Belief Changes and Cognitive Development: Doxastic Logic LCB

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Abstract

We present the logic LCB which is expressed in a propositional language constantly enriched by new atomic expressions. Our formal framework is the propositional doxastic logic KD45 with the belief operator $\mathcal{B}$, extended by the $\mathcal{C}$ operator, to be read *it changes that ...*. We describe the changing beliefs of an agent who uses progressively expanding language. The approach presented here allows us to weaken pragmatic objections to the so-called principle of *negative retrospection* accepted in KD45 and the problem of *logical omniscience*. In what follows, we present the expanding propositional language used in our formalism, interpreted using an epistemic version of Kripke semantics. Next, we give a syntactic characterization of logic LCB and prove the soundness and completeness of LCB in respect to our semantics. Finally, we compare the idea of expanding language with the notion of *agent awareness* and we relate our formalism to two epistemic temporal logics.

Keywords Logic of change · Doxastic logic · Expanding language · Negative introspection · Omniscience problem

1 Introduction

Our aim is to describe the changing beliefs of an agent who uses a progressively expanding language. We choose as the formal framework the propositional doxastic logic KD45 with a primary operator $\mathcal{B}$, to be read: *it is believed that ...*. We extend its vocabulary with the $\mathcal{C}$ operator for change, to be read: *it changes that ...*. It is assumed that modality $\mathcal{C}$ has properties described within the logic of change LC (Świątorzecka 2008; Świątorzecka and Czermak 2012). LC deals with the notion of

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change as primitive, which is used also to characterize temporal notions such as *immediate succession* and *durability* (Świątorzecka and Czermak 2015).

Our attempt is motivated, primarily, by the simple observation that beliefs, in contrast to knowledge, are not always veridical and so may be subject to change. A consistent agent who is unable to verify his beliefs, might take into account, that his beliefs are changing for some reason. Our intention is to provide a formalism that can capture these relationships between beliefs and their variability in a situation of limited access to knowledge about the reality.

The second component we introduce to the assumed doxastic logic is related to a more basic epistemological issue than that of an agent's updating his beliefs. We are also interested in simulating changes that occur in connection with the agent's *cognitive development*. In general, the extension of cognitive experiences leads to an enrichment of the agent's conceptual apparatus (a system of concepts) and hence to the enrichment his language. That is, the agent may expand his linguistic tools by adding new expressions to his language. At the same time, however, he may not know if certain sentences containing these new expressions are actually true. As a result, he may have to update his convictions by using sentences which did not previously belong to his language. The idea of an expanding language comes from the logic of change LC. We believe that it brings a philosophical profit, when it is used to our doxastic background.¹ Let us note, that it can be used to weaken pragmatic objections to the so-called principle of *negative retrospection* accepted in KD45. Given our approach, the negative retrospection principle does not apply to formulas outside of the current language of the agent, e.g., it does not apply to formulas that will belong to his language in connection with some new cognitive circumstances. The same happens with the rule of attaching $\mathcal{B}$, treated as expressing the so-called problem of *logical omniscience* of an agent. In our semantics, even formulas which are logically valid but do not belong to the agent's current language do not enter into the subjects of his beliefs.

The result of our research is modal logic LCB expressed in a propositional language which is constantly enriched by new atomic expressions. This formalism is the marriage of KD45 and LC, in which a number of relationships between modalities $\mathcal{B}$ and $\mathcal{C}$ are taken into account. In setting up LCB, we have ensured that the difference between beliefs-in-changes and changing-of-beliefs is maintained. Additionally, we want to capture at least two relationships which reflect the self-awareness of an agent who decides to change his beliefs or to keep his beliefs constant. The first condition we mean is, that the constant believing in A implies the belief in the constancy of A . Secondly, we want to capture the circumstances where an agent decides to change his point of view regarding his lack of belief in A (and this does not imply that he believes in not- A), then in result he believes in A .

We order our considerations in the following way. We present the expanding propositional language used in our formalism, interpreted in a certain epistemic version of Kripke semantics (1). Next, we give a syntactic characterization of the logic LCB (2). In (3), we prove the soundness and completeness of LCB in respect to our semantics. Finally, we sketch some remarks on the comparison of the

¹ Its epistemic attractiveness is indicated in Świątorzecka (2012).

proposed approach with two other approaches known in the literature (4). We will compare the idea of expanding language used in LCB with the concept of *general awareness* used in the logics of awareness. We will also relate our formalism to the epistemic temporal logic of linear time.

2 The Family of n -languages and Their Interpretation

Let us consider the set of atomic sentences At^n used by an agent who expresses beliefs relating to state n of his cognitive development (his conceptual apparatus). For any $n, j \in \mathbb{N}$ we use α_j^n to symbolize atomic sentences with respect to state n .

At^n is defined as follows: $At^1 = \{\alpha_j^1\}_{j \in J}$, $At^{n+1} = At^n \cup \{\alpha_k^{n+1}\}_{k \in K}$, where J is the set of natural numbers or its initial segment and the same is with K .

In every At^{n+1} there is at least one new atomic sentence which is not a member of At^n .

Elements of At^n are called n -atoms.

As previously mentioned, operators $\mathcal{B}$, $\mathcal{C}$ are read respectively as: *it is believed that ...* (or *the agent believes that ...*); *it changes that ...*.

We define an n -language as follows:

Definition 1 (*n-language*) An n -language comprises formulas built of n -atoms, classical logical connectives, and operators $\mathcal{C}$, and $\mathcal{B}$.

If a formula A belongs to n -language, we say that A is of level n . We also call an n -language: *language of level n* .

Because of our definition of At^n , every formula of level n is also a formula of level $n + 1$.

