



# Rough set approximations based on a matroidal structure over three sets

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## Abstract

Pawlak's classical model of rough set approximations provides an efficient tool for extracting information exactly by employing available knowledge (i.e., known knowledge) in an information system, since many problems in rough set theory are NP-hard and their solution process is therefore greedy and approximate. Many extensions of Pawlak's classical model have been proposed in recent years. Most of them are considered over one or two sets, that is, one- or two-dimensional space or one- or two-dimensional data. Aided by relation-based rough set models, a few of these extensions are considered over three sets. However, the real world is in three-dimensional space. Therefore, it is necessary to solve these problems with other models, such as covering rough set models. For this purpose, we propose the TP-matroid—a matroidal structure over three sets. Employing the family of feasible sets of a TP-matroid as the available knowledge, a pair of rough set approximations—lower and upper approximations—is provided. In addition, for an information system defined over three sets, assisted by formal concept analysis, we establish a pair of rough set approximations. Furthermore, two TP-matroids are established based on the above pair of rough set approximations. The integration between the two pairs of rough set approximations presented here is discussed. The results show that for an information system in three-dimensional space, the rough set approximations provided here can effectively explore unknown knowledge by using available knowledge based on the family of feasible sets of a TP-matroid.

**Keywords** Rough set approximations · TP-matroid · Covering · Three sets · Semiconcept

## 1 Introduction

Rough set theory, proposed by Pawlak [1, 2], addresses the vagueness and uncertainty of data tables. Its basic operators are known as lower and upper approximations. Pawlak's classical rough set approximations are defined by a partition of a universe (i.e., a nonempty set) [1, 2], which restricts the applications of rough sets in real cases. Many researchers have generalized Pawlak's classical rough set model based on more general binary relations [3–8], by employing coverings [4, 8–12], or by combining the model with other theories such as matroid theory [13–17] and others [18–33].

Moreover, Pawlak's classical model is also restricted by the number of universes, which is one. Hence, another interesting type of generalization of Pawlak's classical rough set model is to extend the single universe to more than one universe, which has become a very popular topic in recent years and has yielded fruitful results [34–40]. Among them, it is worth mentioning that based on relations, Sun and Ma [36] generalized Pawlak's classical rough set model from one universe to not only two but three

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universes and considered further multi-universe cases for fuzzy rough sets, even infinite universes. For relation-based fuzzy rough sets, the model in [36] is perfect. However, now, with respect to covering-based rough sets over multiple universes, there are few articles with results as good as those of [36], although some achievements have been made for two different universes [41, 42]. There are differences and connections between the two rough set models—relation-based and covering-based [8]. Therefore, it is necessary to consider generalizing Pawlak's classical rough set model from one to three sets from the perspective of the covering rough set model.

The achievements of rough sets in application fields are low-hanging fruit in many domains [23, 36, 43–55]. They show that the demands of practical use in many real-life fields are one of the driving forces promoting the development of rough set theory.

Matroid theory, proposed by Whitney [56], is used to generalize graph theory and linear algebra [57, 58]. Since its inception, many matroidal structures have been produced by combination with other theories, such as rough sets [13–17, 59, 60]. Matroid theory can be employed to solve combinatorial optimization problems due to its good structure for greedy algorithms [57, 58]. In real life, some information appears with matroid constraints, so problems that involve such information need to be solved with the assistance of matroid theory [61–65].

In what follows, the necessity of studying a covering rough set model over a matroidal structure in reality is illustrated through an example of the biological classification of insects on the basis of morphology. According to the common methods of biological classification of insects, we can see that (1) in research on the classification of insects from morphology, the researcher first collects the insect specimens of some group. Next, for a family of specimens from different locations, or even specimens from the same location, combined with the morphological characteristics that the researcher believes need to be considered, the properties of the specimens in terms of these morphological characteristics are taken as the research content; the researcher will use his or her existing insect morphological knowledge that is closest to the discussed content to approximate the discussed content to obtain the results that the researcher believes are most appropriate. The collected specimens of the insect group are the first factor in analysis and research, the morphological characteristics that the researcher believes should be considered are the second factor, and the collected locations of the specimens in this insect group are the third factor. The three factors belong to three different considered sets. (2) The results of the research that the researcher believes are most appropriate can be obtained only after step-by-step analysis. This is actually a 'greedy' process. Because matroid theory builds a

good platform for greedy algorithms, we can conclude that the known knowledge structure of the researcher related to the research content constitutes a matroidal structure. (3) The approximate inference process of the researcher is also that of approximate inference to unknown knowledge from known knowledge; that is, the lower and upper approximations of the rough set are used to express the unknown knowledge.

By (1) and (2), it is necessary to establish a matroidal structure over three sets. Combining (2) and (3), we conclude that it is necessary to study the lower and upper approximation operators of rough sets based on a matroidal structure over three sets.

In [36], to describe the motivation of the study, an example given in Section 1, of a disease diagnosis decision-making problem in a clinic, illustrates a relation-based rough set model over three universes for realistic decision-making problems. We will look at this problem from the perspective of covering rough set models over three sets. Since each disease must show many basic symptoms and some concrete results of clinical examination, the known knowledge of the doctor is a set consisting of three parts for a disease  $d$ :  $BS$  is the set of basic symptoms of  $d$ ,  $CE$  is the set of concrete clinical examination results, and  $D$  is  $\{d\}$ . The doctor will compare the basic symptoms and the results of the clinical examination of the patient to known diseases and analyze them to finally determine the most likely disease through the approximate inference method. The known knowledge of the doctor relative to his or her known diseases consists of three parts:  $\{BS \mid BS \text{ is relative to a disease } d\}$ ,  $\{CE \mid CE \text{ is relative to a disease } d\}$ , and  $\{D \mid D \text{ is a disease } d\}$ .

The process of comparative analysis by the doctor determines the optimal solution from the knowledge base of the doctor with respect to the diseases that are closest to that of the patient. This process is greedy. Combined with matroid theory, which provides a good platform for greedy algorithms, the structure of the known knowledge of the doctor is related to a matroidal structure. Approximate inference is the doctor's representation of unknown knowledge with his or her known knowledge relative to diseases, which is an approximate representation of a rough set. We should note that if the doctor's known knowledge base with respect to diseases does not completely cover the patient's symptoms and clinical examination results, the inference process must be absolutely approximate. For instance, when COVID-19 first broke out in 2019, no doctor in the world had known knowledge that covered this new disease; only approximative knowledge was available to make inferences regarding this new disease. This type of inference finds an optimal solution from the doctor's known knowledge base with respect to diseases; that is, it is a greedy inference. Therefore, this new disease was called

unexplained pneumonia at the time, although doctors now have knowledge of this disease and some ways to treat it. Hence, it is necessary to discuss rough sets as well as covering rough sets based on a matroidal structure over three sets.

As Ytow et al. [66] discussed, biological classification has an intimate relation to rough set theory. We note that both biological classification and doctors' decision-making are considered in three-dimensional space. Additionally, mining valuable information from an information system expressed in three parts is already being explored by many researchers, such as in [36, 67, 68]. The real world is in three-dimensional space. The human cognitive process moves from lower dimensions to higher dimensions, from one-dimensional to two-dimensional space and then to three-dimensional space. Rough set theory is one of the methods by which human beings understand the world. Constructing a covering rough set model over three universes, or three-dimensional space, has become an urgent task. Completing this work is exactly in line with patterns of human cognition. Additionally, many problems in rough set theory are NP-hard, so solving these problems is often greedy; that is, greedy algorithms often need to be used, equivalently to say, matroid theory often need to be used. Hence, it is necessary to build up a matroidal structure on three-dimensional space, i.e., on the Cartesian product of three sets. Using this new matroidal structure, it is also necessary to construct approximation operators in rough set theory that are expressed in ternary form. For this purpose, we present the following contributions:

- First, we present a matroidal structure over three sets—TP-matroid—and demonstrate that TP-matroid is an extension of Whitney's classical matroid [56–58] under the idea of isomorphisms. Considering approximations of rough sets in knowledge spaces [69] with approximations in covering rough sets [11], we provide a pair of lower and upper approximations using the set of feasible sets of a TP-matroid.
- Second, with the help of formal concept analysis, we explore a pair of lower and upper approximations expressed in ternary form over three sets. Furthermore, we construct two TP-matroids by using this pair of lower and upper approximations. The integration of the two pairs of approximations in this paper is also discussed.

For every structure and some of the definitions and properties presented in this paper, corresponding explanations are given through examples, where the information tables come from biological information systems.

There are two research goals of this paper: one is to theoretically study rough sets, aided by matroid theory over three sets, and the other is for the results provided here to be used in actual practice; we provide some examples with practical information systems.

The rest of this paper is organized as follows: In Section 2, we review some basic definitions and properties of matroids, formal concept analysis, and rough sets. In Section 3, we first provide a matroidal structure over three sets with ternary form, i.e., a TP-matroid, and determine how to find rough set approximations over three sets with a precovering TP-matroid. In Section 4, for information data relative to formal contexts over three sets, we provide a pair of lower and upper approximations expressed in ternary form with the help of formal concept analysis. Using this pair of approximations, two TP-matroids are built. Concluding remarks are given in the last section.

## 2 Some notions and properties

Below, we review some basic notions used in this paper. For more details, matroid theory is referred to in [57, 58], formal concept analysis is seen in [70], semiconcepts are seen in [71], poset theory is referred to in [72], and rough sets are seen in [1, 2]. Since a data table is finite in practice, we assume that all of the discussions are finite in this paper.

### 2.1 Some notations

Let  $U$ ,  $V$  and  $W$  be three sets. Then we will use the following notations in this paper for  $\forall X, X_1, X_2 \subseteq U$ ,  $\forall Y, Y_1, Y_2 \subseteq V$  and  $\forall Z, Z_1, Z_2 \subseteq W$ .

- (1)  $|X|$  stands for the cardinality of  $X \subseteq U$ .
- (2)  $(X_1, Y_1, Z_1) \subseteq (X_2, Y_2, Z_2) : \Leftrightarrow X_1 \subseteq X_2, Y_1 \subseteq Y_2$  and  $Z_1 \subseteq Z_2$ .
- (3)  $(X_1, Y_1, Z_1) \sqsubseteq (X_2, Y_2, Z_2) : \Leftrightarrow X_1 \subseteq X_2, Y_1 \supseteq Y_2$  and  $Z_1 \subseteq Z_2$ .
- (4)  $(X_1, Y_1, Z_1) \cup (X_2, Y_2, Z_2) := (X_1 \cup X_2, Y_1 \cup Y_2, Z_1 \cup Z_2)$ .
- (5)  $(X_1, Y_1, Z_1) \cap (X_2, Y_2, Z_2) := (X_1 \cap X_2, Y_1 \cap Y_2, Z_1 \cap Z_2)$ .
- (6)  $(X_1, Y_1, Z_1) \setminus (X_2, Y_2, Z_2) := (X_1 \setminus X_2, Y_1 \setminus Y_2, Z_1 \setminus Z_2)$ .
- (7)  $|(X, Y, Z)| := |X| + |Y| + |Z|$ , that is, the cardinality of  $(X, Y, Z)$ .
- (8)  $(X_1, Y_1, Z_1) \sqcup (X_2, Y_2, Z_2) := (X_1 \cup X_2, Y_1 \cap Y_2, Z_1 \cup Z_2)$ .
- (9)  $2^S$  represents the power set of a set  $S$ .
- (10) “ $E$  is in unary (binary; ternary) form” means:  $E := X(E := (X, Y); E := (X, Y, Z))$ , where  $X \subseteq U((X, Y) \subseteq (U, V); (X, Y, Z) \subseteq (U, V, W))$ .
- (11) If there is a bijection  $f : U \rightarrow V$ , then we say  $U$  and  $V$  are isomorphic, denoted as  $U \cong V$ .
- (12) A ‘universe’ is a nonempty set.
- (13) The Cartesian product of one set (two sets; three sets)  $U(U, V; U, V, W)$  is  $U(U \times V; U \times V \times$

$W$ ) and is called one- (two-; three-) dimensional space.

*Remark 1* We sometimes write  $y$  for  $\{y\}$  if  $y$  is a singleton set.

### 2.2 Matroid

**Definition 1** (1) [57, p.7][58, p.7] A *matroid*  $M$  is a set  $S$  and a collection  $\mathcal{I}$  of subsets of  $S$  (called *independent sets*) such that (i1)-(i3) are satisfied.

- (i1)  $\emptyset \in \mathcal{I}$ .
  - (i2)  $X \in \mathcal{I}$  and  $Y \subseteq X \Rightarrow Y \in \mathcal{I}$ .
  - (i3)  $X, Y \in \mathcal{I}$  and  $|X| < |Y| \Rightarrow X \cup y \in \mathcal{I}$  for some  $y \in Y \setminus X$ .
- (2) [57, p.11][58, p.9] Two matroids  $M_1$  and  $M_2$  on  $S_1$  and  $S_2$  respectively are *isomorphic* if there is a bijection  $\varphi : S_1 \rightarrow S_2$  that preserves independence. We write  $M_1 \cong M_2$  if  $M_1$  and  $M_2$  are isomorphic.

### 2.3 Formal concept analysis

Formal concept analysis (or a concept lattice), proposed by Wille [73], is a useful and successful tool for dealing with data represented by a kind of information table—a formal context. It is well known that many data tables are similar in form to formal contexts. Hence, to study rough sets and matroids, formal concept analysis is a good tool [18–20, 26, 35, 69].

Next, we review some definitions and lemmas for formal concept analysis.

**Definition 2** (1) [70, pp.17-18] A *formal context* is a set structure  $\mathbb{K} := (O, P, I)$  such that  $O$  and  $P$  are nonempty sets and  $I \subseteq O \times P$ ; the elements of  $O$  and  $P$  are called *objects* and *attributes*, respectively, and *gIm* is  $(g, m) \in I$ . The *derivation operators* of  $\mathbb{K}$  are defined as follows

$$(X \subseteq O, Y \subseteq P): X' = \{m \in P \mid gIm \text{ for all } g \in X\} \text{ and } Y' = \{g \in O \mid gIm \text{ for all } m \in Y\}.$$

- (2) [71] In a formal context  $\mathbb{K} = (O, P, I)$ , a pair  $(X, Y)$  with  $X \subseteq O$  and  $Y \subseteq P$  is called a  $\sqcap$ -*semiconcept* if  $Y = X'$ . Dually, a pair  $(C, D)$  with  $C \subseteq O$  and  $D \subseteq P$  is called a  $\sqcup$ -*semiconcept* if  $C = D'$ .

**Lemma 1** [70, p.19] *The two derivation operators in a formal context  $\mathbb{K} = (O, P, I)$  satisfy the following condition for any  $A_j \subseteq O$  (or  $A_j \subseteq P$ ) where  $j \in J$  and  $J$  is an index set:  $(\cup_{j \in J} A_j)' = \cap_{j \in J} A_j'$ .*

*Remark 2* (1) For a formal context  $\mathbb{K} = (O, P, I)$ , if  $x \in O$  (or  $x \in P$ ), then  $\{x\}'$  is abbreviated as  $x'$ .

- (2) We can easily find that the family of  $\sqcap$ -semiconcepts has the dual property of that of the family of  $\sqcup$ -semiconcepts. Hence, we only consider the family of  $\sqcup$ -semiconcepts and simply use *semiconcept* instead of  $\sqcup$ -semiconcept in what follows.
- (3) All semiconcepts in a formal context  $\mathbb{K}$  are denoted as  $\mathcal{B}(\mathbb{K})$ .

### 2.4 Posets and equivalence relations

**Definition 3** [58, p.45] A *poset* is a set  $S$  together with a binary relation  $\leq$ , i.e., a partial order, such that the following properties hold for  $\forall x, y, z \in S$ :

- (p1)  $x \leq x$ .
- (p2)  $x \leq y$  and  $y \leq x \Rightarrow x = y$ .
- (p3)  $x \leq y$  and  $y \leq z \Rightarrow x \leq z$ .

**Definition 4** [72, pp.2-3] A binary relation  $\varepsilon$  on a nonempty set  $A$  is called an *equivalence relation* if it satisfies the following three properties for  $\forall a, b, c \in A$ :

- (e1)  $(a, a) \in \varepsilon$ .
- (e2)  $(a, b) \in \varepsilon \Rightarrow (b, a) \in \varepsilon$ .
- (e3)  $(a, b) \in \varepsilon$  and  $(b, c) \in \varepsilon \Rightarrow (a, c) \in \varepsilon$ .

### 2.5 Rough set

**Definition 5** [1,2]

- (1) Let  $U$  be a universe,  $R \subseteq U \times U$  be an equivalence relation on  $U$ , and  $[x]_R$  denote the equivalence class involving the element  $x$ . For any  $X \subseteq U$ , we call  $\underline{R}(X) = \{x \in U \mid [x]_R \subseteq X\}$  and  $\overline{R}(X) = \{x \in U \mid [x]_R \cap X \neq \emptyset\}$ , the *lower* and *upper approximations* of  $X$  about the *Pawlak approximation space*  $(U, R)$ , respectively.
- (2) Let  $U/R = \{[x]_R \mid x \in U\}$ . Every element in  $U/R$  is called *R-basic category*.  $X \subseteq U$  is called an *R-definable* if  $X$  is the union of some *R-basic categories*; otherwise,  $X$  is *R-undefinable*.

**Lemma 2** [1,2] *Let  $(U, R)$  be a Pawlak approximation space.*

- (1) *The lower and upper approximations can be described by the following an equivalent form:*  

$$\underline{R}X = \bigcup \{Y \in U/R \mid Y \subseteq X\}, \overline{R}X = \bigcup \{Y \in U/R \mid Y \cap X \neq \emptyset\}.$$
- (2)  $X \subseteq U$  is *R-definable*  $\Leftrightarrow \underline{R}(X) = \overline{R}(X)$ .

**Definition 6** (1) [11] Let  $U$  be a universe, and  $\mathcal{C}$  be a family of subsets of  $U$ . If no subsets in  $\mathcal{C}$  are empty and  $\bigcup \mathcal{C} = U$ , then  $\mathcal{C}$  is called a *covering* of  $U$ .  $(U, \mathcal{C})$  is called a *covering approximation space*.

- (2) [74] Let  $Q$  be a universe. A *knowledge structure* is denoted by a pair  $(Q, \mathcal{K})$ , where  $\mathcal{K} \subseteq 2^Q$ . The only special assumption about  $\mathcal{K}$  is that it must contain the empty set and the full set  $Q$ .

Considering the definition of a covering approximation space in Definition 6, we can state that the expression of Definition 6(1) over three sets is given below, where at least one of  $U, V$  and  $W$  is a universe.

Let  $\mathcal{C}$  be a family of subsets of  $(U, V, W)$ ; i.e.,  $C \in \mathcal{C} \Rightarrow C = (X, Y, Z) \subseteq (U, V, W)$ . If none of the subsets in  $\mathcal{C}$  is  $(\emptyset, \emptyset, \emptyset)$  and  $\bigcup \mathcal{C} = (U, V, W)$ , then  $\mathcal{C}$  is called a covering of  $(U, V, W)$ .  $(U \times V \times W, \mathcal{C})$  is called a covering approximation space.

In the coming Example 2 in Section 3, we will see that  $\mathcal{J} \setminus \{(\emptyset, \emptyset, \emptyset)\}$  is a covering of  $(U, V, W)$ .

We generalize the definition of a knowledge structure in Definition 6 from one set to three sets.

**Definition 7** Let  $U, V$  and  $W$  be three sets such that at least one of  $U, V$  and  $W$  is a universe. Let  $\mathcal{K} \subseteq \{(X, Y, Z) \mid X \subseteq U, Y \subseteq V, Z \subseteq W\}$  and  $\mathcal{K} \neq \emptyset$ . Then,  $(U \times V \times W, \mathcal{K})$  is called a *knowledge space* and  $x \in \mathcal{K}$  is called *basic knowledge*.

Comparing Definition 6(2) with Definition 7, we see that Definition 7 is a generalization of Definition 6(2) since  $\mathcal{K}$  need not satisfy  $\emptyset, U \times V \times W \in \mathcal{K}$  in Definition 7, but the corresponding condition is included in Definition 6(2).

Yao et al. [11] pointed out that when generalizing Pawlak's approximations, one task is to specify a subset of these properties that new approximation operators are required to preserve. Hence, according to Pawlak's approximations, Yao et al. [11] and Yao [74] presented generalized definitions for lower and upper approximation operators, respectively. Considering Definitions 5, 6 and 7, Lemma 2, and the discussion in [11, 74] with the expression of approximations for knowledge spaces in [69], we can present the following definition:

**Definition 8** Let  $S$  be a universe. Suppose that  $(S, \mathcal{J})$  is a knowledge space in which  $\mathcal{J} \subseteq 2^S$  and  $\mathcal{J} \neq \emptyset$ . Then,  $\underline{APR}$  and  $\overline{APR}$ , where  $\underline{APR}, \overline{APR} : 2^S \rightarrow 2^S$ , are a pair of *lower and upper approximations* on  $2^S$  if and only if  $\underline{APR}$  and  $\overline{APR}$  satisfy the following conditions with a partial order  $\leq$  defined on  $2^S$  for any  $X \subseteq S$ :

- (1)  $\underline{APR}(X) \leq X \leq \overline{APR}(X)$ ,
- (2)  $X \in \mathcal{J} \Leftrightarrow \underline{APR}(X) = X = \overline{APR}(X)$ .

### 3 Rough set approximations produced by a new matroidal structure—TP-matroid

To combine rough sets and matroids, we first need to generalize the construction of matroids from one set to three sets, in particular, three universes. Then, we can explore rough set approximations with the new matroidal structure.

#### 3.1 Relationships between TP-matroids and matroids

We generalize the definition of a matroid from one set to three sets.

**Definition 9** (1) Let  $U, V$  and  $W$  be three sets such that at least one of  $U, V$  and  $W$  is not empty. Let  $\mathcal{TI} \subseteq \{(X, Y, Z) \mid (X, Y, Z) \subseteq (U, V, W)\}$ ; i.e., we have a collection of subsets of  $U \times V \times W$  (called *feasible sets*) such that (I1)-(I3) are satisfied for  $\forall (X_j, Y_j, Z_j) \subseteq (U, V, W) (j = 1, 2)$ .

