Applied Intelligence (2023) 53:5126–5145
https://doi.org/10.1007/s10489-022-03672-4

Network rule extraction under the network formal context based on three-way decision

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Accepted: 21 April 2022 / Published online: 18 June 2022

Abstract

Knowledge discovery combined with network structure is an emerging field of network data analysis and mining. Three-way concept analysis is a method that can fit the human mind in uncertain decisions and analysis. In reality, when three-way concept analysis is placed in the background of a network, not only the three-way rules need to be obtained, but also the network characteristic values of these rules should be obtained, which is of great significance for concept cognition in the network. This paper mainly combines complex network analysis with the formal context of three-way decision. Firstly, the network formal context of three-way decision (NFC3WD) is proposed to unify the two studies mentioned above into one data framework. Then, the network weaken-concepts of three-way decision (NWC3WD) and their corresponding sub-networks are studied. Therefore, we can not only find out the network weaken-concepts but also know the average influence of the sub-network, as well as the influence difference within the sub-network. Furthermore, the concept logic of network and the properties of its operators are put forward, which lays a foundation for designing the algorithm of rule extraction. Subsequently, the bidirectional rule extraction algorithm and reduction algorithm based on confidence degree are also explored. Meanwhile, these algorithms are applied to the diagnosis examples of COVID-19 from which we can not only get diagnostic rules, but also know the importance of the population corresponding to these diagnostic rules in the network through network eigenvalues. Finally, experimental analysis is made to show the superiority of the proposed method.

Keywords Granular computing · Three-way decision · Three-way concept analysis · Network rule extraction · Network weaken-concept · Concept logic of network

1 Introduction

With the gradual networking of society, many data collection and analysis usually have corresponding network contexts behind them. Currently, for the research on decision problems in networks, such as diagnosis and epidemic prevention in infectious disease networks, objects can often be divided into positive domain, negative domain and boundary domain by every attribute. Therefore, we can use the method of three-way decision and three-way rule extraction to study them. Moreover, uniting complex networks analysis and formal context of three-way decision, and mining three-way network concepts and rules have become hot research topics with great theoretical and practical significance.

Formal concept analysis theory is based on concepts and their hierarchy through a mathematical and formal representation [1], which could obtain rules through the implication relationship between concepts [2]. Furthermore, the concept cognition and learning were used as the basis of rule extraction in complex systems [3–7]. Kumar et al. [3], Mi et al. [4] and Zhao et al. [5] conducted a series of studies from the perspective of granular concept cognitive learning. Furthermore, in order to improve the efficiency and flexibility of concept learning, Li et al. [6] explored concept learning through granular computing from the perspective of cognitive computing. Based on the concept of granularity structure, Yao [7] combined granular computing and cognitive psychology and proposed the triangle of information computing as well as the triangle theory of granular computing to further incorporate

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granular computing into artificial intelligence. Considering the complexity of concepts in a network, they may not necessarily constitute concepts under Galois connections, but may only satisfy the unilateral mapping of weak concepts. Therefore, based on the above researches, the cognition and learning of weak concepts in a network will be studied in this paper.

Meanwhile, how to do three-way decision making in the real world was also studied. For instance, References [8–11] described in detail and elaborated the cognitive advantages of three-way decision making from different aspects. In addition, based on the results of cognitive science, Yao [12] studied the two fields of three-way decision and granular computing, as well as the interaction between them. To make this theory further meet more needs, many scholars have conducted related researches [13, 14], such as multi-agent three-way decision making based on rough set decision theory [15], which discussed approximate rough sets from three-way decision theory viewpoint [16]. Zou et al. [17] proposed a kind of formal concept analysis method of linguistic-valued data based on lattice implication algebra. These theories and applications have deepened people's understanding of the three-way decision theories and broadened the scope of application.

Due to the wide application of three-way decision and formal concept, Yao [12], Wang et al. [18], and Qi et al. [19] have made three-way concept analysis become an important tool for knowledge discovery and data analysis. Gaeta et al. [20] used temporal information granules and three-way formal concept analysis for spatial and temporal reasoning. For the attribute reduction problem of three-way concept lattice, Ren and Wei [21] proposed four kinds of reductions and gave the reduction calculation method. In addition, Li et al. [22] proposed three-way cognitive concept learning through multiple granularities. Yao and Herbert [23] introduced the attribute selection problem into a three-way decision construction algorithm and applied it to a medical network system. Yao [24] examined geometric structures, graphical representations, and semantical interpretations of triads in terms of basic geometric notions, as well as more complex structures derived from these basic notions. Deng et al. [25] introduced three-way decision into a multi-scale decision information system (MS-DIS) with reference to fuzzy-neighborhood classes, and unravelled the three-way decision rule from MS-DIS. They drove the three-way decision making science towards the field of artificial intelligence. Among them, rule extraction is a key research direction [26–31]. Hu et al. [26] investigated rule acquisition in the generalized one-sided formal context and in the generalized one-sided formal context with multi-scale. Wu et al. [27] studied optimal scale selection and rule acquisition in incomplete multi-scale decision tables.

The above researches are based on three-way formal context, which do not contain information about the network where the object-object relationship is considered. At the same time, on one hand, more and more data has a network background, and on the other hand, in practical applications such as infectious disease networks, the research of three-way decision theory with network background needs to be solved urgently.

What is more, in recent years, complex network analysis about disease diagnosis has yielded fruitful research results on many fields [32–42]. These studies can be divided into three directions: the study of network topological eigenvalues [35, 36, 39], the research of network trends using dynamic system methods [40, 41], and the combination of data driven methods and physical methods [42]. Barabasi [35–37] proposed network investigation to get the functional organization of cells and the identification of new disease genes, and revealed the biological significance of disease-associated mutations. Furthermore, this kind of networks can identify drug targets and biomarkers for complex diseases [35–37]. Especially for the research of infectious disease networks [43–45], Pinto et al. [39] used several complex network models widely known in the literature to verify their topological effects on the propagation of the disease, Salje et al. [40] estimated the impact of lockdown and current population immunity by applying models to hospital and death data. Nande et al. [41] established a random epidemic model to examine the impact of the clinical progress of COVID-19 and the structure of the transmission network on the outcome of social distancing interventions. Gaeta et al. [43] combined the three-way decision model and graph theory to analyze the spread of COVID-19. Despite the effectiveness of the above methods, they still face the challenge of large-scale datasets and multiple parameters determination problems. Moreover, these complex network analysis methods seldom considered the attribute characteristics of network nodes.

To tackle with this problem, there has been some prior work on network cognition and learning in formal contexts. Snasel et al. [46] introduced formal concept analysis into social networks to facilitate understanding of internal structures and discussed the computational complexity of social network analysis. Hao et al. [47, 48] utilized formal concept analysis for efficient k-clique community networks detection. Peters and Ramanna [49] used information obtained from proximal three-way decision to study social networks. Ma et al. [50] linked complex networks and concept cognition through adjacency and association matrices in the same data framework, combining the advantages of the above two research fields, and investigated the generation and propagation of network weaken-concepts. Liu et al. [51] studied the problem of network community division based on the network formal context, which considered the characteristics of both network structure and formal contexts. Yan and Li [52] proposed a dynamic concept updating method based on three-way decision and network formal context.

Motivated by the above analysis, we further combine complex network analysis with three-way concept analysis, and make full use of the advantages of the two methods. For the networks of infectious diseases, we can take the network nodes as objects, and a network edge is viewed as a relation between objects. Moreover, using adjacency matrix to present the structure of a network, the matrix of conditional attributes and that of decision attributes whose attribute vales are 1, -1 and 0, can be combined to form the network formal context of three-way decision (NFC3WD). After that, we get a uniform data framework of the two methods. As a result, we can not only find out the network concept but also know the network eigenvalues of the obtained concept, which will lay a foundation for the further study of network rule extraction.

That is to say, the above mentioned researches to be studied can lay a foundation for concept cognition and three-way decision of the network data analysis and mining. So, it is important to unify complex network analysis and formal context of three-way decision into the same framework, since it can not only build a bridge connecting complex networks and three-way decision, but also can deepen the theoretical research and application research for these two fields.

In summary, the main contributions of this paper can be summarized as follows:

(1) The NFC3WD is proposed that can unify the two studies mentioned above into one data framework.
(2) The network weaken-concepts of NWC3WD are presented. Therefore, we can not only find out the network weaken-concept but also know the average influence of the network, as well as the influence difference within the sub-network.
(3) The bidirectional rule extraction algorithm and reduction algorithm based on network weaken-concept and confidence degree are also explored.

The rest of this paper is organized as follows. In Section 2, basic notions related to three-way concept analysis are introduced. In Section 3, the NFC3WD is put forward, which is explained with a simple example of an infectious disease network. In Section 4, network weaken-concept and their corresponding subnetworks are proposed, and meanwhile, the characteristic values of the subnetworks are given to describe the average influence and the difference in influence within the subnetworks. Furthermore, to obtain the decision rules of NFC3WD, the network weaken-concept logic is presented in Section 5. In Section 6, the extraction algorithm and the reduction algorithm of network rules are investigated. Considering the prevention and control of infectious disease networks, the network structure-based suspected case prevention algorithm and the sequential decision algorithm for infectious disease control are further developed. In Section 7, these algorithms are verified by constructing network adjacency matrices through the UCI databases.

2 Related theoretical foundation

Definition 1 [1]. Let (U, A, I) be a formal context, where $U = \{x_1, x_2, \cdots, x_n\}$ is a set of non-empty finite objects, $A = \{a_1, a_2, \cdots, a_m\}$ is a set of non-empty finite attributes, and I is the binary relation on the Cartesian product $U \times A$. Among them, $(x, a) \in I$ means that the object x has the attribute a, while $(x, a) \notin I$ means that the object x does not have the attribute a. In order to describe the formal concept, the following operators need to be defined: for $\forall X \subseteq U, B \subseteq A$,

$$X^* = \{a \in A | \forall x \in X, (x, a) \in I\},$$

$$B^* = \{x \in U | \forall a \in B, (x, a) \in I\}.$$

Definition 2 [1]. Let (U, A, I) be a formal context. For $\forall X \subseteq U, B \subseteq A$, if $X^* = B$, $B^* = X$, then we call (X, B) as a concept, and X and B are the extent and intent of the concept, respectively.

Definition 3 [19]. Let (U, A, I) be a formal context. For $\forall X, Y \subseteq U, B, C \subseteq A$, a pair of three-way operators derived from the object is defined as: $\lessdot$: $\mathcal{P}(U) \rightarrow \mathcal{DP}(A)$, $X^{\lessdot} = (X^*, X^{\bar{*}})$, $\gtrdot$: $\mathcal{DP}(A) \rightarrow \mathcal{P}(U)$, $(B, C)^{\gtrdot} = \{x \in U | x \in B^*, x \in C^{\bar{*}}\} = B^* \cap C^{\bar{*}}$, and a pair of three-way operators derived from the attribute is defined as: $\lessdot$: $\mathcal{P}(A) \rightarrow \mathcal{DP}(U)$, $B^{\lessdot} = (B^*, B^{\bar{*}})$, $\gtrdot$: $\mathcal{DP}(U) \rightarrow \mathcal{P}(A)$, $(X, Y)^{\gtrdot} = \{a \in A | x \in X^*, x \in Y^{\bar{*}}\} = X^* \cap Y^{\bar{*}}$.

Definition 4 [19]. Let (U, A, I) be a formal context. For $\forall X \subseteq U, B, C \subseteq A$, if $X^{\lessdot} = (B, C)$, $(B, C)^{\gtrdot} = X$, then $(X, (B, C))$ is called a three-way concept derived from the object, or simply OE-concept, where X is the extent of the OE-concept, and (B, C) is the intent of the OE-concept.

Definition 5 [19]. Let (U, A, I) be a formal context. For $\forall X, Y \subseteq U, B \subseteq A$, if $(X, Y)^{\gtrdot} = B$, $B^{\lessdot} = (X, Y)$, then $((X, Y), B)$ is called a three-way concept derived from the attribute, or simply AE-concept, where (X, Y) is the extent of the AE-concept, and B is the intent of the AE-concept.

Definition 6 [50]. The quadruple (U, M, A, I) is called a network formal context if $U = \{x_1, x_2, \cdots, x_n\}$ is a set of network nodes, $A = \{a_1, a_2, \cdots, a_m\}$ is a set of non-empty finite attributes, $M = \{M_1, M_2, \cdots, M_k\}$ is the structure matrix of the network, M_l $(l = 1, 2, \cdots, k)$ is the l-order adjacency matrix of the network, and $I = \{I_1, I_2, \cdots, I_k, I_{k+1}\}$, where $I_1, I_2, \cdots, I_k$ are the binary relations on the Cartesian product $U \times U$, and I_{k+1} is the binary relation on the Cartesian product $U \times A$. Among them, $(x_i, x_j) \in I_l$ means that the nodes x_i and x_j are adjacent to each other through at least l edges, and $(x_i, a_j) \in I_{k+1}$ means that the object x_i has the attribute a_j.

3 The network formal context of three-way decision

Definition 7 The quintuple $(U, M, \mathcal{C}, \mathcal{D}, I)$ is called a network formal context of three-way decision, or simply NFC3WD, if $U = \{x_1, x_2, \cdots, x_n\}$ is a set of network nodes, $M = \{M_1, M_2, \cdots, M_k\}$ is the structure matrix of the network, $\mathcal{C} = \{c_1, c_2, \cdots, c_m\}$ is a set of conditional attributes, $\mathcal{D} = \{d_1, d_2, \cdots, d_r\}$ is a set of decision attributes, and $I = \{I_1, I_2, \cdots, I_k, I_{\mathcal{C}}, I_{\mathcal{D}}\}$. Let $I_l = M_l = (m^l_{ij})_{n\times n}$ denote the l-order adjacency matrix of the network. When the nodes x_i and x_j are adjacent to each other through at least l edges, $m^l_{ij} = 1$; otherwise, $m^l_{ij} = 0$. $I_{\mathcal{C}} : U \times \mathcal{C} \to \{-1, 0, 1\}$, $I_{\mathcal{D}} : U \times \mathcal{D} \to \{-1, 0, 1\}$ are the binary relations on the Cartesian product $U \times \mathcal{C}$ and $U \times \mathcal{D}$, respectively. $I^{c_r}_{\mathcal{C}}(x_i)$, $I^{d_p}_{\mathcal{D}}(x_i)$ represent the values of x_i under the attributes c_r and d_p; $I^{c_r}_{\mathcal{C}}(x_i) = 1$, $I^{d_p}_{\mathcal{D}}(x_i) = 1$ denote that the node x_i has the attributes c_r and d_p; $I^{c_r}_{\mathcal{C}}(x_i) = -1$, $I^{d_p}_{\mathcal{D}}(x_i) = -1$ denote that the node x_i does not have the attributes c_r and d_p; $I^{c_r}_{\mathcal{C}}(x_i) = 0$, $I^{d_p}_{\mathcal{D}}(x_i) = 0$ denote that it is not sure whether the node x_i has the attributes c_r and d_p.

