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On the Structure of an Internal Groupoid

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Abstract
The category of internal groupoids (in an arbitrary category) is shown to be equivalent to the full subcategory of so called involutive-2-links that are unital and associative.

Keywords Internal groupoid · Reflexive graph · Multiplicative structure · Involutive-2-link

The purpose of this note is to show that the category of internal groupoids in an arbitrary category is equivalent to the full subcategory of involutive-2-links that are unital and associative. An involutive-2-link consists of a morphism together with two intertwined involutions on its domain. This approach to internal groupoids contrasts with the one in which a groupoid is seen as a reflexive graph equipped with a multiplicative structure (as e.g. in [5]). Here, the underlying reflexive graph is seen as a unique structure (provided it exists) associated with an involutive-2-link, thus recapturing once again a geometric and differential perspective [3], as briefly explained in [6]. Moreover, since the ambient category is arbitrary, this approach is suitable to study Lie groupoids.

Theorem 1 *Let* **C** *be any category. The category of internal groupoids is equivalent to the full subcategory of unital and associative involutive-2-links.*

The category of *involutive-2-links*, internal to an arbitrary category **C**, consists of triples $(\theta, \varphi, m\colon A \to B)$ where $m\colon A \to B$ is any morphism in **C**, whereas $\theta, \varphi\colon A \to A$ are such that $\theta^2 = \varphi^2 = 1_A$ and $\theta\varphi\theta = \varphi\theta\varphi$. A morphism of involutive-2-links, from $(\theta, \varphi, m\colon A \to B)$ to $(\theta', \varphi', m'\colon A' \to B')$, is a pair of morphisms $(f\colon A \to A', g\colon B \to B')$ such that $f\theta = \theta' f$, $f\varphi = \varphi' f$ and $m' f = gm$.

Definition 1 Let **C** be any category. An *involutive-2-link* structure in **C**, say $(\theta, \varphi, m\colon C_2 \to C_1)$, is said to be:

(1) *associative* when the pair $(m\varphi, m\theta)$ is bi-exact (see diagram (7) bellow with $\pi_1 = m\varphi$ and $\pi_2 = m\theta$) and the induced morphisms $m_1, m_2\colon C_3 \to C_2$, determined by (see

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diagram (8))

$$\pi_1 m_1 = m p_1, \quad \pi_2 m_1 = \pi_2 p_2$$
$$\pi_1 m_2 = \pi_1 p_1, \quad \pi_2 m_2 = m p_2$$

are such that $mm_1 = mm_2$.

(2) *unital* when the two pairs of morphisms $(m, m\theta)$, $(m, m\varphi)$ are jointly monomorphic and there exist morphisms $e_1, e_2 \colon C_1 \to C_2$ such that

$$me_1 = 1_{C_1} = me_2 \tag{1}$$
$$\theta e_2 = e_2, \quad \varphi e_1 = e_1 \tag{2}$$
$$m\theta\varphi e_2 = m\varphi\theta e_1 \tag{3}$$
$$m\theta e_1 m\varphi = m\varphi e_2 m\theta \tag{4}$$
$$m\theta e_1 m = m\theta e_1 m\theta \tag{5}$$
$$m\varphi e_2 m = m\varphi e_2 m\varphi. \tag{6}$$

A pair of parallel morphisms (or a graph) is said to be *bi-exact* if when considered as a span it can be completed into a commutative square which is both a pullback and a pushout and moreover, if considered as a cospan, it can be completed into another commutative square which is both a pullback and a pushout. In other words, a graph such as

$$C_2 \underset{\pi_1}{\overset{\pi_2}{\rightrightarrows}} C_1 \tag{7}$$

is *bi-exact* precisely when there exist p_1, p_2, d, c as displayed

$$\begin{array}{ccc} C_3 & \overset{p_2}{\dashrightarrow} & C_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ C_2 & \underset{\pi_2}{\longrightarrow} & C_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle c} \\ C_1 & \overset{d}{\dashrightarrow} & C_0 \end{array} \tag{8}$$

such that both squares are commutative and simultaneously a pullback and pushout. Such squares are also called exact squares, bicartesian squares, Dolittle diagrams or pulation squares [1].

The functor, say F, from the category of internal groupoids to the category of involutive-2-links, defined via the assignment

$$C_2 \overset{\pi_2}{\underset{\pi_1}{\overset{m}{\Rrightarrow}}} C_1 \circlearrowleft^{i} \; \overset{d}{\underset{c}{\overset{e}{\leftrightarrows}}} C_0 \quad \mapsto \quad {}^{\theta,\varphi}\circlearrowright C_2 \xrightarrow{m} C_1 , \tag{9}$$

