



# The Parthood of Indiscernibles

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## Abstract

In the following work we propose to incorporate the main feature of quantum mechanics, i.e., the concept of indiscernibility. To achieve this goal, first we present two models of set theories: a quasi-set theory (QST) and a *non-antisymmetric mereology* (NAM). Next, we show how specific objects of QST—*m-atoms*—can be defined within NAM. Finally, we introduce a concept of a parthood of indiscernibles and discuss its features in respect to standard notions of indiscernibles and within NAM.

**Keywords** Quasi-set theory · Mereology · Indiscernible objects · Non-identity · Quantum entanglement

## 1 Introduction

For the last decades the questions of quantum indistinguishability and quantum particle number have been widely investigated, not only by philosophers, but also by physicists and mathematicians. Different formal theories trying to explain various aspects of quantum mechanics (QM) were developed. Several works regard non-reflexive logics (Da Costa 1980; Da Costa and Bueno 2009); there are also efforts to incorporate quantum indistinguishability in set theories, in particular in classical ZFC set theory (French and Krause 2006; Dalla Chiara and Di Francia 1995; Dalla Chiara et al. 1998; Santorelli and Sant’Anna 2005; Krause 2003). One important proposal, Decimo Krause, *Quasiset Theory*—(QST) (Devlin 1993; Kunen 2009)—is a formal framework that incorporates indistinguishability of quantum particles.

In quantum mechanics, in general, particles cannot be considered as individuals (French and Krause 2006). There are particles which are intrinsically indistinguishable, i.e., they are ontologically indiscernible. To describe them we need a language, but quantum mechanics does not possess its own language; it instead uses

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instruments of classical mathematics based on ZFC set theory. Hence, QM applies concepts such as individuals (in ZFC—*ur-elemente*) understood in a classical sense. The point is—as the Russian mathematician, Manin suggested, and what Da Costa and Holik (2015) underline—to search for set theories which inspire its concepts in the quantum domain (Manin 1977). QST can be an answer to this request.

QST was developed by Decimo Krause. Krause (1992) tried to represent an idea of quantum indistinguishability within classical set theory. He enlarged ZFC set theory by new objects—*m-atoms*—which could correspond to indiscernible objects. Therefore, within QST, we find two sorts of objects: *ur-elements*, i.e., individuals characteristic of ZFC set theory; and new *m-atoms*, representing singlet states, triplets, etc., i.e., collections of indiscernible particles. However, this version of QST is not sufficient. As Domenech and Holik (2007) showed it could not capture one of the most important features of mathematics and quantum non-individuality: the quantum particle number. There are quantum systems for which a particle number is not well defined.

In 2009 Krause proposed an improved version of QST (French and Krause 2009) in which equality (identity) is not a primitive concept; there exists some kinds of objects for which only an indistinguishability relation holds (*m-atoms*). Therefore, in new QST non-individuality is introduced by an equivalence relation of indiscernibility, and equality is introduced in respect to this relation. In this way, within QST it has no meaning whether some entities are equal or not. However, such a perspective can create problems. If we consider two entangled particles, it was proved that we are able to distinguish one particle from another in a pair by their spin: one particle has a spin up, another particle has a spin down. Therefore, some kind of non-identity or weak identity should be introduced. We know that such a perspective is in contradiction with Schrödinger's radical position that elementary particles are not individuals (Schrödinger 1998), but nonseparability of quantum states forces an opposite approach.

Moreover, QST includes a concept of quantum particle number called *quasicardinality*. However, it can be shown (Wajch 2018) that two different quasicardinals can be assigned to one quaset, which contradicts the idea of a cardinal number. Therefore, QST needs further improvements.