Example 1 Table 1 shows a NFC3WD $(U, M, \mathcal{C}, \mathcal{D}, I)$. The matrices $M_1, M_2, \cdots, M_k$ on the left reflect the adjacency structure between the network nodes. The information of $\mathcal{C}$ and $\mathcal{D}$ on the right reflect the values of the network nodes under the conditional attributes and decision attributes, respectively.

Example 2 Suppose there is an infectious diseases network as Fig. 1. By Definition 7, we can get a NFC3WD from the network as Table 2. In the M_k, let $k = 1$. Then we obtain the first-order adjacency matrix M_1 of the network.

Here, $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ represents 15 patients in the infectious disease network, $\mathcal{C} = \{a, b, c, d, e, f, g, h, i, j, k, l\}$ represents 12 symptoms of patients, and $\mathcal{D} = \{m, n, o\}$ represents 3 infectious diseases. The symptoms from a to l are: stuffy nose, sneeze, cough, dry cough, physical weakness, body aches, headache, positive nucleic acid test, the presence of CT imaging, total white blood cell count is normal or low and lymphocyte count is decreased, fever, viral gene sequencing is highly homologous to COVID-19. In addition, m, n, o mean cold, COVID-19 and flu, respectively.

For the attributes that the nodes are not sure to have, they are followed with "∼" in the figure. Taking the decision attribute "n" as an example, three colors represent three disease conditions. The green node means that it definitely has COVID-19, the orange node means that it is not sure whether it suffers from COVID-19, and the blue node means that it does not suffer from COVID-19. Taking the node 8 as an example, the label $bce \sim fgk, m \sim o$ indicates that it definitely has the conditional attributes b, c, f, g, k, it is not sure whether the node 8 has the conditional attribute e, it definitely has the decision attribute o, and it is not sure whether the node 8 has the decision attribute m.

Take the node 2 as an example. In the matrix M_1, the data (1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 1, 1) in the second line represents that the node 2 is first-order adjacent to the nodes 1, 3, 4, 6, 8, 9, 11, 13, 14 and 15.

Based on the information of $\mathcal{C}$ and $\mathcal{D}$, the data (−1, −1, −1, −1, 1, 1, 1, 0, 0, 0, 1, −1, −1, −1, 1) in the second line represents: the node 2 owns the conditional attributes e, f, g, k, the node 2 does not own the conditional attributes a, b, c, d, l, and it is not sure whether the node 2 owns the conditional attributes h, i, j; the node 2 owns the decision attribute o, and it does not own the decision attributes m, n.

Table 1 A network formal context of three-way decision

	M_1				$\cdots$	M_k				$\mathcal{C}$				$\mathcal{D}$			
	x_1	x_2	$\cdots$	x_n	$\cdots$	x_1	x_2	$\cdots$	x_n	c_1	c_2	$\cdots$	c_m	d_1	d_2	$\cdots$	d_r
x_1	0	1	$\cdots$	1	$\cdots$	0	0	$\cdots$	0	0	1	$\cdots$	-1	0	1	$\cdots$	-1
x_2	1	0	$\cdots$	0	$\cdots$	0	0	$\cdots$	1	1	-1	$\cdots$	0	0	1	$\cdots$	0
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	.	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_n	1	0	$\cdots$	0	$\cdots$	0	1	$\cdots$	0	0	1	$\cdots$	0	-1	0	$\cdots$	1

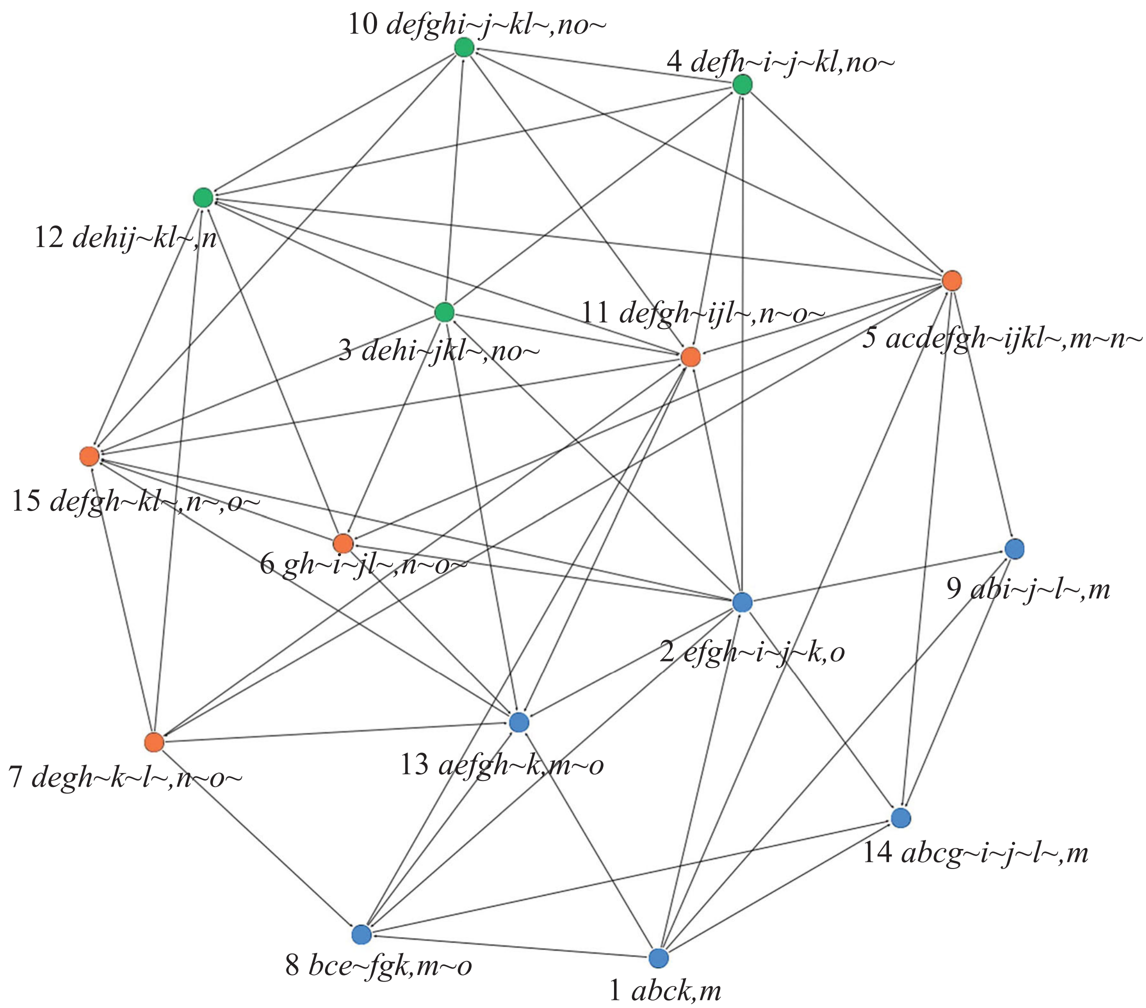

Fig. 1 Infectious disease network

Table 2 The network formal context of three-way decision of infectious diseases

	M_1															$\mathcal{C}$												$\mathcal{D}$		
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*m*	*n*	*o*
1	0	1	0	0	1	0	0	1	1	0	0	0	1	1	0	1	1	1	-1	-1	-1	-1	-1	-1	-1	1	-1	1	-1	-1
2	1	0	1	1	0	1	0	1	1	0	1	0	1	1	1	-1	-1	-1	-1	1	1	1	0	0	0	1	-1	-1	-1	1
3	0	1	0	1	0	1	0	0	0	1	1	1	1	0	1	-1	-1	-1	1	1	-1	-1	1	0	1	1	0	-1	1	0
4	0	1	1	0	1	0	0	0	0	1	1	1	0	0	0	-1	-1	-1	1	1	1	-1	0	0	0	1	1	-1	1	0
5	1	0	0	1	0	1	1	0	1	1	1	1	0	1	0	1	-1	1	1	1	1	1	0	1	1	1	0	0	0	-1
6	0	1	1	0	1	0	0	0	0	0	0	1	1	0	1	-1	-1	-1	-1	-1	-1	1	0	0	1	-1	0	-1	0	0
7	0	0	0	0	1	0	0	1	0	0	1	1	1	0	1	-1	-1	-1	1	1	-1	1	0	-1	-1	0	0	-1	0	0
8	1	1	0	0	0	0	1	0	0	0	1	0	1	1	0	-1	1	1	-1	0	1	1	-1	-1	-1	1	-1	0	-1	1
9	1	1	0	0	1	0	0	0	0	0	0	0	0	1	0	1	1	-1	-1	-1	-1	-1	-1	0	0	-1	0	1	-1	-1
10	0	0	1	1	1	0	0	0	0	0	1	1	0	0	1	-1	-1	-1	1	1	1	1	1	0	0	1	0	-1	1	0
11	0	1	1	1	1	0	1	1	0	1	0	1	1	0	1	-1	-1	-1	1	1	1	1	0	1	1	-1	0	-1	0	0
12	0	0	1	1	1	1	1	0	0	1	1	0	0	0	1	-1	-1	-1	1	1	-1	-1	1	1	0	1	0	-1	1	-1
13	1	1	1	0	0	1	1	1	0	0	1	0	0	0	1	1	-1	-1	-1	1	1	1	0	-1	-1	1	-1	0	-1	1
14	1	1	0	0	1	0	0	1	1	0	0	0	0	0	0	1	1	1	-1	-1	-1	0	-1	0	0	-1	0	1	-1	-1
15	0	1	1	0	0	1	1	0	0	1	1	1	1	0	0	-1	-1	-1	1	1	1	1	0	-1	-1	1	0	-1	0	0

From the above example, it is observed that the NFC3WD can be obtained from a network. Conversely, it is also possible to obtain the corresponding network from a NFC3WD.

4 Network weaken-concept and its sub-networks under NFC3WD

Definition 8 Let $(U, M, \mathcal{C}, \mathcal{D}, I)$ be a NFC3WD. For $\forall X \in 2^U$, $C \in 2^{\mathcal{C}}$, $D \in 2^{\mathcal{D}}$, we define the following notations:

(1) $X^{*<\mathcal{C}_P>} = \{c_r \in \mathcal{C} | \forall x_i \in X, I_{\mathcal{C}}^{c_r}(x_i) = 1\}$ denotes the set of conditional attributes that all the elements in the node set X must have in common.

Here, $* < \mathcal{C}_P >$ is just a symbol, which represents an operator, so are $* < \mathcal{C}_N >$, $* < \mathcal{C}_B >$, $* < \mathcal{D}_P >$, $* < \mathcal{D}_N >$, $* < \mathcal{D}_B >$, $* < \mathcal{X}_P >$, $* < \mathcal{X}_N >$ and $* < \mathcal{X}_B >$.

(2) $X^{*<\mathcal{C}_N>} = \{c_r \in \mathcal{C} | \forall x_i \in X, I_{\mathcal{C}}^{c_r}(x_i) = -1\}$ denotes the set of conditional attributes that all the elements in the node set X don't have in common.

(3) $X^{*<\mathcal{C}_B>} = \{c_r \in \mathcal{C} | \forall x_i \in X, I_{\mathcal{C}}^{c_r}(x_i) = 0\}$ denotes the set of conditional attributes that it is not sure whether all the elements in the node set X have in common.

(4) $X^{*<\mathcal{D}_P>} = \{d_p \in \mathcal{D} | \forall x_i \in X, I_{\mathcal{D}}^{d_p}(x_i) = 1\}$ denotes the set of decision attributes that all the elements in the node set X must have in common.

(5) $X^{*\mathcal{D}_N>} = \{d_p \in \mathcal{D} | \forall x_i \in X, I_{\mathcal{D}}^{d_p}(x_i) = -1\}$ denotes the set of decision attributes that all the elements in the node set X don't have in common.

(6) $X^{*<\mathcal{D}_B>} = \{d_p \in \mathcal{D} | \forall x_i \in X, I_{\mathcal{D}}^{d_p}(x_i) = 0\}$ denotes the set of decision attributes that it is not sure whether all the elements in the node set X have in common.

(7) $C^{*<\mathcal{X}_P>} = \{x_i \in U | \forall c_r \in C, I_{\mathcal{C}}^{c_r}(x_i) = 1\}$ denotes the set of nodes that must have all the conditional attributes in C.

(8) $C^{*<\mathcal{X}_N>} = \{x_i \in U | \forall c_r \in C, I_{\mathcal{C}}^{c_r}(x_i) = -1\}$ denotes the set of nodes that have no conditional attributes in C.

(9) $C^{*<\mathcal{X}_B>} = \{x_i \in U | \forall c_r \in C, I_{\mathcal{C}}^{c_r}(x_i) = 0\}$ denotes the set of nodes that it is not sure whether they have all the conditional attributes in C.

(10) $D^{*<\mathcal{X}_P>} = \{x_i \in U | \forall d_p \in D, I_{\mathcal{D}}^{d_p}(x_i) = 1\}$ denotes the set of nodes that must have all the decision attributes in D.

(11) $D^{*<\mathcal{X}_N>} = \{x_i \in U | \forall d_p \in D, I_{\mathcal{D}}^{d_p}(x_i) = -1\}$ denotes the set of nodes that have no decision attributes in D.