The *minimal level* of a formula A ($Lv(A)$) is the highest upper index in the set of atoms occurring in the formula A .

Actually, the minimal level of a formula A is the language level at which formula A occurs for the first time (the first language of the agent in which A occurs). So, for example, formula $\alpha_j^3 \wedge \alpha_n^4$ has the minimal level 4, and it is not a formula of any language of levels 1, 2, and 3.

Let us interpret our language in a certain kind of structures of possible worlds. The intended interpretation combines elements of the original LC semantics (Świątorzecka 2008) and a doxastic version of possible worlds semantics for KD45 (Fagin et al. 1995). The LC interpretation deals with the so-called *histories of changes* (we call them *LC-histories*) which may be described as left-limited, endless sequences of possible worlds. The latter are sets of atomic expressions, possibly updated by new atoms. We also consider histories of changes which are left-limited, infinite sequences but are formed from non-empty sets of possible worlds. The subsequent sets of possible worlds are domains of *doxastic accessibility* relations. At the same time we follow the doxastic way of talking about possible worlds from (Fagin et al. 1995, 64). We keep the distinction between one *real* world and other worlds, which are possible *states* of the agent's beliefs about the real world. The possible states are not verifiable by the agent due to the lack of their access to the

real world. We enrich the introduced distinction of possible worlds, assuming that the real world may be described more and more fully, because the agent has richer and richer conceptual apparatus. In this situation, we speak about the *actual* world relativized to the n level of the agent's cognition and about the states of his beliefs which are equally possible from the perspective of this actual world. Components of both the actual world and the agent's states of beliefs may be subject to dichotomic changes.

Let w be the actual world, which comprises certain propositions of the n -language, as true in w . Every n -world u consisting of propositions *believable* by the agent in view of the objective truths in w , stands in a doxastic accessibility relation to the world w . We say that such states are possible with respect to the agent's beliefs and assume that all propositions believable from the perspective of the actual world w belong to these states. The expansion of the language of level n to the language of level $n + 1$ results in updating the n -worlds (both n -actual world and n -possible states) to $n + 1$ -worlds.

After expanding of the n -language to $n + 1$ -language, the agent has access to updates of all his n -worlds: if the world u was accepted as believable with respect to w at level n and if w is updated to w' and u is updated to u' (i.e., w', u' are $n + 1$ -worlds), then u' is accessible from w' at $n + 1$.

In our approach we take into account the fact that the agent's conceptual apparatus grows, and thus the agent may consider new possible states that he could not describe before (because his language was too poor).

These introductory remarks are captured more precisely by the following definition:

Definition 2 (*$\mathfrak{B}$ -structure*) $\mathfrak{B}$ -structure is $\mathfrak{B} = \langle L, W, d, u, \{B^n\}_{n \in L} \rangle$, where

1. $L = \{1, 2, \dots\}$ is a set of *language levels*;
2. $W = \{w, u, \dots\}$ is an infinite set of *possible worlds*;
3. $d: L \rightarrow 2^W$ is a function that assigns to each language level n a set of those possible worlds, which the agent can describe in his n language. d fulfills the following conditions:

$$\forall_{n \in L} \exists_{w \in W} (w \in d(n)), \quad (d1)$$

$$\forall_{w \in W} \exists_{n \in L}^1 (w \in d(n)); \quad (d2)$$

4. $u: W \rightarrow W$ is a *updating* function. u fulfills the following condition:

$$\forall_{n \in L} \forall_{w \in d(n)} (u(w) \in d(n + 1)); \quad (u)$$

5. every $B^n \subseteq d(n) \times d(n)$ is a relation between possible worlds describable in the n -language, such that:

$$B^n \text{ is serial, transitive, and Euclidean in } d(n) \times d(n); \quad (B)$$

6. the functions d , u , and B are connected as follows:

$$\forall_{w,u \in \mathbf{d}(n)} (w\mathbf{B}^n v \implies u(w)\mathbf{B}^{n+1}u(v)). \quad (\text{duB})$$

Condition (d1) states that for each level n there is at least one possible world describable in the agent's n -language. Following (d2), we can say that for every possible world, there are no two different states of cognition which entirely describe w . In other words, a complete description of every world $w \in \mathbf{d}(n)$ can only be given by formulas of n -language. For every world w , function u assigns the world which is the update of w . According condition (u), when the agent expands n -language, then every possible world $w \in \mathbf{d}(n)$ must be updated to another possible world describable in his new $n + 1$ -language.

Note, that from conditions (d2), and (u) we get acyclicity for u .

Every $\mathbf{B}^n$ is a doxastic accessibility relation between n -worlds with properties that are semantic counterparts of KD45 axioms.

Finally, condition (duB) states that the agent has access to all updates of his possible worlds.

We shall now introduce the notion of a *history of changes*.

Definition 3 (*History of changes*) For any $\mathfrak{B}$ -structure, the history of changes is $\mathcal{H} = \langle \mathfrak{B}, \psi \rangle$, where $\psi = \bigcup_n \psi_n$ and every $\psi_n: \mathbf{d}(n) \rightarrow 2^{A^n}$ is a function which assigns to every possible world $w \in \mathbf{d}(n)$ a subset of atoms of level n .

Let us take $\langle L, W, \mathbf{d} \rangle$, with $\mathbf{d}(n)$ being a singleton for every n . If we consider $\mathcal{C}$ fragments of our n -languages, LC-history is a structure $\langle \langle L, W, \mathbf{d} \rangle, \psi \rangle$ (cf. Świątorzecka and Czermak 2015).

If $w \in \mathbf{d}(n)$, then we denote w as w^n . The quantifier $\forall_{w^n}$ is restricted to the domain $\mathbf{d}(n)$ of possible worlds.