Da Costa and Holik (2015) proposed another framework of set theory that puts together pieces of mereology and a standard ZFC set theory. The goal of this proposal was to introduce a concept of undefined particle number in quantum mechanics. The authors define a physical object—as understood by Leśniewski and for which a mereology was developed (Surma et al. 1992)—as an object which is not a set, but they must have in mind a classical, distributive set because in mereology every object is a set and vice versa. Hence, this assumption contradicts foundations of mereology in which a physical object must be a collective class. Therefore, their physical object is not a physical object as conceived by Leśniewski. Moreover, since a physical object is a distributive class, they define a concept of *cantorian*, i.e., they claim that a physical object is cantorian if all of its parts form a distributive set. And since we deal with distributive sets by the Axiom of Choice, they assign to it a cardinal number. This is true for distributive sets, but it is not true for mereological sets. However, Da Costa and Holik (2015) put together a concept of cantorian with a physical object; they define a physical object which is cantorian. The advantage of mereology is that allows multiple

ways of conceiving objects. For example, let us take a circle:  $C = \{\theta : 0 \leq \theta < 2\pi\}$ . Mereologically, it can be considered as a set composed of two halves:  $S_1 = \{\theta : 0 \leq \theta < \pi\}$ ,  $S_2 = \{\theta : \pi \leq \theta < 2\pi\}$ ; therefore,  $C_1 = \{S_1, S_2\}$  or as a set composed of four quarters; and  $C_2 = \{Q_1, Q_2, Q_3, Q_4\}$ , where:  $Q_1 = \{\theta : 0 \leq \theta < \frac{\pi}{2}\}$ ,  $Q_2 = \{\theta : \frac{\pi}{2} \leq \theta < \pi\}$ ,  $Q_3 = \{\theta : \pi \leq \theta < \frac{3\pi}{2}\}$ ,  $Q_4 = \{\theta : \frac{3\pi}{2} \leq \theta < 2\pi\}$ . In the case of halves, this set has two elements:  $S_1, S_2$ ; in the case of quarters, the given set has four elements:  $Q_1, Q_2, Q_3, Q_4$ . Therefore, in mereology, it is impossible to assign a cardinal number to any mereological set. Even if we assume an atomic structure, it is always possible to present a given object as an aggregate composed of different number of elements. We can investigate a case, whether a maximal decomposition of a whole is possible, but it is a totally different case to study; hence, putting together a concept of cantorians with physical objects has no sense.

Besides the problem of nonindividuality for quantum particles, the validity of the principle of identity of indiscernibles (PII) in QM is also questioned (French and Krause 2006; French and Redhead 1988; Butterfield 1993). In the *Stanford Encyclopedia of Philosophy*, the following definition of PII is presented:

$$\forall_\alpha (\alpha(x) \iff \alpha(y)) \implies x = y. \quad (1)$$

Therefore, if for every property  $\alpha$ , an object  $x$  has  $\alpha$  if and only if an object  $y$  has  $\alpha$ , then we can state that  $x$  is identical to  $y$ .

Moreover, we can also find the converse definition, i.e.:

$$x = y \implies \forall_\alpha (\alpha(x) = \alpha(y)). \quad (2)$$

Additionally, the conjunction of both definitions (also known as *Leibniz's Law*) we find in foundations of mathematics (in ZFC, ZF, etc.) expressed by the *Extensional-ity Principle* (EP), i.e., for any sets  $A$  and  $B$ :<sup>1</sup>

$$A \subseteq B \wedge B \subseteq A \iff A = B. \quad (3)$$

Hence, it is not possible for two different individuals to possess all the same properties. As French and Redhead (1988) state, if quanta were not individuals, PII would not be either true or false, but simply inapplicable, which is the case in a non-standard mereology, i.e., a mereology without antisymmetry (NAM) presented in Sect. 3.

Finally, some authors also state that indistinguishable particles are not utterly indiscernible, but obey a weaker form of discernibility; they have in mind a weak form of discernibility (Saunders 2006, 1995). In Muller and Saunders (2008), Muller and Seevinck (2009) and Muller (2015), we find out that this weak form of discernibility is achieved by a relational symmetric and non reflexive relation between the relata.<sup>2</sup> Most of these authors are criticized because of a non-physical approach. Perhaps some justification of weak discernibility can be found within NAM, since in

<sup>1</sup> The extensionality principle can also be expressed in terms of the membership relation:  $\forall_z (z \in A \iff z \in B) \implies A = B$ .

