Applied Categorical Structures (2023) 31:37
https://doi.org/10.1007/s10485-023-09738-9

Admissibility of Localizations of Crossed Modules

Olivia Monjon[1] · Jérôme Scherer[1] · Florence Sterck[2]

Received: 23 January 2023 / Accepted: 11 July 2023 / Published online: 8 September 2023

Abstract
The correspondence between the concept of conditional flatness and admissibility in the sense of Galois appears in the context of localization functors in any semi-abelian category admitting a fiberwise localization. It is then natural to wonder what happens in the category of crossed modules where fiberwise localization is not always available. In this article, we establish an equivalence between conditional flatness and admissibility in the sense of Galois (for the class of regular epimorphisms) for regular-epi localization functors. We use this equivalence to prove that nullification functors are admissible for the class of regular epimorphisms, even if the kernels of their localization morphisms are not acyclic.

Keywords Crossed modules · Localization functors · Admissibility · Regular epimorphisms · Conditional flatness · Nullifications

Mathematics Subject Classification 18G45 · 55P60 · 18E50 · 55R70 · 18E13

Introduction

It is a natural question to ask whether the pullback of a nice extension inherits these nice properties. When working with localization functors or reflections one particularly nice feature for an extension is flatness. We say that an extension is L-flat, for a localization functor L, if applying L to the extension yields another extension, see Definition 2.1. The question is thus to understand when the pullback of an L-flat extension is again L-flat.

Communicated by Marino Gran.

✉ Jérôme Scherer
jerome.scherer@epfl.ch

Olivia Monjon
olivia.monjon@gmail.com

Florence Sterck
florence.sterck@uclouvain.be

[1] Mathematics, Ecole Polytechnique Fédérale de Lausanne, EPFL, Lausanne, Switzerland

[2] Institut de Recherche en Mathématique et Physique, Université catholique de Louvain, Louvain-la-Neuve, Belgium

Such questions have been studied first in a homotopical context by Berrick and Farjoun, [2]. For homotopical localization functors in the category of topological spaces (in the sense of Bousfield, [7], see also Farjoun's book [14]), preservation of L-flatness (for fiber sequences) under pullbacks was shown to be equivalent for L to be a so-called nullification functor. The situation is surprisingly more delicate in the category of groups. Farjoun and the second author proved for example that all nilpotent quotient functors have this nice property, which they called *conditional flatness*, see [15].

The standard strategy to establish conditional flatness for a localization functor consists in a few reduction steps culminating in a simpler form, which Gran identified as admissibility in the sense of Galois for the class of regular epimorphisms [19, Proposition 3.3]. This shifted the study of conditional flatness in homotopy theory to that of admissibility in semi-abelian categories, see [16]. Admissibility had been introduced by Janelidze and Kelly in [19] and has since then played a central role in the categorical study of extensions, let us mention for example Everaert, Gran, and Van der Linden's work in [13].

In this article we study admissibility for localization functors in the category of crossed modules (of groups), a category of interest to both topologists due to Whitehead's work on connected 2-types, [27], and algebraists since Brown and Spencer [8] proved the equivalence between crossed modules and internal groupoids in the category of groups (a result that they credit to Verdier). This equivalence relates two interesting notions and allows one to deal with the concept of internal groupoid in an alternative way, which is useful for computations. Moreover, crossed modules form a semi-abelian category in the sense of Janelidze, Márki and Tholen, [20]. We adopt the algebraic point of view here and continue our work started in [24]. Indeed, among the reduction steps we have mentioned above, the first one calls on fiberwise localization techniques. For group theoretical localization and homotopy localization functors, it allows one to reduce the study to extensions with local kernel (fiber). Fiberwise localization techniques are available in the category of groups thanks to work of Casacuberta and Descheemaeker, [11], but we proved in [24] that they are not at hand in general for crossed modules. Our aim in this article is thus to modify the strategy to be able to study admissibility in this setting.

We focus on localization functors such that the co-augmentation morphism $\ell^{\mathsf{T}}\colon \mathsf{T} \to \mathsf{LT}$ is a regular epimorphism for all crossed modules T. We call them *regular-epi localization* and notice that many examples of interest are provided by nullification functors, as defined in Definition 1.10. Any crossed module A determines a nullification functor P_{A} that "kills" all morphisms from A and there are other regular-epi localization functors such as abelianization. One first important observation which makes the reduction strategy viable is that, even though fiberwise localization does not exist in general, even for nullification functors, we can use this tool for *certain* extensions.

Lemma 2.5. *Let* L *be a regular-epi localization. Let*

$$1 \longrightarrow \mathsf{N} \xrightarrow{\kappa} \mathsf{T} \xrightarrow{\alpha} \mathsf{Q} \longrightarrow 1 \qquad (1)$$

be an L*-flat exact sequence of crossed modules and* $g\colon \mathsf{Q}' \to \mathsf{Q}$ *a morphism of crossed modules. Then, we can construct the fiberwise localization of the pullback of (1) along* g:

$$\begin{tikzcd}
1 \arrow[r] & \mathsf{N} \arrow[r, "\kappa'"] \arrow[d, equal] & \mathsf{T}' \arrow[r, "\pi_{\mathsf{Q}'}"] \arrow[d, "\pi_{\mathsf{T}}"] & \mathsf{Q}' \arrow[r] \arrow[d, "g"] & 1 \\
1 \arrow[r] & \mathsf{N} \arrow[r, "\kappa"'] & \mathsf{T} \arrow[r, "\alpha"'] & \mathsf{Q} \arrow[r] & 1
\end{tikzcd}$$

This allows us to relate conditional flatness with admissibility, in the same spirit as what was done in the category of groups, [15], or in the wider context of semi-abelian categories where fiberwise localization exists, [16]. A localization functor L is said to be *admissible* for the class of regular epimorphisms if it preserves any pullback of the form

$$\begin{tikzcd} \mathsf{T}' \arrow[r, "\pi_{\mathsf{Q}}"] \arrow[d, "\pi_{\mathsf{LT}}"'] & \mathsf{Q} \arrow[d, "\ell^{\mathsf{Q}}"] \\ \mathsf{LT} \arrow[r, "\alpha"'] & \mathsf{LQ} \end{tikzcd}$$

where α is a regular epimorphism between L-local objects.

Theorem 3.5. *Let* L *be a regular-epi localization functor. Then the following statements are equivalent*

(1) L *is conditionally flat;*
(2) L *is admissible for the class of regular epimorphisms.*

One difference between groups and crossed modules, which is maybe the main source of complication, is highlighted by the behavior of kernels. This was already the reason why one cannot always construct fiberwise localization and we were also surprised to find examples of nullification functors for which the kernel of the nullification morphism $\ell^{\mathsf{T}}\colon \mathsf{T} \to \mathsf{P_A T}$ is not always $\mathsf{P_A}$-acyclic, see [24, Proposition 4.6]. For groups and spaces, this property actually characterizes nullification functors.

Still we prove here that having acyclic kernels implies admissibility and in Proposition 4.3, that if the kernels of the localization morphisms are L_f-acyclic, then L_f is a nullification functor. Well behaved nullification functors are therefore admissible, but what about arbitrary nullification functors, for which fiberwise localization does not necessarily exist and for which the kernel of the nullification is not necessarily acyclic? By carefully looking at the inductive construction of $\mathsf{P_A T}$ we show our main result, namely that all nullification functors are admissible.

Theorem 5.5. *Let* A *be any crossed module. The nullification functor* $\mathsf{P_A}$ *is admissible for the class of regular epimorphisms.*

We end this introduction with a short outline. The first section consists of preliminaries that we use in the rest of the article. Then in Sect. 1 we introduce L-flat exact sequences and conditionally flat localization functors in the context of crossed modules. We show how to construct fiberwise localization of L-flat exact sequences. The third section is essential in the development of a simpler characterization of conditional flatness: It provides an equivalence with the notion of admissibility in the specific context of regular-epi localization functors. In Sect. 4 the link between L-acyclicity and admissibility is established and the last section is devoted to the proof that every nullification functor is admissible.

1 Preliminaries

1.1 The Semi-Abelian Category of Crossed Modules

In this subsection, following Norrie [26] and Brown-Higgins [4], we provide the basic definitions and notation concerning crossed modules.

Definition 1.1 [27] A *crossed module of groups* is a pair of groups T_1 and T_2, an action by group automorphisms of T_2 on T_1, denoted by $T_2 \times T_1 \to T_1 \colon (b, t) \mapsto {}^{b}t$, together with a group homomorphism $\partial^{\mathsf{T}} \colon \mathrm{T}_1 \to \mathrm{T}_2$ such that for any b in T_2 and any t, s in T_1,

$$\partial^{\mathsf{T}}({}^{b}t) = b\partial^{\mathsf{T}}(t)b^{-1}, \tag{2}$$

$${}^{\partial^{\mathsf{T}}(t)}s = tst^{-1}. \tag{3}$$

Hence we often write a crossed module as a triple $(T_1, T_2, \partial^{\mathsf{T}})$, or simply T for short, and we refer sometimes to ∂^{T} as the *connecting morphism*.

Definition 1.2 Let $\mathsf{N} := (N_1, N_2, \partial^{\mathsf{N}})$ and $\mathsf{M} := (M_1, M_2, \partial^{\mathsf{M}})$ be two crossed modules. A *morphism of crossed modules* $\alpha \colon \mathsf{N} \to \mathsf{M}$ is a pair of group homomorphisms $\alpha_1 \colon N_1 \to M_1$ and $\alpha_2 \colon N_2 \to M_2$ such that the two following diagrams commute

$$\begin{tikzcd} N_1 \arrow[r, "\alpha_1"] \arrow[d, "\partial^{\mathsf{N}}"'] & M_1 \arrow[d, "\partial^{\mathsf{M}}"] \\ N_2 \arrow[r, "\alpha_2"'] & M_2 \end{tikzcd} \qquad \begin{tikzcd} N_2 \times N_1 \arrow[r] \arrow[d, "\alpha_2 \times \alpha_1"'] & N_1 \arrow[d, "\alpha_1"] \\ M_2 \times M_1 \arrow[r] & M_1. \end{tikzcd}$$

where the horizontal arrows in the diagram on the right are the respective group actions of the two crossed modules.

We write XMod for the category of crossed modules of groups.

Remark 1.3 The category of groups embeds in this category via two functors which are respectively left and right adjoint to the truncation functor $Tr \colon \mathsf{XMod} \to \mathsf{Grp}$ that sends a crossed module $\mathsf{T} := (T_1, T_2, \partial^{\mathsf{T}})$ to T_2. The functor $\mathsf{X} \colon \mathsf{Grp} \to \mathsf{XMod}$ which sends a group G to the crossed module $\mathsf{X}G = (1, G, 1)$ reduced to the group G at level 2 is the left adjoint functor and the functor $\mathsf{R} \colon \mathsf{Grp} \to \mathsf{XMod} \colon G \mapsto (G, G, Id_G)$ is the right adjoint functor. This will help us to import group theoretical results into XMod.

