33}$$

To finish the proof, we recall that $\mathbf{q} \in \mathcal{Q}_{\check{\alpha}, S_{11}, r^p}^m(\check{I}) \times \mathcal{Q}_{\check{\alpha}, S_{22}, r^u}^m(\check{I})$. Therefore, by Lemma 3.4 and some elementary algebraic manipulations, we have

$$\begin{aligned} \|\mathbf{q}\|_{0,\check{I}} &= \sqrt{\|q_1\|_{0,\check{I}}^2 + \|q_2\|_{0,\check{I}}^2} && \text{(By definition)} \\ &\leq C_7(\rho, c, m) \sqrt{q_1(0)^2 + q_1(1)^2 + q_2(0)^2 + q_2(1)^2}, && \text{(Using Lemma 3.4)} \\ &\leq C_8(\rho, c, m) (\|\mathbf{q}(0)\| + \|\mathbf{q}(1)\|) \\ \|\mathbf{q}\|_{0,\check{I}} &\leq C_9(\rho, c, m) \|\check{\mathbf{u}}\|_{1,\check{I}}, && \text{(using (5.33))} \end{aligned}$$

which is the desired result. □

By combining Lemmas 5.4 and 5.5, we can show that the norm of $\check{\Pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}}$ can be bounded by a norm of $\check{\mathbf{u}}$ independently of $\check{\alpha}$ as described in the following theorem. We note that $\check{\Pi}_{\check{\alpha}, \mathbf{r}}^m$ maps $\mathbb{H}_{\check{\alpha}, \mathbf{r}}^{m+1}(\check{I})$ to $\mathbb{V}_{\check{\alpha}, \mathbf{r}}^m(\check{I})$. Nevertheless, we shall call $\check{\Pi}_{\check{\alpha}, \mathbf{r}}^m$ a RIFE projection since the results from Sect. 4 in the scalar case apply directly to the vector case here.

Theorem 5.2 *Let $m \geq 1$ and let $\check{\mathbf{u}} \in \mathbb{H}_{\check{\alpha}, \mathbf{r}}^{m+1}(\check{I})$. Then, there is $C(\rho, c, m) > 0$ independent of $\check{\alpha}$ such that*

$$\|\check{\Pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}}\|_{0,\check{I}} \leq C(\rho, c, m) \|\check{\mathbf{u}}\|_{1,\check{I}}$$

That is, $\{\check{\Pi}_{\check{\alpha}, \mathbf{r}}^m\}_{0 < \check{\alpha} < 1}$ is a uniformly bounded collection of RIFE projections.

Proof By definition of \mathbf{p} and \mathbf{q} , we have

$$\|\check{\Pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}}\|_{0,\check{I}} \leq \|\mathbf{p}\|_{0,\check{I}} + \|\mathbf{q}\|_{0,\check{I}},$$

Which, using Lemmas 5.4 and 5.5, leads to

$$\|\check{\Pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}}\|_{0,\check{I}} \leq C(\rho, c, m) \|\check{\mathbf{u}}\|_{1,\check{I}},$$

where $C(\rho, c, m) > 0$ is independent of $\check{\alpha}$. □

Corollary 5.1 *Let $\mathbf{u} \in \mathbb{H}_{\check{\alpha}, \mathbf{r}}^{m+1}(I_{k_0})$, then there is $C(S_{\pm}, A_{\pm}, m) > 0$ independent of $\check{\alpha}$ and h such that*

$$\|\mathbf{u} - \Pi_{\check{\alpha}, \mathbf{r}}^m \mathbf{u}\|_{i, I_{k_0}} < C(S_{\pm}, A_{\pm}, m) h^{m+1-i} |\mathbf{u}|_{m+1, I_{k_0}}, \quad 0 \leq i \leq m.$$

Proof Let $\check{\mathbf{u}} = (\check{p}, \check{u})^T = \mathcal{M}\mathbf{u}$ and let $\check{\pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}} = (\check{\pi}_{\check{\alpha}, r^p}^m \check{p}, \check{\pi}_{\check{\alpha}, r^u}^m \check{u})^T$, where $\check{\pi}_{\check{\alpha}, r^p}^m \check{p}, \check{\pi}_{\check{\alpha}, r^u}^m \check{u}$ are defined in Lemma 4.1. Then by Theorem 4.1, we have

$$\|\check{\mathbf{u}} - \check{\pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}}\|_{i,\check{I}} \leq C(\rho, c, m) |\check{\mathbf{u}}|_{m+1,\check{I}}, \quad i = 0, 1, \dots, m + 1. \tag{5.34}$$

On the other hand, we have

$$\begin{aligned}
 \left| \mathbf{u} - \Pi_{\check{\alpha}, \mathbf{r}}^m \mathbf{u} \right|_{i, I_{k_0}} &= h^{1-i} \left| \check{\mathbf{u}} - \check{\Pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}} \right|_{i, \check{I}} \\
 &\leq h^{1-i} \left(\left| \check{\Pi}_{\check{\alpha}, \mathbf{r}}^m (\check{\mathbf{u}} - \check{\pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}}) \right|_{i, \check{I}} + \left| \check{\mathbf{u}} - \check{\pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}} \right|_{i, \check{I}} \right) \\
 &\leq C_1(\rho, c, m) h^{1-i} \\
 &\quad \left(\left| \check{\Pi}_{\check{\alpha}, \mathbf{r}}^m (\check{\mathbf{u}} - \check{\pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}}) \right|_{0, \check{I}} + \left| \check{\mathbf{u}} - \check{\pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}} \right|_{i, \check{I}} \right) \quad (\text{Using Lemma 3.3}) \\
 &\leq C_2(\rho, c, m) h^{1-i} \left(\left\| \check{\mathbf{u}} - \check{\pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}} \right\|_{1, \check{I}} + \left| \check{\mathbf{u}} - \check{\pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}} \right|_{i, \check{I}} \right) \quad (\text{Using Theorem 5.2}) \\
 &\leq C_3(\rho, c, m) h^{1-i} \left\| \check{\mathbf{u}} - \check{\pi}_{\check{\alpha}, \mathbf{r}}^m \check{\mathbf{u}} \right\|_{m+1, \check{I}} \\
 &\leq C_4(\rho, c, m) h^{1-i} |\check{\mathbf{u}}|_{m+1, \check{I}} \quad (\text{From (5.34)}) \\
 &= C_4(\rho, c, m) h^{m+1-i} |\mathbf{u}|_{m+1, I_{k_0}}.
 \end{aligned}$$