(12) $D^{*<\mathcal{X}_B>} = \{x_i \in U | \forall d_p \in D, I_{\mathcal{D}}^{d_p}(x_i) = 0\}$ denotes the set of nodes that it is not sure whether they have all the decision attributes in D.

(13) $(C^{*<\mathcal{X}_P>})^{*<\mathcal{D}_P>} = \{d_p \in \mathcal{D} | \forall x_i \in C^{*<\mathcal{X}_P>}, I_{\mathcal{D}}^{d_p}(x_i) = 1\}$ denotes the set of decision attributes that all the elements in the node set $C^{*<\mathcal{X}_P>}$ must have in common.

(14) $(C^{*<\mathcal{X}_P>})^{*<\mathcal{D}_N>} = \{d_p \in \mathcal{D} | \forall x_i \in C^{*<\mathcal{X}_P>}, I_{\mathcal{D}}^{d_p}(x_i) = -1\}$ denotes the set of decision attributes that all the elements in the node set $C^{*<\mathcal{X}_P>}$ don't have in common.

(15) $(C^{*<\mathcal{X}_P>})^{*<\mathcal{D}_B>} = \{d_p \in \mathcal{D} | \forall x_i \in C^{*<\mathcal{X}_P>}, I_{\mathcal{D}}^{d_p}(x_i) = 0\}$ denotes the set of decision attributes that it is not sure whether all the elements in the node set $C^{*<\mathcal{X}_P>}$ have in common.

(16) $(D^{*<\mathcal{X}_P>})^{*<\mathcal{C}_P>} = \{c_r \in \mathcal{C} | \forall x_i \in D^{*<\mathcal{X}_P>}, I_{\mathcal{C}}^{c_r}(x_i) = 1\}$ denotes the set of conditional attributes that all the elements in the node set $D^{*<\mathcal{X}_P>}$ must have in common.

(17) $(D^{*<\mathcal{X}_P>})^{*<\mathcal{C}_N>} = \{c_r \in \mathcal{C} | \forall x_i \in D^{*<\mathcal{X}_P>}, I_{\mathcal{C}}^{c_r}(x_i) = -1\}$ denotes the set of conditional attributes that all the elements in the node set $D^{*<\mathcal{X}_P>}$ don't have in common.

(18) $(D^{*<\mathcal{X}_P>})^{*<\mathcal{C}_B>} = \{c_r \in \mathcal{C} | \forall x_i \in D^{*<\mathcal{X}_P>}, I_{\mathcal{C}}^{c_r}(x_i) = 0\}$ denotes the set of conditional attributes that it is not sure whether all the elements in the node set $D^{*<\mathcal{X}_P>}$ have in common.

(19) $(D^{*<\mathcal{X}_N>})^{*<\mathcal{C}_P>} = \{c_r \in \mathcal{C} | \forall x_i \in D^{*<\mathcal{X}_N>}, I_{\mathcal{C}}^{c_r}(x_i) = 1\}$ denotes the set of conditional attributes that all the elements in the node set $D^{*<\mathcal{X}_N>}$ must have in common.

(20) $(D^{*<\mathcal{X}_N>})^{*<\mathcal{C}_N>} = \{c_r \in \mathcal{C} | \forall x_i \in D^{*<\mathcal{X}_N>}, I_{\mathcal{C}}^{c_r}(x_i) = -1\}$ denotes the set of conditional attributes that all the elements in the node set $D^{*<\mathcal{X}_N>}$ don't have in common.

(21) $(D^{*<\mathcal{X}_N>})^{*<\mathcal{C}_B>} = \{c_r \in \mathcal{C} | \forall x_i \in D^{*<\mathcal{X}_N>}, I_{\mathcal{C}}^{c_r}(x_i) = 0\}$ denotes the set of conditional attributes that it is not sure whether all the elements in the node set $D^{*<\mathcal{X}_N>}$ have in common.

Similarly, the following notations can be defined:

$$(C^{*<\mathcal{X}_N>})^{*<\mathcal{D}_P>}, (C^{*<\mathcal{X}_N>})^{*<\mathcal{D}_N>}, (C^{*<\mathcal{X}_N>})^{*<\mathcal{D}_B>},$$
$$(C^{*<\mathcal{X}_B>})^{*<\mathcal{D}_P>}, (C^{*<\mathcal{X}_B>})^{*<\mathcal{D}_N>}, (C^{*<\mathcal{X}_B>})^{*<\mathcal{D}_B>},$$
$$(D^{*<\mathcal{X}_B>})^{*<\mathcal{C}_P>}, (D^{*<\mathcal{X}_B>})^{*<\mathcal{C}_N>}, (D^{*<\mathcal{X}_B>})^{*<\mathcal{C}_B>}.$$

The notations in Definition 8 will be illustrated in Example 3.

From Definition 8, we find that some of operators can also stand for sub-networks, such as $C^{*<\mathcal{X}_P>}$. So the network characteristic values of their corresponding sub-networks should be discussed. In this way, not only can the network weaken-concept be found, but also the characteristics of the sub-network can be described quantitatively.

Definition 9 [50]. Let $\mathfrak{M} = \{\mathfrak{M}_1, \mathfrak{M}_2\}$ be the characteristic values of the network,

$$\mathfrak{M}_1 = \frac{\sum_{i=1}^{N} c_D(i)}{(N-1)(L+1)},$$
$$\mathfrak{M}_2 = \frac{\sum_{i=1}^{N} [c_{D\max} - c_D(i)]}{(N-1)(L+1)}.$$

Here, $c_D(i) = \sum_{j=1}^{N} (|x_j| + \sum_{k=1}^{L} |a_{ijk}|)$ denotes the degree of the node x_i, $|x_j|$ denotes the number of edges adjacent to the node x_i, $\sum_{k=1}^{L} |a_{ijk}|$ denotes the number of attributes shared by the nodes x_i and x_j, N is the number of nodes in the corresponding network, L is the number of attributes in the NFC3WD, and $c_{D_{\max}}$ is the largest value of $c_D(i)$. Then $\mathfrak{M}_1$ is called the average degree, which represents the average influence of the corresponding sub-network, and $\mathfrak{M}_2$ is the difference of the influence, which represents the degree of difference of influence among the nodes within the sub-network.

The larger the $\mathfrak{M}_1$, the greater the influence of the network. The larger the $\mathfrak{M}_2$, the greater the difference of influence between the nodes in the network, that is, the greater the structural heterogeneity. On the contrary, the structural heterogeneity is smaller.

Based on Definitions 8 and 9, the notion of network weaken-concepts of NFC3WD is given below.

Definition 10 Let $(U, M, \mathcal{C}, \mathcal{D}, I)$ be a NFC3WD. For $\forall X \in 2^U, C \in 2^{\mathcal{C}}, D \in 2^{\mathcal{D}}$, we can define the network weaken-concepts of NFC3WD as follows:

(1) $(\mathfrak{M}, X, X^{*<\mathcal{C}_P>}, X^{*<\mathcal{C}_N>}, X^{*<\mathcal{C}_B>}, X^{*<\mathcal{D}_P>}, X^{*<\mathcal{D}_N>}, X^{*<\mathcal{D}_B>})$ is called a network weaken-concept of NFC3WD induced by the node set X, or simply NWC3WD.

(2) $(\mathfrak{M}, C^{*<\mathcal{X}_P>}, (C^{*<\mathcal{X}_P>})^{*<\mathcal{D}_P>}, (C^{*<\mathcal{X}_P>})^{*<\mathcal{D}_N>}, (C^{*<\mathcal{X}_P>})^{*<\mathcal{D}_B>})$ is called a NWC3WD induced by the set of nodes that have all the condition attributes in C.

(3) $(\mathfrak{M}, C^{*<\mathcal{X}_N>}, (C^{*<\mathcal{X}_N>})^{*<\mathcal{D}_P>}, (C^{*<\mathcal{X}_N>})^{*<\mathcal{D}_N>}, (C^{*<\mathcal{X}_N>})^{*<\mathcal{D}_B>})$ is called a NWC3WD induced by the set of nodes that have no condition attributes in C.

(4) $(\mathfrak{M}, D^{*<\mathcal{X}_P>}, (D^{*<\mathcal{X}_P>})^{*<\mathcal{C}_P>}, (D^{*<\mathcal{X}_P>})^{*<\mathcal{C}_N>}, (D^{*<\mathcal{X}_P>})^{*<\mathcal{C}_B>})$ is called a NWC3WD induced by the set of nodes that have all the decision attributes in D.

(5) $(\mathfrak{M}, D^{*<\mathcal{X}_N>}, (D^{*<\mathcal{X}_N>})^{*<\mathcal{C}_P>}, (D^{*<\mathcal{X}_N>})^{*<\mathcal{C}_N>}, (D^{*<\mathcal{X}_N>})^{*<\mathcal{C}_B>})$ is called a NWC3WD induced by the set of nodes that have no decision attributes in D.

(6) $(\mathfrak{M}, D^{*<\mathcal{X}_B>}, (D^{*<\mathcal{X}_B>})^{*<\mathcal{C}_P>}, (D^{*<\mathcal{X}_B>})^{*<\mathcal{C}_N>}, (D^{*<\mathcal{X}_B>})^{*<\mathcal{C}_B>})$ is called a NWC3WD induced by the set of nodes that it is not sure whether they have all the decision attributes in D.

The above notations are further explained through the following example.

Example 3 Using the NFC3WD in Example 2, by Definitions 8, 9 and 10, we can compute the following NWC3WDs.

(1) The NWC3WD induced by the node set X.

Let $X = \{3, 4, 10, 12\}$. Then we have $X^{*<\mathcal{C}_P>} = \{d, e, k\}$, $X^{*<\mathcal{C}_N>} = \{a, b, c\}$, $X^{*<\mathcal{C}_B>} = \varnothing$, $X^{*<\mathcal{D}_P>} = \{n\}$, $X^{*<\mathcal{D}_N>} = \{m\}$, $X^{*<\mathcal{D}_B>} = \varnothing$, $c_D(3) = 17$, $c_D(4) = 16$, $c_D(10) = 18$, $c_D(12) = 17$, $\mathfrak{M}_1 = 1.42$, and $\mathfrak{M}_2 = 0.08$. So, the NWC3WD induced by the node set X is:

$$(\mathfrak{M}, X, X^{*<\mathcal{C}_P>}, X^{*<\mathcal{C}_N>}, X^{*<\mathcal{C}_B>}, X^{*<\mathcal{D}_P>}, X^{*<\mathcal{D}_N>}, X^{*<\mathcal{D}_B>}) = (\{1.42, 0.08\}, \{3, 4, 10, 12\}, \{d, e, k\}, \{a, b, c\}, \varnothing, \{n\}, \{m\}, \varnothing).$$

It shows that in the sub-network $X = \{3, 4, 10, 12\}$ corresponding to this weaken-concept, the average influence of the nodes is 1.42, and the difference of influence between the nodes is 0.08.

(2) The NWC3WD induced by the node set $C^{*<\mathcal{X}_P>}$.

Let $C_1 = \{d, e, h, k\}$. Then $C_1^{*<\mathcal{X}_P>} = \{3, 10, 12\}$, $(C_1^{*<\mathcal{X}_P>})^{*<\mathcal{D}_P>} = \{n\}$, $(C_1^{*<\mathcal{X}_P>})^{*<\mathcal{D}_N>} = \{m\}$, $(C_1^{*<\mathcal{X}_P>})^{*<\mathcal{D}_B>} = \varnothing$, $c_D(3) = 12$, $c_D(10) = 12$, $c_D(12) = 12$, $\mathfrak{M}_1 = 1.13$, and $\mathfrak{M}_2 = 0$. So, the NWC3WD induced by the node set $C_1^{*<\mathcal{X}_P>}$ is:

$$(\mathfrak{M}, C_1^{*<\mathcal{X}_P>}, (C_1^{*<\mathcal{X}_P>})^{*<\mathcal{D}_P>}, (C_1^{*<\mathcal{X}_P>})^{*<\mathcal{D}_N>}, (C_1^{*<\mathcal{X}_P>})^{*<\mathcal{D}_B>}) = (\{1.13, 0\}, \{3, 10, 12\}, \{n\}, \{m\}, \varnothing).$$

It shows that in the sub-network $C_1^{*<\mathcal{X}_P>} = \{3, 10, 12\}$ corresponding to this weaken-concept, the average influence of the nodes is 1.13, and the difference of influence between the nodes is 0.

(3) The NWC3WD induced by the node set $C^{*<\mathcal{X}_N>}$.

Let $C_2 = \{h, l\}$. Then $C_2^{*<\mathcal{X}_N>} = \{1, 8\}$, and we can get $(C_2^{*<\mathcal{X}_N>})^{*<\mathcal{D}_P>} = \varnothing$, $(C_2^{*<\mathcal{X}_N>})^{*<\mathcal{D}_N>} = \{n\}$, $(C_2^{*<\mathcal{X}_N>})^{*<\mathcal{D}_B>} = \varnothing$, $c_D(1) = 4$, $c_D(8) = 4$, $\mathfrak{M}_1 = 0.5$, $\mathfrak{M}_2 = 0$. So, the NWC3WD induced by the node set $C_2^{*<\mathcal{X}_N>}$ is:

$$(\mathfrak{M}, C_2^{*<\mathcal{X}_N>}, (C_2^{*<\mathcal{X}_N>})^{*<\mathcal{D}_P>}, (C_2^{*<\mathcal{X}_N>})^{*<\mathcal{D}_N>}, (C_2^{*<\mathcal{X}_N>})^{*<\mathcal{D}_B>}) = (\{0.5, 0\}, \{1, 8\}, \varnothing, \{n\}, \varnothing).$$

It shows that in the sub-network $C_2^{*<\mathcal{X}_N>} = \{1, 8\}$ corresponding to this weaken-concept, the average influence of the nodes is 0.5, and the difference of influence between the nodes is 0.

(4) The NWC3WD induced by the node set of $D^{*<\mathcal{X}_P>}$.