There is an obvious notion of subcrossed module, see [26]. One simply requires the subobject to be made levelwise of subgroups, the connecting homomorphism and the action are induced by the given connecting homomorphism and action. The notion of normality is less obvious.

Definition 1.4 A subcrossed module $\mathsf{N} := (N_1, N_2, \partial^{\mathsf{N}})$ of $\mathsf{T} := (T_1, T_2, \partial^{\mathsf{T}})$ is *normal* if the following three conditions hold

(1) N_2 is a normal subgroup of T_2;
(2) for any $t_2 \in T_2$ and $n_1 \in N_1$, we have ${}^{t_2}n_1 \in N_1$;
(3) $[N_2, T_1] := \langle {}^{n_2}t_1 t_1^{-1} \mid t_1 \in T_1, n_2 \in N_2 \rangle \subseteq N_1$.

In contrast to limits, which are built component-wise, colimits are generally more delicate to construct. In particular, the construction of cokernels is not straightforward, but when N is a normal subcrossed module of T the cokernel is simply the levelwise quotient by the normal subgroups $N_1 \triangleleft T_1$ and $N_2 \triangleleft T_2$.

The category of crossed modules shares many nice properties with the category of groups. The traditional homological lemmas, [1], the Split Short Five Lemma, [5], and the Noether Isomorphism Theorems, [1], hold. One can recognize pullbacks by looking at kernels or cokernels, [1, Lemmas 4.2.4 and 4.2.5], and in fact Xmod is a semi-abelian category, as introduced by Janelidze, Márki, and Tholen in [20]. This is shown in [20]. There is one result we will use several times in this article, namely [1, Lemma 4.2.5], which we recall now.

Proposition 1.5 *Let $\mathcal{C}$ be a semi-abelian (or homological) category. Consider the following diagram of exact rows:*

$$\begin{tikzcd}
1 \arrow[r] & N' \arrow[r, "\kappa'"] \arrow[d, "u"'] & T' \arrow[r, "\alpha'"] \arrow[d, "v"'] \arrow[dr, phantom, "(2)"] & Q' \arrow[r] \arrow[d, "w"] & 1 \\
1 \arrow[r] & N \arrow[r, "\kappa"] & T \arrow[r, "\alpha"] & Q &
\end{tikzcd}$$

Then the following statements hold.

(1) *If u is an isomorphism then (2) is a pullback.*
(2) *If u and w are regular epimorphisms then v is also a regular epimorphism.*

1.2 Localization Functors

In this subsection we recall the definition of localization functors in the category of crossed modules. We also recall some important properties of such functors as well as some examples.

Definition 1.6 A *localization* functor in the category of crossed modules is a coaugmented idempotent functor $\mathsf{L}\colon \mathsf{XMod} \to \mathsf{XMod}$. The coaugmentation $\ell\colon \mathsf{Id} \to \mathsf{L}$ is a natural transformation such that ℓ^{LX} and $\mathsf{L}\ell^{\mathsf{X}}$ are isomorphisms.

In particular we have $\ell^{\mathsf{LX}} = \mathsf{L}\ell^{\mathsf{X}}$, see [9, Proposition 1.1].

Definition 1.7 Let L be a localization functor. A crossed module T is L-*local* if $\ell^{\mathsf{T}}\colon \mathsf{T} \to \mathsf{LT}$ is an isomorphism. A morphism $f\colon \mathsf{N} \to \mathsf{M}$ is an L-*equivalence* if $\mathsf{L}(f)$ is an isomorphism.

We recall a few basic and useful closure properties of L-equivalences.

Lemma 1.8 (1) *The pushout of an L-equivalence is an L-equivalence.*
(2) *The composition of L-equivalences is an L-equivalence.*
(3) *A κ-filtered colimit of a diagram T_β of L-equivalences $\mathsf{T}_\beta \to T_{\beta+1}$ for all successor ordinals $\beta+1<\kappa$ yields an L-equivalence $\mathsf{T}_0 \to \mathsf{T}_\kappa = \mathsf{colim}_{\beta<\kappa}\mathsf{T}_\beta$.*
(4) *Let F be an I-indexed diagram of L-equivalences in the category of morphisms of crossed modules. Then the colimit $\mathsf{colim}_{\mathsf{I}}\mathsf{F}$ is an L-equivalence.*

Sometimes a localization functor L is associated to a full reflexive subcategory $\mathcal{L}$ of XMod. The pair of adjoint functors $\mathsf{U}\colon \mathcal{L} \leftrightarrows \mathsf{XMod}\colon \mathsf{F}$ provides a localization functor $\mathsf{L} = \mathsf{UF}$, as Cassidy, Hébert, and Kelly do in [12]. Some other times there is a morphism f one wishes to invert so as to construct a localization functor often written L_f.

Definition 1.9 Let f be a morphism of crossed modules. A crossed module T is L_f-*local* if $Hom(f, \mathsf{T})$ is an isomorphism. A morphism g in XMod is an L_f-*equivalence* if $Hom(g, \mathsf{T})$ is an isomorphism for any L_f-local crossed module T.

Such localization functors exist in XMod, see for example Bousfield's foundational work [6]. Local objects and local equivalences coincide then with the notions introduced in Definition 1.7. Lemma 1.8 is the analogue of Hirschhorn's [17, Proposition 1.2.20 and Proposition 1.2.21].

If the codomain of the morphism f is the trivial crossed module, the functor L_f is of particular interest.

Definition 1.10 Let A be a crossed module and f be the morphism $\mathsf{A} \to 1$. The localization functor L_f is then written P_A and is called a *nullification* functor. An L_f-local object is called A-*null*, or A-*local* and a crossed module T is A-*acyclic* if $\mathsf{P}_\mathsf{A}\mathsf{T} = 1$. The localization morphism $\ell^T : \mathsf{T} \to \mathsf{P}_\mathsf{A}\mathsf{T}$ is written p^T.

Proposition 1.11 *Let* A *and* T *be crossed modules. Then there exists an ordinal* λ *depending on* A *such that* $\mathsf{P}_\mathsf{A}\mathsf{T}$ *is constructed as a transfinite filtered colimit of a diagram of the form* $\mathsf{T} = \mathsf{T}_0 \to \mathsf{T}_1 \to \cdots \to \mathsf{T}_\beta \to \dots$ *for* $\beta < \lambda$ *where all morphisms are* P_A*-equivalences and regular epimorphims.*

This inductive construction has been carefully described in [24, Proposition 2.8]. The reason why each step is a P_A-equivalence and a regular epimorphism is that $\mathsf{T}_{\beta+1}$ is constructed from T_β by taking the cokernel of all morphisms $\mathsf{A} \to \mathsf{T}_\beta$. We recall the details and use them in Section 5. There is a larger class of localization functors we investigate in this sequel to [24]. They share with P_A the property that the localization morphism is a regular epimorphism.

Definition 1.12 A localization functor L is a *regular-epi localization* if for any crossed module T the coaugmentation $\ell^\mathsf{T} : \mathsf{T} \to \mathsf{LT}$ is a regular epimorphism.

Remark 1.13 In the category of crossed modules, a morphism $\alpha = (\alpha_1, \alpha_2)$ is a regular epimorphism (a coequalizer of a pair of parallel arrows) if and only if both α_1 and α_2 are surjective group homomorphisms [22, Proposition 2.2]. A surjective homomorphism of crossed modules is an epimorphism but there exist epimorphisms that are not surjective. In a pointed protomodular category such as XMod, regular epimorphisms and normal epimorphisms (the cokernels of arbitrary morphisms) coincide.

We present now some interesting examples of localization functors that will illustrate our results in the rest of the article, see also the end of [24, Section 2].

Example 1.14 The nullification functor $\mathsf{P}_{\mathsf{X}\mathbb{Z}}$ with respect to the crossed module $\mathsf{X}\mathbb{Z}$ is given by:

$$\mathsf{P}_{\mathsf{X}\mathbb{Z}} \left(\begin{array}{c} N_1 \\ \downarrow{\scriptstyle \partial} \\ N_2 \end{array} \right) = \begin{array}{c} N_1/[N_2, N_1] \\ \downarrow \\ 1 \end{array}$$

Example 1.15 The abelianization functor $\mathsf{Ab} : \mathsf{XMod} \to \mathsf{XMod}$ is already described in [25]. It is defined by:

$$\mathsf{Ab} \left(\begin{array}{c} N_1 \\ \downarrow{\scriptstyle \partial} \\ N_2 \end{array} \right) = \begin{array}{c} N_1/[N_2, N_1] \\ \downarrow{\scriptstyle \tilde{\partial}} \\ N_2/[N_2, N_2] \end{array}$$

Example 1.16 Our third and last example of localization functor of crossed modules is $\mathsf{I} : \mathsf{XMod} \to \mathsf{XMod}$, see [24, Example 2.15]:

$$\mathsf{I} \left(\begin{array}{c} N_1 \\ \downarrow{\scriptstyle \partial^\mathsf{N}} \\ N_2 \end{array} \right) = \begin{array}{c} N_2 \\ \downarrow{\scriptstyle Id_{N_2}} \\ N_2 \end{array}$$

This functor is induced by the adjunction between the truncation functor $Tr\colon \mathsf{XMod} \to \mathsf{Grp}$, defined by $Tr(T_1, T_2, \partial^{\mathsf{T}}) = T_2$, see Remark 1.3, and its right adjoint $\mathsf{R}\colon \mathsf{Grp} \to \mathsf{XMod}$ that sends a group T to (T, T, Id_T).

Remark 1.17 The functor considered in Example 1.14 is a regular-epi localization, since all nullification functors are so. However regular-epi localizations are not nullification functors in general as illustrated by the functor Ab in Example 1.15. Indeed, if Ab were a nullification P_{A}, then $\mathsf{A} = (A_1, A_2, \partial^{\mathsf{A}})$ would be a perfect crossed module, i.e. one such that $\mathsf{Ab}(\mathsf{A}) = (1, 1, \mathsf{Id})$. In particular, the group A_2 would be a perfect group. But then $\mathsf{P}_{\mathsf{A}}(\mathsf{X}S_3) = \mathsf{X}S_3$ since there are no non-trivial homomorphisms from a perfect group to the symmetric group S_3. But we know that $\mathsf{Ab}(\mathsf{X}S_3) = \mathsf{X}C_2$, where C_2 is the cyclic group of order two, so abelianization is not a nullification.

We finally note that a localization functor L_f is a regular-epi localization functor if f itself is a regular epimorphism, an analogous observation appears in [10] for groups.

To conclude these preliminaries, let us recall the notion of fiberwise localization. We introduced this for crossed modules in [24, Definition 3.1], but this is not new, for spaces a good reference is [14, Section I.F].