□

By summing over all elements, we get a similar bound for the global Radau projection $\mathcal{R}\mathbf{u}$ with a function $\mathbf{u} \in \mathbb{H}_{\alpha, \mathbf{r}}^{m+1}(I)$

$$\|\mathbf{u} - \mathcal{R}\mathbf{u}\|_{i, I} < C(S_{\pm}, A_{\pm}, m) h^{m+1-i} |\mathbf{u}|_{m+1, I}, \quad 0 \leq i \leq m. \quad (5.35)$$

Theorem 5.3 *Let \mathbf{u} be the solution of problem (5.2) and let $\mathbf{u}_h \in \mathbb{W}_{\alpha, \mathbf{r}}^m(I)$ be the solution of (5.7). If $\mathbf{u} \in C([0, T]; \mathbb{H}_{\alpha, \mathbf{r}}^{m+2}(I))$, then there is $C > 0$ independent of h and α such that*

$$\begin{aligned}
 \|\mathbf{u}(\cdot, T) - \mathbf{u}_h(\cdot, T)\|_{0, I} &\leq Ch^{m+1} \\
 &\quad \left(|\mathbf{u}_0|_{m+1, I} + |\mathbf{u}(\cdot, T)|_{m+1, I} + T \max_{0 \leq t \leq T} |\mathbf{u}(\cdot, t)|_{m+2, I} \right), \quad T > 0.
 \end{aligned}$$

Proof Our proof follows the usual methodology used for the non-interface problem (see [16]). We first note that $\mathcal{R}\mathbf{u}_t = \frac{d}{dt} \mathcal{R}\mathbf{u}$ and split the error $\mathbf{e} = \mathbf{u}_h - \mathbf{u}$ as

$$\mathbf{e} = \mathbf{z} - \mathbf{g}, \quad \mathbf{z} = \mathbf{u} - \mathcal{R}\mathbf{u}, \quad \mathbf{g} = \mathbf{u}_h - \mathcal{R}\mathbf{u}.$$

It follows from the definition of \mathcal{R} in (5.15) that

$$B(\mathbf{g}, \mathbf{g}) = B(\mathbf{g} - \mathbf{z}, \mathbf{g}) = B(\mathbf{u}_h - \mathbf{u}, \mathbf{g}) = B(\mathbf{e}, \mathbf{g}). \quad (5.36)$$

By combining (5.36), (5.7) and (5.14), we get

$$\begin{aligned}
 (\mathbf{S}\mathbf{z}_t(\cdot, t), \mathbf{g}(\cdot, t))_I &= (\mathbf{S}\mathbf{g}_t(\cdot, t), \mathbf{g}(\cdot, t))_I - (\mathbf{S}\mathbf{e}_t(\cdot, t), \mathbf{g}(\cdot, t))_I, \\
 &= \frac{1}{2} \frac{d}{dt} \left\| \sqrt{\mathbf{S}}\mathbf{g}(\cdot, t) \right\|_{0, I}^2 - B(\mathbf{e}(\cdot, t), \mathbf{g}(\cdot, t)),
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \frac{d}{dt} \left\| \sqrt{S} \mathbf{g}(\cdot, t) \right\|_{0,I}^2 - B(\mathbf{g}(\cdot, t), \mathbf{g}(\cdot, t)), \\
 &= \frac{1}{2} \frac{d}{dt} \left\| \sqrt{S} \mathbf{g}(\cdot, t) \right\|_{0,I}^2 + \sigma(t),
 \end{aligned} \tag{5.37}$$

where $\sigma(t) \geq 0$ by (5.14). Let $\kappa(t) = \left\| \sqrt{S} \mathbf{g}(\cdot, t) \right\|_{0,I}$, then by Cauchy–Schwarz inequality,

$$(S \mathbf{z}_t(\cdot, t), \mathbf{g}(\cdot, t))_I \leq \|\mathbf{z}_t(\cdot, t)\|_{0,I} \kappa(t). \tag{5.38}$$

Following the ideas of the proof of Lemma 5.3, we can show that $\mathbf{u}_t(\cdot, t) = -A \mathbf{u}_x(\cdot, t) \in \mathbb{H}_{\alpha, \mathbf{r}}^{m+1}(I)$ since $\mathbf{u}(\cdot, t) \in \mathbb{H}_{\alpha, \mathbf{r}}^{m+2}(I)$. Therefore, by (5.35), there is C independent of h and α such that

$$\|\mathbf{z}_t(\cdot, t)\|_{0,I} \leq Ch^{m+1} |\mathbf{u}(\cdot, t)|_{m+2,I}. \tag{5.39}$$

Now, we use (5.39), (5.38) and integrate (5.37) on $[0, T]$ to get

$$\frac{1}{2} \kappa(T)^2 - \frac{1}{2} \kappa(0)^2 + \sigma(t) \leq Ch^{m+1} \int_0^T \kappa(s) |\mathbf{u}(\cdot, s)|_{m+2,I} ds, \tag{5.40}$$

Using a generalized version of Gronwall’s inequality (see [9, p.24]), we get the following bound on $\kappa(T)$

$$\kappa(T) \leq \kappa(0) + Ch^{m+1} \int_0^T |\mathbf{u}(\cdot, s)|_{m+2,I} ds, \tag{5.41}$$

$$\leq \kappa(0) + Ch^{m+1} T \max_{0 \leq t \leq T} |\mathbf{u}(\cdot, t)|_{m+2,I}. \tag{5.42}$$