We define *satisfaction* as follows:

Definition 4 (*Satisfaction*) If α_j^m is a formula of level n (i.e. when $m \leq n$), then:

$$1. \quad \mathfrak{B}, \psi, w^n \models^n \alpha_j^m \text{ iff } \alpha_j^m \in \psi_n(w^n).$$

If A, B are formulas of level n then:

2. $\mathfrak{B}, \psi, w^n \models^n \neg A$ iff $\mathfrak{B}, \psi, w^n \not\models^n A$,
3. $\mathfrak{B}, \psi, w^n \models^n A \wedge D$ iff $\mathfrak{B}, \psi, w^n \models^n A$ and $\mathfrak{B}, \psi, w^n \models^n D$,
4. $\mathfrak{B}, \psi, w^n \models^n A \vee D$ iff $\mathfrak{B}, \psi, w^n \models^n A$ or $\mathfrak{B}, \psi, w^n \models^n D$,
5. $\mathfrak{B}, \psi, w^n \models^n A \rightarrow D$ iff $\mathfrak{B}, \psi, w^n \not\models^n A$ or $\mathfrak{B}, \psi, w^n \models^n D$,
6. $\mathfrak{B}, \psi, w^n \models^n CA$ iff $(\mathfrak{B}, \psi, w^n \models^n A \text{ and } \mathfrak{B}, \psi, w^{n+1} \not\models^{n+1} A)$ or $(\mathfrak{B}, \psi, w^n \not\models^n A \text{ and } \mathfrak{B}, \psi, w^{n+1} \models^{n+1} A)$,
7. $\mathfrak{B}, \psi, w^n \models^n BA$ iff $\forall_{u^n} (w^n \mathbf{B}^n u^n \implies \mathfrak{B}, \psi, u^n \models^n A)$.

For any formula A with $Lv(A) = k$ and $n < k$, the expression: $\mathfrak{B}, \psi, w^n \models^n A$ is not defined. This simulates a situation in which our agent cannot consider as senseful these expressions which will be added to his linguistic apparatus only after further cognitive development. Indexes occurring in any formula are meant to inform us about the beginning of the process of assigning truth values to the formula. As we

see our proposition is close to a hybrid approach, because in syntax we operate with symbols from the metalanguage. However, in general, our approach is not a hybrid logic because we do not introduce to our language nominals or variables (open to binding) for states.

We shall now define the notions of $\mathcal{H}$ -truthness and *logical truth*.

Definition 5 (*$\mathcal{H}$ -truthness*) For $\mathcal{H} = \langle \mathfrak{B}, \psi \rangle$, formula A is $\mathcal{H}$ -true iff $\forall_{n \geq Lv(A)} (\forall_{w \in d(n)} \mathfrak{B}, \psi, w \models^n A)$.

We always consider truthness of formula A , in some history $\mathcal{H}$ of changes starting from the minimal level of A .

Definition 6 (*Logical truth*) A formula A is logically true iff A is $\mathcal{H}$ -true for every history $\mathcal{H}$ of changes.

3 Axiomatization

Our formalism is based on classical logic. Schemes of specific axioms are ordered in the following groups:

- describing KD45 properties of $\mathcal{B}$:

$$\mathcal{B}(A \rightarrow D) \rightarrow (\mathcal{B}A \rightarrow \mathcal{B}D) \quad (\text{K})$$

$$\mathcal{B}A \rightarrow \neg \mathcal{B}\neg A \quad (\text{D})$$

$$\mathcal{B}A \rightarrow \mathcal{B}\mathcal{B}A \quad (4)$$

$$\neg \mathcal{B}A \rightarrow \mathcal{B}\neg \mathcal{B}A \quad (5)$$

- describing LC properties of $\mathcal{C}$:

$$\mathcal{C}A \rightarrow \mathcal{C}\neg A \quad (\text{C1})$$

$$\mathcal{C}(A \wedge D) \rightarrow \mathcal{C}A \vee \mathcal{C}D \quad (\text{C2})$$

$$A \wedge \neg \mathcal{C}A \wedge \mathcal{C}D \rightarrow \mathcal{C}(A \rightarrow D) \quad (\text{C3})$$

$$\neg A \wedge \neg D \wedge \mathcal{C}A \wedge \mathcal{C}D \rightarrow \mathcal{C}(A \wedge D) \quad (\text{C4})$$

- describing the relationships between $\mathcal{C}$ and $\mathcal{B}$:

$$\neg \mathcal{B}A \wedge \mathcal{C}\neg \mathcal{B}A \rightarrow \mathcal{B}((\neg A \rightarrow \mathcal{C}A) \wedge (A \rightarrow \neg \mathcal{C}A)) \quad (\text{CB1})$$

$$\mathcal{B}A \wedge \neg \mathcal{C}\mathcal{B}A \rightarrow \mathcal{B}\neg \mathcal{C}A \quad (\text{CB2})$$

We accept the following primitive rules: modus ponens, extensionality for the equivalence:

$$\vdash F[A], \vdash A \leftrightarrow D \implies \vdash F[D] \quad (\text{rep})$$

and two rules of adding modalities $\mathcal{B}$ and $\neg \mathcal{C}$:

$$\vdash A \implies \vdash \mathcal{M}A, \text{ where } \mathcal{M} \in \{\mathcal{B}, \neg\mathcal{C}\}. \quad (\text{gen}\mathcal{M})$$

Later we will also use symbol $\mathcal{D}$ for *disbelief in negation* introduced via the following equivalence

$$\mathcal{D}A \leftrightarrow \neg\mathcal{B}\neg A. \quad (\mathcal{D}/\mathcal{B})$$

The system described above is the logic of changing beliefs **LCB**. Schemata: (K), (D), (4), and (5) are intended to express respectively: belief closure principle, consistency of the agent's beliefs, and his positive and negative introspection. Together with rule (gen $\mathcal{B}$), they characterize logic **KD45**, which is considered in the literature as a fairly adequate logical frame for description of *doxa* knowledge (cf. Fagin et al. 1995, 59–60, 105).