Let $D_1 = \{n\}$. Then $D_1^{*<\mathcal{X}_P>} = \{3, 4, 10, 12\}$, and we can get $(D_1^{*<\mathcal{X}_P>})^{*<\mathcal{C}_P>} = \{d, e, k\}$, $(D_1^{*<\mathcal{X}_P>})^{*<\mathcal{C}_N>} = \{a, b, c\}$, $(D_1^{*<\mathcal{X}_P>})^{*<\mathcal{C}_B>} = \varnothing$, $c_D(3) = 17$, $c_D(4) = 16$, $c_D(10) = 18$, $c_D(12) = 17$, $\mathfrak{M}_1 = 1.42$, $\mathfrak{M}_2 = 0.08$. So, the NWC3WD induced by the node set of $D_1^{*<\mathcal{X}_p>}$ is:

$$(\mathfrak{M}, D_1^{*<\mathcal{X}_P>}, (D_1^{*<\mathcal{X}_P>})^{*<\mathcal{C}_P>}, (D_1^{*<\mathcal{X}_P>})^{*<\mathcal{C}_N>}, (D_1^{*<\mathcal{X}_P>})^{*<\mathcal{C}_B>}) = (\{1.42, 0.08\}, \{3, 4, 10, 12\}, \{d, e, k\}, \{a, b, c\}, \varnothing).$$

It shows that in the sub-network $D_1^{*<\mathcal{X}_P>} = \{3, 4, 10, 12\}$ corresponding to this weaken-concept, the average influence of the nodes is 1.42, and the difference of influence between the nodes is 0.08.

(5) The NWC3WD induced by the node set of $D^{*<\mathcal{X}_N>}$.

Let $D_2 = \{n, o\}$. Then $D_2^{*<\mathcal{X}_N>} = \{1, 9, 14\}$, and we can get $(D_2^{*<\mathcal{X}_N>})^{*<\mathcal{C}_P>} = \{a, b\}$, $(D_2^{*<\mathcal{X}_N>})^{*<\mathcal{C}_N>} = \{d, e, f, h\}$, $(D_2^{*<\mathcal{X}_N>})^{*<\mathcal{C}_B>} = \varnothing$, $c_D(1) = 9$, $c_D(9) = 8$, $c_D(14) = 9$, $\mathfrak{M}_1 = 0.81$, $\mathfrak{M}_2 = 0.03$. So, the NWC3WD induced by the node set of $D_2^{*<\mathcal{X}_N>}$ is:

$$(\mathfrak{M}, D_2^{*<\mathcal{X}_N>}, (D_2^{*<\mathcal{X}_N>})^{*<\mathcal{C}_P>}, (D_2^{*<\mathcal{X}_N>})^{*<\mathcal{C}_N>}, (D_2^{*<\mathcal{X}_N>})^{*<\mathcal{C}_B>}) = (\{0.81, 0.03\}, \{1, 9, 14\}, \{a, b\}, \{d, e, f, h\}, \varnothing).$$

It shows that in the sub-network $D_2^{*<\mathcal{X}_N>} = \{1, 9, 14\}$ corresponding to this weaken-concept, the average influence of the nodes is 0.81, and the difference of influence between the nodes is 0.03.

(6) The NWC3WD induced by the node set of $D^{*<\mathcal{X}_B>}$.

Let $D_3 = \{n, o\}$. Then $D_3^{*<\mathcal{X}_B>} = \{6, 7, 11, 15\}$, $(D_3^{*<\mathcal{X}_B>})^{*<\mathcal{C}_P>} = \{g\}$, $(D_3^{*<\mathcal{X}_B>})^{*<\mathcal{C}_N>} = \{a, b, c\}$, $(D_3^{*<\mathcal{X}_B>})^{*<\mathcal{C}_B>} = \{h, l\}$, $c_D(6) = 2$, $c_D(7) = 8$, $c_D(11) = 9$, $c_D(15) = 11$, $\mathfrak{M}_1 = 0.63$, and $\mathfrak{M}_2 = 0.29$. So, the NWC3WD induced by the node set of $D_3^{*<\mathcal{X}_B>}$ is:

$$(\mathfrak{M}, D_3^{*<\mathcal{X}_B>}, (D_3^{*<\mathcal{X}_B>})^{*<\mathcal{C}_P>}, (D_3^{*<\mathcal{X}_B>})^{*<\mathcal{C}_N>}, (D_3^{*<\mathcal{X}_B>})^{*<\mathcal{C}_B>}) = (\{0.63, 0.29\}, \{6, 7, 11, 15\}, \{g\}, \{a, b, c\}, \{h, l\}).$$

It shows that in the sub-network $D_3^{*<\mathcal{X}_B>} = \{6, 7, 11, 15\}$ corresponding to this weaken-concept, the average influence of the nodes is 0.63, and the difference of influence between the nodes is 0.29.

5 The network weaken-concept logic of three-way decision

In order to further discuss the rules between NWC3WDs, we need to introduce logic between the network weaken-concepts of three-way decision.

Note that the logical description language of NWC3WD follows that of the rough set theory. Firstly, we can define the antecedent formula and the consequent formula corresponding to the network weaken-concept. Conversely, the corresponding attribute sets can be obtained by the antecedent formula and the consequent formula. Moreover, the network decision rule extraction algorithm and its rule confidence degree can be investigated.

The logic language of NWC3WD uses φ, ψ to represent the antecedent formula and the consequent formula, and adopts the logical connectors $\wedge$, $\vee$, $\neg$, $\rightarrow$ and $\leftarrow$ to form more complex logical expressions in a recursive manner.

Definition 11 Let $(U, M, \mathcal{C}, \mathcal{D}, I)$ be a NFC3WD. For $\forall C \in 2^{\mathcal{C}}, D \in 2^{\mathcal{D}}$, the antecedent formula and the consequent formula are denoted by $\varphi_x(C) = \underset{c_i \in C}{\wedge} (c_i, I_{\mathcal{C}}^{c_i}(x))$, $\psi_x(D) = \underset{d_i \in D}{\wedge} (d_i, I_{\mathcal{D}}^{d_i}(x))$, respectively.

In particular, $\varphi_P(C) = \underset{c_i \in C}{\wedge} (c_i, 1)$, $\varphi_N(C) = \underset{c_i \in C}{\wedge} (c_i, -1)$, $\varphi_B(C) = \underset{c_i \in C}{\wedge} (c_i, 0)$, $\psi_P(D) = \underset{d_i \in D}{\wedge} (d_i, 1)$, $\psi_N(D) = \underset{d_i \in D}{\wedge} (d_i, -1)$, $\psi_B(D) = \underset{d_i \in D}{\wedge} (d_i, 0)$.

Definition 12 Let $(U, M, \mathcal{C}, \mathcal{D}, I)$ be a NFC3WD. For $\forall X \in 2^U$, the antecedent formula and consequent formula of the node set X are defined as:

$$\begin{aligned}
\varphi_P(X^{*<\mathcal{C}_P>}) &= \underset{c_i \in X^{*<\mathcal{C}_P>}}{\wedge} (c_i, 1), \\
\varphi_N(X^{*<\mathcal{C}_N>}) &= \underset{c_i \in X^{*<\mathcal{C}_N>}}{\wedge} (c_i, -1), \\
\varphi_B(X^{*<\mathcal{C}_B>}) &= \underset{c_i \in X^{*<\mathcal{C}_B>}}{\wedge} (c_i, 0), \\
\psi_P(X^{*<\mathcal{D}_P>}) &= \underset{d_i \in X^{*<\mathcal{D}_P>}}{\wedge} (d_i, 1), \\
\psi_N(X^{*<\mathcal{D}_N>}) &= \underset{d_i \in X^{*<\mathcal{D}_N>}}{\wedge} (d_i, -1), \\
\psi_B(X^{*<\mathcal{D}_B>}) &= \underset{d_i \in X^{*<\mathcal{D}_B>}}{\wedge} (d_i, 0).
\end{aligned}$$

Similarly, for $\forall D \in 2^{\mathcal{D}}$, the antecedent formula and consequent formula of the decision attribute set D can be defined as:

$$\varphi_P((D^{*<\mathcal{X}_P>})^{*<\mathcal{C}_P>}) = \underset{c_i \in (D^{*<\mathcal{X}_P>})^{*<\mathcal{C}_P>}}{\wedge} (c_i, 1),$$

$$\varphi_N((D^{*<\mathcal{X}_N>})^{*<\mathcal{C}_N>}) = \underset{c_i\in(D^{*<\mathcal{X}_N>})^{*<\mathcal{C}_N>}}{\wedge}(c_i,-1),$$
$$\varphi_B((D^{*<\mathcal{X}_B>})^{*<\mathcal{C}_B>}) = \underset{c_i\in(D^{*<\mathcal{X}_B>})^{*<\mathcal{C}_B>}}{\wedge}(c_i,0).$$

Example 4 Let $D_1 = \{n\}$. We can get $\varphi_P((D_1^{*<\mathcal{X}_P>})^{*<\mathcal{C}_P>}) = (d,1)\wedge(e,1)\wedge(k,1)$, $\varphi_N((D_1^{*<\mathcal{X}_N>})^{*<\mathcal{C}_N>}) = (d,-1)$, and $\varphi_B((D_1^{*<\mathcal{X}_B>})^{*<\mathcal{C}_B>}) = (h,0)\wedge(l,0)$.

Definition 13 Let $(U, M, \mathcal{C}, \mathcal{D}, I)$ be a NFC3WD. For $\forall C\in 2^{\mathcal{C}}$, $D\in 2^{\mathcal{D}}$,

$$(\varphi_P(C))^{*<\mathcal{C}>} = \{c_i\in\mathcal{C}|\varphi_P(C) = \underset{c_i\in C}{\wedge}(c_i,1)\}$$

is the set of conditional attributes corresponding to the antecedent formula $\varphi_P(C)$.

Similarly, we can define

$$(\varphi_N(C))^{*<\mathcal{C}>} = \{c_i\in\mathcal{C}|\varphi_N(C) = \underset{c_i\in C}{\wedge}(c_i,-1)\},$$
$$(\varphi_B(C))^{*<\mathcal{C}>} = \{c_i\in\mathcal{C}|\varphi_B(C) = \underset{c_i\in C}{\wedge}(c_i,0)\}.$$

In addition, we can define

$$(\psi_P(D))^{*<\mathcal{D}>} = \{d_i\in\mathcal{D}|\psi_P(D) = \underset{d_i\in D}{\wedge}(d_i,1)\},$$
$$(\psi_N(D))^{*<\mathcal{D}>} = \{d_i\in\mathcal{D}|\psi_N(D) = \underset{d_i\in D}{\wedge}(d_i,-1)\},$$
$$(\psi_B(D))^{*<\mathcal{D}>} = \{d_i\in\mathcal{D}|\psi_B(D) = \underset{d_i\in D}{\wedge}(d_i,0)\}.$$

They are in fact the sets of decision attributes corresponding to the consequent formulas $\psi_P(D)$, $\psi_N(D)$ and $\psi_B(D)$, respectively.

Example 5 Let $\varphi_P(C_1) = (d,1)\wedge(e,1)\wedge(k,1)$. Then it is easy to get $(\phi_P(C_1))^{*<\mathcal{C}>} = \{d,e,k\}$.

Definition 14 Let $(U, M, \mathcal{C}, \mathcal{D}, I)$ be a NFC3WD, and $C^{*<\varphi_P>} = \underset{c_i\in C}{\wedge}(c_i,1)$, $C^{*<\varphi_N>} = \underset{c_i\in C}{\wedge}(c_i,-1)$, $C^{*<\varphi_B>} = \underset{c_i\in C}{\wedge}(c_i,0)$ be the logical formulas of conditional attributes. We rewrite them uniformly as $C^{*<\varphi_x>} = \underset{c_i\in C}{\wedge}(c_i, I_{\mathcal{C}}^{c_i}(x))$. That is, $C^{*<\varphi_x>}$ is equivalent to $\varphi_x(C)$.

Similarly, the logical formulas for decision attributes can also be defined as $D^{*<\psi_x>} = \underset{d_i\in D}{\wedge}(d_i, I_{\mathcal{D}}^{d_i}(x))$. In particular, $D^{*<\psi_P>} = \underset{d_i\in D}{\wedge}(d_i,1)$, $D^{*<\psi_N>} = \underset{d_i\in D}{\wedge}(d_i,-1)$, and $D^{*<\psi_B>} = \underset{d_i\in D}{\wedge}(d_i,0)$.

Example 6 Let $D_1 = \{m,n,o\}$. Then we can get $D_1^{*<\psi_N>} = (m,-1)\wedge(n,-1)\wedge(o,-1)$, $D_1^{*<\psi_P>} = (m,1)\wedge(n,1)\wedge(o,1)$, and $D_1^{*<\psi_B>} = (m,0)\wedge(n,0)\wedge(o,0)$.

Property 1 Let $(U, M, \mathcal{C}, \mathcal{D}, I)$ be a NFC3WD. For $\forall X\in 2^U$, $C\in 2^{\mathcal{C}}$, $D\in 2^{\mathcal{D}}$, $(\varphi_x(C))^{*<\mathcal{C}>}$, $(\psi_x(D))^{*<\mathcal{D}>}$, $C^{*<\varphi_x>}$ and $D^{*<\psi_x>}$ have the following properties:

(1) $((\varphi_P(C))^{*<\mathcal{C}>})^{*<\varphi_P>} = \varphi_P(C)$, $((\psi_P(D))^{*<\mathcal{D}>})^{*<\psi_P>} = \psi_P(D)$;
(2) $((\varphi_N(C))^{*<\mathcal{C}>})^{*<\varphi_N>} = \varphi_N(C)$, $((\psi_N(D))^{*<\mathcal{D}>})^{*<\psi_N>} = \psi_N(D)$;
(3) $((\varphi_B(C))^{*<\mathcal{C}>})^{*<\varphi_B>} = \varphi_B(C)$, $((\psi_B(D))^{*<\mathcal{D}>})^{*<\psi_B>} = \psi_B(D)$;
(4) $(C^{*<\varphi_P>})^{*<\mathcal{C}>} = C$, $(D^{*<\psi_P>})^{*<\mathcal{D}>} = D$;
(5) $(C^{*<\varphi_N>})^{*<\mathcal{C}>} = C$, $(D^{*<\psi_N>})^{*<\mathcal{D}>} = D$;
(6) $(C^{*<\varphi_B>})^{*<\mathcal{C}>} = C$, $(D^{*<\psi_B>})^{*<\mathcal{D}>} = D$.