We also have

$$\begin{aligned}
 \kappa(0) &= \left\| \sqrt{S} (\mathbf{u}_h(\cdot, 0) - \mathcal{R} \mathbf{u}_0) \right\|_{0,I} \leq \left\| \sqrt{S} (\mathbf{u}_h(\cdot, 0) - \mathbf{u}_0) \right\|_{0,I} \\
 &\quad + \left\| \sqrt{S} (\mathbf{u}_0 - \mathcal{R} \mathbf{u}_0) \right\|_{0,I} \leq Ch^{m+1} |\mathbf{u}_0|_{m+1,I}.
 \end{aligned} \tag{5.43}$$

We substitute (5.43) into (5.42), to obtain

$$\kappa(T) = \left\| \sqrt{S} \mathbf{g}(\cdot, T) \right\|_{0,I} \leq Ch^{m+1} \left(|\mathbf{u}_0|_{m+1,I} + T \max_{0 \leq t \leq T} |\mathbf{u}(\cdot, t)|_{m+2,I} \right).$$

To finalize the proof, we use the triangle inequality

$$\begin{aligned}
 \|\mathbf{e}(\cdot, T)\|_{0,I} &\leq \|\mathbf{z}(\cdot, T)\|_{0,I} + \|\mathbf{g}(\cdot, T)\|_{0,I} \leq Ch^{m+1} (|\mathbf{u}_0|_{m+1,I} + |\mathbf{u}(\cdot, T)|_{m+1,I} \\
 &\quad + T \max_{0 \leq t \leq T} |\mathbf{u}(\cdot, t)|_{m+2,I}).
 \end{aligned}$$

□

6 Novel proofs for results already established in the literature

In this section, for demonstrating the versatility of the immersed scaling argument established in Sects. 3 and 4, we redo the error estimation for two IFE methods in the literature. One of them is the IFE space for an elliptic interface problem [5], and the other one is the IFE space for an interface problem of the Euler-Bernoulli Beam [25]. We note that the approximation capability for these IFE spaces were already analyzed, but with complex and lengthy procedures. Our discussions here is to demonstrate that similar error bounds for the optimal approximation capability of these different types of IFE spaces can be readily derived by the unified immersed scaling argument.

6.1 The m -th degree IFE space for an elliptic interface problem

In this subsection, we consider the m -th degree IFE space developed in [5] for solving the following interface problem:

$$\begin{cases} -\beta(x)u''(x) = f(x), & x \in (a, \alpha) \cup (\alpha, b) \\ u(a) = u(b) = 0, \end{cases} \quad \beta(x) = \begin{cases} \beta^- > 0, & x \in (a, \alpha), \\ \beta^+ > 0, & x \in (\alpha, b), \end{cases} \quad [u]_\alpha = [\beta u']_\alpha = 0. \quad (6.1)$$

Assume that f is in $C^{m-1}(I)$ which implies that the solution $u \in \mathcal{H}_{\alpha,r}^{m+1}(I)$ with

$$r_0 = 1, \quad \text{and} \quad r_i = \frac{\beta^-}{\beta^+} \quad \text{for } i = 1, 2, \dots, m. \quad (6.2)$$

The discussion in Sect. 2 suggests the following IFE space for this elliptic interface problem:

$$Z_{\alpha,r}^m(\mathcal{T}_h) = H_0^1(I) \cap \mathcal{W}_{\alpha,r}^m(\mathcal{T}_h) \quad (6.3)$$

which coincides with the one developed in [5] based on the extended jump conditions where it was proved, by an elementary but complicated multi-point Taylor expansion technique, to have the optimal approximation capability with respect to m -th degree polynomials employed in this IFE space. We now reanalyze this IFE space by the immersed scaling argument.

The continuity of functions in the IFE space suggests to consider the following immersed Lobatto projection $\mathcal{L}_{\alpha,r}^m : \mathcal{H}_{\alpha,r}^{m+1}(I_{k_0}) \rightarrow \mathcal{V}_{\alpha,r}^m(I_{k_0})$ defined by

$$\begin{cases} \mathcal{L}_{\alpha,r}^m u(x_{k_0-1}) = u(x_{k_0-1}), \\ \mathcal{L}_{\alpha,r}^m u(x_{k_0}) = u(x_{k_0}), \\ (\mathcal{L}_{\alpha,r}^m u, v_h)_{w, I_{k_0}} = (u, v_h)_{w, I_{k_0}}, \quad \forall v_h \in \mathcal{V}_{\alpha, \tau^2(r)}^{m-2}(I_{k_0}), \end{cases} \quad w(x) = \begin{cases} r_1, & x \in I_{k_0}^-, \\ 1, & x \in I_{k_0}^+, \end{cases} \quad (6.4)$$

where $\tau^2 = \tau \circ \tau$ and τ is the shift operator defined in (3.11). The related reference immersed Lobatto projection $\check{\mathcal{L}}_{\check{\alpha},r}^m : \check{\mathcal{H}}_{\check{\alpha},r}^{m+1}(\check{I}) \rightarrow \check{\mathcal{V}}_{\check{\alpha},r}^m(\check{I})$ is defined by the diagram (4.10), that is, $\check{\mathcal{L}}_{\check{\alpha},r}^m \check{u} = \check{\mathcal{L}}_{\check{\alpha},r}^m u$ where $\check{u} = \mathcal{M}v$.

For simplicity, let $\tilde{u} = \check{\mathcal{L}}_{\check{\alpha},r}^m \check{u}$ for a given $\check{u} \in \check{\mathcal{V}}_{\check{\alpha},r}^m(\check{I})$ and note that the system (6.4) is a square system of $m + 1$ equations since the last line can be written as $m - 1$ equations. Therefore, we only need to show that if $\check{u} \equiv 0$ then $\tilde{u} \equiv 0$ to prove that $\check{\mathcal{L}}_{\check{\alpha},r}^m$ is well defined.