$\mathcal{C}$ operator describes dychotomic changes from A to $\neg A$ or conversely. Its syntactic properties are essentially different from modality $\mathcal{B}$. $\mathcal{C}$ versions of schemata K, D, 4, 5, as well as schema T are not theses of **LC** or **LCB**. In contrast to $\mathcal{B}$, in our logic we get the equivalence $\mathcal{C}A \leftrightarrow \mathcal{C}\neg A$ ((C1), (rep)). **LC** logic is neither monotonic nor anti-monotonic, and so the rules $\vdash A \rightarrow D \implies \vdash \mathcal{C}A \rightarrow \mathcal{C}D$ and $\vdash A \rightarrow D \implies \vdash \mathcal{C}D \rightarrow \mathcal{C}A$ are also not admissible in **LCB**.

The last schemata describe changes of beliefs and their connection with beliefs about change. To describe that the agent is self-aware in relation to his beliefs, we assume in (CB1) that a change of the agent's unbelief in A (to a belief in A) always results in belief in A ; and in (CB2) we assume that the lack of change of belief in A is associated with belief in the non-change of A .

Rule (gen $\mathcal{B}$) expresses the idea that the agent always accepts theses expressed in his current language. Rule (gen $\neg\mathcal{C}$) says that logical theses do not change.

Referring to our initial intuitions we note that beliefs-in-changes are not equivalent to changing-of-belief, and so the formulas

$$\mathcal{C}BA \rightarrow BCA, \quad BCA \rightarrow CBA$$

are not logically true.

Modalities $\mathcal{C}$ and $\mathcal{B}$ are also not mutually cancelled in the sense that formulas

$$XYA \rightarrow YA, \quad YXA \rightarrow YA$$

for any different X and Y being $\mathcal{C}$ or $\mathcal{B}$, are not logically true.

Contexts $\mathcal{B}\mathcal{C}$ and $\mathcal{C}\mathcal{B}$ are interdependent in the way that is described, for example, in the **LCB** theses:

$$\mathcal{B}A \wedge \mathcal{B}CA \rightarrow CBA, \quad \mathcal{B}A \wedge \mathcal{C}\mathcal{B}\neg A \rightarrow BCA.$$

4 Soundness and Completeness of LCB

We begin by proving the adequacy of our formalism for the given semantics.

Theorem 1 (Soundness) *If A is an LCB theorem, then A is logically true.*

Proof The logical truth of all LC axioms (C1), (C2), (C2), (C4), and admissibility of (gen- $\neg$ C) can be proved in analogical way as in (Świątorzecka and Czermak 2012, 5–6).

For axiom (CB1) we assume that there are $\mathfrak{B}, \psi, w, n$ such that $\mathfrak{B}, \psi, w^n \models^n \neg BA \wedge C \neg BA$ and $\mathfrak{B}, \psi, w^n \not\models^n B((\neg A \rightarrow CA) \wedge (A \rightarrow \neg CA))$. We obtain $w^n R^n c^n$ and $\mathfrak{B}, \psi, c^n \models^n \neg A \wedge \neg CA$ or $\mathfrak{B}, \psi, c^n \models^n A \wedge CA$ and next $\mathfrak{B}, \psi, u(c^n) \models^{n+1} \neg A$. From the assumption we have $\mathfrak{B}, \psi, w^n \models^n \neg BA$ and $\mathfrak{B}, \psi, w^n \models^n C \neg BA$, we obtain $\mathfrak{B}, \psi, u(w^n) \models^{n+1} BA$. We have $w^n B^n c^n$, therefore from (duB) we obtain $u(w^n) B^{n+1} u(c^n)$, and so with $\mathfrak{B}, \psi, u(w^n) \models^{n+1} BA$ we get $\mathfrak{B}, \psi, u(c^n) \models^{n+1} A$, which gives a contradiction.

For axiom (CB2) we assume that there are $\mathfrak{B}, \psi, w, n$ such that $\mathfrak{B}, \psi, w^n \models^n BA \wedge \neg CBA$ and $\mathfrak{B}, \psi, w^n \models^n \neg B \neg CA$. We obtain $w^n B^n c^n$ and $\mathfrak{B}, \psi, c^n \models^n CA$. Now by assumption we have $\mathfrak{B}, \psi, w^n \models^n BA$, thus using $w^n B^n c^n$ we have $\mathfrak{B}, \psi, c^n \models^n A$, therefore with $\mathfrak{B}, \psi, c^n \models^n CA$ we obtain $\mathfrak{B}, \psi, u(c^n) \models^{n+1} \neg A$. By assumption we have $\mathfrak{B}, \psi, w^n \models^n BA$ and $\mathfrak{B}, \psi, w^n \models^n \neg CBA$, thus we have $\mathfrak{B}, \psi, u(w^n) \models^{n+1} BA$. We have $w^n B^n c^n$, thus from (duB) we obtain $u(w^n) B^{n+1} u(c^n)$ and next with $\mathfrak{B}, \psi, u(w^n) \models^{n+1} BA$ we get $\mathfrak{B}, \psi, u(c^n) \models^{n+1} A$, which gives a contradiction. $\square$

In the next step, we show completeness of LCB in our semantics using the Henkin style. We proceed in a similar way as Hughes and Cresswell (1968, 155–158), but we use the notion of a maximally n -consistent set.