<sup>2</sup> As Domenech and Holik (2007) observe, in various works we can find different scales of discernibility applied in the standard model of QM and philosophical problems linked to proposed definitions, e.g. Caulton and Butterfield (2012a), Caulton and Butterfield (2012b), Ladyman and Bigaj (2010) and Ladyman and Pettigrew (2012).

this model supplementation principles change their character, i.e., a strong supplementation principle becomes weaker (logically) than a weak supplementation principle, as it was shown in Obojska (2013a).

Therefore, this current paper proposes a non-standard mereology which can be used to describe some quantum phenomena, like entangled states. Yet Einstein et al. (1935) observed this strange quantum phenomena, and began a rich discussion about the nature of quantum mechanics that continues into our own time. The proposed non-standard mereology differs from the work done by Da Costa and Holik (2015) since the authors present a framework in which they incorporate aspects of mereology into ZFC set theory. Instead, this work presents a mereology without antisymmetry, a mereology that can describe quantum entanglement. Moreover, we incorporate classical notions of ZFC set theory by the use of an algebraic, unary operator. In this way, ZFC is no longer a basic theory, but it becomes a theory created on the basis of non-standard mereology.

Hence, in Sect. 2 we will outline basic notions of quasisets and indistinguishability that are characteristic for QST. Section 3 will contain a mathematical framework of NAM. Section 4 will propose how to derive a concept of indiscernibility of quantum particles within NAM.

## 2 Quasisets and Indiscernible Objects of QST

To introduce basic notions of quasi-set theory, we will use the language of second order logic with classical logical connectives and rules. Moreover, QST—as presented by Krause (1992)—is a theory based on ZFU—like axioms, i.e., ZFC axioms that assume existing ur-elements (atoms).

Within the first version of QST (Krause 1992) we find two new predicates for the variable  $x$ , i.e.,  $m(x)$  denotes that  $x$  is an  $m$ -object—an object composed of a finite number of indiscernible objects; and  $M(x)$ — $x$  is an  $M$ -object—a typical ur-element of ZFC set theory (Kunen 2009).

The Axiom of Indistinguishability is defined as follows:

**Axiom 2.1**  $\forall_x (x \equiv x),$   
 $\forall_{x,y} (x \equiv y \implies y \equiv x),$   
 $\forall_{x,y,z} (x \equiv y \wedge y \equiv z \implies x \equiv z),$   
 $\forall_{x,y} (\neg m(x) \wedge \neg m(y) \implies (x \equiv y \implies (A(x, x) \implies A(x, y))))).$

where  $\equiv$  is an equivalence relation, and  $A(x, x)$  is a formula and  $A(x, y)$  arises from  $A(x, x)$  by replacing some, but not necessarily all, free occurrences of  $x$  by  $y$ , provided that  $y$  is free from  $x$  in  $A(x, x)$ .

The equivalence relation applied in Axiom 2.1 is a relation that divides the universe of objects into equivalence classes. Within equivalence classes objects are considered to be equal in respect to this equivalence relation. Example: let us take the operation of modulo 2 over the integer numbers. We will obtain two equivalence

classes: (A) composed of odd numbers; (B) composed of even numbers. Within (A) all numbers are considered to be equivalent since the rest of the division by 2 is equal to 1; for (B) the rest of the division by 2 is equal to 0.

The second part of Axiom 2.1 states that, if we have two objects  $x, y$ , which are not  $m$ -atoms, and they are equivalent, they may differ in some feature. Example: let us take the numbers  $x = 15$  and  $y = 25$  from the equivalence class  $[A]$ , and let the formula  $A(x, x)$  be defined as follows:  $x$  is an integer number, and  $x$  has 5 as a divisor. We can observe that  $A(x, x)$  and  $A(x, y)$  are true,  $x \equiv y$  and  $x \neq y$ .

Quasi-sets are defined as composed objects which contain entities of “mixed” nature, i.e., in a quasi-set we may find both  $ur$ -elements or indistinguishable objects:

**Definition 2.1**  $Q(x) =_{df} \neg(m(x) \vee M(x))$ .

Therefore, nothing is at the same time a quasi-set and an atom ( $M$ -atom or  $m$ -atom).