- (I1)  $\mathcal{TI} \neq \emptyset$ .
- (I2)  $(X_1, Y_1, Z_1) \sqsubseteq (X_2, Y_2, Z_2) \in \mathcal{TI} \Rightarrow (X_1, Y_1, Z_1) \in \mathcal{TI}$ .
- (I3) Let  $(X_j, Y_j, Z_j) \in \mathcal{TI} (j = 1, 2)$ . If at least one of  $X_2, Y_2$  and  $Z_2$  is not empty, and  $|(X_1, Y_1, Z_1)| < |(X_2, Y_2, Z_2)|$ , then  $(X_1, Y_1, Z_1) \cup (x_2, y_2, z_2) \in \mathcal{TI}$  holds for some  $(x_2, y_2, z_2) \in (X_2, Y_2, Z_2) \setminus (X_1, Y_1, Z_1)$  such that at least one of  $x_2, y_2$  and  $z_2$  is not empty.

Then,  $(U \times V \times W, \mathcal{TI})$  is called a *three-partial matroid*, abbreviated as *TP-matroid*.

- (2) Let  $(U \times V \times W, \mathcal{TI})$  be a TP-matroid. If  $\mathcal{TI} = \{(X_\gamma, Y_\gamma, Z_\gamma), \gamma \in \Upsilon\}$  satisfies  $\bigcup_{\gamma \in \Upsilon} X_\gamma = U, \bigcup_{\gamma \in \Upsilon} Y_\gamma = V$  and  $\bigcup_{\gamma \in \Upsilon} Z_\gamma = W$ , then  $(U \times V \times W, \mathcal{TI})$  is called a *precovering TP-matroid*.
- (3) Two TP-matroids  $(U_1 \times V_1 \times W_1, \mathcal{TI}_1)$  and  $(U_2 \times V_2 \times W_2, \mathcal{TI}_2)$  are *isomorphic* if there is a bijection  $\psi : U_1 \times V_1 \times W_1 \rightarrow U_2 \times V_2 \times W_2$  that preserves feasibility. We write  $(U_1 \times V_1 \times W_1, \mathcal{TI}_1) \cong (U_2 \times V_2 \times W_2, \mathcal{TI}_2)$  if  $(U_1 \times V_1 \times W_1, \mathcal{TI}_1)$  and  $(U_2 \times V_2 \times W_2, \mathcal{TI}_2)$  are isomorphic.

**Remark 3** (1) Let  $U$  be a set of collected insect specimens,  $V$  be a set of considered morphological characteristics, and  $W$  be a set of locations of the collected specimens in  $U$ . Let  $(U \times V \times W, \mathcal{TI})$  be a TP-matroid, and let  $(X_1, Y_1, Z_1), (X_2, Y_2, Z_2) \subseteq (U, V, W)$ . Suppose  $X_1 \subseteq X_2$  and  $Z_1 \subseteq Z_2$ . Biologists will consider the common characteristics  $Y$  of  $X \subseteq U$  when they analyze the set  $X$  of specimens during classification. Then,  $X_1 \subseteq X_2$

**Table 1** Characteristics of stridulatory files

Specimen	The number of teeth in the distal part	The number of teeth in the proximate part	Source/specimen, origin (scanning electron microscope, SEM)
Japonica 1	4(9)	61	Kim (2009): Korea, SEM
Japonica 2	6	57-60	CH7421-2: Korea ( $n = 2$ )
Japonica 3	6	66	Wu (2010): China

will imply  $Y_1 \supseteq Y_2$  if  $Y_j$  is the set of common characteristics of  $X_j$  ( $j = 1, 2$ ). This follows from  $(X_1, Y_1, Z_1) \sqsubseteq (X_2, Y_2, Z_2)$ . That is, the order ‘ $\sqsubseteq$ ’ in (I2) is reasonable in some practical cases.

- (2) We first explain some terms in Definition 9.
- (2.1) A matroid  $M$  in Definition 1 is defined on one set. That is, the background set of  $M$  consists of one ‘part’. The background set of a TP-matroid  $(U \times V \times W, \mathcal{TI})$  in Definition 9 consists of three ‘parts’— $U, V$  and  $W$ . In other words,  $(U \times V \times W, \mathcal{TI})$  is an extension of the matroid from one set to three sets. Hence,  $(U \times V \times W, \mathcal{TI})$  is called a three-partial matroid or simply a TP-matroid.
- (2.2) Definition 6(1) and Example 2 below show that for a TP-matroid  $(U \times V \times W, \mathcal{TI})$ , even if  $\bigcup \mathcal{TI} = (U, V, W)$ ,  $\mathcal{TI}$  may not be a covering of  $(U, V, W)$  since  $(\emptyset, \emptyset, \emptyset) \in \mathcal{TI}$  holds for some TP-matroids, such as that in Example 2. However,  $\mathcal{TI} \setminus (\emptyset, \emptyset, \emptyset)$  is a covering of  $(U, V, W)$  if  $\bigcup \mathcal{TI} = (U, V, W)$ . Therefore, it is suitable to call this a ‘precovering’ TP-matroid as described in Definition 9(2). Comparing items (1) and (2) in Definition 9, we assert that the structure of a precovering TP-matroid is a special case of the structure of the TP-matroid.
- (3) We will analyze the existence of  $(x_2, y_2, z_2)$  such that at least one of  $x_2, y_2$  and  $z_2$  is not empty if  $|(X_1, Y_1, Z_1)| < |(X_2, Y_2, Z_2)|$  in (I3).

Let  $U, V$  and  $W$  be three sets such that one of  $U, V$  and  $W$  is a universe. Suppose that  $(X_j, Y_j, Z_j) \subseteq U \times V \times W$  ( $j = 1, 2$ ) satisfy the requirement that at least one of  $X_2, Y_2$  and  $Z_2$  is nonempty. Then, we confirm that:

$$|(X_1, Y_1, Z_1)| < |(X_2, Y_2, Z_2)| \Rightarrow \exists(x_2, y_2, z_2) \in (X_2, Y_2, Z_2) \setminus (X_1, Y_1, Z_1), \text{ where at least one of } x_2, y_2 \text{ and } z_2 \text{ is not empty.}$$

The reason for this is as follows:

**Table 2** Mathematical expression of Table 1

	$b_1$	$b_2$	
$a_1$	4(9)	61	$c_1$
$a_2$	6	57-60	$c_1$
$a_3$	6	66	$c_2$

We know that  $|(X_j, Y_j, Z_j)| = |X_j| + |Y_j| + |Z_j|$  ( $j = 1, 2$ ). Since at least one of  $X_2, Y_2$  and  $Z_2$  is not empty, this implies  $|(X_2, Y_2, Z_2)| \neq 0$ . Therefore,  $|X_2| \neq 0, |Y_2| \neq 0$  and  $|Z_2| \neq 0$  hold.

If  $|(X_1, Y_1, Z_1)| < |(X_2, Y_2, Z_2)|$ , we assert that one of  $|X_1| < |X_2|, |Y_1| < |Y_2|$  and  $|Z_1| < |Z_2|$  holds. If this assertion is not true, then  $|X_2| \leq |X_1|, |Y_2| \leq |Y_1|$  and  $|Z_2| \leq |Z_1|$ . This implies  $|X_2| + |Y_2| + |Z_2| \leq |X_1| + |Y_1| + |Z_1|$ , a contradiction of the known condition  $|(X_1, Y_1, Z_1)| < |(X_2, Y_2, Z_2)|$ .

Suppose  $|X_1| < |X_2| \neq 0$ . Then there is an  $x_2 \in X_2 \setminus X_1 \neq \emptyset$  satisfying  $x_2 \neq \emptyset$ . Therefore,  $(x_2, \emptyset, \emptyset) \in (X_2, Y_2, Z_2) \setminus (X_1, Y_1, Z_1)$  holds, and  $(x_2, \emptyset, \emptyset) \neq (\emptyset, \emptyset, \emptyset)$  is correct.

Similarly, for  $|Y_1| < |Y_2| \neq 0$  or  $|Z_1| < |Z_2| \neq 0$ , the needed results are correct.

The following example shows the existence of a TP-matroid.

*Example 1* Table 1 is an expression of some biological information in [75, Table 4].

Let  $a_j :=$  japonica  $j$ , ( $j = 1, 2, 3$ ),  $b_1 :=$  ‘The number of teeth in the distal part’,  $b_2 :=$  ‘The number of teeth in the proximate part’,  $c_1 :=$  ‘Korea’, and  $c_2 :=$  ‘China’. Then, we obtain the mathematical expression of Tables 1 in Table 2.

Let  $U = \{a_1, a_2, a_3\}, V = \{b_1, b_2\}$  and  $W = \{c_1, c_2\}$ . Let  $\mathcal{TI} = \{(\{a_1, a_2\}, b_1, c_1)\} \cup \{(a_j, b_1, \emptyset), (a_j, b_1, c_1), (a_j, \{b_1, b_2\}, \emptyset), (a_j, \{b_1, b_2\}, c_1), (j = 1, 2)\} \cup \{(\emptyset, b_1, \emptyset), (\emptyset, b_1, c_1), (\emptyset, \{b_1, b_2\}, \emptyset), (\emptyset, \{b_1, b_2\}, c_1)\} \cup \{(\{a_1, a_2\}, b_1, \emptyset), (\{a_1, a_2\}, \{b_1, b_2\}, \emptyset), (\{a_1, a_2\}, \{b_1, b_2\}, c_1)\}$ . Then, we may easily check that  $\mathcal{TI}$  satisfies (II)-(I3). Therefore, using Definition 9(1), we find that  $(U \times V \times W, \mathcal{TI})$  is a TP-matroid.

Here, for  $\forall X \subseteq U, \forall Y \subseteq V$  and  $\forall Z \subseteq W, (X, Y, Z) \in \mathcal{TI}$  means that in researching the japonica population with the biological information shown in Table 1, one of the basic knowledge items of the biologists is that  $X$ , the japonica that comes from location  $Z$ , must have characteristics  $Y$ . For example,  $(X = \{a_1, a_2\}, Y = \{b_1\}, Z = \{c_1\}) \in \mathcal{TI}$  means that the biologist believes the japonica collected in  $c_1$  must possess the common characteristic  $b_1$ .

Simply, we denote  $\mathcal{TI}$  as  $\{(X_\gamma, Y_\gamma, Z_\gamma), \gamma \in \Upsilon\}$ . We find that  $\mathcal{TI}$  satisfies  $\bigcup_{\gamma \in \Upsilon} X_\gamma = \{a_1, a_2\} \subset U, \bigcup_{\gamma \in \Upsilon} Y_\gamma = \{b_1, b_2\} =$

$V$  and  $\bigcup_{\gamma \in \mathcal{Y}} Z_\gamma = \{c_1\} \subset W$ . Considering Definition 6(1) and Definition 9(2), we may easily confirm that

- (1)  $\mathcal{TI}$ , i.e.,  $\mathcal{TI} \setminus (\emptyset, \emptyset, \emptyset)$ , since  $(\emptyset, \emptyset, \emptyset) \notin \mathcal{TI}$ , is not a covering of  $(U, V, W)$ .
- (2)  $(U \times V \times W, \mathcal{TI})$  is not a precovering TP-matroid.
- (3) We also see that the set of specimens of the insect group is  $U = \{a_1, a_2, a_3\}$ , the set of morphological characteristics that the biologist believes need to be considered is  $V = \{b_1, b_2\}$ , and the set of locations of the collected specimens is  $W = \{c_1, c_2\}$ . The available knowledge of the biologist in Example 1 is  $\mathcal{TI}$ .

The next example will show the existence of a precovering TP-matroid.

*Example 2* Let  $U, V$  and  $W$  be given as in Example 1. Let  $\mathcal{J} = \{(X, Y, Z) \subseteq (U, V, W) \mid |X| \leq 1, |Z| \leq 1\} = \{(a_i, b_j, c_k), i = 1, 2, 3; j = 1, 2; k = 1, 2\} \cup \{(a_i, \{b_1, b_2\}, c_k), i = 1, 2, 3; k = 1, 2\} \cup \{(\emptyset, b_j, c_k), j = 1, 2; k = 1, 2\} \cup \{(\emptyset, \{b_1, b_2\}, c_k), k = 1, 2\} \cup \{(\emptyset, b_j, \emptyset), j = 1, 2\} \cup \{(\emptyset, \{b_1, b_2\}, \emptyset)\} \cup \{(a_i, b_j, \emptyset), i = 1, 2, 3; j = 1, 2\} \cup \{(a_i, \{b_1, b_2\}, \emptyset), i = 1, 2, 3\} \cup \{(a_i, \emptyset, c_k), i = 1, 2, 3; k = 1, 2\} \cup \{(\emptyset, \emptyset, c_k), k = 1, 2\} \cup \{(a_i, \emptyset, \emptyset), i = 1, 2, 3\} \cup \{(\emptyset, \emptyset, \emptyset)\}$ . Then we may easily find that

- (1)  $(U \times V \times W, \mathcal{J})$  is a TP-matroid by Definition 9(1).
- (2)  $\mathcal{J} \setminus (\emptyset, \emptyset, \emptyset)$  is a covering of  $(U, V, W)$  since  $\mathcal{J} \setminus (\emptyset, \emptyset, \emptyset)$  satisfies  $\bigcup(\mathcal{J} \setminus (\emptyset, \emptyset, \emptyset)) = (U, V, W)$  using Definition 6(1).
- (3)  $(U \times V \times W, \mathcal{J})$  is a precovering TP-matroid since  $\bigcup \mathcal{J} = \bigcup(\mathcal{J} \setminus (\emptyset, \emptyset, \emptyset)) = (U, V, W)$  by Definition 9(2).
- (4) The known knowledge of the biologist in Example 2 is  $\mathcal{J}$  on  $U \times V \times W$ .

*Remark 4* We next compare the definitions of a matroid and TP-matroid.

**I)** The comparisons of the structures between the two definitions are shown in Table 3.

Using Table 3, we find results (1) and (2).

- (1) Let  $S = U \times V \times W$ . Then,  $(S, \mathcal{TI})$  is not a matroid if  $(U \times V \times W, \mathcal{TI})$  is a TP-matroid, since  $\mathcal{TI} \subseteq 2^U \times 2^V \times 2^W$  according to Definition 9(1). If  $(S, \mathcal{TI})$  is a matroid, then  $\mathcal{TI} \subseteq 2^S = 2^{U \times V \times W}$ . It is easy to see that  $2^{U \times V \times W} \neq 2^U \times 2^V \times 2^W$

in general. For instance, in Example 1,  $|2^{U \times V \times W} = 2^{\{a_1, a_2, a_3\} \times \{b_1, b_2\} \times \{c_1, c_2\}}| \neq |2^U \times 2^V \times 2^W = 2^{\{a_1, a_2, a_3\}} \times 2^{\{b_1, b_2\}} \times 2^{\{c_1, c_2\}}|$  implies  $2^{U \times V \times W} \neq 2^U \times 2^V \times 2^W$ . That is, the range of the family  $\mathcal{I}$  of independent sets of a matroid and that of the family  $\mathcal{TI}$  of feasible sets of a TP-matroid are different in general.

(2) We next compare some relations between the restricted conditions of the matroid and TP-matroid.

(2.1) (i1) means that  $\emptyset \in \mathcal{I}$ . Therefore, it follows that  $\mathcal{I} \neq \emptyset$ .

Considering Example 1, we know  $(\emptyset, \emptyset, \emptyset) \notin \mathcal{TI}$  for some TP-matroid. This indicates that (I1) cannot determine  $\mathcal{TI}$ . It only confirms that  $\mathcal{TI} \neq \emptyset$ .

Hence, (i1) is a special case of (I1).

(2.2) Conditions (i3) and (I3) have some similarity. The similarity suggests that there is a close relation between the matroid and TP-matroid.

(2.3) Let  $(U \times V \times W, \mathcal{TI})$  be defined as in Example 1. Let  $\mathcal{I}_2 = \{\emptyset, b_j, (j = 1, 2)\} \subseteq 2^V$ . We know that  $M_2 = (V, \mathcal{I}_2)$  is a matroid using Definition 1(1).

Let  $X_1 = \emptyset, X_2 = \emptyset \subseteq U, Y_1 = \{b_1, b_2\}, Y_2 = \{b_1\} \subseteq V, Z_1 = \emptyset$ , and  $Z_2 = \emptyset \subseteq W$ . Then, we consider the following two cases:

In one case,

(\*1) (i2) is correct for  $\mathcal{I}_2$ . If (I2) holds for  $\mathcal{I}_2$ , then  $\{b_1, b_2\} \subseteq b_1 \Rightarrow \{b_1, b_2\} \in \mathcal{I}_2$  holds, which contradicts  $\{b_1, b_2\} \notin \mathcal{I}_2$ .

Thus, (\*1) implies that (i2) cannot be replaced by (I2).

In the other case,

(\*2) (I2) is correct for  $\mathcal{TI}$ . If (i2) holds for  $\mathcal{TI}$ , then  $(\emptyset, b_2, \emptyset) \subseteq (\emptyset, \{b_1, b_2\}, \emptyset) \in \mathcal{TI} \Rightarrow (\emptyset, b_2, \emptyset) \in \mathcal{TI}$  holds, which contradicts  $(\emptyset, b_2, \emptyset) \notin \mathcal{TI}$ .

Hence, (\*2) means that (I2) cannot be replaced by (i2).

The above two cases show that (i2) and (I2) are independent.

**II)** To continue the discussion of the definitions of matroid and TP-matroid, we can obtain more results for their relations as follows in (3)-(6).

- (3) We may easily prove  $V \cong \emptyset \times V \times \emptyset$ . We can also easily demonstrate  $M_1 = (\emptyset \times V \times \emptyset, \mathcal{TI}_2 = \{(\emptyset, X, \emptyset) \mid X \in \mathcal{I}_2\})$  to be a matroid such that  $M_1 \cong M_2$ .

**Table 3** Compare the structures between a matroid and a TP-matroid

	dimension of ground set	range of family of independent(feasible) set	restricted conditions
$(S, \mathcal{I})$ , a matroid	one	$2^S$	(i1)-(i3)
$(U \times V \times W, \mathcal{TI})$ , a TP-matroid	three	$2^U \times 2^V \times 2^W$	(I1)-(I3)

Suppose that every matroid is a TP-matroid. Then,  $M_1$  is a TP-matroid. In fact, we know that  $M_1$  is not a TP-matroid since  $(\emptyset, \{b_1, b_2\}, \emptyset) \sqsubseteq (\emptyset, \emptyset, \emptyset) \not\Leftarrow (\emptyset, \{b_1, b_2\}, \emptyset) \in \mathcal{TI}_2$ . Hence, even under isomorphisms of sets and matroids,  $M_2$  is not a TP-matroid. In other words, a matroid may not be a TP-matroid even up to isomorphism.

- (4) Suppose that every TP-matroid is a matroid. From Example 1, we know  $(\{a_1, a_2\}, \{b_1, b_2\}, c_1) \in \mathcal{TI}$ , where  $(U \times V \times W, \mathcal{TI})$  is defined as in Example 1. By (i2), we obtain  $(X, Y, Z) \subseteq (\{a_1, a_2\}, \{b_1, b_2\}, c_1) \Rightarrow (X, Y, Z) \in \mathcal{TI}$ ; in particular,  $(\{a_1, a_2\}, b_2, c_1) \in \mathcal{TI}$  which contradicts Example 1. Thus, not every TP-matroid is a matroid.
- (5) The above items (3) and (4) imply that the TP-matroid is a new structure that is different from the matroid.
- (6) Let  $M_{21} = (U, \mathcal{I}_{21} = \{\emptyset\})$  and  $M_{23} = (W, \mathcal{I}_{23} = \{\emptyset\})$ , where  $U$  and  $W$  are defined as in Example 1. Using Definition 1(1),  $M_{21}$  and  $M_{23}$  are matroids. Let  $M = (M_{21}, M_2, M_{23})$ , i.e.,  $M = (U \times V \times W, \mathcal{T} = \{X \times \emptyset \times \emptyset \mid X \in \mathcal{I}_{21}\} \cup \{\emptyset \times Y \times \emptyset \mid Y \in \mathcal{I}_2\} \cup \{\emptyset \times \emptyset \times Z \mid Z \in \mathcal{I}_{23}\})$ . Then, we obtain that  $M$  is not a TP-matroid since  $(\emptyset, \{b_1, b_2\}, \emptyset) \sqsubseteq (\emptyset, \emptyset, \emptyset) \in \mathcal{T} \not\Leftarrow (\emptyset, \{b_1, b_2\}, \emptyset) \in \mathcal{T}$ . This result indicates that the TP-matroid is not a combination of three matroids. It is a new matroidal structure over three sets.

*Remark 5* If  $\mathcal{TI}$  is the family of feasible sets of a TP-matroid  $(U \times V \times W, \mathcal{TI})$ , then  $(U \times V \times W, \mathcal{TI})$  can be seen as a knowledge space by Definition 7 with  $\mathcal{TI}$  as the family of basic knowledge. Examples 1 and 2 indicate that in biology, some known knowledge on  $U \times V \times W$  may be used to construct the family of feasible sets of a TP-matroid  $(U \times V \times W, \mathcal{P})$ , where  $\mathcal{P} = \mathcal{TI}$  in Example 1 and  $\mathcal{P} = \mathcal{J}$  in Example 2, respectively.

Xu et al. [69] depicted a knowledge space for one universe as one of two types of knowledge structures is a knowledge space and closed under set union. Hence, to extend the rough set model of a knowledge space from one universe to three universes, the known knowledge should have a property similar to being closed under set union. Hence, we give the following definition.

**Definition 10** Let  $U, V$  and  $W$  be three sets such that at least one of  $U, V$  and  $W$  is a universe and  $\mathcal{A} \subseteq 2^U \times 2^V \times 2^W$ . If  $(X_1, Y_1, Z_1), (X_2, Y_2, Z_2) \in \mathcal{A}$  satisfy  $(X_1, Y_1, Z_1) \sqcup (X_2, Y_2, Z_2) \in \mathcal{A}$ , then,  $\mathcal{A}$  is called  $\sqcup$ -closed.