Lemma 6.1 *The reference immersed Lobatto projection $\check{\mathcal{L}}_{\check{\alpha},r}^m$ is well defined.*

Proof Let $\check{u} \equiv 0$, we will show that $\tilde{u} = \check{\mathcal{L}}_{\check{\alpha},r}^m \check{u} \equiv 0$. We have

$$\tilde{u}(0) = \tilde{u}(1) = 0, \quad (\tilde{u}, v_h)_{\check{w},\check{I}} = 0, \quad \forall v \in \check{\mathcal{V}}_{\check{\alpha},\tau^2(r)}^{m-2}(\check{I}),$$

where $\check{w} = \mathcal{M}w$. Using (3.11), $\tilde{u}'' \in \check{\mathcal{V}}_{\check{\alpha},\tau^2(r)}^{m-2}(\check{I})$, then

$$\begin{aligned} 0 &= \int_0^1 w(x)\tilde{u}(x)\tilde{u}''(x) \, dx = r_1 \int_0^{\check{\alpha}} \tilde{u}(x)\tilde{u}''(x) \, dx + \int_{\check{\alpha}}^1 \tilde{u}(x)\tilde{u}''(x) \, dx \\ &= \tilde{u}(\check{\alpha}) [r_1\tilde{u}'(\check{\alpha}^-) - \tilde{u}'(\check{\alpha}^+)] - \int_0^1 w(x)[\tilde{u}'(x)]^2 \, dx \\ 0 &= - \int_0^1 w(x)[\tilde{u}'(x)]^2 \, dx, \end{aligned}$$

which implies that \tilde{u} is zero since $\tilde{u}(0) = \tilde{u}(1) = 0$. □

Next, we will show that $\{\check{\mathcal{L}}_{\check{\alpha},r}^m\}_{0 \leq \check{\alpha} < 1}$ is a uniformly bounded collection of RIFE projections in the following lemma.

Lemma 6.2 *There is a constant $C(\beta^+, \beta^-, m) > 0$ independent of $\check{\alpha}$ such that the following estimate holds for every $\check{u} \in \check{\mathcal{H}}_{\check{\alpha},r}^{m+1}(\check{I})$*

$$\|\tilde{u}\|_{0,\check{I}} \leq C(\beta^+, \beta^-, m) \|\check{u}\|_{1,\check{I}}.$$

Proof We write \tilde{u} as $\tilde{u} = q_1 + q_2$, where $q_1 \in \check{\mathcal{V}}_{\check{\alpha},r}^1(\check{I})$ such that

$$q_1(0) = \check{u}(0), \quad q_1(1) = \check{u}(1),$$

and $q_2 = \check{\mathcal{L}}_{\check{\alpha},r}^m(\check{u} - q_1) \in \check{\mathcal{V}}_{\check{\alpha},r}^m(\check{I})$. The construction of q_1 is straightforward (see [22]) and we have $\|q_1\|_{0,\check{I}} \leq C(\beta^+, \beta^-) \|\check{u}\|_{1,\check{I}}$. Now, the second term q_2 satisfies

$$q_2(0) = q_2(1) = 0, \quad (q_2, v_h)_{\check{w},\check{I}} = (\check{u} - q_1, v_h)_{\check{w},\check{I}}, \quad \forall v_h \in \check{\mathcal{V}}_{\check{\alpha},\tau^2(r)}^{m-2}(\check{I}),$$

where $\check{w} = \mathcal{M}w$. Following the proof of Lemma 6.1, we can choose $v_h = q_2''$ and integrate by parts to get

$$- \|wq_2'\|_{0,\check{I}}^2 = (\check{u} - q_1, q_2'')_{\check{w},\check{I}}.$$

We take the absolute value of each side and apply Cauchy–Schwarz inequality

$$\|q_2'\|_{0,\check{I}}^2 \leq C(\beta^+, \beta^-, m) \|\check{u} - q_1\|_{0,\check{I}} \|q_2''\|_{0,\check{I}}.$$

The inverse inequality in Lemma 3.3 implies that $\|q_2''\|_{0,\check{I}} \leq C(\beta^+, \beta^-, m) \|q_2'\|_{0,\check{I}}$. Hence,

$$\|q_2'\|_{0,\check{I}} \leq C(\beta^+, \beta^-, m) \|\check{u} - q_1\|_{0,\check{I}} \leq C(\beta^+, \beta^-, m) \|\check{u}\|_{1,\check{I}}. \tag{6.5}$$

Since $q_2(0) = q_2(1) = 0$, we can apply Poincaré’s inequality to obtain $\|q_2\|_{0,\check{I}} < C \|q_2'\|_{0,\check{I}}$. Finally, we have

$$\|\check{u}\|_{0,\check{I}} \leq \|q_1\|_{0,\check{I}} + \|q_2\|_{0,\check{I}} \leq C(m, \beta^+, \beta^-) \|\check{u}\|_{1,\check{I}}.$$

□

Then, we can use Theorem 4.3 to derive an error bound for the Lobatto projection $\mathcal{L}_{\alpha,r}^m u$ in the following theorem which confirms the optimal approximation capability of the IFE space established in [5] by a more complex analysis.

Theorem 6.1 *There is $C(\beta^+, \beta^-, m) > 0$ such that the following estimate holds for every $u \in \mathcal{H}_{\alpha,r}^{m+1}(I_{k_0})$*

$$|u - \mathcal{L}_{\alpha,r}^m u|_{i,I_{k_0}} \leq C(\beta^+, \beta^-, m) h^{m+1-i} |u|_{m+1,I_{k_0}}, \quad \forall i = 0, 1, \dots, m.$$