Definition 1.18 Let $\mathsf{L}\colon \mathsf{XMod} \to \mathsf{XMod}$ be a localization functor. An exact sequence

$$1 \longrightarrow \mathsf{N} \xrightarrow{\kappa} \mathsf{T} \xrightarrow{\alpha} \mathsf{Q} \longrightarrow 1$$

admits a fiberwise localization if there exists a commutative diagram of horizontal exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \mathsf{N} \arrow[r, "\kappa"] \arrow[d, "\ell^{\mathsf{N}}"'] & \mathsf{T} \arrow[r, "\alpha"] \arrow[d, "f"'] & \mathsf{Q} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathsf{LN} \arrow[r, "j"] & \mathsf{E} \arrow[r, "p"] & \mathsf{Q} \arrow[r] & 1
\end{tikzcd} \tag{4}$$

where f is an L-equivalence.

The following theorem is a fusion of two results from [24] namely Theorem 3.4 and Corollary 3.7. From now on, every localization functor that we consider is a regular-epi localization.

Theorem 1.19 *Let* $\mathsf{L}\colon \mathsf{XMod} \to \mathsf{XMod}$ *be a regular-epi localization functor. An exact sequence of crossed modules*

$$1 \longrightarrow \mathsf{N} \xrightarrow{\kappa} \mathsf{T} \xrightarrow{\alpha} \mathsf{Q} \longrightarrow 1 \tag{5}$$

admits a fiberwise localization if and only if we have the following inclusion

$$[\kappa_2(ker(\ell_2^{\mathsf{N}})), T_1] \subseteq \kappa_1(ker(\ell_1^{\mathsf{N}})) \tag{6}$$

In this case the morphism $f\colon \mathsf{T} \to \mathsf{E}$ *in (4) is given by the quotient map* $\mathsf{T} \to \mathsf{T}/\kappa(\mathsf{ker}(\ell^{\mathsf{N}}))$.

Remark 1.20 As stated, this theorem seems to give us only an existence result, but we also proved in [24, Proposition 3.3] that if the fiberwise localization of the exact sequence (5) exists, then it is unique and must be given by the quotient of T by the image under κ of the kernel of the localization morphism $\ell^N\colon \mathsf{N} \to \mathsf{LN}$. The condition that appears in the theorem simply guarantees the possibility to construct this quotient.

2 Fiberwise Localization and Flatness

In this section, we investigate the fiberwise localization of L-flat exact sequences and their pullbacks in the context of regular-epi localization functors of crossed modules $\mathsf{L}\colon \mathsf{XMod} \to \mathsf{XMod}$ (even if this notion is not defined only for regular-epi localizations as we will see in Proposition 5.6). This section will be essential to study the link between conditional flatness and admissibility in Sect. 3. First, let us recall the definitions of L-flat and conditional flatness.

Definition 2.1 Let L be a localization functor, a short exact sequence

$$1 \longrightarrow \mathsf{N} \xrightarrow{\kappa} \mathsf{T} \xrightarrow{\alpha} \mathsf{Q} \longrightarrow 1$$

is called L-*flat* if the sequence $\mathsf{LN} \xrightarrow{\mathsf{L}(\kappa)} \mathsf{LT} \xrightarrow{\mathsf{L}(\alpha)} \mathsf{LQ}$ is a short exact sequence.

Remark 2.2 We recall that limits are computed componentwise in the category of crossed modules. In the case of pullbacks in XMod they are built as follows [21]. Let $\alpha\colon \mathsf{T} \to \mathsf{Q}$ and $g\colon \mathsf{Q}' \to \mathsf{Q}$ be two morphisms of crossed modules. Then the pullback of α along g is given by the following square

$$\begin{tikzcd} \mathsf{T}' \arrow[r, "\pi_{\mathsf{Q}'}"] \arrow[d, "\pi_{\mathsf{T}}"'] & \mathsf{Q}' \arrow[d, "g"] \\ \mathsf{T} \arrow[r, "\alpha"'] & \mathsf{Q} \end{tikzcd}$$

The object part T' of the pullback is built component-wise as in the case of groups

$$(T_1 \times_{Q_1} Q_1', T_2 \times_{Q_2} Q_2', \partial'),$$

where ∂' and the action are induced by the universal property of the pullbacks in Grp. The projections are the natural ones, given also component-wise.

Following the terminology introduced in [15] for groups and spaces, we define the notion of conditional flatness for localization functors in crossed modules.

Definition 2.3 Let L be a localization functor. We say that this functor is *conditionally flat* if the pullback of any L-flat exact sequence is L-flat.

In Sect. 3 we provide a characterization of conditional flatness. To achieve this goal we will use a similar strategy to the one applied to groups and topological spaces in [15]. The authors exploit heavily the existence of fiberwise localization in the categories of groups and spaces. However, in our article [24], we observed that fiberwise localization does not always exist for a given localization functor and a given exact sequence in XMod. Fortunately, when we work with L-flat exact sequences we can show that it is always possible to construct a fiberwise localization.

Lemma 2.4 *Let* L *be a regular-epi localization. Then any* L-*flat exact sequence of crossed modules admits a fiberwise localization.*

Proof Let $1 \longrightarrow \mathsf{N} \xrightarrow{\kappa} \mathsf{T} \xrightarrow{\alpha} \mathsf{Q} \longrightarrow 1$ be an L-flat exact sequence of crossed modules. The L-flatness of the sequence implies in particular that $\mathsf{L}(\kappa)$ is a monomorphism. Consider the following diagram of exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & \mathsf{ker}(\ell^{\mathsf{N}}) \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "(1)"] & \mathsf{N} \arrow[r, "\ell^{\mathsf{N}}"] \arrow[d, tail, "\kappa"] & \mathsf{LN} \arrow[d, tail, "\mathsf{L}(\kappa)"] \\
1 \arrow[r] & \mathsf{ker}(\ell^{\mathsf{T}}) \arrow[r, tail] & \mathsf{T} \arrow[r, "\ell^{\mathsf{T}}"'] & \mathsf{LT}
\end{tikzcd}$$

We conclude from [1, Lemma 4.2.4.(1)] that (1) is a pullback since $\mathsf{L}(\kappa)$ is a monomorphism. Then we have that $\kappa(\mathsf{ker}(\ell^{\mathsf{N}}))$ is a normal subcrossed module of T as it can be seen as the intersection of the normal subcrossed modules N and $\mathsf{ker}(\ell^{\mathsf{T}})$ of T. Therefore, we can apply Theorem 1.19 □

To understand conditional flatness we must study the pullback of an L-flat exact sequence. It will thus be very handy in Sect. 3 to know that any such pullback admits a fiberwise localization.

Lemma 2.5 *Let* L *be a regular-epi localization. Let*

$$1 \longrightarrow \mathsf{N} \xrightarrow{\kappa} \mathsf{T} \xrightarrow{\alpha} \mathsf{Q} \longrightarrow 1 \tag{7}$$

be an L*-flat exact sequence of crossed modules and* $g\colon \mathsf{Q}' \to \mathsf{Q}$ *a morphism of crossed modules. Then, we can construct the fiberwise localization of the pullback of (7) along* g

$$\begin{tikzcd}
1 \arrow[r] & \mathsf{N} \arrow[r, "\kappa'"] \arrow[d, equal] & \mathsf{T}' \arrow[r, "\pi_{\mathsf{Q}'}"] \arrow[d, "\pi_{\mathsf{T}}"] & \mathsf{Q}' \arrow[r] \arrow[d, "g"] & 1 \\
1 \arrow[r] & \mathsf{N} \arrow[r, "\kappa"'] & \mathsf{T} \arrow[r, "\alpha"'] & \mathsf{Q} \arrow[r] & 1
\end{tikzcd}$$

Remark 2.6 In the rest of the article, and in particular in the following proof, we identify N with the normal subcrossed module $\kappa(\mathsf{N})$ of T and with $\kappa'(\mathsf{N})$, normal subcrossed module of T'. We will therefore omit the use of κ and κ'. For example an element of the group N_1 that we want to consider in T'_1 will be denoted $(n_1, 1)$ instead of $\kappa'_1(n_1) = (\kappa_1(n_1), 1)$.

Proof of Lemma 2.5 We need to verify that $\mathsf{ker}(\ell^{\mathsf{N}})$ is a normal subcrossed module of T'. Since N is a subcrossed module of T', we just need to verify (6) of Theorem 1.19. Let (t_1, q_1) be an element in T'_1 and $(x_2, 1)$ be an element of $ker(\ell^{\mathsf{N}}_2)$, then we have the following equality

$$^{(x_2,1)}(t_1, q_1)(t_1, q_1)^{-1} = ({}^{x_2}t_1t_1^{-1}, q_1q_1^{-1}) = ({}^{x_2}t_1t_1^{-1}, 1).$$

Indeed, by Lemma 2.4 we know that the original sequence (7) admits a fiberwise localization which then implies by Theorem 1.19 that $[ker(\ell^{\mathsf{N}}_2), T_1] \subset ker(\ell^{\mathsf{N}}_1)$ i.e for any $x_2 \in ker(\ell^{\mathsf{N}}_2)$ and $t_1 \in T_1$ we have ${}^{x_2}t_1t_1^{-1} \in ker(\ell^{\mathsf{N}}_1)$. With the notation introduced in Remark 2.6, this is equivalent to say that the element $({}^{x_2}t_1t_1^{-1}, 1)$ belongs to $ker(\ell^{\mathsf{N}}_1)$. □

This lemma is not trivial since the fiberwise localization of an exact sequence of crossed modules does not always exist as we have proved in [24, Theorem 4.5]. If we want the strategy for groups and spaces to be also viable in the study of conditional flatness for crossed modules, we need a final ingredient, namely a commutation rule for the fiberwise localization and the pullback operations. We speak here about “the” fiberwise localization since it is unique as mentioned in Remark 1.20.