*Remark 6* (1) Let  $\mathcal{TI}$  be as in Example 1. Using Definition 10, we may easily show that  $\mathcal{TI}$  is  $\sqcup$ -

closed, although  $\mathcal{TI}$  is not a covering of  $(U, V, W)$  as shown in Example 1.

- (2) Let  $(U \times V \times W, \mathcal{J})$  be as in Example 2. Using Definition 10, we know that  $\mathcal{J}$  is not  $\sqcup$ -closed since  $(a_1, b_1, c_1) \sqcup (a_2, b_2, c_1) = (\{a_1, a_2\}, \emptyset, c_1) \notin \mathcal{J}$ , although  $\mathcal{J}$  is a covering of  $(U, V, W)$  as shown in Example 2.
- (3) (1) and (2) above imply that the definition of  $\sqcup$ -closed is independent from that of covering.

We will continue to discuss some relationships between matroids and TP-matroids.

**Lemma 3** Let  $U$  be a universe.

- (1) If  $(U \times \emptyset \times \emptyset, \mathcal{TI})$  is a TP-matroid, then  $(U, \mathcal{TI}(1))$  is a matroid in which  $\mathcal{TI}(1) = \{X \mid (X, \emptyset, \emptyset) \in \mathcal{TI}\}$ .
- (2) Let  $(U, \mathcal{I})$  be a matroid and  $U \neq \emptyset$ . If  $\mathcal{I}(3) = \{(X, Y, Z) \subseteq (U, \emptyset, \emptyset) \mid X \in \mathcal{I}\}$ , then  $(U \times \emptyset \times \emptyset, \mathcal{I}(3))$  is a TP-matroid.

The first property of Lemma 3 can be easily verified by Definition 1(1). The second property can be easily proven by Definition 9. These proofs are omitted.

Here, we stress the fact that  $\mathcal{TI}$  as given in Example 1 is  $\sqcup$ -closed, and  $\mathcal{J}$  as given in Example 2 is not  $\sqcup$ -closed. This fact implies that the family of feasible sets of a TP-matroid cannot always have the property of being  $\sqcup$ -closed. Combined with  $(U \times \emptyset \times \emptyset, \mathcal{I}(3))$  in Lemma 3(2), we believe the family of independent sets of a matroid  $(U, \mathcal{I})$  does not always have the property of being  $\sqcup$ -closed; that is,  $\mathcal{I}$  is not  $\sqcup$ -closed. This result is the same as in the discussion of  $\mathcal{I}$  in classical matroid theory [57, 58]. It also hints that there is an intimate relation between matroids and TP-matroids.

Using Lemma 3, we may easily obtain  $(U, \mathcal{I}(3)(1)) = (U, \mathcal{I}(3))$  since  $\mathcal{I}(3)(1) = \{X \mid (X, \emptyset, \emptyset) \in \mathcal{I}(3)\}$ . Furthermore, we obtain the following lemma.

**Lemma 4** Let  $U_j$  be a universe ( $j = 1, 2, 3, 4$ ).

- (1) Let  $(U_1 \times \emptyset \times \emptyset, \mathcal{TI}_1)$  and  $(U_2 \times \emptyset \times \emptyset, \mathcal{TI}_2)$  be two TP-matroids satisfying  $(U_1 \times \emptyset \times \emptyset, \mathcal{TI}_1) \cong (U_2 \times \emptyset \times \emptyset, \mathcal{TI}_2)$ . Then,  $(U_1, \mathcal{TI}_1(1)) \cong (U_2, \mathcal{TI}_2(1))$  holds.
- (2) Let  $(U_3, \mathcal{I}_3)$  and  $(U_4, \mathcal{I}_4)$  be two matroids such that  $(U_3, \mathcal{I}_3) \cong (U_4, \mathcal{I}_4)$ . Then,  $(U_3 \times \emptyset \times \emptyset, \mathcal{I}_3(3)) \cong (U_4 \times \emptyset \times \emptyset, \mathcal{I}_4(3))$  holds.

Lemma 4 can be easily verified with Definitions 1(2) and 9(3) and Lemma 3. The proof is omitted.

*Remark 7* Lemma 3 implies that a matroid on a universe  $U$  corresponds to a TP-matroid on  $U \times \emptyset \times \emptyset$ , and every TP-matroid on  $U \times \emptyset \times \emptyset$  corresponds to a matroid



on  $U$ . Lemma 4 implies that under isomorphism, the correspondences are unique.

Combining Lemmas 3 and 4, we may obtain the following theorem.

**Theorem 1** *The correspondence between a matroid  $(U, \mathcal{I})$  and a TP-matroid  $(U \times \emptyset \times \emptyset, \mathcal{I}(3))$  is a bijection between  $\mathbf{S}_1 = \{(U, \mathcal{I}) \mid U \text{ is a nonempty set}\}$  and  $\mathbf{S}_2 = \{(U \times \emptyset \times \emptyset, \mathcal{I}(3)) \mid U \text{ is a nonempty set}\}$  up to isomorphism for matroids and up to isomorphism for TP-matroids.*

*Remark 8* Since the structure of a TP-matroid  $(U \times \emptyset \times \emptyset, \mathcal{TI})$  is only a special kind of TP-matroid, combining this expression and Theorem 1, we determine that under isomorphism of matroids and isomorphism of TP-matroids, the definition of a TP-matroid is a generalization of the definition of a matroid. Hence, the TP-matroid is a matroidal structure over three sets.

### 3.2 Approximations generalized by TP-matroids

Section 3.1 generalizes the definition of a matroid from one set to three sets. Examples 1 and 2 imply that sometimes, the basic knowledge of some researchers is constructed by the feasible sets of a TP-matroid. In addition, some problems are solved by some matroidal structures [61–65, 76, 77]. Hence, we hope to solve some problems with the new matroidal structure—the TP-matroid. We note that utilizing a set of basic knowledge (or known knowledge) to infer unknown knowledge is a good and natural strategy. In fact, this inference corresponds to rough set theory. Using matroidal structures has already yielded many results on rough sets, and vice versa. Hence, it is necessary to explore the central content of rough sets—approximation operations—with the assistance of TP-matroids.

Considering Definitions 5, 6 and 7, Lemma 2 and Remark 6(1), we provide the following definitions.

**Definition 11** Let  $(U \times V \times W, \mathcal{TI})$  be a TP-matroid. Let  $(A, B, C) \subseteq (U, V, W)$ .

- (1)  $low(A, B, C) = \{(X, Y, Z) \in \mathcal{TI} \mid (X, Y, Z) \sqsubseteq (A, B, C)\}$ .
- (2)  $upr(A, B, C) = \{(X, Y, Z) \in \mathcal{TI} \mid X \cap A \neq \emptyset \text{ or } Y \cap B \neq \emptyset \text{ or } Z \cap C \neq \emptyset\}$ .
- (3)  $apr(A, B, C) = \left( \bigcup_{(X,Y,Z) \in low(A,B,C)} X, \bigcap_{(X,Y,Z) \in low(A,B,C)} Y, \bigcup_{(X,Y,Z) \in low(A,B,C)} Z \right)$ ;
- (4) If one of  $A, B$  and  $C$  is empty, then define  $\overline{apr}(A, B, C) = (U, \emptyset, W)$ .

If any of  $A, B$  and  $C$  is not empty, then define  $\overline{apr}(A, B, C) =$

$$\left( \bigcup_{(X,Y,Z) \in upr(A,B,C)} (X \cap A), \bigcup_{(X,Y,Z) \in upr(A,B,C)} (Y \cap B), \bigcup_{(X,Y,Z) \in upr(A,B,C)} (Z \cap C) \right).$$

*Remark 9* We now analyze Definition 11. Let  $(U \times V \times W, \mathcal{TI})$  be a TP-matroid.

- (1) We analyze items (1) and (3) in Definition 11 as follows.  
By Definition 9(1),  $\mathcal{TI}$  satisfies (I1) and (I2). From (I1), we can suppose  $(X_0, Y_0, Z_0) \in \mathcal{TI}$ . Then,  $(\emptyset, V, \emptyset) \sqsubseteq (X_0, Y_0, Z_0)$  and (I2) together imply  $(\emptyset, V, \emptyset) \in \mathcal{TI}$ . In addition,  $(\emptyset, V, \emptyset) \sqsubseteq (A, B, C)$  holds for any  $(A, B, C) \subseteq (U, V, W)$ . This means that  $(\emptyset, V, \emptyset) \in low(A, B, C)$ . Therefore,  $low(A, B, C) \neq \emptyset$  holds. This implies that the definition of  $apr(A, B, C)$  is well defined.
- (2) We analyze items (2) and (4) in Definition 11 as follows.  
(2.1) By Definition 3, we may easily obtain that  $(\{(A, B, C) \mid (A, B, C) \subseteq (U, V, W)\}, \sqsubseteq)$  is a poset with  $(U, \emptyset, W)$  as the maximum element. As a generalization of the upper approximation expressed in Lemma 2, we define  $\overline{apr}(X, Y, Z) = (U, \emptyset, W)$  for  $(X, Y, Z) \subseteq (U, V, W)$  if one of  $X, Y$  and  $Z$  is empty, in particular, if  $X = U, Y = \emptyset$  and  $Z = W$ . Hence,  $\overline{apr}(X, Y, Z) = (U, \emptyset, W)$  is reasonable in Definition 11 if one of  $X, Y$  and  $Z$  is empty.  
Because we have “ $A \neq \emptyset, B \neq \emptyset, C \neq \emptyset$ ”  $\Rightarrow$  “ $B \cap V \neq \emptyset$  since  $B \subseteq V$ ”, and  $(\emptyset, V, \emptyset) \in \mathcal{TI}$ , we obtain  $(\emptyset, V, \emptyset) \in upr(A, B, C)$  if any of  $A, B$  and  $C$  is not empty. This means that  $\overline{apr}(A, B, C)$  is well defined for the case of  $A \neq \emptyset, B \neq \emptyset$  and  $C \neq \emptyset$ .
- (2.2) Let  $U$  be the set of collected insect specimens of a group,  $V$  be the set of considered morphological characteristics, and  $W$  be the set of sources of collected specimens in  $U$ .

Let  $U = \emptyset$ . This means that a specimen could not be obtained, so no insect specimens were collected for research. This case is not valuable for biologists to research.

If  $V = \emptyset$ . This means that there are no morphological characteristics to be considered for the collected specimens. This will not occur in biological research, since any specimen must possess some morphological characteristics to be considered.

If  $W = \emptyset$ . This means that the sources of all the collected insect specimens in  $U$  are unknown. However, biologists generally know where the researched specimens were collected from. Even in special cases in which the source of a specimen is unknown, biologists will try to infer the source of the specimen. Hence,  $W \neq \emptyset$  holds if  $U \neq \emptyset$ .

The above analysis shows that  $U \neq \emptyset, V \neq \emptyset$  and  $W \neq \emptyset$  generally hold in scientific research.

In addition, if  $(A, B, C) \subseteq (U, V, W)$  is considered by biologists, then in general, we have  $A \neq \emptyset, B \neq \emptyset$  and  $C \neq \emptyset$ . Hence, if any of  $A, B$  and  $C$  is not empty for  $(A, B, C) \subseteq (U, V, W)$ , then biologists infer the properties of  $(A, B, C)$  using their known knowledge  $\mathcal{TI}$ , for example, known specimens, known morphological characteristics or known locations, to approximate  $(A, B, C)$ . This implies that the supposition of  $X \cap A \neq \emptyset$  or  $Y \cap B \neq \emptyset$  or  $Z \cap C \neq \emptyset$  in  $upr(A, B, C)$  is reasonable. Furthermore,  $\overline{apr}(A, B, C)$  is effective.

**Lemma 5** Let  $(U \times V \times W, \mathcal{TI})$  be a precovering TP-matroid. Then,  $upr(A, B, C) \neq \emptyset$  holds for  $\forall(A, B, C) \subseteq (U, V, W)$  such that one of  $A, B$  and  $C$  is not empty.

The proof of Lemma 5 can be found in the Appendix.

*Remark 10* We analyze the supposition in Lemma 5 on the basis of biological ideas.

Let  $U$  be a set of collected insect specimens of a group,  $V$  be the set of the considered morphological characteristics, and  $W$  be the set of locations of the collected specimens in  $U$ . Let  $(U \times V \times W, \mathcal{TI})$  be a TP-matroid, and let  $(A, B, C) \subseteq (U, V, W)$ .

- (1) Using the set  $\mathcal{TI}$  of basic biological knowledge to approximate  $(A, B, C)$  is a common method in biological research. If  $A = B = C = \emptyset$ , then according to the discussion in Remark 9(2), this case is not valuable for biologists. Therefore, we assume that at least one of  $A, B$  and  $C$  is not empty. That is, biologists pay much more attention to  $(A, B, C) \subseteq U \times V \times W \setminus (\emptyset, \emptyset, \emptyset)$ .
- (2) If  $A \cap X_\gamma = B \cap Y_\gamma = C \cap Z_\gamma = \emptyset$  for any  $(X_\gamma, Y_\gamma, Z_\gamma) \in \mathcal{TI}$ , then no known knowledge exists in  $\mathcal{TI}$  to infer the properties of  $(A, B, C)$ . During actual biological research, some known knowledge generally exists to infer the properties of  $(A, B, C)$ , or approximate  $(A, B, C)$ . Hence,  $(U \times V \times W, \mathcal{TI})$  should be precovering. That is, the supposition of the precovering of  $(U \times V \times W, \mathcal{TI})$  in Lemma 5 is suitable for biological research and more generally for research in real life.

We explore some properties of  $apr$  and  $\overline{apr}$  as characterized in Definition 11 to decide whether  $apr$  and  $\overline{apr}$  are a pair of lower and upper approximations defined on  $2^U \times 2^V \times 2^W$  according to Definition 8.

**Lemma 6** Let  $(U \times V \times W, \mathcal{TI})$  be a TP-matroid. Let  $apr$  and  $\overline{apr}$  be given as in Definition 11. Then, the following statements are correct for  $\forall(A, B, C) \subseteq (U, V, W)$ .

- (1) If one of  $A, B$  and  $C$  is empty, then  $(A, B, C) \subseteq \overline{apr}(A, B, C)$  holds.
- (2) Let  $(U \times V \times W, \mathcal{TI})$  be precovering. If any of  $A, B$  and  $C$  is not empty, then  $(A, B, C) \subseteq \overline{apr}(A, B, C)$  holds.
- (3) If any of  $A, B$  and  $C$  is not empty and  $(U \times V \times W, \mathcal{TI})$  is precovering, then  $(A, B, C) \in \mathcal{TI} \Rightarrow \overline{apr}(A, B, C) = (A, B, C)$  holds.
- (4) If  $(A, B, C) \in \mathcal{TI}$  and  $(A, B, C) = (U, \emptyset, W)$ , then  $\overline{apr}(A, B, C) = (A, B, C)$  holds.
- (5) If  $\mathcal{TI}$  is  $\sqcup$ -closed, then  $apr(A, B, C) = (A, B, C) \Rightarrow (A, B, C) \in \mathcal{TI}$  holds.
- (6)  $apr(A, B, C) \subseteq (A, B, C)$ .
- (7)  $(A, B, C) \in \mathcal{TI} \Rightarrow apr(A, B, C) = (A, B, C)$ .

The proof of Lemma 6 can be found in the Appendix.

*Remark 11* Let  $U$  be the set of collected insect specimens in a group,  $V$  be the set of considered morphological characteristics, and  $W$  be the set of the sources of collected specimens in  $U$ . Let  $X_j \subseteq U$  ( $j = 1, 2$ ). Let  $Y_j \subseteq V$  be the set of common morphological characteristics for any  $x \in X_j$  ( $j = 1, 2$ ). Then, the common morphological characteristics of  $X_1 \cup X_2$  must be contained in  $Y_1 \cap Y_2$ . With the increase in the number of locations, the chance of collecting specimens will increase. Thus, for  $(X_j, Y_j, Z_j) \subseteq (U, V, W)$  ( $j = 1, 2$ ), the definition of  $(X_1, Y_1, Z_1) \sqcup (X_2, Y_2, Z_2) = (X_1 \cup X_2, Y_1 \cap Y_2, Z_1 \cup Z_2)$  is useful in biology. Furthermore, the restricted condition of  $\mathcal{TI}$  is  $\sqcup$ -closed is similar to some ideas in biology. Hence, the supposition that  $\mathcal{TI}$  is  $\sqcup$ -closed in Lemma 6(5) is in line with typical biological ideas.

We next use an example to illustrate Definition 11 and Lemma 6.

*Example 3* Let  $(U \times V \times W, \mathcal{TI})$  be as given in Example 1. Let  $A = \{a_3\}, B = \{b_2\}$  and  $C = \{c_1\}$ . Then by Definition 11, we obtain the following results:

- (1)  $\{(X, Y, Z) \in \mathcal{TI} \mid X \cap A \neq \emptyset\} = \emptyset$ .
- (2)  $\{(X, Y, Z) \in \mathcal{TI} \mid Y \cap B \neq \emptyset\} = \{(a_j, \{b_1, b_2\}, \emptyset), j = 1, 2\} \cup \{(a_j, \{b_1, b_2\}, c_1), j = 1, 2\} \cup \{(\emptyset, \{b_1, b_2\}, c_1), (\emptyset, \{b_1, b_2\}, \emptyset), (\{a_1, a_2\}, \{b_1, b_2\}, \emptyset), (\{a_1, a_2\}, \{b_1, b_2\}, c_1)\}$ .
- (3)  $\{(X, Y, Z) \in \mathcal{TI} \mid Z \cap C \neq \emptyset\} = \{(a_j, b_1, c_1), (a_j, \{b_1, b_2\}, c_1), (j = 1, 2)\} \cup \{(\emptyset, b_1, c_1), (\emptyset, \{b_1, b_2\}, c_1), (\{a_1, a_2\}, b_1, c_1), (\{a_1, a_2\}, \{b_1, b_2\}, c_1)\}$ .

Hence, we obtain  $upr(A, B, C) = \{(X, Y, Z) \in \mathcal{TI} \mid Y \cap B \neq \emptyset\} \cup \{(X, Y, Z) \in \mathcal{TI} \mid Z \cap C \neq \emptyset\}$ , and furthermore,  $\overline{apr}(A, B, C) = (\emptyset, b_2, c_1)$ . In addition, using Definition 11, we obtain  $low(A, B, C) = \{(\emptyset, \{b_1, b_2\}, \emptyset), (\emptyset, \{b_1, b_2\}, c_1)\}$  and therefore  $apr(A, B, C) = (\emptyset, \{b_1, b_2\}, c_1)$ .

We have the following results from the above discussions:

$$(**1) \text{apr}(A, B, C) \sqsubseteq (A, B, C).$$

$$(**2) (A, B, C) \not\sqsubseteq \overline{\text{apr}}(A, B, C) \text{ since } A = \{a_3\} \not\subseteq \emptyset.$$

Result (\*\*1) implies the correctness of Lemma 6(6). Result (\*\*2) shows that if none of  $A$ ,  $B$  and  $C$  are empty, this does not imply that  $(A, B, C) \sqsubseteq \overline{\text{apr}}(A, B, C)$ . Using Example 1, we know that  $(U \times V \times W, \mathcal{TI})$  is not precovering. Hence, the supposition that  $(U \times V \times W, \mathcal{TI})$  is precovering is necessary to obtain the consequences in Lemma 6(2) and Lemma 6(3).

Let  $A_1 = \{a_1, a_2\}$ ,  $B_1 = \emptyset$  and  $C_1 = c_1$ , where  $a_1, a_2$  and  $c_1$  are defined as in Example 2. Then, we obtain the following result:

$$(**3) \text{apr}(A_1, B_1, C_1) = (A_1, B_1, C_1) \notin \mathcal{J}.$$

Combined with Example 2, we know that  $\mathcal{J}$  is not  $\sqcup$ -closed. The above result (\*\*3) implies that the condition  $\text{apr}(A, B, C) = (A, B, C)$  is necessary for the consequence of  $(A, B, C)$  to be feasible in Lemma 6(5).

We next perform an analysis combining Example 1 and Example 2 with Example 3.

Because the evolution of natural history is impossible to repeat, entomologists often use their known entomological knowledge to infer unknown content in their own research. Such inference is helpful for studying the distribution of insect populations, the formation of historical developments, and so on. It is particularly important for the targeted collection of specimens. For instance, in Table 2,  $a_3$ , i.e., the specimen japonica 3, is collected in  $c_2$ , i.e., China. Since the Korean Peninsula, to which Korea belongs, and China are connected by land, the entomologists in Example 1 and Example 2 hypothesize that if  $a_3$  is collected in  $c_1$ , i.e., Korea, it may also have the characteristic  $b_2$  that it currently has. This is represented by the set  $(A = a_3, B = b_2, C = c_1)$  in Example 3. We will see that (1) using his or her known knowledge  $\mathcal{TI}$ , the entomologist in Example 1 obtains the pessimistic result  $\text{apr}(A, B, C)$  of the hypothesis as  $(\emptyset, \{b_1, b_2\}, c_1)$  and the optimistic result  $\overline{\text{apr}}(A, B, C)$  as  $(\emptyset, b_2, c_1)$  (see Example 3). Both the first coordinates of  $\text{apr}(A, B, C)$  and  $\overline{\text{apr}}(A, B, C)$  are  $\emptyset$ ; that is, both of the corresponding sets of specimens of  $\text{apr}(A, B, C)$  and  $\overline{\text{apr}}(A, B, C)$  are  $\emptyset$ . This means that no conjectured specimens will appear. Therefore, this entomologist will not go to Korea, i.e.,  $c_1$ , to collect the specimen according to his or her hypothesis. (2) Using his or her known knowledge  $\mathcal{J}$ , the entomologist in Example 2 obtains the pessimistic result  $\text{apr}(A, B, C)$  of the hypothesis as  $(A, B, C)$  and the optimistic result  $\overline{\text{apr}}(A, B, C)$  as  $(A, B, C)$ . In other words, theoretically, he or she is convinced that the hypothesis is correct. (3) From Example 2, we find  $(A, B, C) \in \mathcal{J}$ . That is, the known knowledge of the entomologist in Example 2

completely covers  $(A, B, C)$ , but that of the entomologist in Example 1 does not since  $(A, B, C) \notin \mathcal{TI}$  holds in Example 1. This leads to the different conclusions of the two entomologists regarding the same hypothesis. In fact,  $(A, B, C) \in \mathcal{J}$  and  $(A, B, C) \notin \mathcal{TI}$  imply that the conclusion of the entomologist in Example 2 is more correct than that of the entomologist in Example 1. Therefore, the hypothesis should be true. (4) In fact, a similar analysis can be done for sets that can be represented in a ternary form  $(X, Y, Z)$ , where  $(X, Y, Z) \subseteq (U, V, W)$  and the three sets  $U, V$  and  $W$  are as given in Example 1. (5) Rough sets, an intelligent theory, are an effective tool for intelligent computing. (1)-(4) above show that the method proposed here, i.e., rough set approximation based on the TP-matroidal structure, is helpful and usable for the study of insect systematics, which includes the classification of insects. This also shows a practical application of the rough sets provided in this paper. Therefore, it is necessary to further discuss the rough set approximations provided here.