Definition 7 (Maximal LCB n -consistency) A set X is maximally LCB n -consistent iff X is LCB consistent, all elements of X are formulas of level n , and for every formula A of level n : if $A \notin X$, then $X \cup \{A\}$ is LCB inconsistent.

In what follows, consistent and inconsistent sets are to be understood as LCB consistent and inconsistent. Similarly, a maximally n -consistent set is to be understood as maximally LCB n -consistent.

For any n -consistent sets we have the following variants of the Lindenbaum lemma:

Lemma 1 If a set X is consistent, and all elements of X are formulas of level n , then there exists a set Y such that $X \subseteq Y$ and Y is maximally n -consistent.

Lemma 2 If a set X is consistent and all elements of X are formulas of level n , then there exists a set Y , such that $X \subseteq Y$, and Y is maximally $n + 1$ -consistent.

Lemmas 1 and 2 can be proved in the standard way. We take two enumerations (denoted in the same way): $A_1, A_2, \dots$, for the first lemma formulas of n -language, for the second lemma formulas of $n + 1$ -language. We define two sequences (for the first sequence we take enumeration of n -language, and for the second we take selected $n + 1$ -enumeration): $L_0 = X$, $L_{k+1} = L_k \cup \{A_{k+1}\}$ if L_{k+1} is consistent, if not, then $L_{k+1} = L_k$. Now, we define the sum of the first sequence, and the sum of the second and we prove that they are respectively maximally n -consistent and maximally $n + 1$ -consistent.

For operators C and B we have the following lemmas:

Lemma 3 *If X is maximally n -consistent, then $\{A : A \wedge \neg CD \in X\} \cup \{\neg E : E \wedge CE \in X\}$ is maximally n -consistent set.*

Lemma 4 *If X is maximally n -consistent, and $\mathcal{DA} \in X$ then set $\{A\} \cup \{E : BE \in X\}$ is consistent.*

In case of Lemma 3 we need to use (gen $\neg C$), (C1), (C2) (C3), and (C4) (see fact 6 for logic LCS4² (Świątorzecka and Czermak 2015, 519–520). Proof of Lemma 4 is standard: we need only (K), (gen B), and (B/D).

Now we interpret maximally n -consistent sets as possible worlds of level n .

Let us assume that a formula $\mathbf{A}$ with $Lv(\mathbf{A}) = k$ is not an LCB theorem. In this case $\{\neg \mathbf{A}\}$ is consistent.

Using Lemma 1, we construct a k -maximally consistent set w_0^k with $\neg \mathbf{A} \in w_0^k$.

We build $\mathcal{H}^*$ based on w_0^k , and show that formula $\mathbf{A}$ is false in $\mathcal{H}^*$.

We define the following sequence:

Definition 8 (sequence $W^k, W^{k+1}, \dots$)

(s1) We construct W^k so that:

- (a) $w_0^k \in W^k$;
- (b) $\forall u^k \in W^k (\mathcal{DA} \in u^k \implies \exists s^k \in W^k (\{A\} \cup \{D: BD \in u^k\} \subseteq s^k, \text{ and } s^k \text{ is a maximally } k\text{-consistent set}))$;

(s2) W^{n+1} is such that:

- (a) $\forall w^n \in W^n \exists w^{n+1} \in W^{n+1} (\{A: A \wedge \neg CA \in w^n\} \cup \{\neg E: E \wedge CE \in w^n\} \subseteq w^{n+1}, \text{ and } w^{n+1} \text{ is maximally } n+1\text{-consistent})$;
- (b) $\forall u^{n+1} \in W^{n+1} (\mathcal{DA} \in u^{n+1} \implies \exists s^{n+1} \in W^{n+1} (\{A\} \cup \{E: BE \in u^{n+1}\} \subseteq s^{n+1}, \text{ and } s^{n+1} \text{ is maximally } n+1\text{-consistent set}))$.

Correctness of this definition is guaranteed by Lemmas 1, 2, 3, and 4.

Conditions (s1) and (s2) from Definition 8 guarantee that

for $w^n \in W^n$ with $n \geq Lv(\mathbf{A})$: w^n is a maximally n -consistent. ($n - \max$)

Now we define:

$$\mathbf{b} = \langle L, W, \mathbf{d}, \mathbf{u}, \{\mathbf{B}^n\}_{n \in L} \rangle,$$

where

1. $L = \{k, k+1, k+2, \dots\}$ is the set of language levels,
2. $W = \bigcup_n W^n$,

² Logic LCS4 is an essential extension of LC by $\Box$ operator which expresses unchangeability. One of LCS4 axioms is $\Box A \rightarrow \neg \mathcal{C}\Box A$.

3. $d(n) = W^n$,
4. $u : W \rightarrow W$ is such that

$$\{A : A \wedge \neg CA \in w^n\} \cup \{\neg E : E \wedge CE \in w^n\} \subseteq u(w^n) \quad (u^b)$$

and $u(w^n)$ is maximally $n + 1$ -consistent.