Proposition 2.7 *Let us consider an* L*-flat exact sequence where* L *is a regular-epi localization functor. Then, the pullback of its fiberwise localization is the fiberwise localization of its pullback.*

Proof Let

$$\begin{tikzcd}
1 \arrow[r] & \mathsf{N} \arrow[r, "\kappa'"] \arrow[d, equals] & \mathsf{T}' \arrow[r, "\pi_{\mathsf{Q}'}"] \arrow[d, "\pi_{\mathsf{T}}"] & \mathsf{Q}' \arrow[r] \arrow[d, "g"] & 1 \\
1 \arrow[r] & \mathsf{N} \arrow[r, "\kappa"'] & \mathsf{T} \arrow[r, "\alpha"'] & \mathsf{Q} \arrow[r] & 1
\end{tikzcd}$$

be the pullback of an L-flat exact sequence. Then we construct the fiberwise localizations of the two sequences by quotienting out the kernel of the localization morphism ℓ^{N} as in the following solid arrow diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mathsf{N} \arrow[rr, "\kappa'"] \arrow[dr, "l^{\mathsf{N}}"'] \arrow[dd, equals] & & \mathsf{T}' \arrow[rr, "\pi_{\mathsf{Q}'}"] \arrow[dr, "f'"] \arrow[dd, "\pi_{\mathsf{T}}" near end] & & \mathsf{Q}' \arrow[r] \arrow[dr, equals] \arrow[dd, "g" near end] & 1 & \\
& 1 \arrow[r] & \mathsf{LN} \arrow[rr, "j'"] \arrow[dd, equals] & & \mathsf{T}'/\mathsf{ker}(\ell^{\mathsf{N}}) \arrow[rr, "p'"] \arrow[dd, dashed, "\delta" near start] & & \mathsf{Q}' \arrow[r] \arrow[dd, "g"] & 1 \\
1 \arrow[r] & \mathsf{N} \arrow[rr, "\kappa" near end] \arrow[dr, "l^{\mathsf{N}}"'] & & \mathsf{T} \arrow[rr, "\alpha" near end] \arrow[dr, "f"] & & \mathsf{Q} \arrow[r] \arrow[dr, equals] & 1 & \\
& 1 \arrow[r] & \mathsf{LN} \arrow[rr, "j"'] & & \mathsf{T}/\mathsf{ker}(\ell^{\mathsf{N}}) \arrow[rr, "p"'] & & \mathsf{Q} \arrow[r] & 1
\end{tikzcd}$$

We complete the diagram by defining a dashed morphism $\delta\colon \mathsf{T}'/\mathsf{ker}(\ell^{\mathsf{N}}) \to \mathsf{T}/\mathsf{ker}(\ell^{\mathsf{N}})$ via the universal property of the cokernel since $f \circ \pi_{\mathsf{T}} \circ \kappa'|_{\mathsf{ker}(\ell^{\mathsf{N}})} = 1$, where $\kappa'|_{\mathsf{ker}(\ell^{\mathsf{N}})}$ is the inclusion of the kernel $\mathsf{ker}(\ell^{\mathsf{N}}) \to \mathsf{T}'$.

$$\begin{tikzcd}
1 \arrow[r] & \mathsf{ker}(\ell^{\mathsf{N}}) \arrow[r, "\kappa'|_{\mathsf{ker}(\ell^{\mathsf{N}})}"] \arrow[d, equals] & \mathsf{T}' \arrow[r, "f'"] \arrow[d, "\pi_{\mathsf{T}}"] & T'/\mathsf{ker}(\ell^{\mathsf{N}}) \arrow[r] \arrow[d, dashed, "\delta"] & 1 \\
1 \arrow[r] & \mathsf{ker}(\ell^{\mathsf{N}}) \arrow[r, "\kappa|_{\mathsf{ker}(\ell^{\mathsf{N}})}"'] & \mathsf{T} \arrow[r, "f"'] & T/\mathsf{ker}(\ell^{\mathsf{N}}) \arrow[r] & 1
\end{tikzcd}$$

We can check that δ makes the two front faces commute. Indeed, the right and left faces commute by using the fact that ℓ^{N} and f' are epimorphisms.

The commutativity of the above diagram and Proposition 1.5 imply that

$$1 \longrightarrow \mathsf{LN} \xrightarrow{j'} \mathsf{T}'/\mathsf{ker}(\ell^{\mathsf{N}}) \xrightarrow{p'} \mathsf{Q}' \longrightarrow 1$$

is the pullback of $1 \longrightarrow \mathsf{LN} \xrightarrow{j} \mathsf{T}/\mathsf{ker}(\ell^{\mathsf{N}}) \xrightarrow{p} \mathsf{Q} \longrightarrow 1$ along g. □

Remark 2.8 In [15], the construction of the fiberwise localization in the category of groups is functorial. Therefore, from the morphism $\mathsf{T}' \to \mathsf{T}$ between the pullback sequence and the sequence itself we have directly a morphism between the fiberwise localization of the pullback sequence and the fiberwise localization of the original sequence. In other words the map δ comes for free in contrast to the category of crossed modules where we have to build the map δ explicitly.

3 Conditional Flatness and Admissibility

In this section, we develop a simpler characterization of conditional flatness, thanks to the results of the previous section. We introduce the notion of admissibility for the class of regular epimorphisms and show that it is equivalent to conditional flatness. With this equivalence, we can easily establish conditional flatness for a given localization functor. We observe that some properties of localization functors, such as right-exactness, imply directly admissibility for the class of regular epimorphism.

A first step allows us to restrict the definition of conditional flatness (Definition 2.3) to fiberwise localizations of L-flat exact sequences (Lemma 3.1). More precisely, we show that the pullback of an L-flat exact sequence is L-flat if and only if the pullback of its fiberwise localization is so.

Lemma 3.1 *Let L be a regular-epi localization functor. Then L is conditionally flat if and only if for any L-flat exact sequence $1 \longrightarrow \mathsf{N} \xrightarrow{\kappa} \mathsf{T} \xrightarrow{\alpha} \mathsf{Q} \longrightarrow 1$ with N an L-local crossed module, the pullback sequence along any morphism $\mathsf{Q}' \to \mathsf{Q}$ is L-flat.*

Proof This is clear since f' and ℓ^{N} are L-equivalences in this diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mathsf{N} \arrow[r, "\kappa'"] \arrow[d, "\ell^{\mathsf{N}}"'] & \mathsf{T}' \arrow[r, "\pi_{\mathsf{Q}'}"] \arrow[d, "f'"] & \mathsf{Q}' \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathsf{LN} \arrow[r, "j'"'] & \mathsf{T}'/\ker(\ell^{\mathsf{N}}) \arrow[r, "p'"'] & \mathsf{Q}' \arrow[r] & 1
\end{tikzcd}$$

The top row is thus L-flat if and only if so is the bottom row. We conclude then by Proposition 2.7. □

The previous lemma allows us to follow the approach introduced in [15]. For the sake of completeness, we give an explicit proof of the following results even if the arguments are similar to the group theoretical ones.

Proposition 3.2 *Let L be a regular-epi localization functor. Then L is conditionally flat if and only if the pullback of any exact sequence of L-local objects is L-flat.*

Proof By Lemma 3.1 we need only to consider exact sequences with an L-local kernel LN. Consider thus an L-flat exact sequence $1 \longrightarrow \mathsf{LN} \xrightarrow{j} \mathsf{T} \xrightarrow{p} \mathsf{Q} \longrightarrow 1$. We build the following diagram where $g : \mathsf{Q}' \to \mathsf{Q}$ is any morphism of crossed modules and (1) is a pullback.

$$\begin{tikzcd}
1 \arrow[r] & \mathsf{LN} \arrow[r, "j'"] \arrow[d, equal] & \mathsf{T}' \arrow[r, "\pi_{\mathsf{Q}'}"] \arrow[d, "\pi_{\mathsf{T}}"] \arrow[dr, phantom, "(1)"] & \mathsf{Q}' \arrow[r] \arrow[d, "g"] & 1 \\
1 \arrow[r] & \mathsf{LN} \arrow[r, "j"'] \arrow[d, equal] & \mathsf{T} \arrow[r, "p"'] \arrow[d, "\ell^{\mathsf{T}}"] \arrow[dr, phantom, "(2)"] & \mathsf{Q} \arrow[r] \arrow[d, "\ell^{\mathsf{Q}}"] & 1 \\
1 \arrow[r] & \mathsf{LN} \arrow[r, "L(j)"'] & \mathsf{LT} \arrow[r, "L(p)"'] & \mathsf{LQ} \arrow[r] & 1
\end{tikzcd}$$

We observe that since each row is exact, (2) is a pullback by Proposition 1.5, and then (1)+(2) is also a pullback. Hence, the top row is the pullback of the bottom exact sequence of L-local objects along the map $\ell^{\mathsf{Q}} \circ g$, which shows the claim. □

Definition 3.3 A localization functor L is said to be *admissible* for the class of regular epimorphisms if it preserves any pullback of the form

$$\begin{tikzcd}
\mathsf{T}' \arrow[r, "\pi_{\mathsf{Q}}"] \arrow[d, "\pi_{\mathsf{LT}}"'] & \mathsf{Q} \arrow[d, "\ell^{\mathsf{Q}}"] \\
\mathsf{LT} \arrow[r, "\alpha"'] & \mathsf{LQ}
\end{tikzcd}$$

where α is a regular epimorphism.

We start with a very general observation which relates admissibility with conditional flatness.

Lemma 3.4 *Let* L *be a conditionally flat localization functor. Then* L *is admissible for the class of regular epimorphisms.*

Proof Let us consider a pullback square as in Definition 3.3. Since L-local objects are closed under limits the kernel N of α is also L-local. By conditional flatness of L we infer from the flatness of the bottom exact sequence that of the top exact sequence

$$1 \longrightarrow \mathsf{N} \longrightarrow \mathsf{T}' \xrightarrow{\pi_{\mathsf{Q}}} \mathsf{Q} \longrightarrow 1$$

The universal property of the localization $\ell^{\mathsf{T}'} : \mathsf{T}' \to \mathsf{LT}'$ provides a factorization of π_{LT} through a morphism $\mathsf{LT}' \to \mathsf{LT}$. We conclude by the Short Five Lemma that this morphism is an isomorphism. □

For regular-epi localization functors the previous lemma becomes a characterization of conditional flatness in the category of crossed modules.

