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CORRECTION

Correction to: Multilevel Monte Carlo front-tracking for random scalar conservation laws

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Abstract An error in [4, Theorem 4.1, 4.5, Corollary 4.5] is corrected. There, in the Monte Carlo error bounds for front tracking for scalar conservation laws with random input data, 2-integrability in a Banach space of type 1 was assumed. In providing the corrected convergence rate bounds and error versus work analysis of multilevel Monte Carlo front-tracking methods, we also generalize [4] to q-integrability of the random entropy solution for some $1 < q \leq 2$, allowing possibly infinite variance of the random entropy solutions of the scalar conservation law.

Keywords Multilevel Monte Carlo method · Conservation laws

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1 Multilevel Monte Carlo Error Analysis

The statement and proof of Theorem 4.1 in [4] should account for the *type of Banach space* in which the discretization scheme is analyzed. To render the presentation self-contained, we recapitulate the relevant definitions as used, e.g., in [2]. For a general Banach space E, the *type* of the Banach space is defined as follows (see, e.g., [3, Page 246]).

Definition 1.1 Let $1 \leq p \leq \infty$, and Z_j, $j \in \mathbb{N}$, be a sequence of Bernoulli–Rademacher random variables. A Banach space E is said to be of *type* p if there is a *type constant* $C > 0$ such that for all finite sequences $(x_j)_{j=1}^N \subset E$, $N \in \mathbb{N}$,

$$\left\| \sum_{j=1}^{N} Z_j x_j \right\|_E \leq C \left(\sum_{j=1}^{N} \|x_j\|_E^p \right)^{1/p}.$$

Every Banach space is of type 1; L^p-spaces are of type $\min\{p, 2\}$ for $1 \leq p < \infty$ (see [3]). The following result from [3, Proposition 9.11] for Banach spaces of type p is the basis of the MLMC-FT error analysis.

Proposition 1.1 [2, Prop. 2.4] *Let E be a Banach space of type p with type constant C_t. Then, for every finite sequence $(Y_j)_{j=1}^M$ of independent random variables in $L^p(\Omega; E)$ with zero mean, one has*

$$\mathbb{E}\left[\left\| \sum_{j=1}^{N} Y_j \right\|_E^p \right] \leq (2C_t)^p \sum_{j=1}^{N} \mathbb{E}\left[\|Y_j\|_E^p \right].$$

For the front-tracking error analysis, we need strengthen the assumptions from [4] to either the physical or computational domain is bounded (as, e.g., in the periodic setting) or that all realizations of the random initial data are compactly supported in $\mathbb{R}^d$, *with support contained in one common bounded domain D.*

2 MLMC-FT convergence analysis

We recapitulate notation and basic estimates from the convergence analysis from [4, Section 4.4]. In doing so, we take into account the type of the Banach spaces appearing in the error analysis which was disregarded in [4]. This results in estimates of the combined multilevel Monte Carlo front-tracking (MLMC-FT, for short) errors which differ from those in [4, Theorem 4.5, Corollary 4.5] in that more general q-integrability for some $1 < q \leq 2$ of the random entropy solutions is considered, and that convergence rates of the front-tracking errors are now estimated in L^p for some $p \geq q > 1$. We can write

$$\mathbb{E}\left[u(t)\right] - E_L^{\mathrm{MLMC}}[u^L] = \mathbb{E}\left[u(t) - u^L(t)\right] + \mathbb{E}\left[u^L(t)\right] - E_L^{\mathrm{MLMC}}\left[u^L(t)\right].$$

Hence,

$$\begin{aligned}
&\left\|\mathbb{E}\left[u(t)\right] - E_L^{\mathrm{MLMC}}\left[u^L(t)\right]\right\|_{L^q(\Omega;L^p(\mathbb{R}^d))} \\
&\quad\leq \left\|\mathbb{E}[u(t)] - \mathbb{E}[u^L(t)]\right\|_{L^q(\Omega;L^p(\mathbb{R}^d))} \\
&\qquad + \left\|\mathbb{E}[u^L(t)] - E_L^{\mathrm{MLMC}}[u^L(t)]\right\|_{L^q(\Omega;L^p(\mathbb{R}^d))} := A + B.
\end{aligned}$$