5. $B^n \subseteq d(n) \times d(n)$ is such that

$$wB^nu \iff \{A : BA \in w\} \subseteq u \text{ and } w, u \text{ are maximally } n - \text{consistent.} \quad (B^b)$$

Lemma 5 *In the structure **b** conditions (d1) and (d2) are fulfilled.*

Proof From the definition of sequence $W^k, W^{k+1}, \dots$ we know that at every level $n \geq k$ there is at least one possible world. As a result of our expanded language, we obtain that no possible world w^n can occur on more than one level; w^n is maximally n -consistent, thus it occurs only at level n . Therefore conditions (d1) and (d2) are fulfilled in **b**. $\square$

Lemma 6 *In structure **b** condition (u) is fulfilled.*

Proof The definition (u^b) guarantees that any w^n is maximally n -consistent, and $u(w^n)$ is maximally $n + 1$ -consistent, and therefore $u(w^n) \in d(n + 1)$, and thus condition (u) is fulfilled. $\square$

Lemma 7 *In structure **b** condition (B) is fulfilled.*

Proof First, note that for any B^n , if wB^nu , then w , and u are maximally n -consistent and therefore $w, u \in d(n)$. For any B^n , because of (D), (4), (5), and (B^b) , it is the case that B^n is serial, transitive and Euclidean. This can be proved in the standard way. $\square$

Lemma 8 *In structure **b** condition (duB) is fulfilled.*

Proof Let us take any w^n, v^n , such that (a) $w^nB^nv^n$ and (b) it is not true that $u(w^n)B^{n+1}u(v^n)$. We obtain: (a1) $\forall A (BA \in w^n \rightarrow A \in v^n)$, and there exists formula F^* such that (b1): $BF^* \in u(w^n)$ and (b2): $F^* \notin u(v^n)$.

Firstly we prove that $\neg B((F^* \rightarrow \neg CF^*) \wedge (\neg F^* \rightarrow C\neg F^*)) \in w^n$ by indirect proof. From $B((F^* \rightarrow \neg CF^*) \wedge (\neg F^* \rightarrow C\neg F^*)) \in w^n$ and (a1) we obtain $(F^* \rightarrow \neg CF^*) \in v^n$ and $(\neg F^* \rightarrow C\neg F^*) \in v^n$, thus by classical logic we have $(F^* \rightarrow F^* \wedge \neg CF^*) \in v^n$ and $(\neg F^* \rightarrow \neg F^* \wedge C\neg F^*) \in v^n$. Now we have $F^* \wedge \neg CF^* \in v^n$ or $\neg F^* \wedge C\neg F^* \in v^n$ from maximal consistency of v^n ($F^* \in v^n$ or $\neg F^* \in v^n$), thus $F^* \in u(v^n)$, but it is false (b2) and therefore $B((F^* \rightarrow \neg CF^*) \wedge (\neg F^* \rightarrow C\neg F^*)) \notin w^n$, i.e. $\neg B((F^* \rightarrow \neg CF^*) \wedge (\neg F^* \rightarrow C\neg F^*)) \in w^n$.

Secondly we prove $\neg BCF^* \in w^n$ by indirect proof. From $BF^* \in u(w^n)$ and (b1), we get $\neg CBF^* \in w^n$. Therefore $BF^* \wedge \neg CBF^* \in w^n$. Using (CB2) we obtain $B\neg CF^* \in w^n$. From (a1) and $B\neg CF^* \in w^n$ we get $\neg CF^* \in v^n$. We also have $F^* \in$

v^k (from assumption $\mathcal{B}F^* \in w^n$ and (a1)). Now because $F^* \wedge \neg \mathcal{C}F^* \in v^n$, we obtain that $F^* \in u(v^n)$, but it is false (b2), therefore $\mathcal{B}F^* \notin w^n$ and then $\neg \mathcal{B}F^* \in w^n$.

We have $\neg \mathcal{B}F^* \in w^n$ and $\neg \mathcal{B}((F^* \rightarrow \neg \mathcal{C}F^*) \wedge (\neg F^* \rightarrow \mathcal{C}\neg F^*)) \in w^n$, thus from axiom (CB1) we have $\neg \mathcal{C}\neg \mathcal{B}F^* \in w^n$. Therefore from $\neg \mathcal{B}F^* \wedge \neg \mathcal{C}\neg \mathcal{B}F^* \in w^n$ we get $\neg \mathcal{B}F^* \in u(w^n)$ which contradicts assumption (b1). $\square$

We are now ready to prove our main theorem.

Theorem 2 (Completeness) *If $\mathbf{A}$ is logically true, then $\mathbf{A}$ is an LCB theorem.*

Proof Assume that formula $\mathbf{A}$ with $Lv(\mathbf{A}) = k$ is not an LCB theorem. Using Lemma 1 we can construct a k -maximally consistent set w_0^k with $\neg \mathbf{A} \in w_0^k$. Using sequence $W^k, W^{k+1}, \dots$ we define:

$$\mathbf{b} = \langle L, W, \mathbf{d}, u, \{\mathbf{B}^n\}_{n \in L} \rangle.$$

Because of Lemmas 5, 6, 7, and 8, conditions (d1), (d2), (u), (B), and (duB) are fulfilled in $\mathbf{b}$.

Using $(n\text{-max})$, (u^b) , and (B^b) , we obtain that for every w^n and for every formula B , and formula D of level n , we have: $B \in w^n \iff \neg B \notin w^n$ and also $B \wedge D \in w^n \iff B \in w^n$ and $D \in w^n$. Connectives $\vee, \rightarrow, \leftrightarrow$ have standard properties in every w^n .