**Example 4** Let  $(U \times V \times W, \mathcal{TI})$  be a TP-matroid with  $U \neq \emptyset$  or  $W \neq \emptyset$ . Let  $(U, \emptyset, W) \in \mathcal{TI}$ . Then, we obtain  $(X, Y, Z) \in \mathcal{TI}$  for any  $(X, Y, Z) \subseteq (U, V, W)$  since  $(X, Y, Z) \subseteq (U, \emptyset, W)$  and (I2) holds. In particular, we obtain  $(\emptyset, \emptyset, \emptyset) \in \mathcal{TI}$ . That is,  $\mathcal{TI}$  is the family of all subsets of  $(U, V, W)$ . Thus, it is easy to see that  $\mathcal{TI}$  is  $\sqcup$ -closed and  $(U \times V \times W, \mathcal{TI})$  is precovering.

We may easily obtain  $\text{apr}(U, \emptyset, W) = \overline{\text{apr}}(U, \emptyset, W) = (U, \emptyset, W)$ . We also see that  $\text{apr}(\emptyset, \emptyset, \emptyset) = (\emptyset, \emptyset, \emptyset)$  and  $\overline{\text{apr}}(\emptyset, \emptyset, \emptyset) = (U, \emptyset, W) \neq (\emptyset, \emptyset, \emptyset)$ . Therefore, we have  $\text{apr}(\emptyset, \emptyset, \emptyset) \neq \overline{\text{apr}}(U, \emptyset, W)$  since one of  $U$  and  $W$  is not empty.

**Remark 12** On the one hand, Example 4 examines the correctness of Lemma 6(4). On the other hand, Example 4 shows that if one of  $A, B$  and  $C$  is empty in a precovering TP-matroid  $(U \times V \times W, \mathcal{TI})$  such that  $\mathcal{TI}$  is  $\sqcup$ -closed, then we cannot confirm  $\text{apr}(A, B, C) = \overline{\text{apr}}(A, B, C) = (A, B, C)$  even if  $(A, B, C) \in \mathcal{TI}$ .

By Definition 8 with the relationships between a covering and the feasible sets of a precovering TP-matroid, we obtain the following theorem by Lemmas 5 and 6.

**Theorem 2** Let  $(U \times V \times W, \mathcal{TI})$  be a precovering TP-matroid and  $\mathcal{TI}$  be  $\sqcup$ -closed. Let  $(A, B, C) \subseteq (U, V, W)$  satisfy  $A \neq \emptyset, B \neq \emptyset$  and  $C \neq \emptyset$ . Then,

- (1)  $\text{apr}(A, B, C) \sqsubseteq (A, B, C) \sqsubseteq \overline{\text{apr}}(A, B, C)$ .
- (2)  $(A, B, C) \in \mathcal{TI} \Leftrightarrow \text{apr}(A, B, C) = (A, B, C) = \overline{\text{apr}}(A, B, C)$ .

Using items (2) and (6) in Lemma 6, the proof of item (1) is straightforward. The proof of Theorem 2(2) can be found in the Appendix.

Using Theorem 2 and Definition 8, we find that  $\underline{apr}$  and  $\overline{apr}$  are indeed a pair of rough set approximations based on a precovering TP-matroid with a family of feasible sets that is  $\sqcup$ -closed. In what follows, we describe how to acquire information from  $\underline{apr}$  and  $\overline{apr}$  in Algorithms 1 and 2, respectively. In Algorithms 1 and 2, we need to visit  $n$  feasible sets; that is, the complexity of Algorithm 1 is  $O(n)$ , as is that of Algorithm 2.

---

**Input:**  $(U \times V \times W, \mathcal{TI} = \{(X_i, Y_i, Z_i), i = 1, \dots, n\})$ , a precovering TP-matroid with  $\mathcal{TI}$  being  $\sqcup$ -closed;  $(A, B, C)$ , an element in  $2^U \times 2^V \times 2^W \setminus \{(X, Y, Z) \mid (X = \emptyset) \vee (Y = \emptyset) \vee (Z = \emptyset)\}$ ;

**Output:**  $\underline{apr}(A, B, C)$ ;

- 1:  $i = 0, \underline{apr}(A, B, C) = (\emptyset, V, \emptyset)$ ;
- 2: Do
- 3: if  $(X_i, Y_i, Z_i) \sqsubseteq (A, B, C)$  then
- 4:  $\underline{apr}(A, B, C) = \underline{apr}(A, B, C) \sqcup (X_i, Y_i, Z_i)$ ;
- 5: else
- 6:  $\underline{apr}(A, B, C) = \underline{apr}(A, B, C)$ ;
- 7: end if
- 8:  $i = i + 1$ ;
- 9: DO while  $i \leq n$
- 10: Output  $\underline{apr}(A, B, C)$ ;

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**Algorithm 1** Acquiring lower approximation based on a precovering TP-matroid.

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**Input:**  $(U \times V \times W, \mathcal{TI} = \{(X_i, Y_i, Z_i), i = 1, \dots, n\})$ , a precovering TP-matroid with  $\mathcal{TI}$  being  $\sqcup$ -closed;  $(A, B, C)$ , an element in  $2^U \times 2^V \times 2^W \setminus \{(X, Y, Z) \mid (X = \emptyset) \vee (Y = \emptyset) \vee (Z = \emptyset)\}$ ;

**Output:**  $\overline{apr}(A, B, C)$ ;

- 1:  $i = 0, \overline{apr}(A, B, C) = (\emptyset, \emptyset, \emptyset)$ ;
- 2: Do
- 3: if  $X_i \cap A = \emptyset \& Y_i \cap B = \emptyset \& Z_i \cap C = \emptyset$  then
- 4:  $\overline{apr}(A, B, C) = \overline{apr}(A, B, C)$ ;
- 5: else
- 6:  $\overline{apr}(A, B, C) = \overline{apr}(A, B, C) \cup ((X_i, Y_i, Z_i) \cap (A, B, C))$ ;
- 7: end if
- 8:  $i = i + 1$ ;
- 9: DO while  $i \leq n$
- 10: Output  $\overline{apr}(A, B, C)$ ;

---

**Algorithm 2** Acquiring upper approximation based on a precovering TP-matroid.

*Remark 13* (1) From Definition 8 and Theorem 2, we can find that  $\underline{apr}$  and  $\overline{apr}$  are the lower and upper approximations generated by the family of feasible sets of a precovering TP-matroid  $(U \times V \times W, \mathcal{TI})$  such that  $\mathcal{TI}$  is  $\sqcup$ -closed.

(2) Considering Remark 11, we know that the definition of  $\sqcup$ -closed for  $\mathcal{TI}$  is in line with common ideas. In real cases, biologists and other researchers consider  $(A, B, C) \subseteq (U, V, W)$  satisfying  $A \neq \emptyset, B \neq \emptyset$  and  $C \neq \emptyset$ . Hence, the suppositions in Theorem 2 are valuable according to the ideas of biologists and other researchers.

(3) The outline of the process of searching the lower and upper approximations generated by a TP-matroid in this subsection is shown in Fig. 1.

The process in this section is as follows:

$(U \times V \times W, \mathcal{TI})$ , a TP-matroid  
 $\implies \underline{apr}(A, B, C), \overline{apr}(A, B, C)$ , a pair of operators relative to the approximations, where  $(A, B, C) \subseteq (U, V, W)$ .

$(U \times V \times W, \mathcal{TI})$ , a precovering TP-matroid, and  $\mathcal{TI}$ , a  $\sqcup$ -closed family

$\implies \underline{apr}(A, B, C), \overline{apr}(A, B, C)$ , a pair of approximation operators, where  $(A, B, C) \subseteq (U, V, W)$  and  $A \neq \emptyset, B \neq \emptyset, C \neq \emptyset$ .

The converse of the above process is considered in the next section.

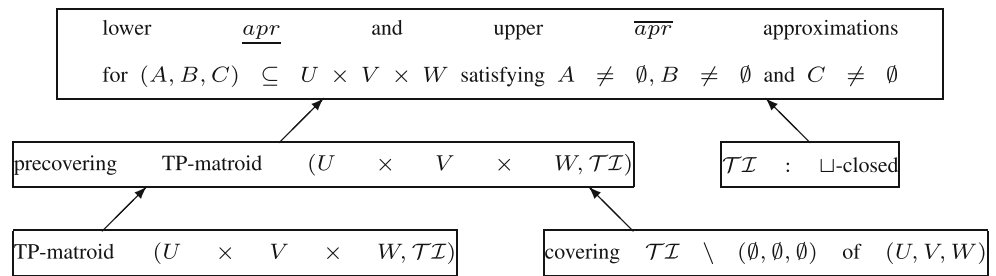
## 4 Approximations related to formal contexts

It is necessary to find matroidal structures with rough sets. This work has been done for a single universe, such as in [17]. The TP-matroid is established over three sets in Section 3, and determining how to build constructions of TP-matroids with rough set theory is now the task that we face. Using rough set theory, the first step of this work is to set up a pair of approximation operators. According to Definitions 5, 6, 7 and 8, the pair of approximation operators is based on a family of basic knowledge. We know that rough set theory and formal concept analysis are two important tools for dealing with data tables. This suggests that formal concept analysis may be helpful in our work. Therefore, in this section, we will construct TP-matroids with the help of some rough set approximations based on a kind of data table—a formal context.

We provide some preliminary definitions.

**Definition 12** Let  $U = U_1 \cup U_2 \cup \dots \cup U_n$  be a universe satisfying  $U_i \neq \emptyset$  and  $U_i \cap U_j = \emptyset (i \neq j; i, j = 1, 2, \dots, n)$ . Let  $V = \{b_j, j = 1, 2, \dots, m\}$  and  $W =$

**Fig. 1** Diagram of searching for lower and upper approximations from TP-matroids



$\{w_j, j = 1, 2, \dots, n\}$  be universes. Any two of  $U, V$  and  $W$  are disjoint.

- (1) For every  $w_j$ , there is a formal context  $\mathbb{K}_j = (U_j, V, R_j)$  relative to  $w_j (j = 1, 2, \dots, n)$ . The derivation operators of  $\mathbb{K}_j$  are denoted as  $'w_j (j = 1, 2, \dots, n)$ .
- (2) Let  $\{i_1, i_2, \dots, i_s\} \subseteq \{1, 2, \dots, n\}$  and  $1 \leq s \leq n$ ; the derivation operators in the formal context  $\mathbb{K}_{i_1 i_2 \dots i_s} = (U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_s}, V, R_{i_1 i_2 \dots i_s})$  are denoted as  $'w_{i_1 i_2 \dots i_s}$ , respectively, where  $R_{i_1 i_2 \dots i_s}$  is defined as: for  $x \in U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_s}$  and  $y \in V, xR_{i_1 i_2 \dots i_s}y \Leftrightarrow xR_jy$  if  $x \in U_j$  satisfies  $xR_jy$  for some  $j \in \{i_1, i_2, \dots, i_s\}$ .

*Remark 14* Let  $U = \bigcup_{j=1}^n U_j, V, W$  and  $\mathbb{K}_j (j = 1, 2, \dots, n)$  be as in Definition 12.

- (1)  $U \times V \times W$  can be decomposed into  $n$  different spaces  $U_j \times V \times w_j (j = 1, 2, \dots, n)$ . In other words,  $U \times V \times W$  is a combination of  $n$  different spaces  $U_j \times V \times w_j (j = 1, 2, \dots, n)$ , where  $(U, V, W) = (\bigcup_{j=1}^n U_j, V, \bigcup_{j=1}^n w_j)$ .
- (2) We analyze the formal context given in Definition 12 as follows.
  - (2.1) For  $w_j \in W$ , there is one and only one formal context  $\mathbb{K}_j = (U_j, V, R_j)$  corresponding to  $w_j$  since  $U_i \cap U_j = w_i \cap w_j = \emptyset (i \neq j; i, j = 1, 2, \dots, n)$ .

If  $w_i \neq w_j$ , then  $'w_i \neq 'w_j$  holds since  $U_i \cap U_j = \emptyset$  implies that  $x'w_j$  is not defined for any  $x \in U_i (i \neq j; i, j = 1, 2, \dots, n)$ . Furthermore, combining Lemma 1 and  $X = \bigcup_{x \in X} x \subseteq U_i$ , we know that  $X'w_j$  is not defined  $(i \neq j; i, j = 1, 2, \dots, n)$ .

(2.2) Let  $\{i_1, i_2, \dots, i_s\} \subseteq \{1, 2, \dots, n\}$ .  $xR_{i_1 i_2 \dots i_s}y$  means that there is one and only one  $j \in \{i_1, i_2, \dots, i_s\}$  such that  $xR_jy$  holds in the formal context  $\mathbb{K}_j = (U_j, V, R_j)$ , since  $U_p \cap U_q = \emptyset (p \neq q; p, q \in \{i_1, \dots, i_s\})$ .

We will use an example to show the existence of the formal contexts in Definition 12.

*Example 5* Table 4 shows some of the biological information in [75, Table 4].

Let  $a_1 := \text{japonica } 1, a_2 := \text{japonica } 2, a_3 := \text{neochlora } 1, a_4 := \text{neochlora } 2, a_5 := \text{neochlora } 3, a_6 := \text{antipoda sp. nov. } 1, a_7 := \text{antipoda sp. nov. } 2; b_1 := \text{'The number of teeth in the distal part'}, b_2 := \text{'The number of teeth in the proximate part'}; w_1 := \text{Korea}, w_2 := \text{China}, \text{ and } w_3 := \text{Australia}$ . Then, the mathematical expression of Table 4 is shown in Table 5.

From Table 5, we can obtain  $T_4$ , as shown in Table 6.

Using Algorithm 2 from [78] on  $T_4$ , we obtain a formal context  $\mathbb{K}^0 = (\{a_j, j = 1, \dots, 7\}, \{b_1, b_2\}, f)$ , where  $f \subseteq \{a_j, j = 1, \dots, 7\} \times \{b_1, b_2\}$  is shown in Table 7.

**Table 4** Some features of stridulatory files

Specimen	The number of teeth in the distal part	The number of teeth in the proximate part	Source/specimen, origin (scanning electron microscope, SEM)
Japonica 1	4(9)	61	Kim (2009): Korea, SEM
Japonica 2	6	57-60	CH7421-2: Korea ( $n = 2$ )
Neochlora 1	10	66	Shi et al. (2003): China, SEM
Neochlora 2	5	72	Shi et al. (2003): China, SEM
Neochlora 3	7	68	CH7670: China
Antipoda sp. nov. 1	12	45	CH4147: Australia
Antipoda sp. nov. 2	12	51	CH4148: Australia

**Table 5** Mathematical expression of Table 4

	$b_1$	$b_2$	
$a_1$	4(9)	61	$w_1$
$a_2$	6	57-60	$w_1$
$a_3$	10	66	$w_2$
$a_4$	5	72	$w_2$
$a_5$	7	68	$w_2$
$a_6$	12	45	$w_3$
$a_7$	12	51	$w_3$

Combining Tables 5 and 7, we obtain the expression of Table 5 with the language related to the formal context; see Table 8.

In Tables 7 and 8, ‘1’ means that  $a_i$  has  $b_j$ , and ‘0’ means that  $a_i$  does not have  $b_j$  ( $i = 1, 2, \dots, 7; j = 1, 2$ ). Let  $U = \{a_j, j = 1, 2, \dots, 7\}$ ,  $V = \{b_1, b_2\}$  and  $W = \{w_1, w_2, w_3\}$ . Then, based on  $w_1, w_2$  and  $w_3$ , we can obtain  $U_1 = \{a_1, a_2\}$ ,  $U_2 = \{a_3, a_4, a_5\}$  and  $U_3 = \{a_6, a_7\}$ , respectively. Hence, we obtain the formal context  $\mathbb{K}_j = (U_j, V, R_j)$  corresponding to  $w_j$  from Table 8; see Tables 9, 10, and 11 ( $j = 1, 2, 3$ ).

It is easy to see that

- (1)  $U = U_1 \cup U_2 \cup U_3 = \{a_j, j = 1, 2, \dots, 7\}; U_i \cap U_j = \emptyset (i \neq j; i, j = 1, 2, 3)$ .
- (2)  $x \in U \Leftrightarrow$  there is a unique  $j$  satisfying  $x \in U_j$  for some  $j \in \{1, 2, 3\}$ .
- (3)  $U \times V \times W = \bigcup_{j=1}^3 (U_j \times V \times w_j) = (\bigcup_{j=1}^3 U_j, V, \bigcup_{j=1}^3 w_j)$ .

In addition, we may easily obtain  $\mathbb{K}_{123} = (U, V, R_{123})$ , i.e., Table 12, such that for  $\forall x \in U$  and  $\forall y \in V, xR_{123}y \Leftrightarrow xR_jy$  if  $x \in U_j$  for some  $j \in \{1, 2, 3\}$ .

*Remark 15* (1) We can use any algorithm to change the information table expressed by  $T_4$  to a formal context and need not always use an algorithm such as the one in [78]. However, it is possible that the obtained formal context will not completely match Table 6. Even so,

**Table 6** A part  $T_4$  of Table 5

	$b_1$	$b_2$
$a_1$	4(9)	61
$a_2$	6	57-60
$a_3$	10	66
$a_4$	5	72
$a_5$	7	68
$a_6$	12	45
$a_7$	12	51

**Table 7** Formal context  $\mathbb{K}^0$

	$b_1$	$b_2$
$a_1$	1	1
$a_2$	1	1
$a_3$	1	1
$a_4$	0	0
$a_5$	1	0
$a_6$	0	0
$a_7$	0	0

this does not affect the research method and results provided in this paper.

- (2) Based on the source of the specimens, Table 4 can produce three formal contexts  $\mathbb{K}_j = (U_j, V, R_j)$  ( $j = 1, 2, 3$ ). In fact, biologists can discuss the relationships among specimens belonging to different locations to determine where their predecessors come from. Furthermore, it may be possible to find other biological content.

**Lemma 7** Let  $U, V, W$  be given as in Definition 12. Then

- (1)  $b'^{w_{i_1 \dots i_s}} = b'^{w_{i_1}} \cup \dots \cup b'^{w_{i_s}}$  for any  $b \in V$  and  $\{i_1, \dots, i_s\} \subseteq \{1, \dots, n\}$ .
- (2)  $Y'^{w_{i_1 \dots i_s}} = \bigcap_{y \in Y} (\bigcup_{j=1}^s y'^{w_{i_j}})$  for any  $Y \subseteq V$ .

The first property of Lemma 7 can be easily verified by Definitions 2 and 12. The second property can be easily verified by the combination of Lemma 1 and item (1). The proofs of these two items are omitted.

**Lemma 8** Let  $U, V$ , and  $W$  be given as in Definition 11. In the formal context  $\mathbb{K}_s = (U_s, V, R_s)$ , where  $s \in \{1, 2, \dots, n\}$ , we define a relation  $\sim_s$  on  $U_s$  as follows:  $a \sim_s b \Leftrightarrow a'^{w_s} = b'^{w_s}$ . Then,  $\sim_s$  is an equivalence on  $U_s$ . We use  $[a]_{R_s}$  to denote a category in  $\sim_s$  containing an element  $a \in U_s$ .

**Table 8** Formal context language’s expression corresponding to Table 5

	$b_1$	$b_2$	
$a_1$	1	1	$w_1$
$a_2$	1	1	$w_1$
$a_3$	1	1	$w_2$
$a_4$	0	0	$w_2$
$a_5$	1	0	$w_2$
$a_6$	0	0	$w_3$
$a_7$	0	0	$w_3$

Lemma 8 can be easily verified by Definition 4, and its proof is omitted.

We will use an example to show Lemma 8.