We estimate the two terms A and B. For the first one, we use the monotonicity of the expectation and the deterministic front-tracking error estimate:

$$A = \left\|\mathbb{E}[u(t)] - \mathbb{E}[u^L(t)]\right\|_{L^p(\mathbb{R}^d)} \leq \left\|u(t) - u^L(t)\right\|_{L^1(\Omega;L^p(\mathbb{R}^d))} \leq C\delta_L^{s/p},$$

where the second inequality follows from the bound on the deterministic approximation error of the FT in $L^p(\mathbb{R}^d)$. Here, δ_L is the FT discretization parameter at refinement level L. In nonadaptive approximations of the flux, $\delta_L = 2^{-L}\delta_0$, where $\delta_0 > 0$ denotes the discretization parameter for the coarsest level. For front tracking, and for Lipschitz regular flux functions, $s = 1$ [1]. To bound the term B, we set $u^{-1} := 0$ and $\Delta u^\ell := u^\ell - u^{\ell-1}$.

$$\begin{aligned}
B^q &= \left\|\mathbb{E}\left[\sum_{\ell=0}^{L}\left(u^\ell - u^{\ell-1}\right)\right] - \sum_{\ell=0}^{L} E_{M_\ell}\left(u^\ell - u^{\ell-1}\right)\right\|_{L^q(\Omega;L^p(\mathbb{R}^d))}^q \\
&= \left\|\sum_{\ell=0}^{L}\left(\mathbb{E}\left[\Delta u^\ell\right] - E_{M_\ell}\left[\Delta u^\ell\right]\right)\right\|_{L^q(\Omega;L^p(\mathbb{R}^d))}^q \\
&= \left\|\sum_{\ell=0}^{L}\sum_{j=1}^{M_\ell}\left(\frac{\mathbb{E}\left[\Delta u^\ell\right] - \Delta u^{\ell,j}}{M_\ell}\right)\right\|_{L^q(\Omega;L^p(\mathbb{R}^d))}^q \\
&\overset{\substack{\text{assume}\\ p\leq q}}{\leq} C_{\mathrm{supp}u}\left\|\sum_{\ell=0}^{L}\sum_{j=1}^{M_\ell}\left(\frac{\mathbb{E}\left[\Delta u^\ell\right] - \Delta u^{\ell,j}}{M_\ell}\right)\right\|_{L^q(\Omega;L^q(\mathbb{R}^d))}^q.
\end{aligned}$$

Define

$$Y^{\ell,j} := \frac{\mathbb{E}\left[\Delta u^\ell\right] - \Delta u^{\ell,j}}{M_\ell}.$$

$Y^{\ell,j}$ are independent, zero-mean random variables. Hence, we can use [3, Prop. 9.11] as stated in Proposition 1.1 (see also [2, Cor. 2.5]) for $q \in (1, 2]$ to estimate B^q by:

$$B^q \leq C_{\mathrm{supp}u} C_q \sum_{\ell=0}^{L} \sum_{j=1}^{M_\ell} \left\| Y^{\ell,j} \right\|^q_{L^q(\Omega;L^q(\mathbb{R}^d))}$$
$$:= \sum_{\ell=0}^{L} \sum_{j=1}^{M_\ell} \left\| \frac{\mathbb{E}\left[\Delta u^\ell\right] - \Delta u^{\ell,j}}{M_\ell} \right\|^q_{L^q(\Omega;L^q(\mathbb{R}^d))},$$

where C_q is a constant depending on the type q of the Banach space L^q. We continue estimating the last term:

$$\begin{aligned}
B^q &\leq C_{\mathrm{supp}u} C_q \sum_{\ell=0}^{L} M_\ell^{1-q} \left\| \mathbb{E}\left[\Delta u^\ell\right] - \Delta u^{\ell,1} \right\|^q_{L^q(\Omega;L^q(\mathbb{R}^d))} \\
&\leq C_{\mathrm{supp}u} C_q \sum_{\ell=0}^{L} M_\ell^{1-q} \left\| \Delta u^{\ell,1} \right\|^q_{L^q(\Omega;L^q(\mathbb{R}^d))} \\
&\leq C_{\mathrm{supp}u,q} \left(M_0^{1-q} \right. \\
&\quad \left. + \sum_{\ell=1}^{L} \left(\frac{\left\| u^\ell(t) - u(t) \right\|^q_{L^q(\Omega;L^q(\mathbb{R}^d))}}{M_\ell^{1-q}} + \frac{\left\| u^{\ell-1}(t) - u(t) \right\|^q_{L^q(\Omega;L^q(\mathbb{R}^d))}}{M_\ell^{1-q}} \right) \right) \\
&\leq C_{\mathrm{supp}u,q} \left(M_0^{1-q} + \sum_{\ell=1}^{L} M_\ell^{1-q} \left\| u^\ell(t) - u(t) \right\|^q_{L^q(\Omega;L^q(\mathbb{R}^d))} \right) \\
&\overset{\text{Hölder}}{\leq} C_{\mathrm{supp}u,q} \left(M_0^{1-q} + C_{\|u_0\|_\infty} \sum_{\ell=1}^{L} M_\ell^{1-q} \left\| u^\ell(t) - u(t) \right\|_{L^q(\Omega;L^1(\mathbb{R}^d))} \right) \\
&\overset{\substack{\text{deterministic}\\ \text{error}\\ \text{estimate}}}{\leq} C(\mathrm{supp}u, q, \|u_0\|_\infty, T, \|f\|_{W^{1,\infty}}) \left(M_0^{1-q} + \sum_{\ell=1}^{L} M_\ell^{1-q} \delta_\ell^s \right),
\end{aligned}$$

where we recall that δ_ℓ is the FT flux discretization parameter at level ℓ, $\ell = 0, \ldots, L$, and that for front tracking, the convergence rate is $s = 1$.

Combining the estimates for terms A and B, we arrive at

$$\left\| \mathbb{E}[u] - E_L^{\mathrm{MLMC}}[u^L] \right\|^q_{L^q(\Omega;L^p(\mathbb{R}^d))} \leq C \left(M_0^{1-q} + \sum_{\ell=1}^{L} M_\ell^{1-q} \delta_\ell^s + \delta_L^{\frac{sq}{p}} \right). \quad (2.1)$$

For $p = 1$, the last term is minimized which results in the bound

$$\left\| \mathbb{E}[u] - E_L^{\mathrm{MLMC}}[u^L] \right\|^q_{L^q(\Omega;L^1(\mathbb{R}^d))} \leq C \left(M_0^{1-q} + \sum_{\ell=1}^{L} M_\ell^{1-q} \delta_\ell^s + \delta_L^{sq} \right). \quad (2.2)$$

3 Choice of MC sample sizes M_ℓ and error versus work bounds

The Monte Carlo sample numbers in the error bound (2.2), i.e., M_ℓ, at FT discretization level ℓ should be chosen such that for a given error tolerance ε, the bound for the required computational work is optimized. We recall, from [4], the work estimates for the deterministic front-tracking method for different scenarios:

Proposition 3.1 *The computational work W_{FT} for the deterministic front-tracking method behaves asymptotically as*

$$W_{FT} \simeq \delta^{-w},$$

where $\delta > 0$ is the discretization parameter and w given by

$$w = \begin{cases} 1, & d = 1, \textit{ convex flux}, \\ 2, & d = 1, \textit{ nonconvex flux}, \\ 2d+1, & d > 1, \textit{ convex flux components}, \\ 2d+2, & d > 1, \textit{ nonconvex flux components}. \end{cases} \tag{3.1}$$

The computational work for the multilevel Monte Carlo front-tracking scheme behaves asymptotically as

$$W_L^{MLMC\text{-}FT} \simeq \sum_{\ell=0}^{L} M_\ell \delta_\ell^{-w}, \tag{3.2}$$

with the work rate w as given in (3.1).