For every $n \geq k$, $\psi_n^*: \mathbf{d}(n) \rightarrow 2^{A^n}$ is a function such that for each atom α_i^j with $j \leq n$ and every $w^n \in \mathbf{d}(n)$:

$$\alpha_i^j \in \psi_n^*(w^n) \iff \alpha_i^j \in w^n.$$

We now define $\psi^* = \bigcup_n \psi_n^*$ and history $\mathcal{H}^* = \langle \mathbf{b}, \psi^* \rangle$ and prove by induction that for every formula A of an n -language, and every possible world w^n , we have:

$$\mathbf{b}, \psi^*, w^n \models^n A \iff A \in w^n.$$

Given that $\neg \mathbf{A} \in w_0^k$, we obtain $\mathbf{b}, \psi^*, w_0^k \not\models^k \mathbf{A}$. Thus A is false in $\mathcal{H}^*$ and therefore $\mathbf{A}$ is not logically true. $\square$

5 LCB, Agent's Awareness and Temporal Beliefs

As we have already noted, the idea of an expanding language combined with the logical background KD45 allows us to weaken a critical standpoint regarding the principle of negative retrospection and the rule responsible for assigning logical omniscience to the agent. The desire to reduce the effect of logical omniscience, as well as the belief in closure axiom K, were motivations for introducing the notion of *awareness* to logical considerations about knowledge and belief in (Fagin and Halpern 1988). In fact, this approach can also be used in discussions about the limitation of other modal principles used in epistemic versions of other modal logics, and so one can address it to the axiom 5. The general idea of this project is to restrict the agent's beliefs to those propositions that the agent is *aware* of. These propositions are *explicitly* believed. All explicitly believed propositions are also

implicitly believed by the agent, but not conversely. Implicit beliefs are a subject of epistemic versions of the assumed modal logic, while explicit beliefs are restricted to formulas that the agent has in his consciousness. The logic of *general awareness* is the weakest epistemic logic of this kind. In its original formulation it is expressed in a standard propositional language formed out of propositional variables: $p, q, \dots$ (*Prop*), classical truth connectives, and two modal operators, noted here as $\mathcal{B}$ and $\mathcal{A}$ to be read respectively as: *the agent implicitly believes that ...* and *the agent is aware of ...*. The considered logic, which we call **LA**, is based on classical propositional logic. It is characterized by **K**, the rule (*genB*), and the specific axioms introducing the notion of explicit belief

$$\mathcal{B}_e A \leftrightarrow BA \wedge AA. \quad (\mathcal{A}0)$$

The logic **LA** may be extended in two directions. One can strengthen its **K** modal basis, so, e.g., extend it to **KD45**. On the other hand, it may be extended by new specific axioms describing different possible interrelations between $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{B}_e$ (cf. Schipper 2015, 87).

In order to compare the idea of awareness with our formalism, let us add symbol $\mathcal{A}$ to the vocabulary of every n -language. Operator $\mathcal{B}$ is now read as *it is implicitly believed that ...*. **LA** expressed in our expanding language is a sublogic of **LCB** extended by ($\mathcal{A}0$). The strength of $\mathcal{B}$ -fragment of **LCB** compared to **LA** results from additional modal axioms for operator $\mathcal{B}$. In our case, there are accepted axioms of the shapes **D**, **4**, and **5**.

Comparing our formalism and the general awareness approach from the semantic perspective generates some further interesting observations. The **LA** language, in its original formulation, is interpreted in a standard possible worlds semantics (Fagin and Halpern 1988, 52). Following this interpretation, there is considered a tuple $\mathcal{M} = \langle S, \mathcal{A}, \mathcal{B}, v \rangle$, where S is a set of states; $\mathcal{B}$ is a serial, transitive, and Euclidean relation on S ; $\mathcal{A}(s) \subseteq \text{For}$ is the set of formulas that the agent is aware-of in s ; function $v : \text{Prop} \rightarrow 2^S$, from the set of atomic propositions, to the power set of states is a valuation.

Satisfaction is defined as follows:

1. $\mathcal{M}, s \models p$ iff $s \in v(p)$, for any $p \in \text{Prop}$,
2. $\mathcal{M}, s \models BA$ iff $\forall s' \in S (sBs' \implies \mathcal{M}, s' \models A)$,
3. $\mathcal{M}, s \models AA$ iff $A \in \mathcal{A}(s)$,
4. $\mathcal{M}, s \models \mathcal{B}_e A$ iff $\mathcal{M}, s \models AA$ and $\forall s' \in S (sBs' \implies \mathcal{M}, s' \models A)$.

The relation $\models$ fulfills standard conditions for classical connectives.

Let us transfer the main idea of this interpretation to **LCB** semantics. We extend the notion of our $\mathfrak{B}$ -structure to

$$\mathfrak{B}^{\mathcal{A}} = \langle L, W, d, u, \{B^n\}_{n \in L}, \{\mathcal{A}^n\}_{n \in L} \rangle, \quad (\mathfrak{B}^{\mathcal{A}})$$

where every $\mathcal{A}^n$ is a function from $d(n)$ to the power set of the formulas of n -language, i.e. $\mathcal{A}^n(w^n) \subseteq \{A : A \text{ is a formula of } n\text{-language}\}$, for every n .

Definition of a history ψ is now addressed to the $\mathfrak{B}^{\mathcal{A}}$ structure and it is understood in the same way as in Definition 3. Satisfaction is extended by the following condition for every formula A with $lv(A) \leq n$:

$$5. \quad \mathfrak{B}^{\mathcal{A}}, \psi, w^n \models^n \mathcal{A}A \text{ iff } A \in \mathcal{A}^n(w^n).$$

The next condition guarantees the logical validity of (A0):

$$6. \quad \mathfrak{B}^{\mathcal{A}}, \psi, w^n \models^n B_e A \text{ iff } A \in \mathcal{A}^n(w^n) \text{ and } \forall u^n (w^n B^n u^n \implies \mathfrak{B}, \psi, u^n \models^n A).$$

We may now compare the notion of awareness and our idea of expanding language following one of two possible paths. If the conceptual state of the agent is to be identified with his awareness, then it is natural to accept that for every A with $lv(A) \leq n$: $A \in \mathcal{A}^n(w^n)$, and so from (6) and (5), we reach the equivalence of the notions of explicit and implicit beliefs, that is, the distinction between them trivializes. The second possible solution is to decide that there can be some n -formulas which are outside of the agent's awareness in n . In this case, we keep the distinction between n -state of conceptual development of the agent and his awareness. This way of speaking could give a new perspective on epistemic considerations, especially if we consider such a formalization to describe the agent's internal logic in connection with the external logic of his wiser observer. Here we wish only to flag such an issue based on linking our concept of an expanding language and the awareness approach. We leave it open in our work.