Proof This follows immediately from Lemma 6.2 and Theorem 4.3. □

6.2 Euler–Bernoulli beam interface problem

In this subsection, we apply the immersed scaling argument to reanalyze the cubic IFE space developed in [27, 36] for solving the following interface problem of the Euler–Bernoulli beam equation:

$$\begin{cases} \beta(x)u^{(4)}(x) = f(x), & x \in (a, \alpha) \cup (\alpha, b) \\ u(a) = u(b) = 0, \\ u'(a) = u'(b) = 0 \end{cases} \quad \beta(x) = \begin{cases} \beta^- > 0, & x \in (a, \alpha), \\ \beta^+ > 0, & x \in (\alpha, b), \end{cases} \tag{6.6}$$

where the solution u satisfies the following jump conditions at α

$$[u]_\alpha = [u']_\alpha = [\beta u'']_\alpha = [\beta u''']_\alpha = 0.$$

First, let $r = \left(1, 1, \frac{\beta^-}{\beta^+}, \frac{\beta^-}{\beta^+}\right)$ be fixed throughout this subsection. Then, the usual weak form of (6.6) suggests to consider the following IFE method:

$$\text{find } u_h \in Q_{\alpha,r}^3(I) \text{ such that } (\beta u_h'', v_h'')_I = (f, v_h)_I, \quad \forall v_h \in Q_{\alpha,r}^3(\mathcal{T}_h), \quad (6.7)$$

where $Q_{\alpha,r}^3(\mathcal{T}_h) = H_0^2(I) \cap W_{\alpha,r}^3(\mathcal{T}_h)$. We note that the IFE space $Q_{\alpha,r}^3(\mathcal{T}_h)$ as well as the method described by (6.7) were discussed in [27, 36], and an error analysis based on a multipoint Taylor expansion was carried out to establish the optimality of this IFE method in [25]. As another demonstration of the versatility of the immerse scaling argument, we now present an alternative analysis for the optimal approximation capability of this IFE space. This new analysis based on the framework developed in Sects. 3 and 4 is shorter and cleaner than the one in the literature.

As usual, for the discussion of the approximation capability of the IFE space, we consider the interpolation on the reference element \check{I} and map it to the physical element I_{k_0} . To define the interpolation, we let $\{\sigma_i\}_{i=1}^4$ be the Hermite degrees of freedom, that is,

$$\sigma_0(v) = v(0), \quad \sigma_1(v) = v(1), \quad \sigma_2(v) = v'(0), \quad \sigma_3(v) = v'(1), \quad \forall v \in H^2(\check{I}).$$

It is known [27, 36] that there is a basis $\{L_{\check{\alpha},r}^i\}_{i=0}^3$ of $\mathcal{V}_{\check{\alpha},r}^3(\check{I})$ that satisfies

$$\sigma_i(L_{\check{\alpha},r}^j) = \delta_{i,j}, \quad i, j = 0, 1, 2, 3. \quad (6.8)$$

These basis functions can then be used to define an immersed Hermite projection/interpolation operator $\check{\mathcal{I}}_{\check{\alpha},r} : \mathcal{H}_{\check{\alpha},r}^4(\check{I}) \rightarrow \mathcal{V}_{\check{\alpha},r}^3(\check{I})$ such that $\check{u}_H = \check{\mathcal{I}}_{\check{\alpha},r} \check{u}$ and

$$\check{u}_H = \sum_{i=0}^3 \sigma_i(\check{u}) L_{\check{\alpha},r}^i. \quad (6.9)$$

Lemma 6.3 *Let $\beta^\pm > 0$ and $\check{\alpha} \in (0, 1)$, then*

$$-1 < L_{\check{\alpha},r}^i(x) < 1, \quad \forall x \in [0, 1], \quad i = 0, 1, 2, 3. \quad (6.10)$$

Proof See ‘‘Appendix B’’. □

Now, we are ready to establish that $\{\check{\mathcal{I}}_{\check{\alpha},r}\}_{0 < \check{\alpha} < 1}$ is a collection of uniformly bounded of RIFE projections.

Lemma 6.4 *Let $\beta^\pm > 0$, $\check{\alpha} \in (0, 1)$. Then there is a constant C independent of $\check{\alpha}$ such that the following estimate holds for every $\check{u} \in \mathcal{H}_{\check{\alpha}, r}^A(\check{I})$*

$$\left\| \check{\mathcal{S}}_{\check{\alpha}, r} \check{u} \right\|_{0, \check{I}} \leq C \|\check{u}\|_{2, \check{I}}.$$

Proof We know that $\sigma_i(\check{u}) \leq C \|\check{u}\|_{2, \check{I}}$ since $\check{u} \in H^2(\check{I})$. Now, we apply the triangle inequality to (6.9) and Lemma 6.3 to get

$$\left\| \check{\mathcal{S}}_{\check{\alpha}, r} \check{u} \right\|_{0, \check{I}} \leq C \|\check{u}\|_{2, \check{I}} \left(\sum_{i=0}^3 \left\| L_{\check{\alpha}, r}^i \right\|_{0, \check{I}} \right) \leq 4C \|\check{u}\|_{2, \check{I}}.$$

□

Now, let $\mathcal{S}_{\alpha, r} = \mathcal{M}^{-1} \circ \check{\mathcal{S}}_{\check{\alpha}, r} \circ \mathcal{M}$ where \mathcal{M} is defined in (4.8). By the commutative diagram in (4.10), $\mathcal{S}_{\alpha, r}$ is the local immersed Hermite interpolation. Then, by Lemma 6.4, $\{\mathcal{S}_{\alpha, r}\}_{\alpha \in I_{k_0}}$ is a collection of uniformly bounded LIFE projections. Hence, the following theorem follows from Theorem 4.3.

Theorem 6.2 *Let $\beta^\pm > 0$, $i \in \{0, 1, 2, 3\}$, $\alpha \in I_{k_0}$. Then, there is a constant C independent of α such that the following estimate holds for every $u \in \mathcal{H}_{\alpha, r}^A(I_{k_0})$*

$$\|u - \mathcal{S}_{\alpha, r} u\|_{i, I_{k_0}} \leq Ch^{3-i} |u|_{4, I_{k_0}}.$$

This theorem establishes the optimal approximation capability of the IFE space $Q_{\alpha, r}^3(\mathcal{T}_h)$ which was first derived in [25] with a lengthy and complex procedure.

7 Conclusion

In this manuscript we developed a framework for analyzing the approximation properties of one-dimensional IFE spaces using the scaling argument. We have applied this IFE scaling argument to establish the optimal convergence of IFE spaces constructed for solving the acoustic interface problem, the elliptic interface problem and the Euler-Bernoulli beam interface problem, respectively. In the future, we intend to extend the results in this paper to higher-dimensional IFEs constructed by employing the Frenet transform recently introduced in [6].