The identity relation is introduced in the following way:

**Definition 2.2**  $x = y =_{df} \neg m(x) \wedge \neg m(y) \wedge x \equiv y$ .

Hence, identity holds for the objects which are not  $m$ -atoms, and in this case it coincides with the concept of indistinguishability.

In 2009 a revised QST was presented (French and Krause 2009). New modifications follow from the fact that classical identity does not hold for  $m$ -atoms—the intuition underlined by Schrödinger (1998). Therefore, this modified version is a theory without identity; in its place, the extensional identity is introduced:

**Definition 2.3**  $(x =_E y) =_{df} [Q(x) \wedge Q(y) \wedge \forall_z (z \in x \iff z \in y)] \vee [M(x) \wedge M(y) \wedge \forall_{Q(z)} (x \in z \iff y \in z)]$ .

Hence, if two objects are quasi-sets, they are extensionally equal if and only if classical extensionality holds for them, i.e., they are uniquely determined by their elements (they have the same elements). This means that they can have collections of indiscernible particles as elements, but they are considered as equal. Therefore, in some way the identity of elements is “hidden” inside this definition. The second part of Definition 2.3 expresses an extensional identity for  $M$ -atoms— $ur$ -elements—and states that if any quasi-set has some  $x$  as an element, it must contain its extensionally equal element.

In this new perspective, the indistinguishability axiom is defined in the following way:

**Axiom 2.2**  $(\equiv_1) \forall_x (x \equiv x),$   
 $(\equiv_2) \forall_{x,y} (x \equiv y \implies y \equiv x),$   
 $(\equiv_3) \forall_{x,y,z} (x \equiv y \wedge y \equiv z \implies x \equiv z),$   
 $(\equiv_4) \forall_{x,y} (x =_E y \implies \alpha(x) \implies \alpha(y)).$

### 3 Foundations of Non-antisymmetric Mereology

#### 3.1 Some Remarks on Classical Mereology

Mathematics has two main set theories: ZFC set theory based on Cantor's concept of a set (Enderton 1977; Devlin 1993); and mereology, based on Leśniewski's concept of a set (Surma et al. 1992). Mereology is also called a collective set theory, and it is a theory where objects are assumed to be real (physical objects); therefore, it is a theory that describes physical phenomena. The invention of mereology was the answer for the set theory paradox discovered by B. Russell in the basis of mathematics. It turned out that apparently casual operations on certain notions, e.g. the notion of a "set" or a "class", may lead to contradictions.

In comparison to ZFC set theory, mereology opens a totally new perspective of conceiving sets. In mereology, first we consider a set as a whole (therefore, we have sums of elements); only from the perspective of the whole do we then distinguish elements. In the above Introduction the abstract example of a circle could correspond, for example, to the problem of division of a cake in shape of a circle. We have shown that a circle can be conceived as a set composed of halves or a set composed of quarters, or yet something different.

Since in mereology the concept of a class (or a set) is synonymous with the concept of sum, in the case of a circle we will have  $C_1 = C_2$ , i.e.,  $\{S_1, S_2\} = \{Q_1, Q_2, Q_3, Q_4\}$ . But in ZFC set theory, since first we take elements, and then form an abstract whole, we have  $C_1 \neq C_2$ , i.e.,  $\{S_1, S_2\} \neq \{Q_1, Q_2, Q_3, Q_4\}$ , by the extensionality principle (3). In fact,  $Q_3 \notin C_1$ . Thus, the novelty of mereology provides a new way of conceiving objects: there are wholes formed of parts—proper or improper<sup>3</sup>; and there are wholes which can be divided into parts in different ways, as exemplified by the decay of a physical particle *meson*  $\pi$  (Patrignani et al. 2016, 2017).

The extensionality principle is crucial for ensuring uniqueness of sum of elements of a given set either in ZFC set theory or in mereology. The extensionality principle describes the conditions under which two sets are considered to be equal. In ZFC this axiom states that a set is uniquely determined by its elements. Hence, a collection containing  $10^6$  elements is different from that containing  $10^6 + 1$  elements.