For a given maximal discretization level L and for prescribed overall error $\varepsilon > 0$, the choice of sample numbers M_ℓ is obtained by optimizing the error bounds, as is by now customary in MLMC error analysis. The following result is analogous to Lemma 4.9 in [4]:

Lemma 3.1 *Let $\{\delta_\ell\}_{\ell \geq 0}$ be a strictly decreasing sequence of positive FT discretization parameters. Assume that the work for the MLMC-FT algorithm with L discretization levels scales asymptotically as in (3.2), that is, there exists a constant $C > 0$ which is independent of $\{\delta_\ell\}_{\ell \geq 0}$ and of $\{M_\ell\}_{\ell \geq 0}$ such that for every $L \geq 1$,*

$$W_L^{MLMC} \leq C \sum_{\ell=0}^{L} M_\ell \delta_\ell^{-w}, \tag{3.3}$$

for $w > 0$, given in (3.1). Given that the MCFT error at FT discretization level L is bounded as [*cf.* (2.2)]

$$\mathrm{Err}_L \leq C \left(\sum_{\ell=0}^{L} M_\ell^{1-q} \delta_\ell^{qs} + \delta_L^{qs} + M_0^{1-q} \right), \tag{3.4}$$

(i.e., $p = 1$), the optimal sample numbers M_ℓ in terms of the work bound (3.3) and of the error bound (3.4) are given by

$$M_\ell \simeq M_0 \delta_0^{-\frac{w}{q}} \delta_\ell^{\frac{s+w}{q}}, \quad \ell = 1, \ldots, L, \tag{3.5}$$

where

$$M_0 \simeq \left(\frac{\delta_0^{\frac{w-wq}{q}} + \sum_{j=1}^{L} \delta_j^{\frac{s+w-wq}{q}}}{\varepsilon - \delta_L^{qs}} \right)^{\frac{1}{q-1}} \delta_0^{\frac{w}{q}}.$$

The symbol $\simeq$ indicates equality up to a constant which may depend on the data (u_0, f) and the domain but which is independent of ℓ and L. If we assume in addition that the δ_ℓ are a geometric sequence, that is, $\delta_\ell = 2^{-\alpha\ell}\delta_0$ for some $\alpha, \delta_0 > 0$, we obtain for $L \to \infty$ that the error of the MLMC algorithm is bounded for any $s > 0$, and for $q \in (1, 2]$ as

$$\begin{aligned} &\left\| \mathbb{E}[u(\cdot, t)] - E_L^{MLMC}[u(\cdot, t)] \right\|_{L^q(\Omega; L^1(\mathbb{R}^d))} \\ &\quad \leq C \begin{cases} (W_L^{MLMC})^{-\frac{s}{s+w}}, & s + w - wq < 0, \\ (W_L^{MLMC})^{-\frac{q-1}{q}} \log(W_L^{MLMC}), & s + w - wq = 0, \\ (W_L^{MLMC} W_{0,\mathrm{det}})^{-\frac{q-1}{q}}, & s + w - wq > 0, \end{cases} \end{aligned}$$

where $W_{0,\mathrm{det}}$ is the work for the deterministic front-tracking algorithm with resolution $\delta_0 > 0$. In particular, the error of the MLMC-FT algorithm ($s = 1$) scales for $p = 1$, $q = 2$ with respect to work as

$$\begin{aligned} &\left\| \mathbb{E}[u(\cdot, t)] - E_L^{MLMC}[u(\cdot, t)] \right\|_{L^2(\Omega; L^1(\mathbb{R}^d))} \\ &\quad \simeq \begin{cases} (W_L^{MLMC})^{-1/2} \log(W_L^{MLMC}), & d = 1, \text{ convex flux}, \\ (W_L^{MLMC})^{-1/3}, & d = 1, \text{ nonconvex flux}, \\ (W_L^{MLMC})^{-\frac{1}{2+2d}}, & d > 1 \text{ convex flux components}, \\ (W_L^{MLMC})^{-\frac{1}{3+2d}}, & d > 1 \text{ nonconvex flux components}. \end{cases} \end{aligned}$$