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Appendix A Proof of Lemma 3.4

Our goal is to show that the ratio $\frac{\sqrt{\check{\varphi}(0)^2 + \check{\varphi}(1)^2}}{\|\check{\varphi}\|_{0,\check{I}}}$ is bounded from below by a constant $c(m, r, w)$ independent of $\check{\alpha}$. For simplicity, let $q_i \in \mathcal{P}^m([0, 1])$ be the monomial basis $q_i(x) = x^i$ for $0 \leq i \leq m$. Using the equivalence of norms, one can show that there is $c_1(m) > 0$ such that

$$\min(|p(0)|, |p(1)|) + \sum_{i=0}^{m-1} |(p, q_i)_{[0,1]}| \geq c_1(m) \|p\|_{0,[0,1]}, \quad \forall p \in \mathcal{P}^m([0, 1]). \tag{A.1}$$

Unfortunately, if we extend (A.1) to $\mathcal{V}_{\check{\alpha},r}^m$, then the constant on the right might depend on $\check{\alpha}$ and might grow unboundedly as $\check{\alpha} \rightarrow 0^+$ or as $\check{\alpha} \rightarrow 1^-$. To circumvent this issue, we will use a scaling trick similar to the one used in the proof of Lemma 3.2. First, we bound $(\check{\varphi}, q_i)_{w_s, \check{I}_s}$ as shown in the following lemma

Lemma A.1 *Let $\check{m} \geq m \geq 0$, $\{r_k\}_{k=0}^m \subset \mathbb{R}_+$ and $\check{\alpha} \in (0, 1)$, there is $C(m, r, w) > 0$ such that if $\check{h}_s > \check{h}_{s'}$, then*

$$\begin{aligned} |(\check{\varphi}_s, q_i)_{\check{I}_s}| &= \left| \int_{\check{I}_s} \check{\varphi}_s(x) x^i dx \right| \leq C(m, r, w) h_{s'} \|\check{\varphi}_s\|_{0, \check{I}_s}, \\ i &= 0, 1, \dots, m-1, \quad \forall \check{\varphi} \in \mathcal{Q}_{\check{\alpha},w,r}^m(\check{I}). \end{aligned}$$

Proof Since $\check{\varphi} \in \mathcal{Q}_{\check{\alpha},w,r}^m(\check{I})$, we have

$$0 = w_s (\check{\varphi}_s, q_i)_{\check{I}_s} + w_{s'} (\check{\varphi}_{s'}, \mathcal{E}_{\check{\alpha},r}^{m,s'}(q_i))_{\check{I}_{s'}}.$$

Then, by Cauchy–Schwarz inequality and (3.3), we have

$$\begin{aligned} |(\check{\varphi}, q_i)_{\check{I}_s}| &= \frac{w_{s'}}{w_s} \left| (\check{\varphi}, \mathcal{E}_{\check{\alpha},r}^{m,s'}(q_i))_{\check{I}_{s'}} \right| \leq C(w) \|\check{\varphi}\|_{0, \check{I}_{s'}} \left\| \mathcal{E}_{\check{\alpha},r}^{m,s'}(q_i) \right\|_{0, \check{I}_{s'}}, \\ &\leq C(m, r, w) \sqrt{h_{s'}} \|\check{\varphi}\|_{0, \check{I}_s} \sqrt{h_{s'}} \|q_i\|_{0, \check{I}_s}, \\ &\leq C(m, r, w) h_{s'} \|\check{\varphi}\|_{\check{I}_s}. \end{aligned}$$

□

The previous lemma shows that $(\check{\varphi}, q_i)_{\check{I}_s}$ will approach 0 if h_s approaches 1. This will allow us to obtain a restricted version of (A.1).

Lemma A.2 *There is $\delta(m, r, w) \in (0, \frac{1}{2})$ and $C(m, r) > 0$ such that if $\min(h_-, h_+) < \delta(m, r, w)$, then*

$$|\check{\varphi}(0)| + |\check{\varphi}(1)| \geq C(m, r) \|\check{\varphi}\|_{0, \check{I}}. \quad \forall \check{\varphi} \in \mathcal{Q}_{\check{\alpha},w,r}^m(\check{I}).$$

Proof We will only discuss the case where $h_- > h_+$, the other case can be proved similarly. We define $\hat{\varphi}_- \in \mathcal{P}^m([0, 1])$ as $\hat{\varphi}_-(\xi) = \check{\varphi}_-(\check{h}_-\xi)$, then by the fact that $h_- \geq 1/2$,

$$\begin{aligned} |\check{\varphi}(0)| + \sum_{i=0}^{m-1} \left| \int_0^{h_-} \check{\varphi}(x)x^i dx \right| &= |\hat{\varphi}_-(0)| + \sum_{i=0}^{m-1} h_-^{i+1} \left| \int_0^1 \hat{\varphi}_-(\xi)\xi^i d\xi \right| \\ &\geq |\hat{\varphi}_-(0)| + \sum_{i=0}^{m-1} 2^{-m-1} |(\hat{\varphi}_-, q_i)_{[0,1]}| \\ &\geq C(m) \left(|\hat{\varphi}_-(0)| + \sum_{i=0}^{m-1} |(\hat{\varphi}_-, q_i)_{[0,1]}| \right). \end{aligned}$$

Then, by (A.1), $h_- \leq 1$, and (3.3), we have

$$\begin{aligned} |\check{\varphi}(0)| + \sum_{i=0}^{m-1} \left| \int_0^{h_-} \check{\varphi}(x)x^i dx \right| &\geq C(m) \|\hat{\varphi}_-\|_{0,[0,1]} \\ &= C(m)h_-^{-1/2} \|\check{\varphi}\|_{0,\check{I}^-} \geq C(m) \|\check{\varphi}\|_{0,\check{I}^-} \geq C_0(m, r) \|\check{\varphi}\|_{0,\check{I}^-}. \end{aligned}$$