Let us look at one more comparison of LCB logic, which may be interesting from the perspective of the relation between modalities of belief and change. Because of the relationship between LC and the linear temporal logic LTL, it is quite natural to compare LCB with certain epistemic extensions of LTL studied in, (Halpern et al. 2004). We focus on two one-agent systems based on LTL, characterizing the notion of knowledge as S5 modality, with axioms expressing the so-called *principle of perfect recall* and *principle of no learning* (cf. Halpern et al. 2004, 681–683). These latter principles are expressed with the use of the temporal operator $\bigcirc$ for *next, it is the case that ...* and the epistemic operator $\mathcal{K}$, which is to be read as *the agent knows that ...*. They are of the following shapes:

$$\mathcal{K} \bigcirc A \rightarrow \bigcirc \mathcal{K}A, \quad (\text{KT2})$$

$$\bigcirc \mathcal{K}A \rightarrow \mathcal{K} \bigcirc A. \quad (\text{KT5})$$

In distributive systems (see Fagin et al. 1995, 109–122) investigated in (Halpern et al. 2004), axiom (KT2) is understood in such a way that *the agent's state at time $m + 1$ contains at least as much information as his state at time m* (c.f. Fagin et al. 1995, 136). We can say that the agent can only reject any possibilities that he previously considered. In turn, (KT2) and (KT5) taken together state that the agent does not learn, because all possibilities he has been taking into account are constantly the same.

Our exposition is based on the fact that the logic of change LC formulated in the language enriched by the operator $\bigcirc$ with definition

$$\bigcirc A \leftrightarrow (A \leftrightarrow \neg CA) \quad (\bigcirc/C)$$

is equivalent to $\bigcirc$ -fragment of LTL extended by the equivalence

$$CA \leftrightarrow (A \leftrightarrow \neg \bigcirc A). \quad (C/\bigcirc)$$

The proof of this fact may be reconstructed on the basis of comparative considerations of LCS4 and LTL, discussed in (Świątorzecka and Czermak 2015, 520–523).

Following notation used in (Halpern et al. 2004), we name the extension of LTL, describing epistemic modality $\mathcal{K}$ by axioms of S5 as $S5^U$, and we consider $\bigcirc$ -fragments of $S5^{U+KT2}$ and $S5^{U+KT2+KT5}$. Expectations addressed to knowledge are now reduced to the so-called *weak S5* requirements characterizing beliefs. Logic, we mean is $\bigcirc$ -fragment of LTL, with KD45 axioms for $\mathcal{B}$, called here $KD45^\bigcirc$.

We focus on the following $\mathcal{B}$ -versions of KT2 and KT5:

$$\mathcal{B} \bigcirc A \rightarrow \bigcirc \mathcal{B} A, \quad (K2)$$

$$\bigcirc \mathcal{B} A \rightarrow \mathcal{B} \bigcirc A. \quad (K5)$$

On our approach, the way in which we understand the above implications remain in contrast with the intuitions mentioned in the case of (K2) and (K5).

Logic $LCB + (\bigcirc/C)$ expressed in a standard language of propositional variables is equivalent to $KD45^\bigcirc + (K5) + (C/\bigcirc)$.

It should be noted that $\mathcal{B}$ -version of the perfect recall postulate is not theorem of LCB. Let us take $At^1 = \{\alpha_1^1, \alpha_2^1\}$, $At^2 = At^1 \cup \{\alpha_1^2\}$, and $\mathfrak{B}^*$ such that $\{w, u, w', u', v\} \subseteq W$, $d(1) = \{w, u\}$, $d(2) = \{w', u', v\}$, $u(w) = w'$, $u(u) = u'$, and $w\mathcal{B}^1 u$, $w'\mathcal{B}^2 u'$, $w'\mathcal{B}^2 v$. We take $\psi^*(w) = \{\alpha_1^1\}$, $\psi^*(u) = \{\alpha_1^1, \alpha_2^1\}$, $\psi^*(w') = \{\alpha_1^1, \alpha_2^2\}$, $\psi^*(u') = \{\alpha_2^1, \alpha_2^2\}$, $\psi^*(v) = \{\alpha_1^1, \alpha_2^1\}$. In this case $\mathfrak{B}^*, \psi^*, w \not\models^1 \mathcal{B} \bigcirc \alpha_2^1 \rightarrow \bigcirc \mathcal{B} \alpha_2^1$ and it falsifies (KT2).

Concerning schema (KT5), we claim that it does not describe the situation wherein the agent is not learning. Because the language is expanding, the agent can always discover new possibilities (new states). In the actual state of the agent's beliefs, he has access to updates of his previous beliefs. We could say that the agent adapts his possible worlds to the current state of his belief expressed in the current language.

Finally, it is worth noting that the addition of (KT2) to LCB would narrow the class of $\mathfrak{B}$ structures to those that meet the condition

$$w^{n+1} \mathcal{B}^{n+1} v^{n+1} \implies \exists_{w^n, v^n} (w^n \mathcal{B}^n v^n \text{ and } u(w) = w^{n+1} \text{ and } u(v) = v^{n+1}).$$

In our semantics, this would mean that the agent cannot consider new possible states, despite the increasingly richer language at his disposal. In other words, the agent could not take advantage of the idea of an expanding language.

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