Also a mereology has an axiom establishing conditions under which two mereological sets can be considered equal. It is called a *Mereological Extensionality Axiom* (MEA), and it is slightly different from the ZFC extensionality principle. Formally, MEA is expressed as follows:  $\forall_z (z \subset x \iff z \subset y) \implies x = y$ .<sup>4</sup> It states that two objects having all the same proper subsets are equal. Let us come back again to the example of a circle, and let  $x = C_1$  and  $y = C_2$ . Since MEA must hold for each proper

<sup>3</sup> Leśniewski called improper parts "ingrediens".

<sup>4</sup> This axiom is defined for the relation of proper inclusion, and it is valid for complex objects, i.e., objects having at least two subsets. Moreover, we adopted the term 'subset' for a mereological part, and the inclusion relation for the mereological relation of ingrediens, because both have the same features as their mereological correspondences, respectively.

subset of  $x$ , we have two such subsets:  $S_1$  and  $S_2$ . We have  $S_1 \subset C_2$ , because  $S_1 \subset C_1$ ; and mereologically  $\{Q_1, Q_2, Q_3, Q_4\} = Q_1 \cup Q_2 \cup Q_3 \cup Q_4 = C_2 = C_1$ ; hence,  $S_1 \subset C_2$ . Analogously, for  $S_2$ . In the opposite direction, i.e.,  $\forall_z (z \subset y \implies z \subset x)$ ; this statement must be true for all subsets  $Q_i, i = 1, \dots, 4$ . Let us take  $Q_1$ .  $Q_1 \subset S_1$ , and by transitivity of  $\subset$ , since  $S_1 \subset C_1$ , we have  $Q_1 \subset C_1$ . Analogously for  $Q_2, Q_3$  and  $Q_4$ . In conclusion, mereologically,  $C_1 = C_2$ . Moreover, we can observe that MEA is conserved also for ZFC sets; in fact,  $C_2$  has different subsets than  $C_1$ , therefore  $C_1 \neq C_2$ .

This difference between the ZFC extensionality axiom and the mereological extensionality axiom follows from the fact that a mereological set is not uniquely determined by its elements. In Obojska (2013b) it was shown that the mereological extensionality axiom follows from the antisymmetry of the inclusion relation  $\subseteq$ . Hence, it is the antisymmetry property for the inclusion relation that is crucial for both theories, for ZFC set theory and mereology.

Antisymmetry is a property which constitutes part of the extensionality principle. It is expressed as follows:  $A \subseteq B \wedge B \subseteq A \implies A = B$ , and it is necessary for the mathematical concept of order. In a way antisymmetry “glues” objects in respect to the investigated relation. For example, the symmetry for the inclusion relation  $A \subseteq B \wedge B \subseteq A$  causes the equality of sets  $A$  and  $B$ . Therefore, antisymmetry excludes symmetry. If we want to maintain symmetry, we have to reject antisymmetry—the purpose of our developing a mereology without antisymmetry.

In a model of non-antisymmetric mereology presented in Obojska (2013b), the relation of division could become a fundamental relation since it is pre-ordering, i.e., it is reflexive and transitive. We chose the division relation because it so naturally corresponds to real phenomena. In fact, in science, we often speak about a division of cells, decomposition of waves into sums of *sine* and *cosine* functions, etc. Moreover, a division relation has the same features as the mereological ingredient (improper part) relation without antisymmetry.

Finally, Clay and Loeb showed that a model of classical mereology with an order relation and two additional postulates forms a model isomorphic to a Boolean algebra without a null element (Clay 1974; Loeb 2014). Likewise, a non-antisymmetric mereology could become a classical model of mereology if we define an order relation in terms of the new ingrediens relation, thus yielding a classical algebraic structure, which will be presented in the next section.