Now, we use Lemma A.1 to estimate the inner product on the left hand side:

$$\sum_{i=0}^{m-1} \left| \int_0^{h_-} \check{\varphi}(x)x^i dx \right| \leq C_1(m, r, w)h_+ \|\check{\varphi}\|_{0,\check{I}^-}.$$

We combine it with the previous inequality to get

$$|\check{\varphi}(0)| \geq \|\check{\varphi}\|_{0,\check{I}^-} (C_0(m, r) - C_1(m, r, w)h_+).$$

Hence, if $h_+ \leq \delta = \min(1, \frac{C_0(m,r)}{2C_1(m,r,w)})$, then

$$|\check{\varphi}(0)| \geq \frac{1}{2}C_0(m, r) \|\check{\varphi}\|_{0,\check{I}^-} \tag{A.2}$$

A similar argument can be used to show that if $h_+ \leq \tilde{\delta}$ (where $\tilde{\delta}$ could be different than the previous δ), then

$$|\check{\varphi}(1)| \geq \frac{1}{2}\tilde{C}_0(m, r) \|\check{\varphi}\|_{0,\check{I}^+}. \tag{A.3}$$

□

So far, we have shown that if one of the sub-elements \check{I}^\pm is small enough, then Lemma 3.4 holds. It remains to show that the lemma holds for $\check{\alpha} \in [\delta, 1-\delta]$, for which, we consider the following sequence $\{\mathcal{O}_{\check{\alpha},w,r}^i\}_{i=0}^m$ by the Gram-Schmidt process:

$$\mathcal{O}_{\check{\alpha},w,r}^0 = \mathcal{N}_{\check{\alpha},r}^0, \quad \mathcal{O}_{\check{\alpha},w,r}^i = \mathcal{N}_{\check{\alpha},r}^i - \sum_{j=0}^{i-1} \frac{(\mathcal{N}_{\check{\alpha},r}^i, \mathcal{O}_{\check{\alpha},w,r}^j)_{w,\check{I}}}{(\mathcal{O}_{\check{\alpha},w,r}^j, \mathcal{O}_{\check{\alpha},w,r}^j)_{w,\check{I}}} \mathcal{O}_{\check{\alpha},w,r}^j, \quad i = 1, 2, \dots, m.$$

$$(A.4)$$

Clearly, we have $\mathcal{O}_{\check{\alpha},w,r}^m \in \mathcal{D}_{\check{\alpha},w,r}^m(\check{I})$. The following lemma shows that when $\mathcal{O}_{\check{\alpha},w,r}^m$ is expressed in terms of the canonical basis $\{\mathcal{N}_{\check{\alpha},r}^i\}_{i=0}^m$, the coefficients of the expansion are rational functions.

Lemma A.3 *Let $\check{m} \geq m \geq 0$, $\{r_k\}_{k=0}^m \subset \mathbb{R}_+$ and $\check{\alpha} \in (0, 1)$, the orthogonal RIFE function $\mathcal{O}_{\check{\alpha},w,r}^m$ defined in (A.4) satisfies*

$$\mathcal{O}_{\check{\alpha},w,r}^m = \sum_{i=0}^m R_{w,r}^{i,m}(\check{\alpha}) \mathcal{N}_{\check{\alpha},r}^i \tag{A.5}$$

for some rational functions $\{R_{w,r}^{i,m}\}_{i=0}^m$ of $\check{\alpha}$.

Proof We will prove Lemma A.3 via strong induction. First, the case $m = 0$ is obvious. Now, we assume that for every $i = 0, 1, \dots, m - 1$, there are rational functions $R_{w,r}^{j,i}$ of $\check{\alpha}$ such that

$$\mathcal{O}_{\check{\alpha},w,r}^i = \sum_{j=0}^i R_{w,r}^{j,i}(\check{\alpha}) \mathcal{N}_{\check{\alpha},r}^j. \tag{A.6}$$

To show that $\mathcal{O}_{\check{\alpha},w,r}^m$ satisfies (A.5), we use the fact that $(\mathcal{N}_{\check{\alpha},r}^i, \mathcal{O}_{\check{\alpha},w,r}^j)_{w,\check{I}}$ is a polynomial in $\check{\alpha}$. Therefore,

$$\frac{(\mathcal{N}_{\check{\alpha},r}^i, \mathcal{O}_{\check{\alpha},w,r}^j)_{w,\check{I}}}{(\mathcal{O}_{\check{\alpha},w,r}^j, \mathcal{O}_{\check{\alpha},w,r}^j)_{w,\check{I}}} \tag{A.7}$$

is a rational function of $\check{\alpha}$. Furthermore, by plugging (A.6) into (A.4) and rearranging the terms, we get

$$\begin{aligned} \mathcal{O}_{\check{\alpha},w,r}^m &= \mathcal{N}_{\check{\alpha},r}^m - \sum_{j=0}^{m-1} \frac{(\mathcal{N}_{\check{\alpha},r}^m, \mathcal{O}_{\check{\alpha},w,r}^j)_{w,\check{I}}}{(\mathcal{O}_{\check{\alpha},w,r}^j, \mathcal{O}_{\check{\alpha},w,r}^j)_{w,\check{I}}} \sum_{i=0}^j R_{w,r}^{i,j}(\check{\alpha}) \mathcal{N}_{\check{\alpha},r}^i \\ &= \mathcal{N}_{\check{\alpha},r}^m - \sum_{i=0}^{m-1} \underbrace{\left(\sum_{j=i}^{m-1} \frac{(\mathcal{N}_{\check{\alpha},r}^m, \mathcal{O}_{\check{\alpha},w,r}^j)_{w,\check{I}}}{(\mathcal{O}_{\check{\alpha},w,r}^j, \mathcal{O}_{\check{\alpha},w,r}^j)_{w,\check{I}}} R_{w,r}^{i,j}(\check{\alpha}) \right)}_{:= R_{w,r}^{j,m}(\check{\alpha})} \mathcal{N}_{\check{\alpha},r}^i. \end{aligned}$$

From the strong induction assumption and (A.7), we conclude that $R_{w,r}^{j,m}$ is a rational function. □