Theorem 3.5 *Let* L *be a regular-epi localization functor. Then the following statements are equivalent*

(1) L *is conditionally flat;*
(2) L *is admissible for the class of regular epimorphisms.*

Proof The implication (1) $\Rightarrow$ (2) is provided by Lemma 3.4, so let us prove (2) $\Rightarrow$ (1). Consider any exact sequence of L-local objects

$$\begin{tikzcd} 1 \arrow[r] & \mathsf{LN} \arrow[r] & \mathsf{LT} \arrow[r, "\alpha"] & \mathsf{LQ} \arrow[r] & 1 \end{tikzcd}$$

and any morphism $g\colon \mathsf{A} \to \mathsf{LQ}$. By Proposition 3.2, conditional flatness is established if we prove that the pullback of the exact sequence along g is L-flat. Let us first observe that this morphism g factors through LA via the universal property of the localization:

$$\begin{tikzcd} \mathsf{A} \arrow[r, "\ell^{\mathsf{A}}"] \arrow[dr, "g"'] & \mathsf{LA} \arrow[d, dashed, "\tilde{g}"] \\ & \mathsf{LQ} \end{tikzcd}$$

Hence, we can first construct the pullback of $1 \longrightarrow \mathsf{LN} \longrightarrow \mathsf{LT} \xrightarrow{\alpha} \mathsf{LQ} \longrightarrow 1$ along $\tilde{g}$ and then pull back the resulting sequence along ℓ^{A}:

$$\begin{tikzcd} 1 \arrow[r] & \mathsf{LN} \arrow[r] \arrow[d, equal] & \mathsf{T}'' \arrow[r, "\pi_{\mathsf{A}}"] \arrow[d] & \mathsf{A} \arrow[r] \arrow[d, "\ell^{\mathsf{A}}"] \arrow[dd, bend left=40, "g"] & 1 \\ 1 \arrow[r] & \mathsf{LN} \arrow[r] \arrow[d, equal] & \mathsf{T}' \arrow[r, "\pi_{\mathsf{LA}}"'] \arrow[d, "\pi_{\mathsf{LT}}"] & \mathsf{LA} \arrow[r] \arrow[d, "\tilde{g}"] & 1 \\ 1 \arrow[r] & \mathsf{LN} \arrow[r] & \mathsf{LT} \arrow[r, "\alpha"'] & \mathsf{LQ} \arrow[r] & 1 \end{tikzcd}$$

Since the category of L-local objects is closed under pullbacks, T' is L-local and we can apply condition (2) to conclude that the upper row is L-flat. This observation implies that the pullback of $1 \longrightarrow \mathsf{LN} \longrightarrow \mathsf{LT} \xrightarrow{\alpha} \mathsf{LQ} \longrightarrow 1$ along g is an L-flat sequence as desired. □

Remark 3.6 Admissibility for the class of regular epimorphisms in the context of semi-abelian categories is studied in [16]. Similar results are proven for localization functors that admit a functorial fiberwise localization. Note that their result does not imply Theorem 3.5 since localization functors of crossed modules do not admit functorial fiberwise localizations in general. However, the implication "(1) implies (2)", in Theorem 3.5, holds even for not necessarily regular-epi localization functors as shown in Lemma 3.4.

Proposition 3.7 *If* $\mathsf{L}\colon \mathsf{XMod} \to \mathsf{XMod}$ *is a localization functor that is right exact in* XMod*, then* L *is admissible for the class of regular epimorphisms.*

Proof Let us consider the following pullback of an L-flat exact sequence of crossed modules along a morphism $g\colon \mathsf{Q}' \to \mathsf{Q}$.

$$\begin{tikzcd}
1 \arrow[r] & \mathsf{N} \arrow[r, "\kappa'"] \arrow[d, equal] & \mathsf{T}' \arrow[r, "\pi_{\mathsf{Q}'}"] \arrow[d, "\pi_{\mathsf{T}}"'] \arrow[dr, phantom, "(1)"] & \mathsf{Q}' \arrow[r] \arrow[d, "g"] & 1 \\
1 \arrow[r] & \mathsf{N} \arrow[r, "\kappa"] & \mathsf{T} \arrow[r, "f"'] & \mathsf{Q} \arrow[r] & 1
\end{tikzcd}$$

We show next that the top exact sequence is L-flat, proving that L is conditionally flat, which implies admissibility by Lemma 3.4. By applying L to this diagram, we obtain (since L is right exact) the following diagram

$$\begin{tikzcd}
 & \mathsf{LN} \arrow[r, "\mathsf{L}(\kappa')"] \arrow[d, equal] & \mathsf{LT}' \arrow[r, "\mathsf{L}(\pi_{\mathsf{Q}'})"] \arrow[d, "\mathsf{L}(\pi_{\mathsf{T}})"'] & \mathsf{LQ}' \arrow[r] \arrow[d, "\mathsf{L}(g)"] & 1 \\
1 \arrow[r] & \mathsf{LN} \arrow[r, "\mathsf{L}(\kappa)"'] & \mathsf{LT} \arrow[r, "\mathsf{L}(f)"'] & \mathsf{LQ} \arrow[r] & 1
\end{tikzcd}$$

Since $\mathsf{L}(\kappa) = \mathsf{L}(\pi_{\mathsf{T}}) \circ \mathsf{L}(\kappa')$ is a (normal) monomorphism, we conclude that $\mathsf{L}(\kappa')$ is a monomorphism. Normality follows then by right-exactness. □

Note that this proof holds in any semi-abelian category. In the next corollary we highlight the admissibility of the abelianization functor in crossed modules. This is not new and has been used by Bourn and Gran in [3, Section 4] to study the relationship with central extensions of crossed modules. They developed the theory in a more general context, that of internal groupoids in a semi-abelian category. In any semi-abelian category, internal groupoids are equivalent to internal crossed modules [18, (3.15)], then their result applies in XMod seen as internal groupoids in Grp.

Corollary 3.8 *The abelianization functor* $\mathsf{Ab}\colon \mathsf{XMod} \to \mathsf{XMod}$ *is admissible for the class of regular epimorphisms.*

Proof The abelianization functor $\mathsf{Ab}\colon \mathsf{XMod} \to \mathsf{XMod}$, see Example 1.15, is right exact. Since, exactness can be shown component-wise the result follows by Proposition 3.7. □

Sometimes it is handy to rely on our group theoretical knowledge to construct simple examples of localization functors and how they behave on crossed modules. The proof of the following proposition is based on a counter-example coming from groups via the functor X defined in Remark 1.3.

Proposition 3.9 *There are regular-epi localization functors* $\mathsf{L}\colon \mathsf{XMod} \to \mathsf{XMod}$ *that are not admissible for the class of regular epimorphisms.*

Proof We export via $\mathsf{X}\colon \mathsf{Grp} \to \mathsf{XMod}$ the example in [15, Theorem 5.1] of a localization functor in groups that is not admissible for the class of regular epimorphisms.

Let L_ϕ be the localization functor induced by the projection $\phi\colon C_4 \to C_2$, where C_n denotes a cyclic group of order n. It gives rise to a localization functor $\mathsf{L}_{X\phi}\colon \mathsf{XMod} \to \mathsf{XMod}$.

In particular, if we apply X to the extension of L_ϕ-local groups considered in [15], we obtain an exact sequence of $\mathsf{L}_{\mathsf{X}\phi}$-local crossed modules:

$$1 \longrightarrow (1, \mathbb{Z}) \longrightarrow (1, \mathbb{Z}) \longrightarrow (1, C_2) \longrightarrow 1$$

If we pull it back along the morphism of crossed modules $\mathsf{X}\phi$, we obtain the following exact sequence

$$1 \longrightarrow (1, \mathbb{Z}) \longrightarrow (1, \mathbb{Z} \times C_2) \longrightarrow (1, C_4) \longrightarrow 1$$

We conclude from [24, Lemma 1.4] that this exact sequence is not $\mathsf{L}_{\mathsf{X}\phi}$-flat. Indeed, if it was the case we would have a contradiction with the group theoretical observation in [15]. □

4 Admissibility and Acyclicity

In the categories of groups and topological spaces, a localization functor L is a nullification functor if and only if the kernels of the localization morphisms are L-acyclic (which means that $\mathsf{L}\mathsf{ker}(\ell^{\mathsf{M}})$ is trivial for any object M). This characterization implies in particular that any nullification functor is admissible for the class of regular epimorphisms. It is interesting to notice that even if nullification functors of crossed modules do not have acyclic kernels, we have a similar result in XMod: the L-acyclicity of the kernels of localization morphisms implies the admissibility of L.

Proposition 4.1 *Let* $\mathsf{L}\colon \mathsf{XMod} \to \mathsf{XMod}$ *be a regular-epi localization functor such that* $\mathsf{ker}(\ell^{\mathsf{M}}\colon \mathsf{M} \to \mathsf{LM})$ *is* L*-acyclic for any* $\mathsf{M} \in \mathsf{XMod}$*. Then* L *is admissible for the class of regular epimorphisms.*

Proof Consider the pullback of $1 \longrightarrow \mathsf{LN} \xrightarrow{\kappa} \mathsf{LT} \xrightarrow{f} \mathsf{LQ} \longrightarrow 1$ along the coaugmentation morphism $\ell^{\mathsf{Q}}\colon \mathsf{Q} \to \mathsf{LQ}$:

$$\begin{tikzcd}
1 \arrow[r] & \mathsf{LN} \arrow[r, "\kappa'"] \arrow[d, equal] & \mathsf{T}' \arrow[r, "\pi_Q"] \arrow[d, "\pi_{\mathsf{LT}}"] & \mathsf{Q} \arrow[r] \arrow[d, "\ell^{\mathsf{Q}}"] & 1 \\
1 \arrow[r] & \mathsf{LN} \arrow[r, "\kappa"'] & \mathsf{LT} \arrow[r, "f"'] & \mathsf{LQ} \arrow[r] & 1
\end{tikzcd}$$

We need to prove that π_{LT} is an L-equivalence. Since XMod is a pointed protomodular category and ℓ^{Q} is a regular epimorphism by assumption, we know that π_{LT} is the cokernel of $\mathsf{ker}(\ell^{\mathsf{Q}}) \cong \mathsf{ker}(\pi_{\mathsf{LT}}) \to \mathsf{T}'$. Let Y be a local object. For any $g\colon \mathsf{T}' \to \mathsf{Y}$ we have the following diagram:

$$\begin{tikzcd}
\mathsf{ker}(\ell^{\mathsf{Q}}) \arrow[r] \arrow[d] & \mathsf{T}' \arrow[r, "\pi_{\mathsf{LT}}"] \arrow[d, "g"] & \mathsf{LT} \arrow[dl, dashed, "\tilde{g}"] \\
\mathsf{Lker}(\ell^{\mathsf{Q}}) = 1 \arrow[r, "g'"] & \mathsf{Y} &
\end{tikzcd}$$

By the universal property of the localization there exists $g'\colon 1 \to \mathsf{Y}$ that makes the left square commute. Hence, by the universal property of the cokernel there exists a unique $\tilde{g}\colon \mathsf{LT} \to \mathsf{Y}$ such that the triangle commutes and we conclude that π_{LT} is an L-equivalence. □

However, localization functors of crossed modules do not behave like localization functors of groups. As explained above, in the category of groups (but also of topological spaces), the kernels of the localization morphisms are L-acyclic if and only if L is a nullification functor [15]. In the context of crossed modules, we do not have such a characterization of nullification functors.

Remark 4.2 We know by [24, Proposition 4.6] that there are nullification functors, for example $\mathsf{P}_{\mathsf{X}\mathbb{Z}}$ defined in Example 1.14, such that the kernels of their localization morphisms are not acyclic in general. Still, in the next proposition, we prove that if the kernel of the localization morphism is L-acyclic, as in Proposition 4.1, then the localization functor is a nullification.

The cardinal in the next proof is chosen exactly as in Bousfield's [7, Theorem 4.4] for spaces.