*Example 6* Let  $U_1, U_2, U_3, U, V, W, \mathbb{K}_1, \mathbb{K}_2,$  and  $\mathbb{K}_3$  be defined as in Example 5. Using Definition 2(1) on  $\mathbb{K}_1, \mathbb{K}_2,$  and  $\mathbb{K}_3,$  we obtain  $a_1^{w_1} = \{b_1, b_2\} = a_2^{w_1}, a_3^{w_2} = \{b_1, b_2\}, a_4^{w_2} = \emptyset, a_5^{w_2} = \{b_1\},$  and  $a_6^{w_3} = \emptyset = a_7^{w_3}.$  Combining Lemma 8, we obtain the following results:

- (1) on  $U_1 : [a_1]_{R_1} = \{a_1, a_2\} = [a_2]_{R_1};$
- (2) on  $U_2 : [a_3]_{R_2} = \{a_3\}, [a_4]_{R_2} = \{a_4\},$  and  $[a_5]_{R_2} = \{a_5\};$
- (3) on  $U_3 : [a_6]_{R_3} = \{a_6, a_7\} = [a_7]_{R_3}.$

**Definition 13** Let  $U_j, U, V, W,$  and  $\mathbb{K}_j (j = 1, \dots, n)$  be defined as in Definition 12. In  $\mathbb{K}_j, [a]_{R_j}$  is defined as in Lemma 8 for  $\forall a \in U_j (j = 1, \dots, n).$  Let  $(A, B, C) \subseteq (U, V, W).$  Let  $\mathcal{S} = \{(a, b_{l_0}, w_{i_0}) \mid \text{there are } w_{i_0} \in C \text{ and } b_{l_0} \in B \text{ such that } a \in b_{l_0}^{w_{i_0}} \neq \emptyset \text{ for some } a \in U_{i_0}, \text{ some } i_0 \in \{1, \dots, n\} \text{ and some } l_0 \in \{1, \dots, m\}\}.$  Let  $(\bigcup \mathcal{S}) \cap (\emptyset, B, \emptyset) = (\emptyset, \{b_{l_i}, i = 1, 2, \dots, t\}, \emptyset)$  and  $(\bigcup \mathcal{S}) \cap (\emptyset, \emptyset, C) = (\emptyset, \emptyset, \{w_{\alpha_j}, j = 1, \dots, \delta\}).$  If  $(a, b_{l_0}, w_{i_0}) \in \mathcal{S},$  then  $[a]_{R_{i_0}} \cap A$  and  $[a]_{R_{i_0}} \cup A$  are denoted as  $([a]_{R_{i_0}} \cap A)_{b_{l_0}}$  and  $([a]_{R_{i_0}} \cup A)_{b_{l_0}},$  respectively. We give the following definitions:

- (1)  $Low(A, B, C) = \{([a]_{R_{i_0}} \cap A)_{b_{l_0}}, b_{l_0}, w_{i_0} \mid (a, b_{l_0}, w_{i_0}) \in \mathcal{S}\}.$
- (2)  $Upr(A, B, C) = \{([a]_{R_{i_0}} \cup A)_{b_{l_0}}, b_{l_0}, w_{i_0} \mid (a, b_{l_0}, w_{i_0}) \in \mathcal{S}\}.$
- (3) If  $Low(A, B, C) = \emptyset,$  then define  $\underline{APr}(A, B, C) = (\emptyset, V, \emptyset).$
- (4) If  $Low(A, B, C) \neq \emptyset,$  then define  $\underline{APr}(A, B, C) = (\bigcap_{i=1}^t \bigcup_{j=1}^{\delta} ([a]_{R_{\alpha_j}} \cap A)_{b_{l_i}}, B, \bigcup_{j=1}^{\delta} w_{\alpha_j}).$
- (5) If  $Upr(A, B, C) = \emptyset,$  then define  $\overline{APr}(A, B, C) = (U, \emptyset, W).$
- (6) If  $Upr(A, B, C) \neq \emptyset,$  then define  $\overline{APr}(A, B, C) = (\bigcap_{i=1}^t \bigcup_{j=1}^{\delta} ([a]_{R_{\alpha_j}} \cup A)_{b_{l_i}}, \bigcup_{i=1}^t b_{l_i}, \bigcup_{j=1}^{\delta} w_{\alpha_j}).$

*Remark 16* Let  $U_j, U, V, W,$  and  $\mathbb{K}_j (j = 1, 2, \dots, n)$  be defined as in Definition 12. Let  $(A, B, C = \{w_{i_j}, j = 1, 2, \dots, |C|\}) \subseteq (U, V, W).$  Using Definition 13, we obtain the following:

- (1)  $Low(A, B, C) = \emptyset$  means that for any  $w_{i_j} \in C$  and every  $x \in U_{i_j},$  there is an  $x \notin b^{w_{i_j}}$  for any  $b \in B (j = 1, \dots, |C|).$  Combining Definition 2(1), we obtain  $B^{w_{i_j}} = \emptyset (j = 1, \dots, |C|).$  Therefore,  $B^{w_{i_1 \dots i_{|C|}}} = \emptyset$  holds by Lemma 7.

Similarly, we find that  $B^{w_{i_j}} = \emptyset$  and  $B^{w_{i_1 i_2 \dots i_{|C|}}} = \emptyset$  if  $Upr(A, B, C) = \emptyset.$

- (2) Clearly,  $(\emptyset, V, \emptyset)$  is the minimum element in the poset  $(\{(X, Y, Z) \mid (X, Y, Z) \subseteq (U, V, W)\}, \sqsubseteq)$  according to Definition 3 and the definition of  $\sqsubseteq.$  Hence, we define  $\underline{APr}(A, B, C) = (\emptyset, V, \emptyset)$  as reasonable in the case  $Low(A, B, C) = \emptyset$  by means of the definition of the lower approximation operator in Yao [74].  $(U, \emptyset, W)$  is the maximum element in the poset  $(\{(X, Y, Z) \mid (X, Y, Z) \subseteq (U, V, W)\}, \sqsupseteq)$  by Definition 3 and the definition of  $\sqsupseteq.$  Hence,  $\overline{APr}(A, B, C) = (U, \emptyset, W)$  is reasonable in the case of  $Upr(A, B, C) = \emptyset$  by the definition of the upper approximation operator in Yao [74].

We give an example of Definition 13 and Lemma 7.

*Example 7* Let  $U_j (j = 1, 2, 3), U, V,$  and  $W$  be given as in Example 5. Let  $A = \{a_2, a_3, a_6\} \subseteq U, B = \{b_1, b_2\} \subseteq V$  and  $C = \{w_1, w_2\} \subseteq W.$  By Example 5, we know that  $a_2 \in U_1, a_3 \in U_2$  and  $a_6 \in U_3.$  Since  $C = \{w_1, w_2\},$  we only consider  $\mathbb{K}_1$  and  $\mathbb{K}_2,$  which are given in Example 5.

In  $\mathbb{K}_1 = (U_1, V, R_1),$  we know that  $b_1^{w_1} = \{a_1, a_2\}$  and  $b_2^{w_1} = \{a_1, a_2\}.$

In  $\mathbb{K}_2 = (U_2, V, R_2),$  we know that  $b_1^{w_2} = \{a_3, a_5\}$  and  $b_2^{w_2} = a_3.$

Thus, we obtain

$$S = \{(a_i, b_1, w_1), i = 1, 2\} \cup \{(a_i, b_2, w_1), i = 1, 2\} \cup \{(a_3, b_1, w_2), (a_5, b_1, w_2), (a_3, b_2, w_2)\}.$$

By Definition 13 and Example 6, we know that  $[a_1]_{R_1} \cap A = [a_2]_{R_1} \cap A = a_2, [a_3]_{R_2} \cap A = a_3$  and  $[a_5]_{R_2} \cap A = \emptyset.$  Therefore, we obtain  $([a_2]_{R_1} \cap A)_{b_1} = a_2, ([a_2]_{R_1} \cap A)_{b_2} = a_2, ([a_3]_{R_2} \cap A)_{b_1} = a_3, ([a_5]_{R_2} \cap A)_{b_1} = \emptyset,$  and  $([a_3]_{R_2} \cap A)_{b_2} = a_3.$  In addition,  $b_{l_j} = b_j$  and  $w_{\alpha_j} = w_j (j = 1, 2).$  Thus,  $t = 2$  and  $\delta = 2.$  Hence, we obtain the following:

$$(\diamond 1) \text{ For } b_1: ([a_2]_{R_1} \cap A)_{b_1} \cup ([a_3]_{R_2} \cap A)_{b_1} \cup ([a_5]_{R_2} \cap A)_{b_1} = a_2 \cup a_3 \cup \emptyset = \{a_2, a_3\} = \bigcup_{j=1}^{\delta} ([a_2]_{R_{\alpha_j}} \cap A)_{b_1}.$$

$$(\diamond 2) \text{ For } b_2: ([a_2]_{R_1} \cap A)_{b_2} \cup ([a_3]_{R_2} \cap A)_{b_2} = a_2 \cup a_3 = \{a_2, a_3\} = \bigcup_{j=1}^{\delta} ([a_2]_{R_{\alpha_j}} \cap A)_{b_2}.$$

Furthermore, we obtain  $\bigcap_{i=1}^t (\bigcup_{j=1}^{\delta} ([a]_{R_{\alpha_j}} \cap A)_{b_i}) = \{a_2, a_3\} \cap \{a_2, a_3\} = \{a_2, a_3\}.$

In addition, we easily find that  $\bigcup_{j=1}^{\delta} w_{\alpha_j} = \{w_1, w_2\}.$  Therefore, we have  $\underline{APr}(A, B, C) = (\{a_2, a_3\}, \{b_1, b_2\}, \{w_1, w_2\}).$

Additionally, it is easy to see that  $([a_1]_{R_1} \cup A)_{b_1} = ([a_2]_{R_1} \cup A)_{b_1} = \{a_1, a_2, a_3, a_6\}, ([a_1]_{R_1} \cup A)_{b_2} = ([a_2]_{R_1} \cup A)_{b_2} = \{a_1, a_2, a_3, a_6\}.$

**Table 9** Formal context  $\mathbb{K}_1$

	$b_1$	$b_2$
$a_1$	1	1
$a_2$	1	1

$A)_{b_2} = \{a_1, a_2, a_3, a_6\}$ ,  $([a_3]_{R_2} \cup A)_{b_1} = \{a_2, a_3, a_6\}$ ,  $([a_5]_{R_2} \cup A)_{b_1} = \{a_2, a_3, a_5, a_6\}$ , and  $([a_3]_{R_2} \cup A)_{b_2} = \{a_2, a_3, a_6\}$ .

Considering the above, we obtain the following:

( $\diamond\Diamond 1$ ) For  $b_1$ :  $\bigcup_{j=1}^{\delta} ([a]_{R_{\alpha_j}} \cup A)_{b_1} = \{a_1, a_2, a_3, a_6\} \cup \{a_2, a_3, a_6\} \cup \{a_2, a_3, a_5, a_6\} = \{a_1, a_2, a_3, a_5, a_6\}$ .

( $\diamond\Diamond 2$ ) For  $b_2$ :  $\bigcup_{j=1}^{\delta} ([a]_{R_{\alpha_j}} \cup A)_{b_2} = \{a_1, a_2, a_3, a_6\} \cup \{a_2, a_3, a_6\} = \{a_1, a_2, a_3, a_6\}$ .

Furthermore, we obtain  $\bigcap_{i=1}^t (\bigcup_{j=1}^{\delta} ([a]_{R_{\alpha_j}} \cup A)_{b_i}) = \{a_1, a_2, a_3, a_5, a_6\} \cap \{a_1, a_2, a_3, a_6\} = \{a_1, a_2, a_3, a_6\}$ .

In addition, we easily find that  $\bigcup_{i=1}^t b_i = \{b_1, b_2\}$

and  $\bigcup_{j=1}^{\delta} w_{\alpha_j} = \{w_1, w_2\}$ . Hence,  $\overline{APr}(A, B, C) = (\{a_1, a_2, a_3, a_6\}, \{b_1, b_2\}, \{w_1, w_2\})$  holds. From the above results, we can obtain the following:

- (1)  $Low(A, B, C) \neq \emptyset$  and  $Upr(A, B, C) \neq \emptyset$ .
- (2)  $\underline{APr}(A, B, C) \sqsubseteq (A, B, C) \sqsubseteq \overline{APr}(A, B, C)$  holds since  $\{a_2, a_3\} \subseteq \{a_2, a_3, a_6\} = A \subseteq \{a_1, a_2, a_3, a_6\}$ ,  $\{b_1, b_2\} \supseteq \{b_1, b_2\} = B \supseteq \{b_1, b_2\}$ , and  $\{w_1, w_2\} \subseteq \{w_1, w_2\} = C \subseteq \{w_1, w_2\}$ .
- (3)  $B'^{w_{12}} = \{b_1, b_2\}'^{w_{12}} = (b_1'^{w_1} \cup b_1'^{w_2}) \cap (b_2'^{w_1} \cup b_2'^{w_2}) = \{a_1, a_2, a_3\}$ .

**Lemma 9** Let  $U, V$ , and  $W$  be given as in Definition 12. Let  $S, \{w_{\alpha_1}, \dots, w_{\alpha_\delta}\}, \{b_i, i = 1, \dots, t\}, \underline{APr}, \overline{APr}$  be given as in Definition 13. Let  $\mathbb{K}_{\alpha_1 \dots \alpha_\delta} = (\bigcup_{j=1}^{\delta} U_{\alpha_j}, V, R_{\alpha_1 \dots \alpha_\delta})$  be given as in Definition 12(2), and let  $\delta$  be given as in Definition 13. Then, we can obtain the following results for any  $(A, B, C) \sqsubseteq (U, V, W)$  such that  $A \neq \emptyset, B \neq \emptyset$ , and  $C \neq \emptyset$ :

- (1)  $Low(A, B, C) \neq \emptyset \Leftrightarrow Upr(A, B, C) \neq \emptyset$ .

**Table 10** Formal context  $\mathbb{K}_2$

	$b_1$	$b_2$
$a_3$	1	1
$a_4$	0	0
$a_5$	1	0

**Table 11** Formal context  $\mathbb{K}_3$

	$b_1$	$b_2$
$a_6$	0	0
$a_7$	0	0

- (2) Let  $B = \{b_{B_i}, i = 1, \dots, |B|\} \neq \emptyset$  and  $C = \{w_{C_j}, j = 1, \dots, |C|\} \neq \emptyset$ . Then,  $b_{B_i}^{w_{C_j}} = \emptyset, (i = 1, \dots, |B|; j = 1, \dots, |C|) \Leftrightarrow Low(A, B, C) = \emptyset \Leftrightarrow Upr(A, B, C) = \emptyset$ .
- (3) If  $Low(A, B, C) \neq \emptyset$ , then  $\underline{APr}(A, B, C) \cap (U, V, \emptyset) = (A \cap B'^{w_{\alpha_1 \dots \alpha_\delta}}, B, \emptyset)$  and  $\overline{APr}(A, B, C) \cap (U, \emptyset, \emptyset) = (A \cup B'^{w_{\alpha_1 \dots \alpha_\delta}}, \emptyset, \emptyset)$ .
- (4)  $\underline{APr}(A, B, C) = \overline{APr}(A, B, C) = (A, B, C) \Rightarrow (A, B) \in \mathcal{B}(\mathbb{K}_{\alpha_1 \dots \alpha_\delta})$ .
- (5) If  $(A, B) \in \mathcal{B}(\mathbb{K}_{\alpha_1 \dots \alpha_\delta}), Low(A, B, C) \neq \emptyset, t = |B|$  and  $\delta = |C|$ , then  $\underline{APr}(A, B, C) = \overline{APr}(A, B, C) = (A, B, C)$ .
- (6)  $\underline{APr}(A, B, C) \sqsubseteq (A, B, C); (A, B, C) \sqsubseteq \overline{APr}(A, B, C)$  if  $\delta = |C|$ .

The proof of Lemma 9 can be found in the Appendix.

**Remark 17** Let  $U_j, U, V, W, S, \{w_{\alpha_j}, j = 1, \dots, \delta\}, \{b_i, i = 1, \dots, t\}$ , and  $\mathbb{K}_{\alpha_1 \dots \alpha_\delta}$  be as in Lemma 9. Let  $(A, B, C) \sqsubseteq (U, V, W)$ .

If  $B = \emptyset$ , then  $S = \emptyset$ . This implies  $Low(A, B, C) = \emptyset$  and  $Upr(A, B, C) = \emptyset$ . If  $C = \emptyset$ , then  $\mathbb{K}_{\alpha_1 \dots \alpha_\delta}$  is not defined by Definitions 12 and 13.

In biology research,  $A = \emptyset$  means that no biological specimens are considered by biologists.  $B = \emptyset$  means that no biological characteristics are considered by biologists. These two cases do not have any value for biological research.  $C = \emptyset$  means that no locations of specimens are chosen. This has no value for biologists since  $W \neq \emptyset$  and  $C \subseteq W$ .

Hence, in the suppositions of Lemma 9, we require  $A \neq \emptyset, B \neq \emptyset$  and  $C \neq \emptyset$ .

**Table 12** Formal context  $\mathbb{K}_{123}$

	$b_1$	$b_2$
$a_1$	1	1
$a_2$	1	1
$a_3$	1	1
$a_4$	0	0
$a_5$	1	0
$a_6$	0	0
$a_7$	0	0



**Theorem 3** Let  $U_j$  ( $j = 1, \dots, n$ ),  $U, V$ , and  $W = \{w_j, j = 1, \dots, n\}$  be as given in Definition 12. Let  $\underline{APr}$  and  $\overline{APr}$ ,  $\{\alpha_1, \dots, \alpha_\delta\}$ , be as given in Definition 13, and let  $\mathbb{K}_{\alpha_1 \dots \alpha_\delta}$  be as given in Definition 12(2).

If for any  $w_j \in W$  and  $\mathbb{K}_j = (U_j, V, R_j)$  satisfies  $b'^{w_j} \neq \emptyset$  for every  $b \in V$  ( $j \in \{1, \dots, n\}$ ), then the following statements are correct for  $\forall (A, B, C) \subseteq (U, V, W)$  with  $A \neq \emptyset, B \neq \emptyset$  and  $C \neq \emptyset$ .

- (1)  $\underline{APr}(A, B, C) = \overline{APr}(A, B, C) = (A, B, C) \Leftrightarrow (A, B) \in \mathcal{B}(\mathbb{K}_{\alpha_1 \dots \alpha_\delta})$ .
- (2)  $\underline{APr}(A, B, C) \sqsubseteq (A, B, C) \sqsubseteq \overline{APr}(A, B, C)$ .

The proof of Theorem 3 can be found in the Appendix.

Considering Definition 8 and Theorem 3, we can determine that  $\underline{APr}$  and  $\overline{APr}$  are a pair of approximation operators.

We give an example to explain Theorem 3.

*Example 8* Let  $U_1 = \{a_1, a_2\}, U_2 = \{a_3, a_4, a_5\}, V = \{b_1, b_2\}, W = \{w_1, w_2\}$  be given as in Example 5. Let  $A = \{a_1, a_2, a_3\}, B = \{b_1, b_2\}$  and  $C = \{w_1, w_2\}$ . Then by Example 7, we know the following: for  $w_1: b_1'^{w_1} = \{a_1, a_2\} = b_2'^{w_1}$ ; for  $w_2: b_1'^{w_2} = \{a_3, a_5\}$  and  $b_2'^{w_2} = a_3$ .

Considering the above, we obtain  $\mathcal{S} = \{(a, b, w) \mid w \in C \text{ and } b \in B \text{ satisfy } a \in b'^w \neq \emptyset\} = \{(a_1, b_1, w_1), (a_1, b_2, w_1), (a_2, b_1, w_1), (a_2, b_2, w_1), (a_3, b_1, w_2), (a_5, b_1, w_2), (a_3, b_2, w_2)\}$ . Hence, we have  $\{b_i, i = 1, \dots, t\} = \{b_1, b_2\}$  and  $\{w_{\alpha_1}, \dots, w_{\alpha_\delta}\} = \{w_1, w_2\}$ .

Using Lemma 7(2), we have  $B'^{w_{12}} = \{a_1, a_2, a_3\}$  since  $\alpha_1 = 1$  and  $\alpha_\delta = 2$ . Hence,  $A = B'^{w_{12}}$  holds. Therefore, we confirm  $(A, B) \in \mathcal{B}(\mathbb{K}_{12})$  and  $\mathbb{K}_{12} = (U_1 \cup U_2, V, R_{12})$ . By Lemma 9(3), we obtain  $\underline{APr}(A, B, C) = (A, B, \bigcup_{j=1}^{\delta} w_{\alpha_j}) = (A, B, C)$  and  $\overline{APr}(A, B, C) = (A, \bigcup_{j=1}^t b_{l_j}, \bigcup_{j=1}^{\delta} w_{\alpha_j}) = (A, B, C)$ . This means that  $\underline{APr}(A, B, C) = \overline{APr}(A, B, C) = (A, B, C)$ .

*Remark 18* We analyze Lemma 9 and Theorem 3.

- (1) Considering Lemma 9(3),  $B'^{w_{\alpha_1 \dots \alpha_\delta}}$  plays an important role in determining  $\underline{APr}(A, B, C)$  and  $\overline{APr}(A, B, C)$ . For a given  $C \subseteq W$ ,  $\{w_{\alpha_1}, \dots, w_{\alpha_\delta}\} \subseteq C$  is known immediately. Furthermore,  $\mathbb{K}_{\alpha_1 \dots \alpha_\delta}$  is found at the same time. Hence, finding  $\underline{APr}(A, B, C)$  and  $\overline{APr}(A, B, C)$  relies on finding  $B'^{w_{\alpha_1 \dots \alpha_\delta}}$ . Combining items (4) and (5) in Lemma 9, we know that  $\underline{APr}$  and  $\overline{APr}$  can characterize the family  $\mathcal{B}(\mathbb{K}_{\alpha_1 \dots \alpha_\delta})$  of basic knowledge under some preconditions.

Using Definitions 7 and 13 with Theorem 3, we can say that  $\underline{APr}$  and  $\overline{APr}$  are the lower and upper approximations with respect to formal contexts  $\mathbb{K}_{\alpha_1 \dots \alpha_\delta}$  for  $(A, B, C) \subseteq (U, V, W) \setminus \{(X, Y, Z) \subseteq (U, V, W) \mid \text{at least one of } X, Y \text{ and } Z \text{ is } \emptyset\}$ .

Therefore, under some preconditions on  $(U, V, W)$ , we provide the lower and upper approximations in a ternary form to characterize  $\mathcal{B}(\mathbb{K}_{\alpha_1 \dots \alpha_\delta})$ .

- (2) Using Definition 8 and Theorem 3, we can say that  $\mathcal{B}(\mathbb{K}_{\alpha_1 \dots \alpha_\delta})$  is the family of basic knowledge used to approximate  $(A, B, C) \subseteq (U, V, W)$  for  $A \neq \emptyset, B \neq \emptyset$  and  $C \neq \emptyset$  with the rough set approximations  $\underline{APr}$  and  $\overline{APr}$ .
- (3) Using Theorem 3 and Lemma 9, we can roughly say that the definitions of  $\underline{APr}$  and  $\overline{APr}$  in Definition 13 are the generalizations of lower and upper approximations in Definition 8 from one universe to three sets with respect to formal contexts. We can also roughly say that  $\underline{APr}$  and  $\overline{APr}$  generalize the rough set approximations in [50] from two sets to three sets with respect to the family of semiconcepts in formal contexts.
- (4) Let  $U_j$  be the set of insect specimens of a group ( $j = 1, \dots, n$ ),  $V$  be the set of morphological characteristics considered by biologists, and  $W$  be the set of sources of specimens in  $U = \bigcup_{j=1}^n U_j$ .