Proof We only prove (1) and (4) since the rest can be similarly proved.

(1) According to Definition 13, we can get $(\varphi_P(C))^{*<\mathcal{C}>} = \{c_i\in\mathcal{C}|\varphi_P(C) = \underset{c_i\in C}{\wedge}(c_i,1)\} = C$, then $((\varphi_P(C))^{*<\mathcal{C}>})^{*<\varphi_P>} = (C)^{*<\varphi_P>} = \underset{c_i\in C}{\wedge}(c_i,1) = \varphi_P(C)$.

In a similar way, we can obtain $((\psi_P(D))^{*<\mathcal{D}>})^{*<\psi_P>} = \psi_P(D)$.

(4) By Definition 14, we can get $C^{*<\varphi_P>} = \underset{c_i\in C}{\wedge}(c_i,1)$, and $(C^{*<\varphi_P>})^{*<\mathcal{C}>} = (\underset{c_i\in C}{\wedge}(c_i,1))^{*<\mathcal{C}>} = C$.

In a similar way, we can obtain $(D^{*<\psi_P>})^{*<\mathcal{D}>} = D$. □

Note that $\varphi_x(C)^{*<\mathcal{C}>}$ and $C^{*<\varphi_x>} = \varphi_x(C)$ are a pair of operators, so are $\psi_x(D)^{*<\mathcal{D}>}$ and $D^{*<\psi_x>} = \psi_x(D)$. The properties of these operators are discussed below.

For convenience, we do not distinguish the operators φ_P, φ_N, φ_B, and use φ_x to represent them uniformly. For the operators ψ_P, ψ_N, ψ_B, we use ψ_x to represent them uniformly.

Property 2 Let $(U, M, \mathcal{C}, \mathcal{D}, I)$ be a NFC3WD. For $\forall X\in 2^U$, $C_1, C_2\in 2^{\mathcal{C}}$, $D_1, D_2\in 2^{\mathcal{D}}$, the attribute sets $\varphi_x(C)^{*<\mathcal{C}>}$ and $\psi_x(D)^{*<\mathcal{D}>}$ for logical formulas satisfy the following properties:

(1) $(\varphi_x(C_1))^{*<\mathcal{C}>}\cup(\varphi_x(C_2))^{*<\mathcal{C}>} = (\varphi_x(C_1)\wedge\varphi_x(C_2))^{*<\mathcal{C}>}$;
(2) $(\psi_x(D_1))^{*<\mathcal{D}>}\cup(\psi_x(D_2))^{*<\mathcal{D}>} = (\psi_x(D_1)\wedge\psi_x(D_2))^{*<\mathcal{D}>}$.

Proof By Definition 14 and Property 1, we can get $\varphi_x(C_1) = \underset{c_i\in C_1}{\wedge}(c_i, I_{\mathcal{C}}^{c_i}(x))$, $\varphi_x(C_2) = \underset{c_k\in C_2}{\wedge}(c_k, I_{\mathcal{C}}^{c_k}(x))$, then $(\varphi_x(C_1))^{*<\mathcal{C}>} = C_1$, $(\varphi_x(C_2))^{*<\mathcal{C}>} = C_2$. As a result, $(\varphi_x(C_1))^{*<\mathcal{C}>}\cup(\varphi_x(C_2))^{*<\mathcal{C}>} = C_1\cup C_2$. On the other hand, $(\varphi_x(C_1)\wedge\varphi_x(C_2))^{*<\mathcal{C}>} = ((\underset{c_i\in C_1}{\wedge}(c_i, I_{\mathcal{C}}^{c_i}(x)))\wedge$

$(\underset{c_k\in C_2}{\wedge}(c_k, I_{\mathcal{C}}^{c_k}(x))))^{*<\mathcal{C}>} = \underset{c_k\in C_1\cup C_2}{\wedge}(c_k, I_{\mathcal{C}}^{c_k}(x)) = C_1 \cup C_2$. So, $(\varphi_x(C_1))^{*<\mathcal{C}>} \cup (\varphi_x(C_2))^{*<\mathcal{C}>} = (\varphi_x(C_1) \wedge \varphi_x(C_2))^{*<\mathcal{C}>}$.

In a similar way, we can prove $(\psi_x(D_1))^{*<\mathcal{D}>} \cup (\psi_x(D_2))^{*<\mathcal{D}>} = (\psi_x(D_1) \wedge \psi_x(D_2))^{*<\mathcal{D}>}$. □

Property 3 Let $(U, M, \mathcal{C}, \mathcal{D}, I)$ be a NFC3WD. For $\forall C_1, C_2 \in 2^{\mathcal{C}}$, $D_1, D_2 \in 2^{\mathcal{D}}$, the logical formula $C^{*<\varphi_x>}$ of conditional attributes and the logical formula $D^{*<\psi_x>}$ of decision attributes satisfy the following properties:

(1) $(C_1 \cup C_2)^{*<\varphi_x>} = C_1^{*<\varphi_x>} \wedge C_2^{*<\varphi_x>}$, $(D_1 \cup D_2)^{*<\psi_x>} = D_1^{*<\psi_x>} \wedge D_2^{*<\psi_x>}$;
(2) $C_1^{*<\varphi_x>} \wedge C_2^{*<\varphi_x>} \Rightarrow (C_1 \cap C_2)^{*<\varphi_x>}$, $D_1^{*<\psi_x>} \wedge D_2^{*<\psi_x>} \Rightarrow (D_1 \cap D_2)^{*<\psi_x>}$.

Proof (1) By Definition 14, we obtain $C_1^{*<\varphi_x>} = \underset{c_i\in C_1}{\wedge}(c_i, I_{\mathcal{C}}^{c_i}(x))$, $C_2^{*<\varphi_x>} = \underset{c_i\in C_2}{\wedge}(c_i, I_{\mathcal{C}}^{c_i}(x))$. So $C_1^{*<\varphi_x>} \wedge C_2^{*<\varphi_x>} = (\underset{c_i\in C_1}{\wedge}(c_i, I_{\mathcal{C}}^{c_i}(x))) \wedge (\underset{c_i\in C_2}{\wedge}(c_i, I_{\mathcal{C}}^{c_i}(x))) = \underset{c_i\in C_1\cup C_2}{\wedge}(c_i, I_{\mathcal{C}}^{c_i}(x)) = (C_1 \cup C_2)^{*<\varphi_x>}$. Similarly, we can prove $(D_1 \cup D_2)^{*<\psi_x>} = D_1^{*<\psi_x>} \wedge D_2^{*<\psi_x>}$.

(2) According to the first item of Property 3, we get $C_1^{*<\varphi_x>} \wedge C_2^{*<\varphi_x>} = (C_1 \cup C_2)^{*<\varphi_x>} = \underset{c_k\in C_1\cup C_2}{\wedge}(c_k, I_{\mathcal{C}}^{c_k}(x)) \Rightarrow \underset{c_k\in C_1\cap C_2}{\wedge}(c_k, I_{\mathcal{C}}^{c_k}(x)) = (C_1 \cap C_2)^{*<\varphi_x>}$. Similarly, we can prove $D_1^{*<\psi_x>} \wedge D_2^{*<\psi_x>} \Rightarrow (D_1 \cap D_2)^{*<\psi_x>}$.

□

In order to extract the rules between NWC3WDs and analyze them later, we continue to give some operators related to the rules.

Definition 15 Let $r_{ij}^{*<\mathcal{X}>} = \{x_k \in U | x_k \vDash r_{ij}\}$, where $x_k \vDash r_{ij}$ means that the node x_k satisfies the rule $r_{ij} : \varphi_i \to \psi_j$. All the rules satisfied by x_k are represented by $\mathcal{R}(x_k) = \{r_{ij} \in R_u | x_k \vDash r_{ij}\}$, where $R_u = \{r_{ij} | r_{ij} : \varphi_i \to \psi_j\}$ denotes a set of the rules extracted from the network weaken-concepts of three-way decision.

Definition 16 For $r \subseteq R_u$, let $r^{*<\mathcal{X}>} = \{x_k \in U | \forall r_{ij} \in r, x_k \vDash r_{ij}\} = \bigcap_{r_{ij}\in r} r_{ij}^{*<\mathcal{X}>}$. That is, $r^{*<\mathcal{X}>}$ is the set of the nodes in U that satisfy all the rules in r.

Definition 17 For $X \subseteq U$, let $X^{*<\mathcal{R}>} = \{r_{ij} \in R_u | \forall x_k \in X, x_k \vDash r_{ij}\} = \bigcap_{x_k\in X} \mathcal{R}(x_k)$. That is, $X^{*<\mathcal{R}>}$ is the set of the rules which are commonly satisfied by all the nodes in X.

Property 4 Let $(U, M, \mathcal{C}, \mathcal{D}, I)$ be a NFC3WD. For $\forall X_1, X_2 \in 2^U$, the following properties hold:

(1) $(X_1 \cup X_2)^{*<\mathcal{R}>} = X_1^{*<\mathcal{R}>} \cap X_2^{*<\mathcal{R}>}$;
(2) $(X_1 \cap X_2)^{*<\mathcal{R}>} \supseteq X_1^{*<\mathcal{R}>} \cup X_2^{*<\mathcal{R}>}$.

Proof (1) As $X_1^{*<\mathcal{R}>} = \bigcap_{x_k\in X_1} \mathcal{R}(x_k)$ and $X_2^{*<\mathcal{R}>} = \bigcap_{x_k\in X_2} \mathcal{R}(x_k)$, we can get $X_1^{*<\mathcal{R}>} \cap X_2^{*<\mathcal{R}>} = \left(\bigcap_{x_k\in X_1} \mathcal{R}(x_k)\right) \bigcap \left(\bigcap_{x_k\in X_2} \mathcal{R}(x_k)\right) = \bigcap_{x_k\in X_1\cup X_2} \mathcal{R}(x_k) = (X_1 \cup X_2)^{*<\mathcal{R}>}$.

(2) As $X_1 = (X_1 - X_2) \cup (X_1 \cap X_2)$, $X_2 = (X_2 - X_1) \cup (X_1 \cap X_2)$, according to the first item of Property 4, we can get $X_1^{*<\mathcal{R}>} = \bigcap_{x_k\in(X_1-X_2)\cup(X_1\cap X_2)} \mathcal{R}(x_k) = (X_1 - X_2)^{*<\mathcal{R}>} \cap (X_1 \cap X_2)^{*<\mathcal{R}>}$. Similarly, we can obtain $X_2^{*<\mathcal{R}>} = (X_2 - X_1)^{*<\mathcal{R}>} \cap (X_1 \cap X_2)^{*<\mathcal{R}>}$.

To sum up, $X_1^{*<\mathcal{R}>} \cup X_2^{*<\mathcal{R}>} = ((X_1 - X_2)^{*<\mathcal{R}>} \cup (X_2 - X_1)^{*<\mathcal{R}>}) \cap (X_1 \cap X_2)^{*<\mathcal{R}>} \subseteq (X_1 \cap X_2)^{*<\mathcal{R}>}$. □

Property 5 Let $(U, M, \mathcal{C}, \mathcal{D}, I)$ be a NFC3WD and $R_u = \{r_{ij} | r_{ij} : \varphi_i \to \psi_j\}$ be a set of the rules extracted from the network weaken-concepts of three-way decision. For $\forall r_1, r_2 \subseteq R_u$, the following properties hold:

(1) $(r_1 \cup r_2)^{*<\mathcal{X}>} = r_1^{*<\mathcal{X}>} \cap r_2^{*<\mathcal{X}>}$;
(2) $(r_1 \cap r_2)^{*<\mathcal{X}>} \supseteq r_1^{*<\mathcal{X}>} \cup r_2^{*<\mathcal{X}>}$.

Proof (1) According to $r_1^{*<\mathcal{X}>} = \bigcap_{r_{ij}\in r_1} r_{ij}^{*<\mathcal{X}>}$ and $r_2^{*<\mathcal{X}>} = \bigcap_{r_{ij}\in r_2} r_{ij}^{*<\mathcal{X}>}$, we can get $r_1^{*<\mathcal{X}>} \cap r_2^{*<\mathcal{X}>} = \left(\bigcap_{r_{ij}\in r_1} r_{ij}^{*<\mathcal{X}>}\right) \bigcap \left(\bigcap_{r_{ij}\in r_2} r_{ij}^{*<\mathcal{X}>}\right) = \bigcap_{r_{ij}\in r_1\cup r_2} r_{ij}^{*<\mathcal{X}>} = (r_1 \cup r_2)^{*<\mathcal{X}>}$.

(2) As $r_1 = (r_1 \cap r_2) \cup (r_1 - r_2)$, $r_2 = (r_1 \cap r_2) \cup (r_2 - r_1)$, according to the first item of Property 5, we can get $r_1^{*<\mathcal{X}>} = (r_1 \cap r_2)^{*<\mathcal{X}>} \cap (r_1 - r_2)^{*<\mathcal{X}>}$, $r_2^{*<\mathcal{X}>} = (r_1 \cap r_2)^{*<\mathcal{X}>} \cap (r_2 - r_1)^{*<\mathcal{X}>}$. Then we obtain $r_1^{*<\mathcal{X}>} \cup r_2^{*<\mathcal{X}>} = \left((r_1 \cap r_2)^{*<\mathcal{X}>} \cap (r_1 - r_2)^{*<\mathcal{X}>}\right) \bigcup \left((r_1 \cap r_2)^{*<\mathcal{X}>} \cap (r_2 - r_1)^{*<\mathcal{X}>}\right) = (r_1 \cap r_2)^{*<\mathcal{X}>} \cap ((r_1 - r_2)^{*<\mathcal{X}>} \cup (r_2 - r_1)^{*<\mathcal{X}>}) \subseteq (r_1 \cap r_2)^{*<\mathcal{X}>}$.