Proposition 4.3 *Let $f\colon \mathsf{B} \to \mathsf{C}$ be a morphism of crossed modules and $\mathsf{L}_f\colon \mathsf{XMod} \to \mathsf{XMod}$ be a regular-epi localization functor. If the kernels of the localization morphisms are L_f-acyclic, then L_f is a nullifcation functor.*

Proof Our strategy is to construct a crossed module A such that we can compare the functor L_f with the nullification functor P_A (Definition 1.10) via a natural transformation ψ. We choose κ to be the first infinite cardinal greater than the number of chosen generators of B and C, i.e., generators of the groups B_1, B_2, C_1 and C_2. We construct the crossed module $\mathsf{A} := \coprod \mathsf{A}_\alpha$, where A_α are all the L_f-acyclic crossed modules with less than 2^κ generators, see [7, Theorem 4.4].

The first step of this proof is to show that if a crossed module X is L_f-local then it is A-local. Let ϕ be a morphism in $Hom(\mathsf{A}, \mathsf{X})$ and construct by naturality the following commutative diagram

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\phi"] \arrow[d] & \mathsf{X} \arrow[d, "\cong"] \\
1 = \mathsf{L}_f\mathsf{A} \arrow[r, "\mathsf{L}_f(\phi)"'] & \mathsf{L}_f\mathsf{X}
\end{tikzcd}$$

By hypothesis, we have an isomorphism between X and $\mathsf{L}_f\mathsf{X}$ and by construction of A, we obtain $\mathsf{L}_f\mathsf{A} = 1$. Therefore, ϕ factors through the zero object and hence $Hom(\mathsf{A}, \mathsf{X}) = 1$, which is equivalent to say that X is A-local. Now consider the P_A-equivalence $p^\mathsf{T}\colon \mathsf{T} \to \mathsf{P}_\mathsf{A}\mathsf{T}$ and the L_f-local object $\mathsf{L}_f\mathsf{T}$. By the above observation, we have that $\mathsf{L}_f\mathsf{T}$ is A-local and by the universal property we have the desired morphism ψ_T

$$\begin{tikzcd}
\mathsf{T} \arrow[r, "p^\mathsf{T}"] \arrow[dr, "\ell^\mathsf{T}"'] & \mathsf{P}_\mathsf{A}\mathsf{T} \arrow[d, dashed, "\psi_\mathsf{T}"] \\
& \mathsf{L}_f\mathsf{T}
\end{tikzcd}$$

We construct next the fiberwise A-nullification of the following exact sequence

$$1 \longrightarrow \ker(\ell^\mathsf{T}) \longrightarrow \mathsf{T} \xrightarrow{\ell^\mathsf{T}} \mathsf{L}_f\mathsf{T} \longrightarrow 1$$

By assumption $\mathsf{ker}(\ell^{\mathsf{T}})$ is L_f-acyclic, hence also P_{A}-acyclic by design. This implies that $\mathsf{ker}\Big(p^{\mathsf{T}} \colon \mathsf{ker}(\ell^{\mathsf{T}}) \to \mathsf{P}_{\mathsf{A}}\mathsf{ker}(\ell^{\mathsf{T}})\Big)$ is equal to $\mathsf{ker}(\ell^{\mathsf{T}})$. Hence, the exact sequence satisfies condition (6) of Theorem 1.19 and we obtain the following fiberwise nullification

$$\begin{tikzcd}
1 \arrow[r] & \mathsf{ker}(\ell^{\mathsf{T}}) \arrow[r] \arrow[d, "p^{\mathsf{T}}"] & \mathsf{T} \arrow[r, "\ell^{\mathsf{T}}"] \arrow[d, "f"'] & \mathsf{L}_f\mathsf{T} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & 1 \arrow[r] & \mathsf{T}/\mathsf{ker}(\ell^{\mathsf{T}}) \arrow[r, "\cong"] & \mathsf{L}_f\mathsf{T} \arrow[r] & 1
\end{tikzcd}$$

Since f is a P_{A}-equivalence, so is ℓ^{T}. Hence, we obtain a morphism φ^{T} in the following commutative diagram:

$$\begin{tikzcd}
\mathsf{T} \arrow[r, "\ell^{\mathsf{T}}"] \arrow[dr, "p^{\mathsf{T}}"'] & \mathsf{L}_f\mathsf{T} \arrow[d, dashed, bend left, "\varphi^{\mathsf{T}}"] \\
& \mathsf{P}_{\mathsf{A}}\mathsf{T} \arrow[u, bend left, "\psi^{\mathsf{T}}"]
\end{tikzcd}$$

By universal property, we can conclude that the two compositions of ψ^{T} and φ^{T} are isomorphic to identities so that $\mathsf{L}_f\mathsf{T} \cong \mathsf{P}_{\mathsf{A}}\mathsf{T}$. A similar argument shows the naturality of ψ and φ and therefore L_f is a nullification functor, namely P_{A}. □

5 Nullification Functors and Admissibility

In the category of groups, the fact that kernels of localization morphisms are L-acyclic was fundamental to prove that nullification functors are admissible for the class of regular epimorphisms. This fact is not true in general for nullification functors in the category of crossed modules as shown in [24, Proposition 4.6], it is thus natural to ask whether nullification functors are admissible. We provide an affirmative answer in this final section, but let us first prove that our counter-example $\mathsf{P}_{\mathsf{X}\mathbb{Z}}$ is admissible.

Proposition 5.1 *The nullification functor* $\mathsf{P}_{\mathsf{X}\mathbb{Z}}$ *is admissible for the class of regular epimorphisms.*

Proof Theorem 5.1 in [16] implies that $\mathsf{P}_{\mathsf{X}\mathbb{Z}}$ is admissible provided that the reflective category of $\mathsf{X}\mathbb{Z}$-local objects is a Birkhoff subcategory, i.e., it is closed under regular quotients and subobjects. Here $\mathsf{X}\mathbb{Z}$-local objects are crossed modules of the form $A \to 1$ where A is any abelian group and the connecting homomorphism is the trivial homomorphism. Therefore it is clearly closed under subobjects. Moreover, the quotient of $A \to 1$ by a normal subcrossed module $N \to 1$ is the crossed module $A/N \to 1$, which is $\mathsf{X}\mathbb{Z}$-local. □

The remaining part of the section is devoted to the proof that *all* nullification functors are admissible for the class of regular epmorphisms. Consider a nullification functor P_{A} where $\mathsf{A} = (A_1, A_2, \partial)$ is a crossed module. To show the admissibility, it is enough to prove that the pullback of an exact sequence of A-local crossed modules along the coaugmentation map is P_{A}-flat, in other words that the map f in the following commutative diagram of crossed modules is a $\mathsf{P_A}$-equivalence

$$\begin{tikzcd}
1 \arrow[r] & \mathsf{P_A N} \arrow[r] \arrow[d, equal] & \mathsf{W} \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "(1)"] & \mathsf{Q} \arrow[r] \arrow[d, "p^{\mathsf{Q}}"] & 1 \\
1 \arrow[r] & \mathsf{P_A N} \arrow[r] & \mathsf{P_A T} \arrow[r, "h"'] & \mathsf{P_A Q} \arrow[r] & 1
\end{tikzcd}$$

where (1) is a pullback and g and h are regular epimorphisms. To do so we follow step by step the inductive construction of $\mathsf{P_A Q} = \mathsf{colim Q}_\beta$ as presented in [24, Proposition 2.8], see also Proposition 1.11. For each successor ordinal $\beta + 1$ we obtain $\mathsf{Q}_{\beta+1}$ from Q_β by killing all morphisms out of A so let us start with the construction of Q_1 from $\mathsf{Q}_0 = \mathsf{Q}$.

Remark 5.2 Let $\varphi : \mathsf{A} \to \mathsf{Q}$ denote a morphism of crossed modules. The crossed module Q_1 is the quotient of Q by the normal closure $\mathsf{K_Q}$ in Q of the image of

$$ev \colon \mathsf{M} := \coprod_{\varphi \in Hom(\mathsf{A},\mathsf{Q})} \mathsf{A} \longrightarrow \mathsf{Q}$$

which is defined by φ on the copy of A indexed by φ. The idea behind the construction we perform next is that we do not need to kill all morphisms from A to the extension W in order to construct its nullification $\mathsf{P_A W}$, it is sufficient to take care of those factoring through Q. Beware that given an extension $\mathsf{N} \to \mathsf{T} \to \mathsf{Q}$ with N an A-acyclic crossed module, it is not true in general that all morphisms from A to T factor through Q.

By definition of p^{Q} we have the following equality for the composition $p^{\mathsf{Q}} \circ \varphi = 1 = h \circ 1$ as below. Therefore, any morphism from A to Q induces one from A to W:

$$\begin{tikzcd}
A \arrow[dr, dashed, "\exists ! \psi"] \arrow[drr, bend left, "\varphi"] \arrow[ddr, bend right, "1"'] & & \\
& \mathsf{W} \arrow[r, "g"] \arrow[d, "f"] & \mathsf{Q} \arrow[d, "p^{\mathsf{Q}}"] \\
& \mathsf{P_A T} \arrow[r, "h"'] & \mathsf{P_A Q}
\end{tikzcd} \tag{8}$$

We call ψ the morphism determined by φ and it makes sense now to consider $\mathsf{K_W}$, the normal closure in W of the image of $\mathsf{M} \to \mathsf{W}$.

Lemma 5.3 *With the same notation as in Remark 5.2, we have an isomorphism* $\mathsf{K_W} \cong \mathsf{K_Q}$.

Proof Limits are computed levelwise for crossed modules, so the pullback W consists of compatible pairs (x, q) for $x \in (\mathsf{P_A T})_{\mathsf{i}}$ and $q \in \mathsf{Q_i}$ for $i = 1, 2$. By construction of ψ we have $\psi(a) = (1, \varphi(a))$.

Now, we compute the kernels of the cokernels of $ev \colon \mathsf{M} \to \mathsf{Q}$ and $(1, ev) \colon \mathsf{M} \to \mathsf{W}$. We have the two following descriptions of the kernels:

$$\mathsf{K_Q} = \Big(ev_1(M_1)_{Q_2}[ev_2(M_2)_{Q_2}, Q_1],\ ev_2(M_2)_{Q_2}, \partial \Big)$$

$$\mathsf{K_W} = \Big((1, ev_1)(M_1)_{W_2}[(1, ev_2)(M_2)_{W_2}, W_1],\ (1, ev_2)(M_2)_{W_2}, \partial' \Big)$$

The second group of the crossed module K_W is the easier one:

$$\begin{aligned}(1, ev_2)(M_2)_{W_2} &= \{{}^{(t_2,q_2)}(1, ev_2(m_2)) \mid (t_2, q_2) \in W_2,\ m_2 \in M_2\}\\ &= \{(1,\ {}^{q_2}ev_2(m_2)) \mid q_2 \in Q_2,\ m_2 \in M_2\}\\ &= 1 \times ev_2(M_2)_{Q_2}.\end{aligned}$$

where the second equality holds since h is surjective. Via similar computations, we see that $(1, ev_1)(M_1)_{W_2} = 1 \times ev_1(M_1)_{Q_2}$, so we are left with proving that

$$[(1, ev_2)(M_2)_{W_2}, W_1] = 1 \times [ev_2(M_2)_{Q_2}, Q_1]$$

This we do via the following equalities:

$$\begin{aligned}[(1, ev_2)(M_2)_{W_2}, W_1] &= [(1 \times ev_2(M_2)_{Q_2}), W_1]\\ &= \{{}^{(1,x_2)}(t_1, q_1)(t_1, q_1)^{-1} \mid x_2 \in ev_2(M_2)_{Q_2},\ (t_1, q_1) \in W_1\}\\ &= \{(1, {}^{x_2} q_1 q_1^{-1} \mid x_2 \in ev_2(M_2)_{Q_2},\ q_1 \in Q_1\}\\ &= 1 \times [(ev_2(M_2)_{Q_2}, Q_1]\end{aligned}$$

So finally we can conclude that $K_\mathsf{W} = 1 \times K_\mathsf{Q}$, in particular K_W and K_Q are isomorphic.