Proof We use a Lagrange multiplier argument to determine the optimal M_ℓ. Since the assertion is about an asymptotically optimal error versus work bound, we require an (eventually small) error tolerance $\varepsilon > 0$ and assume all constants in the error bounds to take the value 1. This assumption results in the following asymptotic work and error measures:

$$W_L := \sum_{\ell=0}^{L} M_\ell \delta_\ell^{-w},$$

$$\mathrm{Err}_L := M_0^{1-q} + \delta_L^{qs} + \sum_{\ell=1}^{L} M_\ell^{1-q} \delta_\ell^{s}.$$

To minimize the overall work subject to prescribed error $\varepsilon > 0$, consider the Lagrangian

$$\mathscr{L} := W_L - \lambda(\varepsilon - \mathrm{Err}_L),$$

where λ is the Lagrange multiplier. Taking the derivative of $\mathscr{L}$ with respect to M_ℓ, we have

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial M_\ell} &= \frac{\partial W_L}{\partial M_\ell} + \lambda \frac{\partial \mathrm{Err}_L}{\partial M_\ell}, \\
\frac{\partial W_L}{\partial M_\ell} &= \delta_\ell^{-w}, \\
\frac{\partial \mathrm{Err}_L}{\partial M_\ell} &= \begin{cases} (1-q) M_\ell^{-q} \delta_\ell^s, & \ell > 0, \\ (1-q) M_0^{-q}, & \ell = 0. \end{cases}
\end{aligned}$$

Hence,

$$\delta_\ell^{-w} = \begin{cases} \lambda(q-1) M_\ell^{-q} \delta_\ell^s, & \ell > 0, \\ \lambda(q-1) M_0^{-q}, & \ell = 0, \end{cases}$$

or

$$M_\ell = \begin{cases} \left(\delta_\ell^{s+w} \lambda(q-1)\right)^{\frac{1}{q}}, & \ell > 0, \\ \left(\delta_0^{w} \lambda(q-1)\right)^{\frac{1}{q}}, & \ell = 0. \end{cases}$$

Now insert $\varepsilon = \mathrm{Err}_L$ to get an expression for λ:

$$\begin{aligned}
\varepsilon = \mathrm{Err}_L &= \delta_L^{qs} + \left(\delta_0^w \lambda(q-1)\right)^{\frac{1-q}{q}} + (\lambda(q-1))^{\frac{1-q}{q}} \left(\sum_{\ell=1}^{L} \delta_\ell^{\frac{s+w}{q}(1-q)} \delta_\ell^s \right) \\
&= \delta_L^{qs} + (\lambda(q-1))^{\frac{1-q}{q}} \left(\delta_0^{\frac{w(1-q)}{q}} + \sum_{\ell=1}^{L} \delta_\ell^{\frac{s+w-wq}{q}} \right).
\end{aligned}$$

We solve this for λ:

$$\lambda = \frac{1}{q-1} \left(\frac{\delta_0^{\frac{w-wq}{q}} + \sum_{\ell=1}^{L} \delta_\ell^{\frac{s+w-wq}{q}}}{\varepsilon - \delta_L^{qs}} \right)^{\frac{q}{q-1}}.$$

The sample numbers M_ℓ are given by

$$M_\ell = \begin{cases} \left(\frac{\delta_0^{\frac{w-wq}{q}} + \sum_{j=1}^{L} \delta_j^{\frac{s+w-wq}{q}}}{\varepsilon - \delta_L^{qs}} \right)^{\frac{1}{q-1}} \delta_\ell^{\frac{s+w}{q}}, & \ell > 0, \\ \left(\frac{\delta_0^{\frac{w-wq}{q}} + \sum_{j=1}^{L} \delta_j^{\frac{s+w-wq}{q}}}{\varepsilon - \delta_L^{qs}} \right)^{\frac{1}{q-1}} \delta_0^{\frac{w}{q}}, & \ell = 0, \end{cases}$$