By Lemma 9(2),  $Low(A, B, C) = \emptyset$  implies that  $b'^{w_j} = \emptyset$  for every  $b \in B$  and any  $w_j \in C$ . This implies that no specimen in  $A$  has any of the considered morphological characteristics in  $B$  for every specimen location in  $C$ . In this case, biologists will change their ideas, such as by changing the set of considered morphological characteristics, since they hope to obtain the real phylogenetic relationships or other biological relationships among the specimens. This requires  $Low(A, B, C) \neq \emptyset$ .

- (5) In a formal context  $\mathbb{K}$ ,  $(A, B) \in \mathcal{B}(\mathbb{K})$  means that  $A$  is the set of objects having the attributes in  $B$ . In biology, if  $A$  is a set of insect specimens in a group and  $B$  is a set of morphological characteristics considered by biologists, then  $(A, B) \in \mathcal{B}(\mathbb{K})$  means that every specimen in  $A$  jointly has every morphological characteristic in  $B$ . That is, every specimen in  $A$  jointly has the set of ancestral morphological characteristics in  $B$  if the biologists are studying biological properties such as phylogenesis for  $A$ . This demonstrates the importance of discussing  $\mathcal{B}(\mathbb{K})$  and of researching  $\underline{APr}(A, B, C) = \overline{APr}(A, B, C) = (A, B, C)$  according to Theorem 3.

In Section 3.2, we discuss how to construct a pair of approximation operators with the basic knowledge  $\mathcal{TL}$ ,

which is the feasible set of a TP-matroid. Now, we consider the converse, i.e., how to establish a TP-matroid with respect to the rough set approximation operators  $\underline{APr}$  and  $\overline{APr}$ .

**Theorem 4** Let  $U_j (j = 1, \dots, n), U, V, W$  and  $\mathbb{K}_{\alpha_1 \dots \alpha_\delta}$  be described as in Definition 12, in which  $\{\alpha_1, \dots, \alpha_\delta\}$  is as given in Definition 13. Let  $(A, B, C) \subseteq (U, V, W)$  satisfy  $Low(A, B, C) \neq \emptyset$ . Define  $\mathcal{TI}(\underline{APr}(A, B, C)) = \{(X, Y, Z) \subseteq (U, V, W) \mid (X, Y, Z) \subseteq \underline{APr}(A, B, C)\}$  and  $\mathcal{TI}(\overline{APr}(A, B, C)) = \{(X, Y, Z) \subseteq (U, V, W) \mid (X, Y, Z) \subseteq \overline{APr}(A, B, C)\}$ . Then,  $(U \times V \times W, \mathcal{TI}(\underline{APr}(A, B, C)))$  and  $(U \times V \times W, \mathcal{TI}(\overline{APr}(A, B, C)))$  are two TP-matroids.

Theorem 4 can be easily verified by combining Definition 9 and Definition 13, and its proof is omitted.

We discuss some properties of the two TP-matroids given in Theorem 4.

**Theorem 5** Let  $U = \bigcup_{j=1}^n U_j, V, W, \mathbb{K}_{\alpha_1 \dots \alpha_\delta}, \mathcal{TI}(\underline{APr}(A, B, C))$  and  $\mathcal{TI}(\overline{APr}(A, B, C))$  be given as in Theorem 4, in which  $(A, B, C) \subseteq (U, V, W)$  satisfies  $A \neq \emptyset, B \neq \emptyset, C \neq \emptyset$  and  $Low(A, B, C) \neq \emptyset$ . Then, we have the following:

- (1)  $\mathcal{TI}(\underline{APr}(A, B, C)) \subseteq \mathcal{TI}(\overline{APr}(A, B, C))$ .
- (2) If  $\mathbb{K}_j = (U_j, V, R_j)$  satisfies  $b^{w_j} \neq \emptyset$  for every  $b \in V (j = 1, \dots, n)$ , then  $(A, B) \in \mathcal{B}(\mathbb{K}_{\alpha_1 \dots \alpha_\delta}) \Leftrightarrow \mathcal{TI}(\underline{APr}(A, B, C)) = \mathcal{TI}(\overline{APr}(A, B, C))$ .

The proof of Theorem 5 can be found in the Appendix.

**Remark 19** (1) Example 7 shows  $\overline{APr}(A, B, C) \not\subseteq \underline{APr}(A, B, C)$  and  $\underline{APr}(A, B, C) \neq \overline{APr}(A, B, C)$  for some  $(A, B, C) \subseteq (U, V, W)$ . This implies

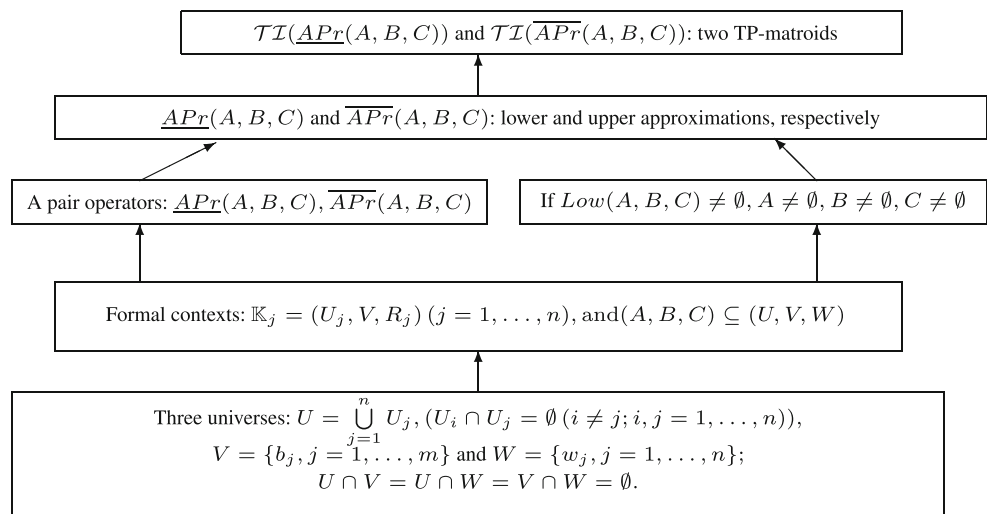
$\mathcal{TI}(\underline{APr}(A, B, C)) \neq \mathcal{TI}(\overline{APr}(A, B, C))$  since  $\overline{APr}(A, B, C) \notin \mathcal{TI}(\underline{APr}(A, B, C))$  holds by Example 7 and the definition of  $\mathcal{TI}(\underline{APr}(A, B, C))$  for  $(A, B, C) \subseteq (U, V, W)$  in Example 7. This demonstrates that the converse of Theorem 5(1) is not correct and shows the importance of Theorem 5(2).

- (2) Theorem 5 implies that the set of semiconcepts in the formal context  $\mathbb{K}_{\alpha_1 \dots \alpha_\delta}$  is characterized by the families of feasible sets of two TP-matroids  $(U \times V \times W, \mathcal{TI}(\underline{APr}(A, B, C)))$  and  $(U \times V \times W, \mathcal{TI}(\overline{APr}(A, B, C)))$ . The two TP-matroids are determined by the lower and upper approximations  $\underline{APr}(A, B, C)$  and  $\overline{APr}(A, B, C)$ , respectively. These facts indicate that studies of TP-matroids and approximation operators will have similar positions in research on knowledge-based fields. They also demonstrate the intimate relationships between matroid theory and rough set theory.
- (3) A sketch of the process of searching for TP-matroids in formal contexts is shown in Fig. 2.

In this paper, we present two pairs of operators related to rough set approximations over three sets:  $(\underline{apr}, \overline{apr})$  and  $(\underline{APr}, \overline{APr})$ . Next, we will explore the relationships between  $(\underline{apr}, \overline{apr})$  and  $(\underline{APr}, \overline{APr})$ , and we aim to determine under what conditions they are the same. Considering Remark 18(3) and Theorems 3, 4 and 5, we can obtain the following corollary.

**Corollary 1** Let  $U = U_1 \cup \dots \cup U_n, V, W = \{w_1, \dots, w_n\}$  and  $\mathbb{K}_j$  be defined as in Definition 12 ( $j = 1, \dots, n$ ). Let  $(A, B, C) \subseteq (U, V, W)$  satisfy  $A \neq \emptyset, B \neq \emptyset$  and  $C \neq \emptyset$ . Let  $\mathcal{S}, \{b_i, i = 1, \dots, t\}, \{w_{\alpha_1}, \dots, w_{\alpha_\delta}\}, \underline{APr}, \overline{APr}$  be as given in Definition 13, and let  $\mathbb{K}_{\alpha_1 \dots \alpha_\delta}$  be as given in Definition 12.

**Fig. 2** Diagram of searching for TP-matroids in formal contexts



Suppose that  $\mathcal{S}$  is a covering of  $(U, V, W)$ . If  $\mathbb{K}_j = (U_j, V, R_j)$  satisfies  $b^{w_j} \neq \emptyset$  for every  $b \in V$  ( $j = 1, \dots, n$ ), then the following statements are correct.

- (1) Let  $\underline{apr}$  and  $\overline{apr}$  be the rough set approximations generated by  $(U \times V \times W, \mathcal{TI}(\underline{APr}(A, B, C)))$  as given in Definition 11. Then, they satisfy  $\overline{apr}(A, B, C) = \underline{apr}(A, B, C) = \underline{APr}(A, B, C)$ .
- (2) Let  $(\underline{apr}, \overline{apr})$  be the pair of rough approximations generated by  $(U \times V \times W, \mathcal{TI}(\overline{APr}(A, B, C)))$  as given in Definition 11. Then,  $(A, B) \in \mathcal{B}(\mathbb{K}_{\alpha_1 \dots \alpha_n}) \Leftrightarrow \overline{apr}(A, B, C) = \underline{apr}(A, B, C) = \overline{APr}(A, B, C) = \underline{APr}(A, B, C) = \overline{APr}(A, B, C) = (A, B, C)$ .

The proof of Corollary 1 can be found in the Appendix. We will use an example to illustrate Corollary 1.

**Example 9** Let  $U_1 = \{a_1, a_2\}, U_2 = \{a_3, a_5\}, V = \{b_1, b_2\}$ , and  $W = \{w_1, w_2\}$  be given in Example 5. It is clear that (1)  $b^{w_j} \neq \emptyset$  for every  $b \in V$  ( $j = 1, 2$ ) and (2) aided by Example 8, we obtain  $\mathcal{S} = \{(a, b, w) \mid w \in C \text{ and } b \in B \text{ satisfy } a \in b^w \neq \emptyset\} = \{(a_1, b_1, w_1), (a_1, b_2, w_1), (a_2, b_1, w_1), (a_2, b_2, w_1), (a_3, b_1, w_2), (a_3, b_2, w_2), (a_5, b_1, w_2)\}$ . Therefore, it follows that  $\cup \mathcal{S} = (\{a_1, a_2, a_3, a_5\}, \{b_1, b_2\}, \{w_1, w_2\})$ . That is,  $\mathcal{S}$  is a covering of  $(U = U_1 \cup U_2, V, W)$ .

Let  $A = \{a_1, a_2, a_3\}, B = \{b_1, b_2\}$  and  $C = \{w_1, w_2\}$ . Considering Example 8, we may easily obtain  $\underline{APr}(A, B, C) = (A, B, C) = \overline{APr}(A, B, C)$ . Thus, using Theorem 4, we obtain  $\mathcal{TI}(\underline{APr}(A, B, C)) = \{(X, Y, Z) \subseteq (U = U_1 \cup U_2, V, W) \mid (X, Y, Z) \sqsubseteq \underline{APr}(A, B, C)\} = \{(X, \{b_1, b_2\}, Z) \mid X \subseteq A, Z \subseteq C\}$ . Furthermore, considering Definition 11, we confirm that  $low(A, B, C) = \{(X, Y, Z) \in \mathcal{TI}(\underline{APr}(A, B, C)) \mid (X, Y, Z) \sqsubseteq (A, B, C)\} = \mathcal{TI}(\underline{APr}(A, B, C))$  and  $upr(A, B, C) = \{(X, \{b_1, b_2\}, Z) \in \mathcal{TI}(\underline{APr}(A, B, C)) \mid (X \subseteq A \text{ and } X \neq \emptyset) \text{ or } (Z \subseteq C \text{ and } Z \neq \emptyset)\}$ .

Using Definition 11, we obtain  $\underline{apr}(A, B, C) = (\bigcup_{(X,Y,Z) \in low(A,B,C)} X, \bigcap_{(X,Y,Z) \in low(A,B,C)} Y, \bigcup_{(X,Y,Z) \in low(A,B,C)} Z) = (\bigcup_{X \subseteq A} X, \{b_1, b_2\}, \bigcup_{Z \subseteq C} Z) = (A, \{b_1, b_2\}, C) = (A, B, C)$  and  $\overline{apr}(A, B, C) = (\bigcup_{X \neq \emptyset, X \subseteq A} X, \{b_1, b_2\}, \bigcup_{Z \neq \emptyset, Z \subseteq C} Z) = (A, B, C)$ . Hence, we obtain  $\underline{apr}(A, B, C) = \overline{apr}(A, B, C) = \underline{APr}(A, B, C)$ . That is, item (1) in Corollary 1 is confirmed.

By Theorem 4, we obtain  $\mathcal{TI}(\overline{APr}(A, B, C)) = \{(X, Y, Z) \subseteq (U, V, W) \mid (X, Y, Z) \sqsubseteq \overline{APr}(A, B, C) = (A, B, C)\} = \{(X, \{b_1, b_2\}, Z) \mid X \subseteq A, Z \subseteq C\}$ .

$C\}$ . In view of Definition 11, we may easily obtain  $\underline{apr}(A, B, C) = \overline{apr}(A, B, C) = (A, B, C)$ . Moreover, we arrive at  $\underline{apr}(A, B, C) = \overline{apr}(A, B, C) = \overline{APr}(A, B, C) = \underline{APr}(A, B, C) = \overline{APr}(A, B, C) = \underline{APr}(A, B, C) = (A, B, C)$ . Considering  $w_{\alpha_1} = w_1, w_{\alpha_2} = w_2, \alpha_1 = 1, \alpha_2 = 2$  and the formal context language expression corresponding to  $(U, V, W)$  with Example 5, we obtain  $U = \{a_1, a_2, a_3, a_5\}, V = \{b_1, b_2\}, W = \{w_1, w_2\}$  and the formal context  $\mathbb{K}_{\alpha_1 \alpha_2}$  in Table 13 below.

We may easily show that  $B^{w_{12}} = \{b_1, b_2\}^{w_{12}} = \{a_1, a_2, a_3\} = A$ . By Definition 2(2) and Remark 2, this means that  $(A, B) \in \mathcal{B}(\mathbb{K}_{12})$ . Therefore, item (2) in Corollary 1 is confirmed.

**Remark 20** Let  $U, V$ , and  $W$  be as given in Definition 12, and let  $\mathcal{S}$  be as given in Definition 13. Let  $(A, B, C) \subseteq (U, V, W)$ .

- (1) If there is an  $a_0 \in U$  satisfying  $a_0 \notin b^{w_0}$  for any  $b \in V$  and every  $w_0 \in W$ , then  $(a_0, b, w_0) \notin \mathcal{S}$  holds. Therefore,  $\mathcal{S}$  is not a covering of  $(U, V, W)$ .

If  $a_0 \in A \neq \emptyset$  satisfies  $a_0 \notin b^{w_0}$  for any  $b \in V$  and every  $w_0 \in W$ , then we obtain  $Low(A, B, C) = Low(A \setminus a_0, B, C)$ , and furthermore,  $\underline{APr}(A, B, C) = \underline{APr}(A \setminus a_0, B, C)$ . Therefore, we determine that  $\mathcal{TI}(\underline{APr}(A, B, C)) = \mathcal{TI}(\underline{APr}(A \setminus a_0, B, C))$ . Hence,  $(U \times V \times W, \mathcal{TI}(\underline{APr}(A, B, C)))$  is not a precovering TP-matroid.

In addition, the above results show that  $a_0 \in A \neq \emptyset$  is not reasonable for the properties of  $\mathcal{TI}(\underline{APr}(A, B, C))$ . Therefore, in the assumptions of Corollary 1, we suppose  $\mathcal{S}$  to be a covering of  $(U, V, W)$ .

- (2) Let  $A \neq \emptyset, B \neq \emptyset$ , and  $C \neq \emptyset$ , and let  $\mathcal{S}$  be a covering of  $(U, V, W)$ . Suppose that every  $\mathbb{K}_j$  satisfies the given condition as in Corollary 1 ( $j = 1, \dots, n$ ).

On the one hand, according to Theorem 4, we know that  $\mathcal{TI}(\underline{APr}(A, B, C)) = \{(X, Y, Z) \mid (X, Y, Z) \sqsubseteq \underline{APr}(A, B, C)\}$ . Considering the proof in Corollary 1, we know that  $\underline{APr}(A, B, C) = (A \cap B^{w_{\alpha_1 \dots \alpha_n}}, B, C)$ . Let  $(X_1, Y_1, Z_1), (X_2, Y_2, Z_2) \in \mathcal{TI}(\underline{APr}(A, B, C))$ . Then, we have  $(X_j, Y_j, Z_j) \sqsubseteq \underline{APr}(A, B, C)$  ( $j = 1, 2$ ). We

**Table 13** Formal context  $\mathbb{K}_{12}$

	$b_1$	$b_2$
$a_1$	1	1
$a_2$	1	1
$a_3$	1	1
$a_5$	1	0

obtain  $(X_1, Y_1, Z_1) \sqcup (X_2, Y_2, Z_2) = (X_1 \cup X_2, Y_1 \cap Y_2, Z_1 \cup Z_2) \sqsubseteq (A \cap B^{w_{\alpha_1 \dots \alpha_\delta}}, B, C)$  since  $X_j \subseteq A \cap B^{w_{\alpha_1 \dots \alpha_\delta}}, Y_j \supseteq B$  and  $Z_j \subseteq C (j = 1, 2)$ . Thus,  $\mathcal{TI}(\underline{APr}(A, B, C))$  is  $\sqcup$ -closed by Definition 10.

Analogously, we determine  $\mathcal{TI}(\overline{APr}(A, B, C))$  to be  $\sqcup$ -closed according to Theorem 4 and Definition 10.

However, from the proof of Corollary 1, we know that  $Low(A, B, C) \neq \emptyset$ . Taking this result and Lemma 9(3), we conclude that  $\underline{APr}(A, B, C) = (A \cap B^{w_{\alpha_1 \dots \alpha_\delta}}, B, C) \subseteq (A, B, C) \subseteq (U, V, W)$  and  $\overline{APr}(A, B, C) =$

$(A \cup B^{w_{\alpha_1 \dots \alpha_\delta}}, B, C) \not\subseteq (A, B, C)$  since  $A \subseteq A \cup B^{w_{\alpha_1 \dots \alpha_\delta}}$ . Thus, we determine that

(†1)  $\mathcal{TI}(\underline{APr}(A, B, C))$  is not a covering of  $(U, V, W)$  since it generally does not satisfy  $A \cap B^{w_{\alpha_1 \dots \alpha_\delta}} = U$ .

(†2)  $\mathcal{TI}(\overline{APr}(A, B, C))$  is not a covering of  $(U, V, W)$  since it generally does not satisfy  $A \cup B^{w_{\alpha_1 \dots \alpha_\delta}} = U$ .

Therefore,  $(U \times V \times W, \mathcal{TI}(\underline{APr}(A, B, C)))$  and  $(U \times V \times W, \mathcal{TI}(\overline{APr}(A, B, C)))$  may not be precovering TP-matroids.

The above analysis of two cases with Corollary 1 indicates that for a TP-matroid  $(U \times V \times W, \mathcal{TI})$  and  $(A, B, C) \subseteq (U, V, W)$  satisfying  $A \neq \emptyset, B \neq \emptyset$  and  $C \neq \emptyset$ , if we assume that results (1) and (2) in Theorem 2 are correct, then we cannot determine  $(U \times V \times W, \mathcal{TI})$  to be a precovering TP-matroid. That is, we cannot determine the correctness of the converse proposition of Theorem 2. Hence, we cannot use Theorem 2 in the proof of Corollary 1. This result demonstrates that the two pairs of rough set approximations provided in this paper are different. Each of them has its own distinguishing features. They are two different kinds of generalizations of Pawlak’s classical rough set approximations.

(3) Using the analysis in Remark 19(1) and Theorem 5, we believe that in general,  $\underline{apr}(A, B, C) \neq \overline{APr}(A, B, C)$  and  $\overline{\underline{apr}}(A, B, C) \neq \overline{\overline{APr}}(A, B, C)$  hold since  $low(\overline{A, B, C}) = \{(X, Y, Z) \in \mathcal{TI}(\overline{APr}(A, B, C)) \mid (X, Y, Z) \sqsubseteq (A, B, C)\}$  and  $upr(\underline{A, B, C}) = \{(X, Y, Z) \in \mathcal{TI}(\underline{APr}(A, B, C)) \mid X \cap A \neq \emptyset \text{ or } Y \cap B \neq \emptyset \text{ or } Z \cap C \neq \emptyset\}$ .

(‡1)  $\underline{APr}(A, B, C)$  is the maximum element in  $(\mathcal{TI}(\underline{APr}(A, B, C)), \sqsubseteq)$ .