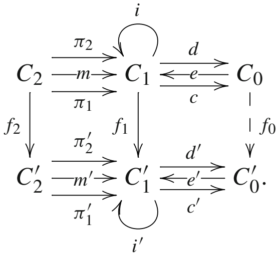
with $\theta = \langle i\pi_1, m\rangle$, $\varphi = \langle m, i\pi_2\rangle$ is full and faithful. This functor takes an internal groupoid (see e.g. [2], Section 7.1), forgets its underlying reflexive graph, keeps the multiplicative structure $m \colon C_2 \to C_1$ and contracts the morphisms $i\pi_1$ and $i\pi_2$ in the form of the two endomorphisms $\theta, \varphi \colon C_2 \to C_2$. As a consequence,

$$m\varphi = \pi_1, \quad m\theta = \pi_2 \tag{10}$$

$$\pi_1\varphi = m, \quad \pi_1\theta = i\pi_1 \tag{11}$$

$$\pi_2\varphi = i\pi_2, \quad \pi_2\theta = m. \tag{12}$$

Conditions $\theta^2 = \varphi^2 = 1_{C_2}$ and $\theta\varphi\theta = \varphi\theta\varphi$ are easily verified and it is easy to see that the functor is faithful. In order to see that the functor F is full let us consider two internal groupoids, say C and C', as illustrated in diagram (13) below. Let us assume the existence of a morphism of involutive-2-links from $F(C)$ to $F(C')$, that is, a pair of morphisms $f_i \colon C_i \to C'_i$, with $i = 1, 2$ such that $\theta' f_2 = f_2\theta$, $\varphi' f_2 = f_2\varphi$ and $m' f_2 = f_1 m$, with $\theta, \varphi, \theta', \varphi'$ the respective involutions associated with $F(C)$ and $F(C')$. We need to show that the pair (f_2, f_1) can be extended to a morphism of internal groupoids



(13)

First observe that $f_2(x, y) = (f_1(x), f_1(y))$ since $\pi'_1 f_2 = m'\varphi' f_2 = m' f_2\varphi = f_1 m\varphi = f_1\pi_1$ and similarly $\pi'_2 f_2 = f_1\pi_2$. This means that the hypotheses $\theta' f_2 = f_2\theta$, $\varphi' f_2 = f_2\varphi$ and $m' f_2 = f_1 m$ are translated, respectively, as

$$(f_1(x)^{-1}, f_1(x) f_1(y)) = (f_1(x^{-1}), f_1(xy))$$

$$(f_1(x) f_1(y), f_1(y)^{-1}) = (f_1(xy), f_1(y^{-1}))$$

$$f_1(x) f_1(y) = f_1(xy)$$

from which we conclude $i' f_1 = f_1 i$. We also have $f_1 ed(x)) = f_1(x^{-1}x) = f_1(x^{-1}) f_1(x) = f_1(x)^{-1} f_1(x) = e'd' f_1(x)$ and $f_1 ec(x) = e'c' f_1(x)$, which give

$$\langle 1, e'd' \rangle f_1 = f_2 \langle 1, ed \rangle$$

$$\langle 1, e'c' \rangle f_1 = f_2 \langle 1, ec \rangle$$

and permits the definition of f_0 either as $d' f_1 e$ or as $c' f_1 e$. Hence, the triple (f_2, f_1, f_0) is a morphism of internal groupoids from C to C', showing that the functor F is full. Note that this part of the proof is *Yoneda invariant*, that is, it would be sufficient to make it for ordinary groupoids.

The unitary and associativity conditions of Definition 1 characterize those involutive-2-links that are of the form $F(C)$ for some internal groupoid C. Indeed, if (θ, φ, m) is a unital and associative involutive-2-link, then, the fact that the pairs $(m, m\theta)$ and $(m, m\varphi)$ are jointly monomorphic uniquely determines the morphisms e_1 and e_2 which are required to exist by the unitary property and must verify the axioms for an internal groupoid if interpreted as $e_1(x) = (x, ed(x))$ and $e_2(x) = (ec(x), x)$. Moreover, the morphism $i \colon C_1 \to C_1$ is obtained by condition (3) either as $i = m\theta\varphi e_2$ or as $i = m\varphi\theta e_1$. The morphism $e \colon C_0 \to C_1$ is uniquely determined by condition (4) as such that $ed = m\theta e_1$ and $ec = m\varphi e_2$ where d and c are obtained as in diagram (8) with $\pi_1 = m\varphi$ and $\pi_2 = m\theta$. Conditions (1), (5) and

(6) assert the contractibility of the pairs $(m, m\theta)$ and $(m, m\varphi)$ in the sense of Beck (see [4], p. 150). Condition (2) is a central ingredient and gives $e_1 e = e_2 e$ from which the conditions $dm = d\pi_2$ and $cm = c\pi_1$ are deduced, thus permitting to define the two morphisms m_1 and m_2 from the fact that the pair $(m\theta, m\varphi)$ is bi-exact. Associativity then follows from $mm_1 = mm_2$. The remaining details of the proof are easily obtained and an example which extends the notion of a crossed module from groups to magmas is illustrated in [7].

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Data Availability Not applicable.

Declarations

Conflict of interest There are no conflicts of interest.

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