□

Based on the above properties and the network weaken-concept logic, the extraction and simplification of the rules are ready to be discussed.

6 The bidirectional rule extraction and simplification for the network weaken-concepts of three-way decision

6.1 The bidirectional rule extraction based on NWC3WD

Definition 18 For the rule $\varphi \xrightarrow{\mu} \psi$, its confidence degree is defined as

$$\mu = \frac{|\varphi^{*<\mathcal{X}>} \bigcap \psi^{*<\mathcal{X}>}|}{|\varphi^{*<\mathcal{X}>}|}.$$

It represents the ratio of the number of nodes that satisfy both the antecedent φ and the consequent ψ to the number of nodes that only satisfy the antecedent φ.

Definition 19 We call $\psi \xrightarrow{\omega} \varphi$ the reverse rule of $\varphi \xrightarrow{\mu} \psi$, and its confidence degree is defined as

$$\omega = \frac{|\varphi^{*<\mathcal{X}>} \bigcap \psi^{*<\mathcal{X}>}|}{|\psi^{*<\mathcal{X}>}|}.$$

It represents the ratio of the number of nodes that satisfy both the formulas φ and ψ to the number of nodes that only satisfy the consequent ψ.

By Definitions 18 and 19, the larger ω and μ, the higher the homogeneities of their rules. And at the same time, it is not difficult to find that the reverse rule of $\psi \xrightarrow{\omega} \varphi$ is $\varphi \xrightarrow{\mu} \psi$.

According to the above definitions, we can first get $\psi \xrightarrow{\omega} \varphi$, and then get $\varphi \xrightarrow{\mu} \psi$. For the meaning and need of mining diagnostic rules, we only discuss the following five kinds of types. To achieve this task, the idea of bidirectional rule extraction is adopted. For the sake of simplicity, let $D_i = \{d_i\}$. We compute $D_i^{*<\mathcal{X}_P>}$, $D_i^{*<\mathcal{X}_N>}$, $D_i^{*<\mathcal{X}_B>}$, and further find $(D_i^{*<\mathcal{X}_P>})^{*<\mathcal{C}_P>}$, $(D_i^{*<\mathcal{X}_N>})^{*<\mathcal{C}_N>}$, $(D_i^{*<\mathcal{X}_B>})^{*<\mathcal{C}_B>}$ which are the conditional attribute sets corresponding to $D_i^{*<\mathcal{X}_P>}$, $D_i^{*<\mathcal{X}_N>}$ and $D_i^{*<\mathcal{X}_B>}$, respectively. Thus, we obtain $r : \psi \xrightarrow{\omega} \varphi$ as follows:

$$\begin{aligned}
\psi_P(D_i) &= (d_i, 1) \xrightarrow{\omega_P} \varphi_P((D_i^{*<\mathcal{X}_P>})^{*<\mathcal{C}_P>}),\\
\psi_N(D_i) &= (d_i, -1) \xrightarrow{\omega_N} \varphi_N((D_i^{*<\mathcal{X}_N>})^{*<\mathcal{C}_N>}),\\
\psi_B(D_i) &= (d_i, 0) \xrightarrow{\omega_B} \varphi_B((D_i^{*<\mathcal{X}_B>})^{*<\mathcal{C}_B>}).
\end{aligned}$$

Then, for $x_j \in D_i^{*<\mathcal{X}_P>}$ and $x_k \in D_i^{*<\mathcal{X}_N>}$, we can further get $\{x_j\}^{*<\mathcal{C}_P>}$ and $\{x_k\}^{*<\mathcal{C}_N>}$. Finally, taking $\{x_j\}^{*<\mathcal{C}_P>}$ and $\{x_k\}^{*<\mathcal{C}_N>}$ as prior information to mine the reverse rule $r : \varphi \xrightarrow{\mu} \psi$, we obtain

$$\begin{aligned}
&\varphi_P(\{x_j\}^{*<\mathcal{C}_P>}) \xrightarrow{\mu_P} (d_i, 1),\\
&\varphi_N(\{x_k\}^{*<\mathcal{C}_N>}) \xrightarrow{\mu_N} (d_i, -1).
\end{aligned}$$

For $r_{ij} : \psi_i \xrightarrow{\omega_{ij}} \varphi_j$ and its reverse rule $r_{ji} : \varphi_j \xrightarrow{\mu_{ji}} \psi_i$, if $\omega_{ij} = \mu_{ji} = 1$, it is said that the rules $r_{ij} : \psi_i \xrightarrow{\omega_{ij}} \varphi_j$ and $r_{ji} : \varphi_j \xrightarrow{\mu_{ji}} \psi_i$ are equivalent, and they are coordination rules.

The above idea of rule extraction is a kind of bidirectional mining. In other words, we get the rule r_{ij}, and further we get the reverse rule r_{ji}. The schematic diagram of bidirectional rule extraction can be shown in Fig. 2.

Based on the above discussion, the bidirectional rule extraction algorithm based on NFC3WD is given in Algorithm 1, where Step 1 is the initialization process, Steps 2-5 are to divide the object set into three categories according to different decision attribute values, Steps 6-14 are to extract the corresponding rules from the decision attributes, and Steps 15-28 are to traverse each object to obtain the rules. In addition, the time complexity of Algorithm 1 in the worst case is $O(|U|^2|\mathcal{C}||\mathcal{D}|)$.

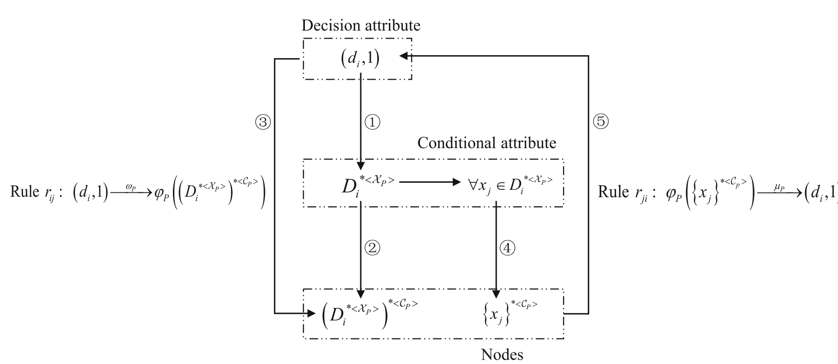

Fig. 2 The main idea of bidirectional rule extraction

Algorithm 1 The bidirectional rule extraction algorithm (BiR).

Input: The network formal context of three-way decision $(U, M, \mathcal{C}, \mathcal{D}, I)$.
Output: The rules for NWC3WDs and their confidence degrees.

Set Rule $= \varnothing$, Rule$_1 = \varnothing$, Rule$_2 = \varnothing$, Rule$_3 = \varnothing$, Rule$_4 = \varnothing$, Rule$_5 = \varnothing$.
For each $D_i = \{d_i\}$ do
  $D_i^{*<\mathcal{X}_P>} = \{x_j \in U | I_{\mathcal{D}}^{d_i}(x_j) = 1\}$.
  $D_i^{*<\mathcal{X}_N>} = \{x_j \in U | I_{\mathcal{D}}^{d_i}(x_j) = -1\}$.
  $D_i^{*<\mathcal{X}_B>} = \{x_j \in U | I_{\mathcal{D}}^{d_i}(x_j) = 0\}$.
  If $(D_i^{*<\mathcal{X}_P>})^{*<\mathcal{C}_P>} \neq \varnothing$
    Rule$_1$ = Rule$_1 \cup \{(d_i, 1) \xrightarrow{\omega_P} \varphi_P((D_i^{*<\mathcal{X}_P>})^{*<\mathcal{C}_P>})\}$.
  End if
  If $(D_i^{*<\mathcal{X}_N>})^{*<\mathcal{C}_N>} \neq \varnothing$
    Rule$_2$ = Rule$_2 \cup \{(d_i, -1) \xrightarrow{\omega_N} \varphi_N((D_i^{*<\mathcal{X}_N>})^{*<\mathcal{C}_N>}\}$.
  End if
  If $(D_i^{*<\mathcal{X}_B>})^{*<\mathcal{C}_B>} \neq \varnothing$
    Rule$_3$ = Rule$_3 \cup \{(d_i, 0) \xrightarrow{\omega_B} \varphi_B((D_i^{*<\mathcal{X}_B>})^{*<\mathcal{C}_B>})\}$.
  End if
  For each $x_j \in D_i^{*<\mathcal{X}_P>}$ do
    $\{x_j\}^{*<\mathcal{C}_P>} = \{c_r \in \mathcal{C} | I_{\mathcal{C}}^{c_r}(x_j) = 1\}$.
    If $\{x_j\}^{*<\mathcal{C}_P>} \neq \varnothing$
      $\varphi_P(\{x_j\}^{*<\mathcal{C}_P>}) \xrightarrow{\mu_P} (d_i, 1)$.
      Rule$_4$ = Rule$_4 \cup \{\varphi_P(\{x_j\}^{*<\mathcal{C}_P>}) \xrightarrow{\mu_P} (d_i, 1)\}$.
    End if
  End For
  For each $x_k \in D_i^{*<\mathcal{X}_N>}$ do
    $\{x_k\}^{*<\mathcal{C}_N>} = \{c_r \in \mathcal{C} | I_{\mathcal{C}}^{c_r}(x_k) = -1\}$.
    If $\{x_k\}^{*<\mathcal{C}_N>} \neq \varnothing$
      $\varphi_N(\{x_k\}^{*<\mathcal{C}_N>}) \xrightarrow{\mu_N} (d_i, -1)$.
      Rule$_5$ = Rule$_5 \cup \{\varphi_N(\{x_k\}^{*<\mathcal{C}_N>}) \xrightarrow{\mu_N} (d_i, -1)\}$.
    End if
  End For
  Rule = Rule$_1 \cup$ Rule$_2 \cup$ Rule$_3 \cup$ Rule$_4 \cup$ Rule$_5$.
End For
Return Rule.

Example 7 Based on the NFC3WDs in Example 2 and Algorithm 1, the diagnostic rules for COVID-19 can be obtained. Considering the medical common knowledge of COVID-19, we select the appropriate conditional attributes d, e, f, g, h, i, j, k and l for rule extraction. It starts with the decision attribute n which means COVID-19. Then we get $\{n\}^{*<\mathcal{C}_P>}$, $\{n\}^{*<\mathcal{C}_N>}$, $\{n\}^{*<\mathcal{C}_B>}$, and we further obtain the following rules:

Rule 1: $(n, 1) \xrightarrow{\omega_P=1} (d, 1) \wedge (e, 1) \wedge (k, 1)$.
Rule 2: $(n, -1) \xrightarrow{\omega_N=1} (d, -1)$.
Rule 3: $(n, 0) \xrightarrow{\omega_B=1} (h, 0) \wedge (l, 0)$.

On the contrary, starting with $x_i \in n^{*<\mathcal{X}_P>}$ and $x_j \in n^{*<\mathcal{X}_N>}$, we can obtain $\{x_i\}^{*<\mathcal{C}_P>}$ and $\{x_j\}^{*<\mathcal{C}_N>}$. Then, we get the corresponding rules as follows:

Rule 4: $(d, 1) \wedge (e, 1) \wedge (h, 1) \wedge (j, 1) \wedge (k, 1) \xrightarrow{\mu_P=1} (n, 1)$.
Rule 5: $(d, 1) \wedge (e, 1) \wedge (f, 1) \wedge (k, 1) \wedge (l, 1) \xrightarrow{\mu_P=1} (n, 1)$.
Rule 6: $(d, 1) \wedge (e, 1) \wedge (f, 1) \wedge (g, 1) \wedge (h, 1) \wedge (k, 1) \xrightarrow{\mu_P=1} (n, 1)$.
Rule 7: $(d, 1) \wedge (e, 1) \wedge (h, 1) \wedge (i, 1) \wedge (k, 1) \xrightarrow{\mu_P=1} (n, 1)$.
Rule 8: $(d, -1) \wedge (e, -1) \wedge (f, -1) \wedge (g, -1) \wedge (h, -1) \wedge (i, -1) \wedge (j, -1) \wedge (l, -1) \xrightarrow{\mu_N=1} (n, -1)$.
Rule 9: $(d, -1) \wedge (l, -1) \xrightarrow{\mu_N=1} (n, -1)$.
Rule 10: $(d, -1) \wedge (h, -1) \wedge (i, -1) \wedge (j, -1) \wedge (l, -1) \xrightarrow{\mu_N=1} (n, -1)$.
Rule 11: $(d, -1) \wedge (e, -1) \wedge (f, -1) \wedge (g, -1) \wedge (h, -1) \wedge (k, -1) \xrightarrow{\mu_N=1} (n, -1)$.
Rule 12: $(d, -1) \wedge (i, -1) \wedge (j, -1) \wedge (l, -1) \xrightarrow{\mu_N=1} (n, -1)$.
Rule 13: $(d, -1) \wedge (e, -1) \wedge (f, -1) \wedge (h, -1) \wedge (k, -1) \xrightarrow{\mu_N=1} (n, -1)$.

6.2 Reduction of three-way decision network rules

This subsection is to discuss the issue of reduction of three-way decision network rules. That is, we find and remove some reducible conditional attributes from the antecedent of a rule while preserving the confidence degree of the rule.

Definition 20 For the rule $(c_1, I_{\mathcal{C}}^{c_1}(x)) \wedge \cdots \wedge (c_i, I_{\mathcal{C}}^{c_i}(x)) \wedge \cdots \wedge (c_m, I_{\mathcal{C}}^{c_m}(x)) \xrightarrow{\mu} (d_1, I_{\mathcal{D}}^{d_1}(x)) \wedge \cdots \wedge (d_r, I_{\mathcal{D}}^{d_r}(x))$, we remove the conditional attribute c_i and get the rule $(c_1, I_{\mathcal{C}}^{c_1}(x)) \wedge \cdots \wedge (c_{i-1}, I_{\mathcal{C}}^{c_{i-1}}(x)) \wedge (c_{i+1}, I_{\mathcal{C}}^{c_{i+1}}(x)) \wedge \cdots \wedge (c_m, I_{\mathcal{C}}^{c_m}(x)) \xrightarrow{\mu_i} (d_1, I_{\mathcal{D}}^{d_1}(x)) \wedge \cdots \wedge (d_r, I_{\mathcal{D}}^{d_r}(x))$. If $\mu = \mu_i$, it is said that c_i is reducible in the rule.