□

Proposition 5.4 *For any ordinal β, we have a commutative diagram*

$$\begin{tikzcd}
\mathsf{W} \arrow[r, "f_\beta"] \arrow[d, "g"'] & \mathsf{W}_\beta \arrow[r] \arrow[d, "h_\beta"'] \arrow[dr, phantom, "(2)"] & \mathsf{P_A T} \arrow[d, "h"] \\
\mathsf{Q} \arrow[r, "p_\beta^\mathsf{Q}"] & \mathsf{Q}_\beta \arrow[r] & \mathsf{P_A Q}
\end{tikzcd}$$

where (2) *is a pullback square, the maps $f_\beta \colon \mathsf{W} \to \mathsf{W}_\beta$ and $p_\beta^\mathsf{Q} \colon \mathsf{Q} \to \mathsf{Q}_\beta$ are P_A-equivalences, and h_β is a regular epimorphism.*

Proof We prove it by induction. Since the nullification uses possibly a transfinite construction we have to initialize the induction, but the case $\beta = 0$ holds by assumption, and then check the statement for successor and limit ordinals.

<u>The successor case</u> Suppose that for an ordinal β the lemma is proved. Then we consider the kernels $\mathsf{K}_\beta^\mathsf{W}$ and $\mathsf{K}_\beta^\mathsf{Q}$ of the cokernels of the evaluation maps $ev : \coprod_{Hom(\mathsf{A},\mathsf{Q}_\beta)} \mathsf{A} \longrightarrow \mathsf{Q}_\beta$ and $ev : \coprod_{Hom(\mathsf{A},\mathsf{Q}_\beta)} \mathsf{A} \longrightarrow \mathsf{W}_\beta$ respectively. They fit in the following diagram of exact rows:

$$\begin{tikzcd}
\mathsf{K}_\beta^\mathsf{W} \arrow[r, "i_\mathsf{W}"] \arrow[d, "\cong"] & \mathsf{W}_\beta \arrow[r, "f_{(\beta\to\beta+1)}"] \arrow[d, "h_\beta"] \arrow[dr, phantom, "(2)"] & \mathsf{W}_{\beta+1} \arrow[d, dashed, "\exists! h_{\beta+1}"] \\
\mathsf{K}_\beta^\mathsf{Q} \arrow[r, "i_\mathsf{Q}"] & \mathsf{Q}_\beta \arrow[r, "p^\mathsf{Q}_{(\beta\to\beta+1)}"] & \mathsf{Q}_{\beta+1}
\end{tikzcd}$$

Lemma 5.3 applies here and gives us the isomorphism between $\mathsf{K}_\beta^\mathsf{W}$ and $\mathsf{K}_\beta^\mathsf{Q}$. The composition

$$p^\mathsf{Q}_{(\beta\to\beta+1)} \circ h_\beta \circ i_\mathsf{W} \colon \mathsf{K}_\beta^\mathsf{W} \to \mathsf{Q}_{\beta+1}$$

is zero by commutativity, yielding by the universal property of the cokernel the morphism $h_{\beta+1}\colon \mathsf{W}_{\beta+1} \to \mathsf{Q}_{\beta+1}$. The isomorphism between the kernels implies that (2) is a pullback (see Proposition 1.5). By induction hypothesis h_β is a regular epimorphism and the composition $p^{\mathsf{Q}}_{(\beta\to\beta+1)} \circ h_\beta\colon \mathsf{W}_\beta \to \mathsf{Q}_{\beta+1}$ is also a regular epimorphism, hence so is $h_{\beta+1}$. We show now that $p^{\mathsf{Q}}_{(\beta\to\beta+1)}$ and $f_{(\beta\to\beta+1)}$ are P_{A}-equivalences.

For the first one we write the cokernel $\mathsf{Q}_{\beta+1}$ as the pushout along the evaluation morphism:

$$\begin{tikzcd}
\coprod \mathsf{A} \arrow[r, "\varphi"] \arrow[d, "1"'] & \mathsf{Q}_\beta \arrow[d, "p^{\mathsf{Q}}_{(\beta\to\beta+1)}"] \\
1 \arrow[r, "inc"'] & \mathsf{Q}_{\beta+1}
\end{tikzcd}$$

where the coproduct is taken over $Hom(\mathsf{A}, \mathsf{Q}_\beta)$. The trivial map $\mathsf{A} \to 1$ is a P_{A}-equivalence, thus so is the pushout $p^{\mathsf{Q}}_{(\beta\to\beta+1)}\colon \mathsf{Q}_\beta \to \mathsf{Q}_{\beta+1}$ by Lemma 1.8 (1). By composing with the P_{A}-equivalence $\mathsf{Q} \to \mathsf{Q}_\beta$ we see that $p^{\mathsf{Q}}_{\beta+1}\colon Q \to Q_{\beta+1}$ is a P_{A}-equivalence as well. The same argument shows that $f_{\beta+1} : \mathsf{W} \to \mathsf{W}_{\beta+1}$ is also a P_{A}-equivalence. By the universal property of the localization, we obtain two maps, one from $\mathsf{W}_{\beta+1}$ to $\mathsf{P}_{\mathsf{A}}\mathsf{T}$ and the other from $\mathsf{Q}_{\beta+1}$ to $\mathsf{P}_{\mathsf{A}}\mathsf{Q}$ such that (2) commutes:

$$\begin{tikzcd}
\mathsf{W} \arrow[r, "f_{\beta+1}"] \arrow[rr, bend left, "f"] \arrow[d, "g"] \arrow[dr, phantom, "(1)"] & \mathsf{W}_{\beta+1} \arrow[r, dashed] \arrow[d, "h_{\beta+1}"] \arrow[dr, phantom, "(2)"] & \mathsf{P}_{\mathsf{A}}\mathsf{T} \arrow[d, "h"] \\
\mathsf{Q} \arrow[r, "p^{\mathsf{Q}}_{\beta+1}"] \arrow[rr, bend right, "p^{\mathsf{Q}}"'] & \mathsf{Q}_{\beta+1} \arrow[r, dashed] & \mathsf{P}_{\mathsf{A}}\mathsf{Q}
\end{tikzcd}$$

Since (1) and the outer rectangle are pullbacks and $h_{\beta+1}$ is a regular epimorphism, we can conclude by [1, Proposition 4.1.4] that (2) is a pullback.

The limit case To prove the statement for a general transfinite induction we need to prove it for a limit ordinal as well. Let γ be a limit ordinal and

$$\mathsf{Q}_\gamma = \operatorname{colim}_{\alpha<\gamma}\mathsf{Q}_\alpha \quad \mathsf{W}_\gamma = \operatorname{colim}_{\alpha<\gamma}\mathsf{W}_\alpha$$

We assume that $p^{\mathsf{Q}}_{(\alpha-1\to\alpha)}\colon \mathsf{Q}_{\alpha-1} \to \mathsf{Q}_\alpha$ is a P_{A}-equivalence for all $\alpha < \gamma$. Hence the composition $p^{\mathsf{Q}}_\alpha\colon \mathsf{Q} \to \mathsf{Q}_\alpha$ is also a P_{A}-equivalence and Lemma 1.8 (3), implies that $p^{\mathsf{Q}}_\gamma\colon \mathsf{Q} \to \mathsf{Q}_\gamma$ is a P_{A}-equivalence. The same reasoning holds for $f_\gamma\colon \mathsf{W} \to \mathsf{W}_\gamma$. The existence of the maps $f : \mathsf{W} \to \mathsf{P}_{\mathsf{A}}\mathsf{T}$ and $p^{\mathsf{Q}} : \mathsf{Q} \to \mathsf{P}_{\mathsf{A}}\mathsf{Q}$ gives us two maps $\mathsf{W}_\gamma \to \mathsf{P}_{\mathsf{A}}\mathsf{T}$ and $\mathsf{Q}_\gamma \to \mathsf{P}_{\mathsf{A}}\mathsf{Q}$ as shown on the diagram below (9).

The nullification $\mathsf{P}_{\mathsf{A}}\mathsf{Q}$ is constructed as filtered colimit of the Q_α's, see Proposition 1.11. Filtered colimits commute with finite limits, in particular with kernels. Therefore

$$\mathsf{K}^{\mathsf{Q}}_\gamma := \mathsf{ker}(\mathsf{Q} \to \mathsf{Q}_\gamma) \cong \operatorname{colim}_{\alpha<\gamma}\mathsf{ker}(\mathsf{Q} \to \mathsf{Q}_\alpha)$$

where $\mathsf{ker}(\mathsf{Q} \to \mathsf{Q}_\alpha)$ will be denoted $\mathsf{K}^{\mathsf{Q}}_\alpha$. The category XMod is a variety of algebras (also called algebra category of fixed type). Hence, by [23, Proposition IX.1.2], we know that the forgetful functor $\mathcal{U}\colon \mathsf{XMod} \to \mathsf{Set}$ creates filtered colimits. In other words we have :

$$\mathcal{U}(\mathrm{colim}_{\alpha<\gamma}\mathsf{K}^{\mathsf{Q}}_{\alpha}) = \mathrm{colim}_{\alpha<\gamma}\mathcal{U}\mathsf{K}^{\mathsf{Q}}_{\alpha} = \bigcup_{\alpha<\gamma}\mathcal{U}\mathsf{K}^{\mathsf{Q}}_{\alpha}$$

where the colimit in the first term lies in the category of crossed modules and the second colimit in the category of sets. This means that we know the structure of $\mathrm{colim}_{\alpha<\gamma}\mathsf{K}^{\mathsf{Q}}_{\alpha}$ as a set. Now since $\mathsf{K}^{\mathsf{Q}}_{\alpha} \cong \mathsf{K}^{\mathsf{W}}_{\alpha}$ for all $\alpha < \gamma$ and $\mathsf{K}^{\mathsf{Q}}_{\gamma}$ can be written as a union of $\mathsf{K}^{\mathsf{Q}}_{\alpha}$ (as well as $\mathsf{K}^{\mathsf{W}}_{\gamma}$) we conclude that $\mathsf{K}^{\mathsf{Q}}_{\gamma} \cong \mathsf{K}^{\mathsf{W}}_{\gamma}$. We consider now the diagram:

$$\begin{tikzcd}
\mathsf{W} \arrow[r, "f_\gamma"] \arrow[d, "g"'] \arrow[rr, bend left, "f"] \arrow[dr, phantom, "(2)"] & \mathsf{W}_\gamma \arrow[r, dashed] \arrow[d, "h_\gamma"] \arrow[dr, phantom, "(1)"] & \mathsf{P}_{\mathsf{A}}\mathsf{T} \arrow[d, "h"] \\
\mathsf{Q} \arrow[r, "p^{\mathsf{Q}}_\gamma"] \arrow[rr, bend right, "p^{\mathsf{Q}}"'] & \mathsf{Q}_\gamma \arrow[r, dashed] & \mathsf{P}_{\mathsf{A}}\mathsf{Q}
\end{tikzcd} \tag{9}$$