(‡2)  $\overline{APr}(A, B, C) \not\subseteq (A, B, C) \Rightarrow \overline{\overline{APr}}(A, B, C) \notin low(\overline{A, B, C})$

$$\Rightarrow \underline{\underline{apr}}(A, B, C) \cap (U, \emptyset, \emptyset) \sqsubseteq (A, \emptyset, \emptyset) \sqsubseteq \overline{\overline{APr}}(A, B, C) \cap$$

$$(U, \emptyset, \emptyset) = \left( \bigcap_{i=1}^t \bigcup_{j=1}^\delta ([a]_{R_{\alpha_j}} \cup A)_{b_i}, \emptyset, \emptyset \right),$$

$$\left( \bigcup_{(X,Y,Z) \in upr(A,B,C)} (X \cap A), \emptyset, \emptyset \right) \sqsubseteq (A, \emptyset, \emptyset) \sqsubseteq$$

$$\left( \bigcap_{i=1}^t \bigcup_{j=1}^\delta ([a]_{R_{\alpha_j}} \cup A)_{b_i}, \emptyset, \emptyset \right) = \overline{\overline{APr}}(A, B, C) \cap (U, \emptyset, \emptyset),$$

$$\left( \bigcup_{(X,Y,Z) \in upr(A,B,C)} (X \cap A), \emptyset, \emptyset \right) = \overline{\underline{\underline{apr}}}(A, B, C) \cap$$

$$(U, \emptyset, \emptyset) \sqsubseteq \overline{\underline{\underline{apr}}}(A, B, C) \cap (U, \emptyset, \emptyset).$$

Corollary 1 demonstrates that if  $(A, B)$  is a semiconcept in the formal context  $\mathbb{K}_{\alpha_1 \dots \alpha_\delta} = (\bigcup_{j=1}^\delta U_j, V, R_{\alpha_1 \dots \alpha_\delta})$ , then

$$\overline{\underline{\underline{apr}}}(A, B, C) = \underline{\underline{apr}}(A, B, C) = \overline{\overline{APr}}(A, B, C) \text{ holds.}$$

Corollary 1 also shows the linkage between semiconcepts and the two kinds of rough set approximations provided in this paper. Since the theory of semiconcepts belongs to the research field of formal concept analysis, we use formal concept analysis to build TP-matroids based on a pair of approximation operators. Therefore, the work described at the beginning of this subsection is completed.

### 5 Conclusion and future work

This paper provides a new mathematical structure—the TP-matroid. It shows that a TP-matroid is a generalization of a matroid from one set to three sets up to isomorphism. Furthermore, using the structure of the TP-matroid and the covering of a set, we provide a precovering TP-matroid over three sets. To precover TP-matroids over three sets, we search for a pair of rough set approximations in Section 3.2. The method used here is different from already existing methods of establishing rough set approximations with matroidal structures [13, 14, 17, 59, 60], since those methods consider matroidal structures over one set, and our structures are over three sets  $U, V$  and  $W$ ; that is, their structures are in one-dimensional space, and ours are in three-dimensional space. In fact, one set  $U$  is a subset of three sets  $(U, V, W)$  up to set isomorphism since  $U \cong (U, \emptyset, \emptyset) \subseteq (U, V, W)$ . Under this idea, we can say that TP-matroids are a generalization of the matroids in [13, 14, 17, 59, 60]. In Section 4, we study some properties of rough set approximations over three sets with respect to formal contexts. All expressions here are different from those in [34–40] since our expressions are in ternary form and theirs are over two universes; our expressions are also different from those in [36] since the model of rough sets in [36] is relation-based and ours is covering-based. However, both the results here and the research results in [34–40] are based on some practical needs and are generalizations of Pawlak’s classical rough set approximations. That is, the research here may be applied in more practical studies, which is one of the goals of this paper. Furthermore, the proposal of TP-matroids enabling some rough set approximations to extract information on three sets with the help of the covering idea is a highlight of the paper.

Using a pair of approximation operators aided by formal concept analysis, we build up two TP-matroids. Regarding other pairs of approximation operators such as that used in [34] to build up TP-matroids, we hope that the ideas presented here can assist in exploring these researches.

Im et al. [79] discussed a new matroidal structure—the matroid cup game on  $\mathbb{R}^n$ , or on a kind of  $n$ -dimensional space. How can TP-matroids be generalized to an  $n$ -dimensional space  $U_1 \times \dots \times U_n$ , where  $U_i$  is a set ( $i = 1, \dots, n > 3$ ) such that at least one of  $U_i$  is a universe ( $i = 1, \dots, n$ )? We now try to solve this problem as follows. Let  $\mathcal{I} \subseteq \{(X_1, \dots, X_n) \mid X_j \subseteq U_j, j = 1, \dots, n\}$  satisfy the following conditions:

- (n1)  $\mathcal{I} \neq \emptyset$ .
- (n2) Let  $(X_1, \dots, X_n), (Y_1, \dots, Y_n) \subseteq U_1 \times \dots \times U_n$ .  $(X_1, \dots, X_n) \sqsubseteq \sqsubseteq (Y_1, \dots, Y_n) \in \mathcal{I} \Rightarrow (X_1, \dots, X_n) \in \mathcal{I}$ , where  $(X_1, \dots, X_n) \sqsubseteq \sqsubseteq (Y_1, \dots, Y_n)$  if and only if  $X_i \subseteq Y_i, X_j \supseteq Y_j$  for  $(i, j \in \{1, \dots, n\}; i$  is odd, and  $j$  is even).
- (n3) Let  $(X_1, \dots, X_n), (Y_1, \dots, Y_n) \in \mathcal{I}$ . Then  $|(X_1, \dots, X_n)| = \sum_{i=1}^n |X_i| < \sum_{i=1}^n |Y_i| = |(Y_1, \dots, Y_n)|$

$\Rightarrow \exists (y_1, \dots, y_n) \in (Y_1, \dots, Y_n) \setminus (X_1, \dots, X_n) = ((Y_1 \setminus X_1, \dots, Y_n \setminus X_n)$  satisfies  $(X_1, \dots, X_n) \cup (y_1, \dots, y_n) \in \mathcal{I}$ , where  $(y_1, \dots, y_n) \neq \emptyset$ . Then,  $(U_1 \times \dots \times U_n, \mathcal{I})$  is a matroidal structure, called an  $n$ -partial matroid or simply an  $np$ -matroid.

Comparing the  $np$ -matroid with matroid cup game, we find the following:

- (1)  $U_i$  can be different, but every  $\mathbb{R}_i$  is the same as  $\mathbb{R}$  ( $i = 1, \dots, n$ ).
- (2) The matroid cup game solves the  $n$ -cup game used in practice. What are the practical needs of the  $np$ -matroid?
- (3) How can  $np$ -matroids be used to simulate a continuous process such as that of Im et al. [79]?

The questions raised in (2) and (3) will be answered in the future.

Additionally, in the future, we will consider the following work:

(\*) It is well known that matroid theory provides a good platform for designing greedy algorithms, which are used widely in practice. How can a greedy algorithm be designed for a TP-matroid? How can this greedy algorithm be used to solve some problems in rough set theory that are NP-hard?

(\*\*) Sun and Ma [36] set up a fuzzy rough set model over multiple universes based on relations. How can we establish a covering-based rough set model over multiple universes

and explore the relationships between the covering-based rough set model over multiple universes and that in [36]?

## Appendix A

### A.1: Proof of Lemma 5

*Proof* The precovering of  $(U \times V \times W, \mathcal{TI})$  and Definition 9 together imply  $\bigcup \mathcal{TI} = (U, V, W)$  and  $U \neq \emptyset$ , or  $V \neq \emptyset$ , or  $W \neq \emptyset$ . Thus, we find  $A \cap X_{\gamma_1} \neq \emptyset$  if  $A \neq \emptyset$ ,  $B \cap Y_{\gamma_2} \neq \emptyset$  if  $B \neq \emptyset$ , and  $C \cap Z_{\gamma_3} \neq \emptyset$  if  $C \neq \emptyset$ , for some  $(X_{\gamma_1}, Y_{\gamma_2}, Z_{\gamma_3}) \in \mathcal{TI}$  such that  $A \neq \emptyset$  and  $i = 1$ , or  $B \neq \emptyset$  and  $i = 2$ , or  $C \neq \emptyset$  and  $i = 3$ . This means that  $(X_{\gamma_1}, Y_{\gamma_2}, Z_{\gamma_3}) \in \text{upr}(A, B, C)$  by Definition 11(2). Thus,  $\text{upr}(A, B, C) \neq \emptyset$  follows.  $\square$

### A.2: Proof of Lemma 6

*Proof* Let  $\text{low}(A, B, C) = \{(X_\alpha, Y_\alpha, Z_\alpha), \alpha \in \Lambda\}$  and  $\text{upr}(A, B, C) = \{(X_\beta, Y_\beta, Z_\beta), \beta \in \Gamma\}$ . According to  $|U|, |V|, |W| < \infty$ , we confirm  $|\text{low}(A, B, C)| < \infty$  and  $|\text{upr}(A, B, C)| < \infty$ .

The proof of item (1) is as follows:

One of  $A, B$  and  $C$  is empty  
 $\Rightarrow \overline{\text{apr}}(A, B, C) = (U, \emptyset, W)$  (Definition 11(4))  
 $\Rightarrow (A, B, C) \subseteq \overline{\text{apr}}(A, B, C)$  ( $A \subseteq U, B \supseteq \emptyset, C \subseteq W$  and Definition of  $\subseteq$ ).

The proof of item (2) is as follows:

According to Definition 11(4), the precovering of  $(U \times V \times W, \mathcal{TI})$ ,  $A \neq \emptyset, B \neq \emptyset, C \neq \emptyset$  and Lemma 5, we obtain  $\text{upr}(A, B, C) \neq \emptyset$  and

$$\overline{\text{apr}}(A, B, C) = \left( \bigcup_{\beta \in \Gamma} (X_\beta \cap A), \bigcup_{\beta \in \Gamma} (Y_\beta \cap B), \bigcup_{\beta \in \Gamma} (Z_\beta \cap C) \right).$$

Suppose  $a \in A$  since  $A \neq \emptyset$ . The precovering of  $(U \times V \times W, \mathcal{TI})$  implies  $a \in X_a$  for some  $(X_a, Y_a, Z_a) \in \mathcal{TI}$ . This implies  $X_a \cap A \neq \emptyset$ , and furthermore,  $(X_a, Y_a, Z_a) \in \text{upr}(A, B, C)$  holds in light of Definition 11(2). Therefore,  $A = \bigcup_{a \in A} a \subseteq \bigcup_{a \in A} X_a \subseteq \bigcup_{\beta \in \Gamma} X_\beta$  holds. Thus, we obtain  $A \subseteq \left( \bigcup_{\beta \in \Gamma} X_\beta \right) \cap A = \bigcup_{\beta \in \Gamma} (X_\beta \cap A)$ .

Similarly, using  $C \neq \emptyset$ , we obtain  $C \subseteq \bigcup_{\beta \in \Gamma} (Z_\beta \cap C)$ .

In addition,  $B \supseteq (Y_\beta \cap B)$  holds for any  $(X_\beta, Y_\beta, Z_\beta) \in \text{upr}(A, B, C)$ ,  $(\beta \in \Gamma)$ . Therefore,  $B \supseteq \bigcup_{\beta \in \Gamma} (Y_\beta \cap B)$  holds

Therefore, we confirm  $(A, B, C) \sqsubseteq \overline{\text{apr}}(A, B, C)$ .

The proof of item (3) is as follows:

$A \neq \emptyset, B \neq \emptyset, C \neq \emptyset$  and the precovering of  $(U \times V \times W, \mathcal{TI})$

$$\begin{aligned} &\Rightarrow \text{upr}(A, B, C) \neq \emptyset && \text{(Lemma 5)} \\ &\Rightarrow (A, B, C) \in \text{upr}(A, B, C) && ((A, B, C) \in \mathcal{TI} \text{ and } A \cap A = A \neq \emptyset) \\ &\Rightarrow A \subseteq \bigcup_{\beta \in \Gamma} X_\beta, B \subseteq \bigcup_{\beta \in \Gamma} Y_\beta, C \subseteq \bigcup_{\beta \in \Gamma} Z_\beta \\ &\Rightarrow A = A \cap \left(\bigcup_{\beta \in \Gamma} X_\beta\right) = \bigcup_{\beta \in \Gamma} (X_\beta \cap A), \\ &B = B \cap \left(\bigcup_{\beta \in \Gamma} Y_\beta\right) = \bigcup_{\beta \in \Gamma} (Y_\beta \cap B), \\ &C = C \cap \left(\bigcup_{\beta \in \Gamma} Z_\beta\right) = \bigcup_{\beta \in \Gamma} (Z_\beta \cap C) \\ &\Rightarrow \overline{\text{apr}}(A, B, C) = (A, B, C) && \text{(Definition 11(4)).} \end{aligned}$$

The proof of item (4) is as follows:

“(A, B, C) = (U, ∅, W)” ⇒ “B = ∅” ⇒ “ $\overline{\text{apr}}(A, B, C) = (U, \emptyset, W)$  according to Definition 10(4)” ⇒ “ $\overline{\text{apr}}(A, B, C) = (A, B, C)$ ”.

The proof of item (5) is as follows:

According to Definition 11(1) and  $\sqcup$ -closed of  $\mathcal{TI}$ , it is easily obtained that  $\text{apr}(A, B, C) \in \mathcal{TI}$  by Definition 10. Thus,  $(A, B, C) \in \overline{\mathcal{TI}}$  holds since  $\overline{\text{apr}}(A, B, C) = (A, B, C)$ .

The proof of item (6) is as follows:

$$\begin{aligned} &(X_\alpha, Y_\alpha, Z_\alpha) \in \text{low}(A, B, C) (\forall \alpha \in \Lambda) \\ &\Rightarrow (X_\alpha, Y_\alpha, Z_\alpha) \sqsubseteq (A, B, C) (\forall \alpha \in \Lambda) && \text{(Definition 11(1))} \\ &\Rightarrow X_\alpha \subseteq A, Y_\alpha \supseteq B, Z_\alpha \subseteq C (\forall \alpha \in \Lambda) && \text{(Definition of } \sqsubseteq) \\ &\Rightarrow \bigcup_{\alpha \in \Lambda} X_\alpha \subseteq A, \bigcap_{\alpha \in \Lambda} Y_\alpha \supseteq B, \bigcup_{\alpha \in \Lambda} Z_\alpha \subseteq C \\ &\Rightarrow \left(\bigcup_{\alpha \in \Lambda} X_\alpha, \bigcap_{\alpha \in \Lambda} Y_\alpha, \bigcup_{\alpha \in \Lambda} Z_\alpha\right) \sqsubseteq (A, B, C) && \text{(Definition of } \sqsubseteq) \\ &\Rightarrow \overline{\text{apr}}(A, B, C) \sqsubseteq (A, B, C) && \text{(Definition 11(3)).} \end{aligned}$$

The proof of item (7) is as follows:

$$\begin{aligned} &(A, B, C) \in \mathcal{TI} \\ &\Rightarrow (A, B, C) \in \text{low}(A, B, C) \quad ((A, B, C) \sqsubseteq (A, B, C), \text{Definition 11(1)}) \\ &\Rightarrow (A, B, C) = (X_{\alpha_1}, Y_{\alpha_1}, Z_{\alpha_1}) \text{ for some } \alpha_1 \in \Lambda \\ &\Rightarrow A \subseteq \bigcup_{\alpha \in \Lambda} X_\alpha, B \supseteq B \cap \left(\bigcap_{\alpha \in \Lambda} Y_\alpha\right), C \subseteq \bigcup_{\alpha \in \Lambda} Z_\alpha \quad (\alpha \in \Lambda) \\ &\Rightarrow (A, B, C) \sqsubseteq \text{apr}(A, B, C) \quad \text{(Definition 11(3) and Definition of } \sqsubseteq) \\ &\Rightarrow (A, B, C) = \overline{\text{apr}}(A, B, C) \quad \text{(Combining item (6))} \end{aligned}$$

□

### A.3: Proof of Theorem 2(2)

*Proof* We prove the two parts  $(\Rightarrow)$  and  $(\Leftarrow)$ .

$(\Rightarrow)$ : Let  $(A, B, C) \in \mathcal{TI}$ . Using Lemma 6(7), we obtain  $\overline{\text{apr}}(A, B, C) = (A, B, C)$ . Combining the precovering of  $(U \times V \times W, \mathcal{TI})$  and Lemma 6(3), we obtain  $\overline{\text{apr}}(A, B, C) = (A, B, C)$ .

$(\Leftarrow)$ : Let  $\overline{\text{apr}}(A, B, C) = (A, B, C) = \overline{\text{apr}}(A, B, C)$ . Considering Lemma 6(5), we obtain  $(A, B, C) \in \mathcal{TI}$ . □

### A.4: Proof of Lemma 9

*Proof* The proof of item (1) is as follows:

$$\begin{aligned} &\text{Low}(A, B, C) \neq \emptyset \\ &\Leftrightarrow (([a]_{R_{i_0}} \cap A)_{b_{i_0}}, b_{i_0}, w_{i_0}) \in \text{Low}(A, B, C) \\ &\text{for every } (a, b_{i_0}, w_{i_0}) \in S \neq \emptyset && \text{(Definition 13(1))} \\ &\Leftrightarrow (([a]_{R_{i_0}} \cup A)_{b_{i_0}}, b_{i_0}, w_{i_0}) \in \text{Upr}(A, B, C) \\ &\text{for every } (a, b_{i_0}, w_{i_0}) \in S \neq \emptyset && \text{(Definition 13(2))} \\ &\Leftrightarrow \text{Upr}(A, B, C) \neq \emptyset. \end{aligned}$$

The proof of item (2) is as follows:

$$\begin{aligned} &b_{B_i}^{i w_{c_j}} = \emptyset \text{ for every } i = 1, 2, \dots, |B| \text{ and } j = 1, 2, \dots, |C| \\ &\Leftrightarrow S = \emptyset && \text{(Definition of } S) \\ &\Leftrightarrow \text{Low}(A, B, C) = \emptyset && \text{(Definition 13(1))} \\ &\Leftrightarrow \text{Upr}(A, B, C) = \emptyset && \text{(item (1))} \end{aligned}$$

The proof of item (3) is as follows:

$\text{Low}(A, B, C) \neq \emptyset$  and item (2) together imply that there is a

$j_0 \in \{\alpha_1, \alpha_2, \dots, \alpha_\delta\}$  satisfying  $b^{i w_{\alpha_j_0}} \neq \emptyset$  for some  $b \in B$ .

We obtain  $A \cap B^{i w_{\alpha_1 \alpha_2 \dots \alpha_\delta}} = A \cap \left(\bigcap_{b \in B} \left(\bigcup_{j=1}^\delta b^{i w_{\alpha_j}}\right)\right) = A \cap$

$$\left(\bigcap_{i=1}^t \left(\bigcup_{j=1}^\delta b_{I_i}^{i w_{\alpha_j}}\right)\right) = \bigcap_{b \in B} \left(\bigcup_{j=1}^\delta ([a]_{R_{\alpha_j}} \cap A)_b\right) \text{ since Lemma 7(2)}$$

implies  $B^{w_{\alpha_1 \alpha_2 \dots \alpha_\delta}} = \bigcap_{b \in B} (\bigcup_{j=1}^\delta b^{w_{\alpha_j}})$ . Thus,  $\bigcup_{j=1}^\delta w_{\alpha_j} \subseteq C$  holds since  $w_{\alpha_j} \in C$  ( $j = 1, 2, \dots, \delta$ ). Hence, in  $2^U \times 2^V \times 2^W$ , the first coordinate of  $\underline{APr}(A, B, C)$  is  $A \cap B^{w_{\alpha_1 \alpha_2 \dots \alpha_\delta}}$ . Moreover, we obtain that  $\underline{APr}(A, B, C) \cap (U, V, \emptyset) = (A \cap B^{w_{\alpha_1 \alpha_2 \dots \alpha_\delta}}, B, \emptyset)$  holds according to Definition 13(4) and  $B \subseteq V$ .

Using item (1) and  $Low(A, B, C) \neq \emptyset$ , we know that  $Upr(A, B, C) \neq \emptyset$ . Similarly to the above, using Definition 13(6), we obtain the first coordinate of  $\overline{APr}(A, B, C)$  in  $2^U \times 2^V \times 2^W$  is  $A \cup B^{w_{\alpha_1 \alpha_2 \dots \alpha_\delta}}$ . Moreover, we obtain  $\overline{APr}(A, B, C) \cap (U, \emptyset, \emptyset) = (A \cup B^{w_{\alpha_1 \alpha_2 \dots \alpha_\delta}}, \emptyset, \emptyset)$ .

The proof of item (4) is as follows:

If  $Low(A, B, C) = \emptyset$ , then  $Upr(A, B, C) = \emptyset$  holds by item (1). Combining items (3) and (5) in Definition 13, we know that  $\underline{APr}(A, B, C) = (\emptyset, V, \emptyset)$  and  $\overline{APr}(A, B, C) = (U, \emptyset, W)$ . The given  $\underline{APr}(A, B, C) = \overline{APr}(A, B, C)$  implies  $(\emptyset, V, \emptyset) = (U, \emptyset, W)$ . Therefore,  $U = V = W = \emptyset$  follows. This result contradicts the suppositions of  $U \neq \emptyset, V \neq \emptyset$  and  $W \neq \emptyset$ . Hence, we confirm  $Low(A, B, C) \neq \emptyset$ . We also confirm  $Upr(A, B, C) \neq \emptyset$  by item (1).

Considering  $Low(A, B, C) \neq \emptyset$ , item (3) and Definition 13 with Lemmas 7 and 8, we obtain  $\underline{APr}(A, B, C) = (A \cap B^{w_{\alpha_1 \alpha_2 \dots \alpha_\delta}}, B, \bigcup_{j=1}^\delta w_{\alpha_j})$  and  $\overline{APr}(A, B, C) = (A \cup B^{w_{\alpha_1 \alpha_2 \dots \alpha_\delta}}, \bigcup_{j=1}^t b_{l_j}, \bigcup_{j=1}^\delta w_{\alpha_j})$ . Therefore,  $\underline{APr}(A, B, C) = \overline{APr}(A, B, C)$  implies  $A \cap B^{w_{\alpha_1 \alpha_2 \dots \alpha_\delta}} = A \cup B^{w_{\alpha_1 \alpha_2 \dots \alpha_\delta}}$  and  $B = \bigcup_{j=1}^t b_{l_j}$ .