Corollary A.1 *Given m, w_{\pm} and r , the function $\mathcal{J}_{w,r}^m : (0, 1) \rightarrow \mathbb{R}_+$ defined in (A.8) is a rational function.*

Now, we are ready to prove Lemma 3.4. We can rewrite Lemma A.2 as: there is $\delta \in (0, \frac{1}{2})$ that depends on m, w and r , and a constant $C_1(m, r)$ such that

$$\sqrt{\mathcal{O}_{\check{\alpha},w,r}^m(0)^2 + \mathcal{O}_{\check{\alpha},w,r}^m(1)^2} \geq C_1(m, r) \left\| \mathcal{O}_{\check{\alpha},w,r}^m \right\|_{0,\check{I}}, \quad \check{\alpha} \in (0, \delta) \cup (1 - \delta, 1).$$

For $\delta \in [\delta, 1 - \delta]$, the following function is continuous

$$\mathcal{J}_{w,r}^m : \check{\alpha} \mapsto \frac{\mathcal{O}_{\check{\alpha},w,r}^m(0)^2 + \mathcal{O}_{\check{\alpha},w,r}^m(1)^2}{\left\| \mathcal{O}_{\check{\alpha},w,r}^m \right\|_{0,\check{I}}^2} \tag{A.8}$$

because both of its numerator and denominator are rational functions of $\check{\alpha}$ and the denominator is not zero. Therefore, there is $C_2(m, w, r) > 0$ such that

$$\sqrt{\mathcal{O}_{\check{\alpha},w,r}^m(0)^2 + \mathcal{O}_{\check{\alpha},w,r}^m(1)^2} \geq C_2(m, w, r) \left\| \mathcal{O}_{\check{\alpha},w,r}^m \right\|_{0,\check{I}}, \quad \check{\alpha} \in [\delta, 1 - \delta].$$

By letting $C(m, w, r) = \min(C_1(m, r), C_2(m, w, r))$, we know that $\mathcal{O}_{\check{\alpha},w,r}^m$ satisfies inequality (3.18) stated in Lemma 3.4. Consequently, the estimates in (3.18) of Lemma 3.4 is true for every function in $\mathcal{Q}_{\check{\alpha},w,r}^m(\check{I})$ because it is a one-dimensional space, and Lemma 3.4 is proven.

B Proof of Lemma 6.3

Let us start with $p = L_{\check{\alpha},r}^0$, we have

$$p(0) = 1, \quad p(1) = 0, \quad p'(0) = 0, \quad p'(1) = 0.$$

By Theorem 3.1, p' does not change sign in $(0, 1)$ since $p' \in \mathcal{V}_{\check{\alpha},\tau(r)}^2(\check{I})$ and $p'(0) = p'(1) = 0$. Therefore, p is monotonically decreasing from $p(0) = 1$ to $p(1) = 0$. The same argument applies to $L_{\check{\alpha},r}^1$.

Next, we show that $q = L_{\check{\alpha},r}^2$ is bounded between 0 and 1. We have

$$q(0) = 0, \quad q(1) = 0, \quad q'(0) = 1, \quad q'(1) = 0.$$

By Rolle’s theorem, there is $c \in (0, 1)$ such that $q'(c) = 0$. By Theorem 3.1, c the only root of q' in $(0, 1)$. Now, by the generalized Rolle’s theorem, there is $d \in (c, 1)$ such that $q''(d^-)q''(d^+) \leq 0$. If $d \neq \check{\alpha}$, then $q''(d^-) = q''(d^+) = q''(d)$ because q is a polynomial on either sides of $\check{\alpha}$. In this case we have $q''(d) = 0$. If $d = \check{\alpha}$, then $q''(d^-)q''(d^+) \leq 0$ and jump condition implies

$$\frac{\beta^-}{\beta^+} (q''(\check{\alpha}^-))^2 \leq 0$$

from which we have $q''(\check{\alpha}^-) = 0 = q''(\check{\alpha}^+)$. Hence, $q''(d) = 0$. Furthermore, by Theorem 3.1, d is the only root of q'' since $q'' \in \mathcal{V}_{\check{\alpha}, \tau^2(r)}^1(I)$. Since q'' is a linear polynomial on either sides of $\check{\alpha}$, the jump condition satisfied by q further implies that q'' does not change its sign $(0, d)$ and $(d, 1)$. Because $q'(0) = 1$ and $q'(c) = 0$ and $0 < c < d$, we know that q' is decreasing on $(0, d)$ but increasing on $(d, 1)$. These further imply $q'(x) \in [0, 1]$ for $x \in [0, c]$ and $q'(x) \leq 0$ for $x \in [c, d]$; hence, $q(x) \leq q(c)$ for all $x \in [0, d]$. Furthermore, since $q'(d) \leq 0$, $q'(1) = 0$ and q' is monotonic on $[d, 1]$, we know that $q'(x) \leq 0$ for all $x \in [d, 1]$. Hence, $0 = q(1) \leq q(x) \leq q(d) \leq q(c)$ for all $x \in [d, 1]$. Consequently, $q(c) \geq q(x)$ for $x \in [0, 1]$. In addition, since q has no local minimum point on $(0, d)$, we have $q(x) \geq \min\{q(0), q(d)\} \geq 0$ for all $x \in [0, d]$. Thus, $q(x) \geq 0 \forall x \in [0, 1]$. On the other hand,

$$q(x) \leq q(c) = \int_0^c q'(x)dx \leq \int_0^c 1dx = c < 1 \quad \forall x \in [0, 1],$$

The last two estimates lead us to conclude that $0 \leq q(x) = L_{\check{\alpha}, r}^2(x) < 1$. As for $L_{\check{\alpha}, r}^3$, we note that

$$L_{\check{\alpha}, r}^3(x) = -L_{1-\check{\alpha}, \tilde{r}}^2(1-x), \quad \text{where } \tilde{r} = \{r_i^{-1}\}_{i=0}^3$$

which leads to $L_{\check{\alpha}, r}^3(x) \in [-1, 0]$.

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