Definition 21 If each conditional attribute in the rule $(c_1, I_{\mathcal{C}}^{c_1}(x)) \wedge \cdots \wedge (c_m, I_{\mathcal{C}}^{c_m}(x)) \xrightarrow{\mu} (d_1, I_{\mathcal{D}}^{d_1}(x)) \wedge \cdots \wedge (d_r, I_{\mathcal{D}}^{d_r}(x))$ is irreducible, it is called as a simplest rule.

The rule reduction algorithm is given as Algorithm 2, where Step 1 is the initialization process, Steps 3-8 are to obtain the reducible attribute, and Steps 9-11 are to obtain the reduction rule according to Definition 20 and Definition 21. In addition, the time complexity of Algorithm 2 in the worst case is $O(|C||U|)$.

Algorithm 2 The rule reduction algorithm based on confidence degree.

Input: The network formal context of three-way decision $(U, M, \mathcal{C}, \mathcal{D}, I)$ and the rule $\varphi_x(C) \xrightarrow{\mu} \psi_x$.
Output: The reduction rule and its confidence degree.
1: Initialize the set of reducible attributes $P = \varnothing$, and Rule $= \varnothing$.
2: **While** $C \neq \varnothing$
3: **For** each attribute $c_i \in C$
4: Calculate the confidence degree μ_i of $\varphi_x(C \backslash \{c_i\}) \xrightarrow{\mu_i} \psi_x$.
5: **If** $\mu_i = \mu$
6: $P = P \cup \{c_i\}$, $C = C \backslash \{c_i\}$, and go to Step 12.
7: **End If**
8: **End For**
9: **If** P is not increased compared to the last iteration
10: Rule $=$ Rule $\cup \{\varphi_x(C) \xrightarrow{\mu} \psi_x\}$ and break.
11: **End If**
12: Continue.
13: **End While**
14: **Return** Rule.

Example 8 Take the obtained rules (Rules 4-13) in Example 7 as an example to illustrate the idea of rule reduction since the antecedents of Rules 1-3 have already been the simplest forms.

For Rule 4: $(d, 1) \wedge (e, 1) \wedge (h, 1) \wedge (j, 1) \wedge (k, 1) \xrightarrow{\mu_P=1} (n, 1)$, the conditional attributes d, e, j and k are removed one by one, and the new confidence degree μ_P is still equal to 1. So, the simplest form of this rule after reduction is

$(h, 1) \xrightarrow{\mu_P=1} (n, 1)$.

A similar simplification of Rule 6 and Rule 7 can be performed, and the final reduction results are the same as that of Rule 4. Similarly, the reduction of Rule 5 is performed, and its simplest form is

$(l, 1) \xrightarrow{\mu_P=1} (n, 1)$.

For Rule 10, we get two simplest rules:

$(h, -1) \xrightarrow{\mu_N=1} (n, -1)$,

$(l, -1) \xrightarrow{\mu_N=1} (n, -1)$.

A similar simplification can also be performed for Rules 8, 9, 11, 12, 13, and the final reduction results are the same as that of Rule 10.

According to the above rules, COVID-19 can be diagnosed according to whether nucleic acid test is positive and the virus gene sequencing of the node is highly homologous to COVID-19.

Note that the symptoms of COVID-19 and flu are very similar. So, it is natural for us to add these two conditional attributes into the antecedents of the rules for flu diagnosis, and the following rules are obtained.

Rule 14: $(e, 1) \wedge (f, 1) \wedge (g, 1) \wedge (k, 1) \wedge (l, -1) \xrightarrow{\mu_P=1} (o, 1)$.

Rule 15: $(f, 1) \wedge (g, 1) \wedge (k, 1) \wedge (h, -1) \wedge (l, -1) \xrightarrow{\mu_P=1} (o, 1)$.

In a similar manner, Rule 14 and Rule 15 can be reduced as follows:

$$(f, 1) \wedge (h, -1) \xrightarrow{\mu_P=1} (o, 1),$$
$$(g, 1) \wedge (h, -1) \xrightarrow{\mu_P=1} (o, 1),$$
$$(f, 1) \wedge (l, -1) \xrightarrow{\mu_P=1} (o, 1),$$
$$(g, 1) \wedge (l, -1) \xrightarrow{\mu_P=1} (o, 1),$$
$$(e, 1) \wedge (l, -1) \xrightarrow{\mu_P=1} (o, 1).$$

6.3 The rule extraction based on three-way decision network structure

Considering the characteristics of infectious diseases spreading on the network, the spread of infectious diseases is often different under different network structures, so are the prevention and control measures.

Definition 22 Let $Nb_k(x_i)$ be the set of all k-order neighbors of the node x_i, and we call $Nb_k(x_i)$ the k-order adjacency set of x_i. In particular, the set of first-order neighbors of x_i is denoted by $Nb(x_i)$.

When a case x_i is found in an infectious disease network, the node connected to the case x_i becomes a suspected case.

The main framework of suspected case recognition based on the network structure is given in Algorithm 3, where $X_i = \bigcup_j Nb(x_j)$ indicates that the set of nodes who are contacted with the patients definitely suffering from the disease d_i. When the target case has contact with a confirmed case, it is put into the set of suspected cases and further the rules of suspected cases can be mined. In addition, the time complexity of Algorithm 3 in the worst case is $O(|\mathcal{D}||U|^2 + |\mathcal{C}||\mathcal{D}||U|)$.

Algorithm 3 Algorithm for mining suspected cases and rules based on the network structure.

Input: The network formal context of three-way decision $(U, M, \mathcal{C}, \mathcal{D}, I)$.

Output: Suspected cases and rules based on the structure of NFC3WD.

Initialize Rule $= \varnothing$, Suspected_case_set $= \varnothing$.
For each $D_i = \{d_i\}$ do
  $D_i^{*<\mathcal{X}_P>} = \{x_i \in U | I_{\mathcal{D}}^{d_i}(x_i) = 1\}$.
  For each node x_j in $D_i^{*<\mathcal{X}_P>}$ do
    Calculate the first-order adjacency set $Nb(x_j)$.
  End for
  $X_i = \bigcup_j Nb(x_j)$.
  For each node x_k in $X_i \backslash D_i^{*<\mathcal{X}_P>}$ do
    Suspected_case_set = Suspected_case_set $\cup \{x_k\}$.
    $\{x_k\}^{*<\mathcal{C}_P>} = \{c_r \in \mathcal{C} | I_{\mathcal{C}}^{c_r}(x_k) = 1\}$.
    Rule = Rule $\cup \{\varphi_P(\{x_k\}^{*<\mathcal{C}_P>}) \xrightarrow{\mu_P} (d_i, 0)\}$.
  End for
End for
Return Suspected_case_set and Rule.

Example 9 According to the network data in Example 2 and Algorithm 3, the set of suspected cases of COVID-19 in the network structure is $\{5, 6, 7, 11, 15\}$. The detailed analysis is given below.

The nodes 3, 4, 10 and 12 are diagnosed with COVID-19, so the set of nodes who have contact with these patients is $\{2, 5, 6, 7, 11, 13, 15\}$. However, the viral gene sequencing of the nodes 2 and 13 is not highly homologous to COVID-19, which means that they are not suspected cases of COVID-19. Node 5 shows that the total white blood cell count is normal or low and lymphocyte count is decreased, and there are CT imaging findings, which is a suspected case of COVID-19. Node 6 shows headache and the total white blood cell count is normal or low and lymphocyte count is decreased, so it is a suspected case of COVID-19. Nodes 7, 11 and 15 have dry cough, fatigue, body aches, headaches and other symptoms, so they are suspected cases of COVID-19.

6.4 The sequential decision making based on the network structure

In Subsection 6.3, we have discussed how to find the suspected cases of COVID-19. So, it is necessary to take additional measures to control them. To achieve this task, we need to further investigate sequential decision making based on the network structure.

Definition 23 For $\forall x_j \in Nb_k(x_i)$, let $d_t(x_i) = v_t$. Then we denote

$$(d_t(x_i) = v_t) \rightarrow S(d_{t+1}(x_j)),$$

where $d_t(x_i)$ is the value of the decision attribute of x_i at time t, and $S(d_{t+1}(x_j))$ is the measure taken for the node x_j at time $t+1$ that has contact with x_i.

Example 10 Additional measures are taken for the nodes x_j who are first-order adjacent to x_i when x_i takes different values under COVID-19. The details are shown in Fig. 3.

Suppose the node 3 in Example 2 is the target node. Since the node 3 is diagnosed as COVID-19 at time t, we have $d_t(3) = 1$. The set of the first-order adjacent nodes is $Nb(3) = \{2, 4, 6, 10, 11, 12, 13, 15\}$, so these nodes need to be centralized isolation at time $t + 1$.

7 Experiments and results

In this section, some numerical experiments are conducted to evaluate the performances of Algorithms 1 and 2, and to illustrate the effectiveness of the algorithm and the rationality of the concept cognition method under NFC3WD.

Fig. 3 Sequential decision making for COVID-19

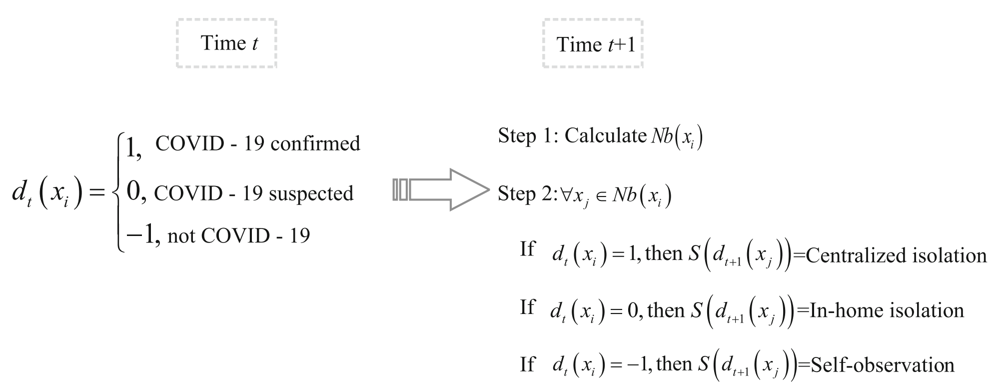

Table 3 The basic information of experimental data sets

Data sets	Nodes	Attributes
Heart disease	303	14
Breast cancer wisconsin (Original)	699	11
Iris	150	5
Acute inflammations	120	8
Seeds	210	7
Breast cancer coimbra	116	11
Hepatitis	155	20
Lymphography	148	19
Spect heart	267	23

7.1 Experimental environment and related descriptions

The machine used for experiments is a WIN 10 operating system with a 16.0 GB of RAM, a 2.10 GHz CPU, and the coding language is Matlab.

Due to the confidentiality of the infectious disease data, we can only test our algorithm by combining the UCI data sets and their induced networks. Meanwhile, nine UCI datasets [1] were selected for our methods in the experiments: Heart Disease, Breast Cancer Wisconsin (Original), Iris, Acute Inflammations, Seeds, Breast Cancer Coimbra, Hepatitis, Lymphography, Spect Heart. The details of the chosen data sets are shown in the Table 3.

Since these nine data sets are presented in the form of multi-valued attributes or continuous attributes, by considering the need for creating the network formal contexts of three-way decision, the data should be pre-processed before analysis. For convenience, the attribute labels in each data set are represented by letters $a, b, \cdots$ in turn.

7.2 Data preprocessing

Firstly, three-way value conversion. Taking Heart Disease data set as an example, the original data was transformed into the network formal context of three-way decision. Note that the original data has two types of continuous attributes and multi-valued attributes. Then, combining general medical knowledge, the test data were divided into three parts: Normal, Abnormal, and Uncertain. Then, the Normal part was set to -1, the Abnormal part was set to 1, and the rest to 0. In particular, for the attribute Sex, the male value was set to 1, and the female value was set to -1. For example, the values of the attributes in the Heart disease data set were transformed by the data as shown in Table 4.

Then, adjacency matrix was constructed by using the similarity between nodes. Here, the similarity degree of nodes was defined as the ratio of the number of attributes that take the same value under the same attribute to the total number of attributes between nodes. By setting the similarity threshold, when the similarity degree between x_i and x_j is more than 0.5, then $m_{ij} = 1$; otherwise, $m_{ij} = 0$. Finally, the adjacency matrix $M = (m_{ij})_{303\times303}$ between nodes was generated.

7.3 Experimental results

Based on the rule extraction algorithm and the rule reduction algorithm (see Algorithms 1 and 2 for details), we can calculate the values $\mu, \omega, \mathfrak{M}_1, \mathfrak{M}_2$ of each rule, and the results are shown in Tables 5 and 6.

In order to understand the data in Tables 5 and 6, additional explanations are given below. For the sake of brevity, the 14 attributes corresponding to the Heart Disease data set are denoted by letters $a, b, \cdots, n$, and n is used as the decision attribute. Firstly, we get the rules from back to front: $\psi_P(d_i, 1) \rightarrow \varphi_P((D_i^{*<\mathcal{X}_P>})^{*<\mathcal{C}_P>})$, $\psi_N(d_i, -1) \rightarrow \varphi_N((D_i^{*<\mathcal{X}_N>})^{*<\mathcal{C}_N>})$, and then the rules from front to back are $\varphi_P(\{x_j\}^{*<\mathcal{C}_P>}) \rightarrow (d_i, 1)$, $\varphi_N(\{x_j\}^{*<\mathcal{C}_N>}) \rightarrow (d_i, -1)$. Thus, we can generate the bidirectional rule extraction results, and the main rules in Table 5 are the three rules with the largest ω.