On the one hand,  $\underline{APr}(A, B, C) = (A, B, C)$  implies  $A \cap B^{w_{\alpha_1 \alpha_2 \dots \alpha_\delta}} = A$ , and so  $A \subseteq B^{w_{\alpha_1 \alpha_2 \dots \alpha_\delta}}$ . On the other hand,  $\overline{APr}(A, B, C) = (A, B, C)$  implies  $A \cup B^{w_{\alpha_1 \alpha_2 \dots \alpha_\delta}} = A$ , and so  $B^{w_{\alpha_1 \alpha_2 \dots \alpha_\delta}} \subseteq A$ . Hence,  $A = B^{w_{\alpha_1 \alpha_2 \dots \alpha_\delta}}$  holds.

$\overline{APr}(A, B, C) = (A, B, C)$  also implies  $C = \bigcup_{j=1}^\delta w_{\alpha_j}$ , so  $\delta = |C|$  holds. Using Definition 2(2) and Remark 2, we confirm  $(A, B) \in \mathcal{B}(\mathbb{K}_{\alpha_1 \alpha_2 \dots \alpha_\delta})$ .

The proof of item (5) is as follows:

By  $(A, B) \in \mathcal{B}(\mathbb{K}_{\alpha_1 \alpha_2 \dots \alpha_\delta})$  and Definitions 2 and 12, we obtain  $A = B^{w_{\alpha_1 \alpha_2 \dots \alpha_\delta}}$ . Combining  $Low(A, B, C) \neq \emptyset$  and item (3) with items (4) and (6) in Definition 13, we obtain  $\underline{APr}(A, B, C) = (A, B, \bigcup_{j=1}^\delta w_{\alpha_j})$  and  $\overline{APr}(A, B, C) = (A, \bigcup_{j=1}^t b_{l_j}, \bigcup_{j=1}^\delta w_{\alpha_j})$ .

Because of  $|B| = t$  and the definition of  $\{b_{l_i}, i = 1, 2, \dots, t\}$  in Definition 13, we know  $\bigcup_{i=1}^t b_{l_i} = B$ . According to  $\delta = |C|$  and the definition of  $\{w_{\alpha_j}, j =$

$1, \dots, \delta\}$  in Definition 13, we know that  $\bigcup_{j=1}^\delta w_{\alpha_j} = C$ . Hence,  $\underline{APr}(A, B, C) = \overline{APr}(A, B, C) = (A, B, C)$  is correct.

The proof of item (6) is as follows:

We distinguish two cases in the proof.

Case 1.  $Low(A, B, C) = \emptyset$ .

Combined with item (1), we know that  $Upr(A, B, C) = \emptyset$ . Using items (3) and (5) in Definition 13, we obtain  $\underline{APr}(A, B, C) = (\emptyset, V, \emptyset)$  and  $\overline{APr}(A, B, C) = (U, \emptyset, W)$ . In view of  $\emptyset \subseteq A \subseteq U, V \supseteq B \supseteq \emptyset$  and  $\emptyset \subseteq C \subseteq W$ , we confirm that  $\underline{APr}(A, B, C) \sqsubseteq (A, B, C) \sqsubseteq \overline{APr}(A, B, C)$ .

Case 2.  $Low(A, B, C) \neq \emptyset$ .

Using item (1) in Definition 13, we obtain the following three results:

- (1)  $\bigcap_{i=1}^t \bigcup_{j=1}^\delta ([a]_{R_{\alpha_j}} \cap A)_{b_{l_i}} \subseteq A$  since  $([a]_{R_{\alpha_j}} \cap A)_{b_{l_i}} \subseteq A$  for any  $(([a]_{R_{\alpha_j}} \cap A)_{b_{l_i}}, b_{l_i}, w_{\alpha_j}) \in Low(A, B, C)$  ( $i = 1, \dots, t; j = 1, \dots, \delta$ );
- (2)  $B \supseteq B$ ;
- (3)  $\bigcup_{j=1}^\delta w_{\alpha_j} \subseteq C$  since  $w_{\alpha_j} \in C$  ( $j = 1, \dots, \delta$ ).

Thus,  $\underline{APr}(A, B, C) \sqsubseteq (A, B, C)$  holds according to Definition 13(4) and the definition of  $\sqsubseteq$ .

Additionally, considering item (1) above and  $Low(A, B, C) \neq \emptyset$ , we know that  $Upr(A, B, C) \neq \emptyset$ . Using Definition 13(6), we obtain the following consequences:

- (1)  $A \subseteq ([a]_{R_{\alpha_j}} \cup A)_{b_{l_i}}$  for some  $a \in U_{\alpha_j}$  and every  $b_{l_i}$  ( $i = 1, \dots, t; j = 1, \dots, \delta$ )  
 $\Rightarrow A \subseteq \bigcup_{j=1}^\delta ([a]_{R_{\alpha_j}} \cup A)_{b_{l_i}}$  for every  $(a, b_{l_i}, w_j) \in \mathcal{S}$  and any  $b_{l_i}$  ( $i = 1, \dots, t; j = 1, \dots, \delta$ )  $\Rightarrow A \subseteq \bigcap_{i=1}^t \bigcup_{j=1}^\delta ([a]_{R_{\alpha_j}} \cup A)_{b_{l_i}}$ ;
- (2)  $\bigcup_{i=1}^t b_{l_i} \subseteq B$  is correct since  $b_{l_i} \in B$  ( $i = 1, \dots, t$ );
- (3)  $C = \bigcup_{j=1}^\delta w_{\alpha_j}$  is correct since  $w_{\alpha_j} \in C$  ( $j = 1, \dots, \delta$ ) and  $\delta = |C|$ .

Hence,  $(A, B, C) \sqsubseteq \overline{APr}(A, B, C)$  holds according to Definition 13(6) and the definition of  $\sqsubseteq$ .  $\square$

### A.5: Proof of Theorem 3

*Proof* The condition  $b^{w_j} \neq \emptyset$  for  $\forall b \in V$  and  $\forall j \in \{1, \dots, n\}$  yields the following:

- (1)  $\delta = |C|$  holds since  $\{w_{\alpha_1}, \dots, w_{\alpha_\delta}\} \subseteq C$ ,  $\{\alpha_1, \dots, \alpha_\delta\} \subseteq \{1, \dots, n\}$  and the  $\alpha_j$  is arbitrary.
- (2)  $t = |B|$  holds since  $\{b_{l_i}, i = 1, \dots, t\} \subseteq B \subseteq V$  and  $b$  is arbitrary.
- (3)  $a \in b^{w_j}$  holds for some  $a \in U_j$  and some  $j \in \{1, \dots, n\}$ .

We have that (3) above implies  $\mathcal{S} \neq \emptyset$  according to the definition of  $\mathcal{S}$  in Definition 13. Furthermore, it follows that  $Low(A, B, C) \neq \emptyset$  according to Definition 13(1).

If  $\underline{APr}(A, B, C) = \overline{APr}(A, B, C) = (A, B, C)$ , then  $(A, B) \in \mathcal{B}(\mathbb{K}_{\alpha_1 \alpha_2 \dots \alpha_\delta})$  holds by Lemma 9(4).

If  $(A, B) \in \mathcal{B}(\mathbb{K}_{\alpha_1 \alpha_2 \dots \alpha_\delta})$ , then combining the above results of  $Low(A, B, C) \neq \emptyset$ ,  $t = |B|$  and  $\delta = |C|$  with Lemma 9(5), we obtain  $\underline{APr}(A, B, C) = \overline{APr}(A, B, C) = (A, B, C)$ .

Taking the above and  $\delta = |C|$  with Lemma 9(6), we confirm that  $\underline{APr}(A, B, C) \sqsubseteq (A, B, C) \sqsubseteq \overline{APr}(A, B, C)$ . □

### A.6: Proof of Theorem 5

*Proof*  $Low(A, B, C) \neq \emptyset$  and Lemma 9(1) imply  $Upr(A, B, C) \neq \emptyset$ .

The proof of item (1) is as follows:

Using Definition 13, we know that  $\underline{APr}(A, B, C) = (\bigcap_{i=1}^t \bigcup_{j=1}^\delta ([a]_{R_{\alpha_j}} \cap A)_{b_{l_i}}, B, \bigcup_{j=1}^\delta w_{\alpha_j})$  and  $\overline{APr}(A, B, C) = (\bigcap_{i=1}^t \bigcup_{j=1}^\delta ([a]_{R_{\alpha_j}} \cup A)_{b_{l_i}}, \bigcup_{i=1}^t b_{l_i}, \bigcup_{j=1}^\delta w_{\alpha_j})$ . This implies  $\bigcap_{i=1}^t \bigcup_{j=1}^\delta ([a]_{R_{\alpha_j}} \cap A)_{b_{l_i}} \subseteq \bigcap_{i=1}^t \bigcup_{j=1}^\delta ([a]_{R_{\alpha_j}} \cup A)_{b_{l_i}}$  since  $([a]_{R_{\alpha_j}} \cap A)_{b_{l_i}} \subseteq ([a]_{R_{\alpha_j}} \cup A)_{b_{l_i}}$ , and  $B \supseteq \bigcup_{i=1}^t b_{l_i}$  since  $b_{l_i} \in B$  ( $i = 1, \dots, t$ ). Combined with the definition of  $\sqsubseteq$ , we obtain  $\underline{APr}(A, B, C) \sqsubseteq \overline{APr}(A, B, C)$ . Hence, we obtain:

$$\begin{aligned} (X, Y, Z) &\in \mathcal{TI}(\underline{APr}(A, B, C)) \\ \Rightarrow (X, Y, Z) &\sqsubseteq \underline{APr}(A, B, C) \quad (\text{Definition of } \mathcal{TI}(\underline{APr}(A, B, C))) \\ \Rightarrow (X, Y, Z) &\sqsubseteq \overline{APr}(A, B, C) \quad (\underline{APr}(A, B, C) \sqsubseteq \overline{APr}(A, B, C)) \\ \Rightarrow (X, Y, Z) &\in \mathcal{TI}(\overline{APr}(A, B, C)) \quad (\text{Definition of } \mathcal{TI}(\overline{APr}(A, B, C))) \\ \Rightarrow \mathcal{TI}(\underline{APr}(A, B, C)) &\subseteq \mathcal{TI}(\overline{APr}(A, B, C)). \end{aligned}$$

The proof of item (2) is as follows:

$(\Rightarrow)$ : “ $(A, B) \in \mathcal{B}(\mathbb{K}_{i_1 \dots i_\delta})$ ”  $\Rightarrow$  “ $\underline{APr}(A, B, C) = \overline{APr}(A, B, C)$  according to the supposition for  $\mathbb{K}_j$  ( $j = 1, \dots, n$ ),  $Low(A, B, C) \neq \emptyset$ , Lemma 9(1) and Theorem 3”  $\Rightarrow$  “ $\mathcal{TI}(\underline{APr}(A, B, C)) = \mathcal{TI}(\overline{APr}(A, B, C))$  according to the definitions of  $\mathcal{TI}(\underline{APr}(A, B, C))$  and  $\mathcal{TI}(\overline{APr}(A, B, C))$ ”.

$(\Leftarrow)$ : Using the definitions of  $\mathcal{TI}(\underline{APr}(A, B, C))$  and  $\mathcal{TI}(\overline{APr}(A, B, C))$ , we obtain  $\underline{APr}(A, B, C) \in \mathcal{TI}(\underline{APr}(A, B, C))$  and  $\overline{APr}(A, B, C) \in \mathcal{TI}(\overline{APr}(A, B, C))$ . Considering  $\mathcal{TI}(\underline{APr}(A, B, C)) = \mathcal{TI}(\overline{APr}(A, B, C))$ , we obtain  $\underline{APr}(A, B, C) \in \mathcal{TI}(\overline{APr}(A, B, C))$  and  $\overline{APr}(A, B, C) \in \mathcal{TI}(\underline{APr}(A, B, C))$ . Hence, we infer

$$\begin{aligned} \underline{APr}(A, B, C) &\sqsubseteq \overline{APr}(A, B, C) \sqsubseteq \underline{APr}(A, B, C). \\ \text{Therefore, } \underline{APr}(A, B, C) &= \overline{APr}(A, B, C). \end{aligned}$$

Combining the above with Lemma 9(3) and  $Low(A, B, C) \neq \emptyset$ , we confirm that  $A \cap B^{w_{\alpha_1 \dots \alpha_\delta}} = A \cup B^{w_{\alpha_1 \dots \alpha_\delta}}$ . Thus, we obtain

$$\begin{aligned} A &\subseteq A \cup B^{w_{\alpha_1 \dots \alpha_\delta}} = A \cap B^{w_{\alpha_1 \dots \alpha_\delta}} \subseteq B^{w_{\alpha_1 \dots \alpha_\delta}}; \\ B^{w_{\alpha_1 \dots \alpha_\delta}} &\subseteq A \cup B^{w_{\alpha_1 \dots \alpha_\delta}} = A \cap B^{w_{\alpha_1 \dots \alpha_\delta}} \subseteq A. \end{aligned}$$

Moreover, we obtain  $A = B^{w_{\alpha_1 \dots \alpha_\delta}}$ . That is,  $(A, B) \in \mathcal{B}(\mathbb{K}_{\alpha_1 \dots \alpha_\delta})$  holds. □

### A.7: Proof of Corollary 1

*Proof* If there exists an  $a_0 \in U$  such that for any  $j$  and every  $b \in V$ ,  $a_0 \notin b^{w_j}$  holds, then in view of  $a_0 \in U = \bigcup_{j=1}^n U_j$ , we obtain  $a_0 \in U_{j_0}$  for some  $j_0 \in \{1, \dots, n\}$ . This means  $a_0 \notin b^{w_{j_0}}$  for any  $b \in V$  since  $U_i \cap U_j = \emptyset$  ( $i \neq j; i, j = 1, \dots, n$ ). That is,  $(a_0, b, w_{j_0}) \notin \mathcal{S}$  holds for any  $b \in V$ . Hence,  $\mathcal{S}$  is not a covering of  $(U, V, W)$ . This is a contradiction to the supposition of  $\mathcal{S}$ . In other words, for any  $a \in U$ , there must exist  $j_a \in \{1, \dots, n\}$  and  $b_a \in V$  satisfying  $(a, b_a, w_{j_a}) \in \mathcal{S}$ . This implies  $Low(A, B, C) \neq \emptyset$  by Definition 13(1).

The given condition that “ $b^{w_j} \neq \emptyset$  for every  $b \in V$  in  $\mathbb{K}_j = (U_j, V, R_j)$  ( $j = 1, \dots, n$ )” yields the following results:

- (1)  $\{b_{l_i}, i = 1, \dots, t\} = B$  holds by  $B \subseteq V$  and the definition of  $\{b_{l_i}, i = 1, \dots, t\}$  in Definition 13.
- (2)  $\delta = |C|$  holds by  $C \subseteq W$  and the definition of  $\{w_{\alpha_1}, \dots, w_{\alpha_\delta}\}$  in Definition 13.

Using results (1) and (2) above, we obtain  $\bigcup_{i=1}^t b_{l_i} = B$  and  $\bigcup_{j=1}^\delta = C$ . Furthermore, we obtain of  $\underline{APr}(A, B, C) \cap (\emptyset, V, W) = (\emptyset, B, C)$  and  $\overline{APr}(A, B, C) \cap (\emptyset, V, W) = (\emptyset, B, C)$  from Definition 13. We obtain  $\underline{APr}(A, B, C) \cap (U, \emptyset, \emptyset) = (A \cap B^{w_{\alpha_1 \dots \alpha_\delta}}, \emptyset, \emptyset)$ , and therefore,  $\bigcap_{i=1}^t \bigcup_{j=1}^\delta ([a]_{R_{\alpha_j}} \cap A)_{b_{l_i}} = A \cap B^{w_{\alpha_1 \dots \alpha_\delta}} \subseteq A$  by Lemma 9(3) and  $Low(A, B, C) \neq \emptyset$ . Hence, it follows that

$$\underline{APr}(A, B, C) = (A \cap B^{w_{\alpha_1 \dots \alpha_\delta}}, B, C) \sqsubseteq (A, B, C).$$

Utilizing the definition of  $\mathcal{TI}(\underline{APr}(A, B, C))$  in Theorem 4, we find that



$\underline{APr}(A, B, C) \in \mathcal{TI}(\underline{APr}(A, B, C))$ . Therefore,  $\underline{APr}(A, B, C) \in \text{low}(A, B, C) = \{(X, Y, Z) \in \mathcal{TI}(\underline{APr}(A, B, C)) \mid (X, Y, Z) \sqsubseteq (A, B, C)\}$  holds.

Combining the above and Definition 11 with  $A \neq \emptyset, B \neq \emptyset$  and  $C \neq \emptyset$ , we obtain  $\underline{apr}(A, B, C) = (\bigcup_{(X, Y, Z) \in \text{low}(A, B, C)} X, B, C)$ . On the one hand, we have  $(X, Y, Z) \in \text{low}(A, B, C) \Rightarrow \text{"}(X, Y, Z) \in \mathcal{TI}(\underline{APr}(A, B, C)) \text{ and } (X, Y, Z) \sqsubseteq (A, B, C)\text{"} \Rightarrow (X, Y, Z) \sqsubseteq \underline{APr}(A, B, C), Y \supseteq B,$

$Z \subseteq C$ , and  $X \subseteq \bigcap_{i=1}^t \bigcup_{j=1}^{\delta} ([a]_{R_{\alpha_j}} \cap A)_{b_i}$  from Definition 13(4). However,  $\underline{APr}(A, B, C) \in \mathcal{TI}(\underline{APr}(A, B, C))$  and  $\underline{APr}(A, B, C) \sqsubseteq (A, B, C)$  together imply  $\underline{APr}(A, B, C) \in \text{low}(A, B, C)$ . Furthermore, we obtain  $\bigcap_{(X, Y, Z) \in \text{low}(A, B, C)} Y = B, \bigcup_{(X, Y, Z) \in \text{low}(A, B, C)} Z = C$  and

$$\bigcup_{(X, Y, Z) \in \text{low}(A, B, C)} X = \bigcap_{i=1}^t \bigcup_{j=1}^{\delta} ([a]_{R_{\alpha_j}} \cap A)_{b_i}.$$

Thus, we obtain  $\underline{apr}(A, B, C) = \underline{APr}(A, B, C)$ .

Additionally, by Definition 3, it is easy to see that  $(\mathcal{TI}(\underline{APr}(A, B, C)), \sqsubseteq)$  is a poset with  $\underline{APr}(A, B, C)$  as the maximum element. This implies that  $\underline{APr}(A, B, C)$  is the maximum element in the poset  $(\text{upr}(A, B, C), \sqsubseteq)$  since  $B \cap B = B \neq \emptyset$ , where  $\text{upr}(A, B, C) = \{(X, Y, Z) \in \mathcal{TI}(\underline{APr}(A, B, C)) \mid X \cap A \neq \emptyset \text{ or } Y \cap B \neq \emptyset \text{ or } Z \cap C \neq \emptyset\}$ . Therefore, we obtain  $\bigcup_{(X, Y, Z) \in \text{upr}(A, B, C)} (X \cap$

$$A) = \bigcap_{i=1}^t \bigcup_{j=1}^{\delta} ([a]_{R_{\alpha_j}} \cap A)_{b_i}, \bigcup_{(X, Y, Z) \in \text{upr}(A, B, C)} (Y \cap B) = B$$

and  $\bigcup_{(X, Y, Z) \in \text{upr}(A, B, C)} (Z \cap C) = C$ . Hence, we confirm that

$\overline{\text{apr}}(A, B, C) = \overline{APr}(A, B, C)$  according to Definition 11(4) with  $A \neq \emptyset, B \neq \emptyset$  and  $C \neq \emptyset$ .

The above implies that item (1) is correct.

Next, we prove item (2).

If  $(A, B) \in \mathcal{B}(\mathbb{K}_{\alpha_1 \dots \alpha_{\delta}})$ , then by Definition 2(2) and Remark 2, we know that  $A = B'^{w_{\alpha_1 \dots \alpha_{\delta}}}$ . Therefore, considering Lemma 9,  $t = |B|, \delta = |C|$  and  $\text{Low}(A, B, C) \neq \emptyset$ , we obtain  $\underline{APr}(A, B, C) = \overline{APr}(A, B, C) = (A, B, C)$ . We also obtain  $\mathcal{TI}(\underline{APr}(A, B, C)) = \mathcal{TI}(\overline{APr}(A, B, C))$  by Theorem 5. Let  $\text{low}(A, B, C) = \{(X, Y, Z) \in \mathcal{TI}(\overline{APr}(A, B, C)) \mid (X, Y, Z) \sqsubseteq (A, B, C)\}$  and  $\text{upr}(A, B, C) = \{(X, Y, Z) \in \mathcal{TI}(\overline{APr}(A, B, C)) \mid X \cap A \neq \emptyset \text{ or } Y \cap B \neq \emptyset \text{ or } Z \cap C \neq \emptyset\}$ . Then, we obtain  $\text{low}(A, B, C) = \text{low}(A, B, C)$  and  $\text{upr}(A, B, C) = \text{upr}(A, B, C)$  since  $\mathcal{TI}(\underline{APr}(A, B, C)) = \mathcal{TI}(\overline{APr}(A, B, C))$ . Combined with Definition 11, we determine that  $\overline{\text{apr}}(A, B, C) = \overline{\text{apr}}(A, B, C)$  and  $\underline{apr}(A, B, C) = \overline{\text{apr}}(A, B, C)$ . Therefore, using the above and item (1), we confirm that  $\overline{\text{apr}}(A, B, C) = \overline{\text{apr}}(A, B, C) = \underline{apr}(A, B, C) = \overline{\text{apr}}(A, B, C) = \underline{APr}(A, B, C) = \overline{APr}(A, B, C) = (A, B, C)$ .

Conversely, if the given expression is correct, then by Theorem 3, we know that  $(A, B) \in \mathcal{B}(\mathbb{K}_{\alpha_1 \dots \alpha_{\delta}})$ .  $\square$

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## Declarations

**Ethics approval** This article does not contain any studies with human participants or animals performed by any of the authors.

**Consent to participate** Informed consent was not required, as no humans or animals were involved.

**Conflict of Interests** The authors declare that they have no conflicts of interest.

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