and the bound for the overall work becomes

$$\begin{aligned} W_L &:= \sum_{\ell=0}^{L} M_\ell \delta_\ell^{-w} \\ &= \left(\frac{\delta_0^{\frac{w-wq}{q}} + \sum_{j=1}^{L} \delta_j^{\frac{s+w-wq}{q}}}{\varepsilon - \delta_L^{qs}} \right)^{\frac{1}{q-1}} \left(\delta_0^{\frac{w-wq}{q}} + \sum_{\ell=1}^{L} \delta_\ell^{\frac{s+w-wq}{q}} \right) \\ &= \left(\frac{1}{\varepsilon - \delta_L^{qs}} \right)^{\frac{1}{q-1}} \left(\delta_0^{\frac{w-wq}{q}} + \sum_{\ell=1}^{L} \delta_\ell^{\frac{s+w-wq}{q}} \right)^{\frac{q}{q-1}} . \end{aligned}$$

Now choose $\varepsilon \simeq \delta_L^{qs}$, i.e., $\varepsilon = 2\delta_L^{qs}$, to balance the two error contributions and obtain

$$W_L = \delta_L^{-\frac{qs}{q-1}} \left(\delta_0^{\frac{w-wq}{q}} + \sum_{\ell=1}^{L} \delta_\ell^{\frac{s+w-wq}{q}} \right)^{\frac{q}{q-1}} .$$

Now let us assume again that $\delta_\ell \simeq 2^{-\alpha\ell}$ with some $\alpha > 0$, i.e., the δ_ℓ are geometrically decreasing to zero. We distinguish three cases:

1. *Case* $s + w - wq < 0$ $(\Leftrightarrow q > \frac{s}{w} + 1)$: Then

$$W_L \simeq \delta_L^{-\frac{qs}{q-1}} \delta_L^{\frac{s+w-wq}{q-1}} = \delta_L^{-(s+w)};$$

hence

$$(\mathrm{Err}_L)^{1/q} \simeq \delta_L^s \simeq W_L^{-\frac{s}{s+w}} .$$

2. *Case* $s + w - wq = 0$ $(\Leftrightarrow q = \frac{s}{w} + 1)$: Then

$$W_L \simeq \delta_L^{-\frac{qs}{q-1}} L^{\frac{q}{q-1}};$$

hence

$$W_L (\log W_L)^{-\frac{q}{q-1}} \simeq \delta_L^{-\frac{qs}{q-1}}$$

$$(\mathrm{Err}_L)^{1/q} \simeq \delta_L^s \simeq \left(W_L (\log W_L)^{-\frac{q}{q-1}} \right)^{-\frac{q-1}{q}} \simeq W_L^{-\frac{q-1}{q}} \log W_L .$$

3. *Case* $s + w - wq > 0$ $(\Leftrightarrow q < \frac{s}{w} + 1)$: Then

$$W_L \simeq \delta_L^{-\frac{qs}{q-1}} \left(\delta_0^{\frac{w(1-q)}{q}} \right)^{\frac{q}{q-1}} = \delta_L^{-\frac{qs}{q-1}} \delta_0^{-w},$$

Note that $\delta_0^{-w} \simeq W_{0,\mathrm{det}}$. Hence,

$$(\mathrm{Err}_L)^{1/q} \simeq \delta_L^s \simeq (W_L W_{0,\mathrm{det}}^{-1})^{-\frac{q-1}{q}}.$$

For the MLMC front-tracking method, this implies according to Proposition 3.1 for $q = 2$,

$$\mathrm{Err}_L^{1/q} \simeq \begin{cases} W_L^{-1/2} \log W_L, & d = 1, \text{ convex flux}, \\ W_L^{-1/3}, & d = 1, \text{ nonconvex flux}, \\ W_L^{-\frac{1}{2+2d}}, & d > 1, \text{ convex flux components}, \\ W_L^{-\frac{1}{3+2d}}, & d > 1, \text{ nonconvex flux components}. \end{cases}$$

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