In Table 5, the rule with the greatest ω is r_1, r_2, and the rule with the greatest μ is also r_1, r_2. For r_1, r_2, $\mathfrak{M}_1$ is largest, while $\mathfrak{M}_2$ is small. That is, the sub-network corresponding to the rules has large average importance but small difference of the importance, and the rules satisfy structural homogeneity. We can also find $\omega = \mu$, so r_1, r_2 satisfy the rule coordination. The rest rules can be similarly analyzed.

The experimental results of the rest 4 data sets are shown in Table 6. According to Table 6, we can obtain the following conclusions:

In the Breast Cancer data set, the rule with the largest ω is $(f, 1) \rightarrow (j, 1)$, where $\mu = 0.8127$ indicates that the ratio of the number of nodes that satisfy this rule $(f, 1) \rightarrow (j, 1)$ to the number of nodes that must have conditional attributes f. Meanwhile, $\omega = 0.8967$ indicates that the ratio of the number of nodes that satisfy this rule $(j, 1) \rightarrow (f, 1)$ to the number of nodes that must have decision attributes j. $\mathfrak{M}_1 = 90.7643$ indicates that in the sub-network corresponding to the node that satisfies this rule, the average influence of the nodes is 90.7643. $\mathfrak{M}_2 = 46.8699$ indicates that the difference of the influence between nodes in this sub-network is 46.8699. The rest rules can be similarly explained.

[1] https://archive.ics.uci.edu/ml/index.php

Table 4 The transformation of attribute values of the Heart Disease data set

Attribute name	Marker name	Attribute value partition	Mapped values
Age	*a*	age<52	-1
		age>65	1
		52≤age≤65	0
Sex	*b*	sex=1	1
		sex=0	-1
cp	*c*	cp=0	-1
		cp=1	1
		cp=2,3	0
trestbps	*d*	trestbps≥90	1
		trestbps<90	-1
chol	*e*	chol ≥250	1
		chol <200	-1
		200≤chol<250	0
fbs	*f*	fbs ≥200	1
		fbs<200	-1
restecg	*g*	restecg=1,2	1
		restecg=0	-1
thalach	*h*	thalach>100,thalach<60	1
		60≤ thalach≤100	-1
exang	*i*	exang=1	1
		exang=0	-1
oldpeak	*j*	oldpeak≥0.5	1
		oldpeak<0.5	-1
slope	*k*	slope=1	1
		slope=2	-1
		slope=3	0
ca	*l*	ca=1,2,3,4	1
		ca=0	-1
thal	*m*	thal=0,1,2	1
		thal=3	-1
target	*n*	target=1	1
		target=0	-1

Table 5 Rules for Heart Disease data set

Rule type	Total number of rules	Main rules		ω	μ	$\mathfrak{M}_1$	$\mathfrak{M}_2$
$\psi_P(d_i, 1) \to \varphi_P((D_i^{*<\mathcal{X}_P>})^{*<\mathcal{C}_P>})$	1	r_1	$(n, 1) \to (d, 1)$	1	1	39.7057	14.8919
$\psi_N(d_i, -1) \to \varphi_N((D_i^{*<\mathcal{X}_N>})^{*<\mathcal{C}_N>})$	1	r_2	$(n, -1) \to (f, -1) \wedge (h, -1)$	1	1	39.462	14.4613
$\varphi_P(\{x_j\}^{*<\mathcal{C}_P>}) \to (d_i, 1)$	40	r_3	$(d, 1) \wedge (m, 1) \to (n, 1)$	0.8303	0.7366	37.4108	11.2108
		r_4	$(g, 1) \to (n, 1)$	0.5879	0.6218	29.3431	8.3118
		r_5	$(g, 1) \wedge (m, 1) \to (n, 1)$	0.4727	0.8041	26.8052	5.7455
$\varphi_N(\{x_j\}^{*<\mathcal{C}_N>}) \to (d_i, -1)$	71	r_6	$(c, -1) \to (n, -1)$	0.7536	0.7273	35.3282	7.6181
		r_7	$(m, -1) \to (n, -1)$	0.6449	0.7607	32.3924	7.7250
		r_8	$(g, -1) \to (n, -1)$	0.5725	0.5374	28.5641	7.0872

Table 6 Experimental results of the chosen data sets

Data sets	Total number of rules	Reduction rules	Main rules	ω	μ	$\mathfrak{M}_1$	$\mathfrak{M}_2$
Breast cancer wisconsin	133	70	$(f, 1) \rightarrow (j, 1)$	0.8967	0.8127	90.7643	46.8699
			$(c, 1) \wedge (f, 1) \wedge (g, 1) \rightarrow (j, 1)$	0.6405	0.9568	92.2786	23.8341
			$(a, 1) \wedge (f, 1) \rightarrow (j, 1)$	0.6157	0.9675	76.5135	24.6198
Iris	23	12	$(c, 1) \wedge (d, 1) \rightarrow$ (setosa, 1)	1	1	9.619	5.6871
			$(c, 1) \wedge (d, 1) \rightarrow$ (versicolor, 1)	0.98	0.875	8.5139	2.2049
			$(a, 1) \rightarrow$ (virginica, 1)	0.82	0.6721	5.8417	1.8458
Acute inflammations	28	18	$(d, 1) \wedge (e, 1) \rightarrow (g, 1)$	0.8305	1	14.2865	2.8125
			$(c, 1) \wedge (d, 1) \rightarrow (h, 1)$	0.8	1	15.9808	1.1987
			$(c, 1) \wedge (f, 1) \rightarrow (h, 1)$	0.6	1	13.3448	2.17241
Seeds	58	55	$(a, -1) \rightarrow (h, -1)$	0.9286	0.8667	46.0313	8.2483
			$(a, -1) \wedge (e, -1) \rightarrow (h, -1)$	0.8857	0.8732	47.7632	6.5574
			$(e, -1) \wedge (f, -1) \rightarrow (h, -1)$	0.7143	0.9615	40.0952	3.6689

In the Breast Cancer and Acute Inflammations data sets, the rule with medium ω has the largest $\mathfrak{M}_1$ and smallest $\mathfrak{M}_2$. That is to say, in its corresponding sub-network, the average influence of the nodes is relatively large, but the difference is small. So, the rule satisfies medium rule homogeneity and higher structural homogeneity.

In the Iris data set, the rule with largest ω has the largest $\mathfrak{M}_1$ and $\mathfrak{M}_2$. That is to say, in its corresponding sub-network, the average influence of the nodes is large, and the difference is large too. So, the rule satisfies higher rule homogeneity and structural heterogeneity.

In the Seeds data set, it is observed that the rule with medium ω has the largest $\mathfrak{M}_1$ and the medium $\mathfrak{M}_2$. So, the rule satisfies medium rule homogeneity and structural heterogeneity.

7.4 Comparison of the proposed algorithm BiR with some machine learning methods

In this subsection, we compare the proposed algorithm BiR (exactly the classifier induced by BiR) with the method in [25] and some common machine learning methods: Bayes net, Random forest, and Decision tree algorithms. The algorithm in [25] is a multi-scale rule extraction algorithm which has been applied to every property with multiple scales. Bayes net algorithm uses a graphical method to describe the relationship between data, with clear semantics and easy to understand. Random forest algorithm trains and predicts samples from multiple trees. Decision tree is an error rate reduction pruning method. In order to evaluate the performance of the proposed algorithms, in the experiments we selected two evaluation indicators: accuracy and Auc. The larger the values of the two evaluation indices, the better the performances of the algorithm to be evaluated. The detailed evaluation results are reported in Table 7 and the best results are shown in bold.

As listed in Table 7 and Fig. 4, the Auc and accuracy values of BiR in all the data sets are greater than those of the compared algorithms except the Breast Cancer Wisconsin data set. At the same time, its average accuracy is 91.50%, which is 4.79% higher than the second ranked Bayes net and 8.52% higher than the last ranked Decision tree. Besides, it seems that BiR has better overall performance and more stable than the selected other algorithms.

Table 7 The experimental results for comparison

Dataset	Our algorithm (BiR)		Literature [25]		Bayes net		Random forest		Decision tree	
	Auc	Accuracy(%)	Auc	Accuracy(%)	Auc	Accuracy(%)	Auc	Accuracy(%)	Auc	Accuracy(%)
Acute inflammations	**1**	**100**	**1**	**100**	**1**	**100**	**1**	**100**	**1**	**100**
Breast cancer wisconsin	0.99	95.85	0.99	**97.89**	**1**	**97.89**	0.99	97.14	0.98	93.14
Breast cancer coimbra	**0.77**	**77.45**	0.72	72.45	0.64	62.07	0.68	58.62	0.70	62.07
Heart disease	**0.88**	**84.59**	0.78	80.26	**0.88**	82.89	0.84	76.32	0.81	78.95
Hepatitis	**0.99**	**94.19**	0.98	80.68	0.91	84.62	0.87	82.05	0.79	71.79
Lymphography	**1**	**100**	**1**	90.60	0	**100**	**1**	97.30	0.99	97.30
Spect heart	**0.94**	**88.39**	0.93	84.59	0.82	79.10	0.83	91.04	0.75	77.61

Bold entries represent better Auc and accuracy values

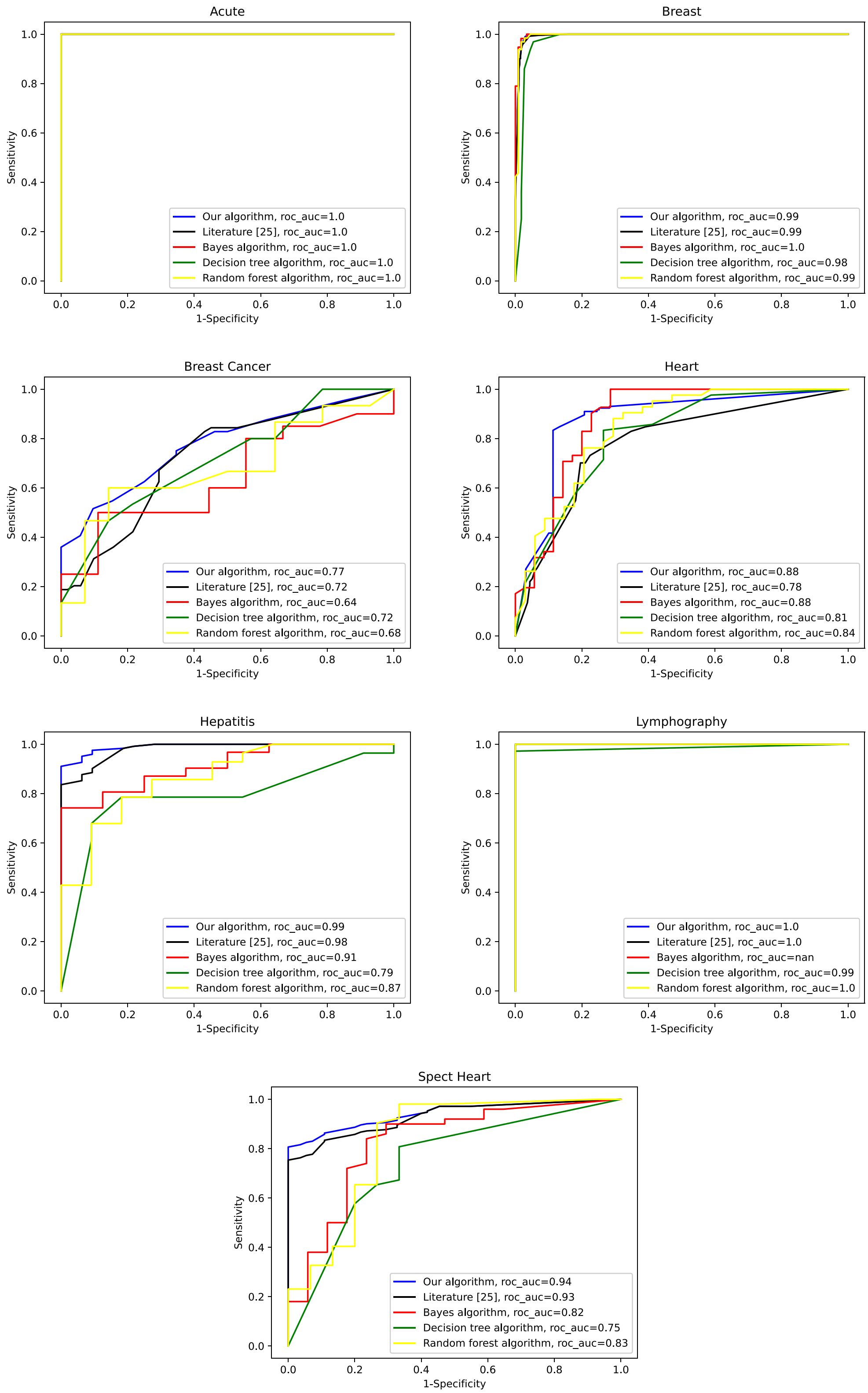

Fig. 4 Roc curve comparison chart

Specifically, we compare our algorithm with the one in [25]. The advantage of the method in [25] is that it is suitable for decision tables with multiple scales of conditional attributes and only one scale of decision attributes. But the final rules have only one scale for each conditional attribute. We discretized each conditional attribute into three values with three different scales, and compared our algorithm with it. Our algorithm is still superior to the algorithm in [25].

All of the comparison results vividly illustrate that the idea of bidirectional rule extraction can greatly improve the rule extraction ability of the proposed model.

8 Conclusion

The study of network formal context can make it possible to combine complex network analysis with formal concept analysis. In this paper, the network formal context of three-way decision (NFC3WD) has made the network data with three-way decision not only obtain the cognition of network weaken-concepts but also obtain the network characteristic values. The bidirectional rule extraction and reduction algorithms for the network weaken-concepts of three-way decision have been developed to demonstrate the effectiveness of the NFC3WD method.

In addition, there are some interesting problems that need to be further investigated. For example, under the NFC3WD background, how to learn the way and speed of the spread of network weaken-concepts, the mutual influence of viewpoints under the network weaken-concepts, and the final formation of such kinds of viewpoints.

Acknowledgements This work was supported by the National Natural Science Foundation of China (Nos. 11971211 and 12171388)

Declarations

Conflict of Interests The authors declare that they have no conflict of interest.

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