### 3.2 Non-antisymmetric Mereology

Let  $M$  be a space of physical objects, which are collective sets (classes), and  $\sqsubseteq$  the relation of ingrediens fulfilling the following two axioms:

$$\forall_{x \in M} (x \sqsubseteq x), \quad (\text{R})$$

$$\forall_{x, y, z \in M} (x \sqsubseteq y \wedge y \sqsubseteq z \implies x \sqsubseteq z). \quad (\text{TR})$$

Therefore, the relation of ingrediens is reflexive and transitive; it is pre-ordering.

In classical mereology (EM)<sup>5</sup> the antisymmetry principle holds, but our proposal rejects antisymmetry. This means that in our mereological space there exist objects for which reciprocal parthood holds, and they are different. A singlet state of quantum particles with the relation of division, which is preordering, falls under this postulate (Obojska 2018).

Additionally, in EM, the relation of  $\sqsubseteq$  is reflexive, transitive and antisymmetric; therefore, it is an order relation; let us denote it by  $\subseteq$ . The pair  $(M, \subseteq)$ , together with two additional axioms forms a model of classical mereology.<sup>6</sup> Since in our model  $\sqsubseteq$  is not an ordering, but pre-ordering, we have to introduce order in another way to create a model of EM isomorphic to  $(M, \subseteq)$ . In this way we will be allowed to apply all statements of classical mereology.

If the relation of mereological Sum is defined in terms of  $\sqsubseteq$ , the rejection of antisymmetry causes the loss of the uniqueness of mereological Sum:

**Definition 3.1**  $a \text{ Sum } x =_{df} \forall_{y \in x} y \sqsubseteq a \wedge \forall_{z \in U} (z \sqsubseteq a \implies \exists_w (w \varepsilon x \implies w \circ z))$ .<sup>7</sup>

**Example 3.1** Let  $x = [a, b, c, d]$  be a collective class,<sup>8</sup> and  $a \sqsubseteq b$ ,  $b \sqsubseteq a$ ,  $a \sqsubseteq c$ ,  $b \sqsubseteq c$ ,  $a \sqsubseteq d$ ,  $b \sqsubseteq d$  and  $c \sqsubseteq d$ ,  $d \sqsubseteq c$ . Moreover,  $a \neq b$  and  $c \neq d$ . We have:  $c \text{ Sum } x$  and  $d \text{ Sum } x$ .

Hence, we can observe that the above relation of Sum is a kind of an upper bound. We need the smallest  $a$  fulfilling Definition 3.1 if we would like to have the classical concept of sum corresponding to the totality of elements; therefore, we need an order relation. However, at this stage, we do not want to introduce the order, because doing so we would merely obtain the classical model of mereology. The innovation of this proposed theory depends on delaying the introduction of the order relation; in this way we will obtain a complete pre-ordered structure.

In Obojska (2013a) it was shown that in the absence of antisymmetry, we obtain two different definitions of proper part:

$$x \sqsubset y \iff (x \sqsubseteq y \wedge x \neq y) \quad (\text{PP1})$$

$$x \sqsubset y \iff (x \sqsubseteq y \wedge y \not\sqsubseteq x) \quad (\text{PP2})$$

<sup>5</sup> This abbreviation follows from the fact that classical mereology is extensional, i.e., (EP) holds.

<sup>6</sup> One of these axioms, called a strong supplementation principle, assures the possibility of separating objects:  $\forall_{x,y \in M} (x \not\sqsubseteq y \implies \exists_{z \in M} z \sqsubseteq y \wedge z \sqcup x)$ , where:  $x \not\sqsubseteq y = \neg(x \circ y) = \neg[\exists_{z \in M} (z \sqsubseteq x \wedge z \sqsubseteq y)]$ ; the other axiom assures the existence of sum for any plurality of objects:  $\forall_{\emptyset \neq z} \exists_{x \in M} x \text{ Sum } z$ .

<sup>7</sup> where:  $x \circ y =_{df} \exists_{z \in U} z \sqsubseteq x \wedge z \sqsubseteq y$ .

<sup>8</sup> We will use square brackets to denote a mereological class.