Since the kernels of f_γ and p^{Q}_{γ} are isomorphic we deduce that (2) is a pullback. As we have shown that every map $p^{\mathsf{Q}}_{(\alpha\to\alpha+1)}\colon \mathsf{Q}_\alpha \to \mathsf{Q}_{\alpha+1}$ is a regular epimorphism, the morphism $p^{\mathsf{Q}}_{\alpha}\colon \mathsf{Q} \to \mathsf{Q}_\alpha$ is also a regular epimorphism, being a composition of regular epimorphisms in a regular category. The colimit functor being a left adjoint functor, it preserves colimits and in particular cokernels. In a pointed protomodular category, any regular epimorphism is a cokernel, therefore

$$p^{\mathsf{Q}}_{\gamma}\colon \mathsf{Q} \to \mathsf{Q}_\gamma$$

is a regular epimorphism. The composition $p^{\mathsf{Q}}_{\gamma} \circ g$ is also a regular epimorphism, and we conclude that so is h_γ. With the same argument as for the successor step, we get that (1) is a pullback, which ends the induction proof. □

We are ready now for the main result of this section.

Theorem 5.5 *Let* A *be any crossed module. The nullification functor* P_{A} *is admissible for the class of regular epimorphisms.*

Proof Let W be the pullback of a regular epimorphism $h\colon \mathsf{P}_{\mathsf{A}}\mathsf{T} \to \mathsf{P}_{\mathsf{A}}\mathsf{Q}$ between A-local crossed modules along the localization morphism $p^{\mathsf{Q}}\colon \mathsf{Q} \to \mathsf{P}_{\mathsf{A}}\mathsf{Q}$. Let λ be the ordinal such that $\mathsf{Q}_\lambda \cong \mathsf{P}_{\mathsf{A}}\mathsf{Q}$ (see Proposition 1.11). By Proposition 5.4 we have a diagram:

$$\begin{tikzcd}
\mathsf{W} \arrow[r, "f_\lambda"] \arrow[d, "g"'] & \mathsf{W}_\lambda \arrow[r] \arrow[d, "h_\lambda"] \arrow[dr, phantom, "(2)"] & \mathsf{P}_{\mathsf{A}}\mathsf{T} \arrow[d, "h"] \\
\mathsf{Q} \arrow[r, "p^{\mathsf{Q}}_\lambda"] & \mathsf{Q}_\lambda \arrow[r, "\cong"] & \mathsf{P}_{\mathsf{A}}\mathsf{Q}
\end{tikzcd}$$

where the outer rectangle is a pullback, the morphisms f_λ and p^{Q}_{λ} are P_{A}-equivalences, and (2) is a pullback. Since isomorphisms are stable under pullbacks, we have an isomorphism

$\mathsf{W}_\lambda \cong \mathsf{P_A T}$. We have thus proved that the map $f: \mathsf{W} \to \mathsf{P_A T}$ is a $\mathsf{P_A}$-equivalence, which means that the functor $\mathsf{P_A}$ is admissible. □

In this article we have focused on regular-epi localization functors because they appear naturally when studying conditional flatness and admissibility in the category of groups, crossed modules, or more general semi-abelian categories. We conclude this section by observing that the notion of conditional flatness can also be defined for non regular-epi localization functors. The next proposition gives an example of such a localization functor which is conditionally flat. Let us stress that we will not a priori have an equivalence with admissiblity, as was the case for regular-epi localization functors by Theorem 3.5. In the proof of the following proposition we have thus to verify the more general condition for conditional flatness, as in Definition 2.3.

Proposition 5.6 *There exists a non regular-epi localization functor which is nevertheless conditionally flat and therefore admissible for the class of regular epimorphisms.*

Proof We consider the functor I defined in Example 1.16 which sends any crossed module $(N_1, N_2, \partial^{\mathsf{N}})$ to (N_2, N_2, Id_{N_2}). This functor is not regular-epi because if we consider a crossed module for which the connecting morphism is not surjective then the localization morphism will not be a regular epimorphism.

We prove now that I is conditionally flat. Let $1 \longrightarrow \mathsf{N} \xrightarrow{\kappa} \mathsf{T} \xrightarrow{\alpha} \mathsf{Q} \longrightarrow 1$ be any exact sequence of crossed modules. We see that $\mathsf{I}((N_1, N_2, \partial^{\mathsf{N}})) = (N_2, N_2, Id_{N_2})$ is a normal subcrossed module of $(T_2, T_2, Id_{T_2}) = \mathsf{I}((T_1, T_2, \partial^{\mathsf{T}})$ and that $\mathsf{I}((Q_1, Q_2, \partial^{\mathsf{Q}})) = (Q_2, Q_2, Id_{Q_2})$ is the cokernel of $\kappa: \mathsf{N} \to \mathsf{T}$. Therefore any exact sequence of crossed modules is I-flat. In particular any pullback along any morphism of crossed modules of an I-flat exact sequence is I-flat, hence I is conditionally flat. □

Acknowledgements We would like to thank Marino Gran for sharing his insight about admissibility, and the referee for their kind comments and helpful suggestions.

Author Contributions All authors (Olivia Monjon, Jérôme Scherer, and Florence Sterck) contributed to the study and wrote the first draft of the manuscript together. All authors read and approved the final manuscript.

Funding Open access funding provided by EPFL Lausanne No funding has been obtained for this study.

Availability of Data and Materials Not applicable.

Declarations

Conflict of interest Not applicable.

Consent for publication All three authors approved the version to be published.

References

1. Borceux, F., Bourn, D.: Mal'cev, protomodular, homological and semi-abelian categories. Vol. 566. Springer Science & Business Media, (2004)
2. Berrick, A.J., Dror Farjoun, E.: Fibrations and nullifications. In: Israel J. Math., 135, 205-220 , (2003)
3. Bourn, D., Gran, M.: Central extensions in semi-abelian categories. J. Pure Appl. Algebra **175**, 31–44 (2002)
4. Brown, R., Higgins, P.: On the connection between the second relative homotopy groups of some related spaces. In: Proceedings of The London Mathematical Society, pp. 193-212, (1978)
5. Bourn, D.: Normalization Equivalence, Kernel Equivalence, and Affine Categories. In: 2006, pp. 43-62
6. Bousfield, A. K.: Constructions of factorization systems in categories. In: J. Pure Appl. Algebra 9.2, pp. 207-220, (1976)
7. Bousfield, A.K.: Homotopical localizations of spaces. In: Amer. J. Math. 119.6, pp. 1321-1354, (1997)
8. Brown, R., Spencer, C.: G-groupoids, crossed modules and the fundamental groupoid of a topological group. In: Indag. Math. (Proceedings), 79.4, 296-302, (1976)
9. Casacuberta, C.: On structures preserved by idempotent transformations of groups and homotopy types. In: Crystallographic groups and their generalizations (Kor- trijk, 1999). Vol. 262. Contemp. Math. Amer. Math. Soc., Providence, RI, pp. 39-68 (2000)
10. Casacuberta, C.: Anderson localization from a modern point of view. Con- temp. Math. **181**, 35–46 (1995)
11. Casacuberta, C., Descheemaeker, A.: Relative group completions. In: Journal of Algebra 285.2, pp. 451-469, (2005)
12. Cassidy, C., Hébert, M., Kelly, G.M.: Reflective subcategories, localizations and factorization systems. In: J. Austral. Math. Soc. Ser. A, 38(3), 87-329, (1985)
13. Everaert, T., Gran, M., Van der Linden, T.: Higher Hopf formulae for Homology via Galois Theory. Adv. Math. **217**, 2231–2267 (2008)
14. Dror Farjoun, E.: Cellular spaces, null spaces and homotopy localization. Vol. 1622. Lecture Notes in Mathematics. Springer-Verlag, Berlin, pp. xiv+199 (1996)
15. Dror Farjoun, E., Scherer, J.: Conditionally flat functors on spaces and groups. In: Collect. Math. 66(1), (2015), 149-160
16. Gran, M., Scherer, J.: Conditional flatness, fiberwise localizations, and admissible reflections. In: J. Aust. Math. Soc., to appear, 23 pages. (2023) https://doi.org/10.1017/S1446788723000046
17. Philip, S.: Hirschhorn. Model categories and their localizations. Vol. 99. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003, pp. xvi+457
18. Janelidze, G.: Internal crossed modules. In: Georgian Math. J. 10.1 (2003), pp. 99-114
19. Janelidze, G., Kelly, G.M.: Galois theory and a general notion of central extension. In: J. Pure Appl. Algebra 97.2 (1994), pp. 135-161
20. Janelidze, G., Márki, L., Tholen, W.: Semi-abelian categories. J. Pure Appl. Algebra **168**, 367–386 (2002)
21. Ladra, M., Grandjean, A.R.: Crossed modules and homology. In: Journal of Pure and Applied Algebra 95.1 (1994), pp. 41-55
22. Ladra, M., López López, M.P., Rodeja, E.: Epimorphisms of crossed modules. In: Southeast Asian Bulletin of Mathematics 28 (2004)
23. Mac Lane, S.: Categories for the Working Mathematicians. 2nd ed. Springer, (1997)
24. Monjon, O., Scherer, J., Sterck, F.: Non-existence of fiberwise localization for crossed modules. In: Israel J. Math., to appear (2023). https://arxiv.org/abs/2207.097
25. Norrie, K.: Crossed modules and analogues of group theorems. PhD thesis. King's College, University of London, (1987)
26. Norrie, K.: Actions and automorphisms of crossed modules. In: Bulletin de la Société Mathématique de France 118(2), pp. 129-146, (1990)
27. Whitehead, J.H.C.: Combinatorial homotopy II. Bull. Amer. Math. Soc **55**, 453–496 (1949)