To define an order relation, (PP2) will be applied, since  $\sqsubset$  is transitive, while (PP1) is not. The order relation can be introduced as follows:  $x \leq y \iff (x \sqsubset y \vee x = y)$ .<sup>9</sup> In this way  $(M, \leq)$  forms a model of EM.<sup>10</sup> We can also introduce an ordering in another way, by the use of an equivalence relation, but since we want to maintain the possibility of separating objects, we opted for the proposal presented above.

The advantage of NAM is that it opens totally new possibilities of investigation. Since we do not assume the order axiomatically, but it is introduced by definition, we are able to take into consideration systems where full symmetry or partial symmetry holds. If we look at the formal presentation of the antisymmetry condition, we can observe that antisymmetry excludes symmetry; therefore, its rejection can be highly useful.

Regarding the mereological extensionality principle, in NAM, in general, it does not hold, but since it depends on the definition of proper part, we obtain two different possibilities of its definition. In Obojska (2013a) it was shown that, by the use of (PP2) we obtain an extensional model.

How can distributive sets be defined in our proposed mereological framework? Let us assume that for any two elements of our mereological domain— $M$ , their sum exists.<sup>11</sup> Classically, we introduce the unary operator in the following manner:

$$\forall_{x \in M} \bigsqcup X := (ix) x \text{ Sum } X, \quad (4)$$

where  $\bigsqcup : X \longrightarrow M$ .

The description operator  $(ix)$  originating from Russell is a one-argument name-forming operator interpreted as “the only  $x$  such that...”. Due to the fact that in a non-antisymmetric metrology *Sum* is not uniquely determined, we have to modify this notation. We propose the following:

$$\forall_{X \subseteq M} \bigsqcup X := \{x \in M : x \text{ Sum } X\}, \quad (5)$$

where  $\bigsqcup : X \longrightarrow 2^M \setminus \{\emptyset\}$ .

In this manner, we assign to any collective set exactly one element—a distributive set, as we will see in Example 3.2. For this reason we can use the concept of an empty set in establishing the range of values for  $\bigsqcup$ .

**Example 3.2** Let  $M = \{0, 1, 2, 12, 21, 3\} = [3]$ , where:  $0 \sqsubseteq 1$ ,  $1 \sqsubseteq 0$ ,  $1 \sqsubseteq 2$ ,  $2 \sqsubseteq 12$ ,  $2 \sqsubseteq 21$ ,  $12 \sqsubseteq 3$ ,  $21 \sqsubseteq 3$ , and  $X = \{0, 1, 2\} = [2]$ . We have:  $\bigsqcup [2] = \{2, 12, 21, 3\}$ . If the set  $\{2, 12, 21, 3\}$  were a collective set, then it would be identical to the set  $\{0, 1, 2, 12, 21, 3\}$ , since 0 and 1 are parts of 2, but  $2 \not\sqsubseteq 0$  [by (5)], therefore  $\bigsqcup \{0, 1, 2\} \neq \{0, 1, 2, 12, 21, 3\}$ .

<sup>9</sup> The proof is very elementary, that is why it is not reported, here.

<sup>10</sup> With additional axioms analogous for  $(M, \subseteq)$ , but defined in terms of  $\leq$ .

<sup>11</sup> This assumption is necessary for the unary operation, we would like to introduce, to be always feasible.

**Corollary 3.1** Objects created by the use of the  $\sqcup$  operator are distributive sets.

We can continue by introducing a concept of being a part of a distributive set, etc., but it is not necessary for the purpose of this paper. One can find further developments of NAM in Obojska (2013b).

Finally, we have shown that NAM is an algebraic structure which corresponds to a special type of a BCI algebra<sup>12</sup>; hence, it is governed by a non-classical logic.

## 4 The Parthood of Indiscernibles and Concluding Remarks

As we have seen in Sect. 2, two versions of QST—(Krause 1992; French and Krause 2009) propose totally different points of view regarding the identity between quantum objects. The version from 2009 is a theory without identity; the version from 1992 is a theory with identity. In spite of further developments of the later version of QST, we opt for the first version since it seems more adequate for indiscernible objects, in particular for entangled particles. Our choice is justified because a singlet state is formed of two indiscernible, yet not identical particles; the particles that are non-identical because we are able to state which particle has a spin up, and which particle has a spin down (Horodecki et al. 2007; Danila 2017). Regarding indiscernibility, for entangled particles it is impossible to determine with certainty which is which. If we measure them, the process of measurement influences the results, but the intimate relationship between our particles makes their spin always opposite. Hence, we should introduce a concept of non-identity. It turns out that within NAM, by rejecting antisymmetry, we maintain non-identity, and in this way we can introduce a concept of *m-atom* in the following way:

**Definition 4.1** We will call  $x$  an *m-atom* if there exists  $y$  such that:  $x \sqsubseteq y \wedge y \sqsubseteq x \wedge x \neq y$ ; and there is no other  $z$  such that:  $z \sqsubset x$ .

In the above definition we apply (PP2) statement for the proper part relation. As a consequence, these atoms, which do not have corresponding parts due to Definition 4.1, we will call *M-atoms*. Moreover, if  $x$  is an *m-atom*, then  $y$  is also an *m-atom*, i.e.,  $m(y)$ . Additionally, we can observe that since we assume that  $x \neq y$ , we have non-identical objects.

Because we assumed that  $\sqsubseteq$  is pre-ordering, we can define an equivalence relation as follows:

**Definition 4.2**  $x \cong y \iff_{df} x \sqsubseteq y \wedge y \sqsubseteq x$ .

Hence,  $\cong$  is reflexive, transitive and symmetric, and  $\cong$  divides the whole universe into equivalence classes within which objects are considered indiscernible in respect

<sup>12</sup> The paper is under review.

to this equivalence relation (Birkhoff and Mac Lane 1977). Moreover, due to Definitions 4.1, 4.2 if  $x$  and  $y$  are  $m$ -atoms then they are indiscernible. Hence, when  $x$  is part of  $y$  and vice versa, we can speak of mutual parthood or indiscernibility of parthood. We do not necessarily have to think about a spatial parthood. Such a mutual parthood should be rather assigned to a special relationship which can hold, for example, in the case of quanta on the level of their energy; therefore, this kind of relationship can be linked to non-material and non-spatial properties of physical objects. Finally, Definition 4.1 can be also considered as a mereological representative of Axiom 2.1, and it is a formal expression of the rejection of antisymmetry. In fact,  $\neg(\forall_{x,y} (x \sqsubseteq y \wedge y \sqsubseteq x \implies x = y))$  is equivalent to the statement:  $\exists_{x,y} (x \sqsubseteq y \wedge y \sqsubseteq x \wedge x \neq y)$ .

In the *Stanford Encyclopedia of Philosophy* (SEP), in respect to the identity of indiscernibles, Black (1952) assumes the existence of a universe containing nothing else but two exactly resembling spheres. In such a completely symmetrical universe these two spheres would be indiscernible because there is nothing else that could differentiate them. However, this example would not be good for the case of mutual parts since nothing is known about their mutual parthood; therefore, within NAM these spheres are only symmetric, and they are not indiscernible. Perfect symmetry alone is insufficient for parthood indiscernibility.

Moreover, Della Rocca (2005) invites us to consider another example of many identical collocated spheres, made up of precisely the same parts. Della Rocca rightly contends that there cannot be two or more indiscernible things with all the same parts in precisely the same place at the same time. But let us adopt the example of two entangled electrons. Both have the same properties, and they both have spins equal to  $+1/2$  and  $-1/2$  at the same time. However, at the moment of measurement, they assign a value exactly correlated, i.e., an opposite value of one feature, the spin. Hence, before measurement they are identical: they have the same parts because they are atoms; they are at the same place at the same time; they behave like one particle. Once a measurement influences their behavior, they become two non-identical particles. They become mutually indiscernible, because we cannot determine which is which. Since each of them possesses both spin values, we cannot determine which particle should be assigned a positive spin value.

Therefore, in conclusion, this is the parthood of indiscernibles, i.e., a symmetric and reflexive relationship that makes objects non-identical and indiscernible.

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