



# Yoneda Lemma for Simplicial Spaces

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Received: 15 July 2022 / Accepted: 4 June 2023 / Published online: 11 July 2023  
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## Abstract

We study the Yoneda lemma for arbitrary simplicial spaces. We do that by introducing *left fibrations* of simplicial spaces and studying their associated model structure, the *covariant model structure*. In particular, we prove a *recognition principle* for covariant equivalences over an arbitrary simplicial space and *invariance* of the covariant model structure with respect to complete Segal space equivalences.

**Keywords** Higher category theory · Simplicial spaces · Complete Segal spaces · Left fibrations · Yoneda lemma

**Mathematics Subject Classification** 18N60 · 18N40 · 18N50 · 18F20

## 0 Introduction

### 0.1 Fibrations and the Yoneda Lemma

The Yoneda lemma is a fundamental result in classical category theory. It states that the value of a set-valued functor  $F : \mathcal{C} \rightarrow \text{Set}$  at an object  $c$  is uniquely determined by natural transformations  $\text{Hom}_{\mathcal{C}}(c, -) \Rightarrow F$ . Using the Yoneda lemma we can embed every category into a larger category via the *Yoneda embedding*, that shares many pleasant features with the category of sets. As a concrete example, we can use the Yoneda lemma and our understanding of limits in the category of sets, to give a precise description of limits in an arbitrary category [45, 63].

In recent decades there has been an effort to generalize the notion of a category to an  $(\infty, 1)$ -category, which satisfies the conditions of a category only up to coherent homotopies and is thus better suited to study objects that arise naturally in homotopy theory. This first happened via several models: *quasi-categories* [11], *complete Segal spaces* [58], *Kan enriched categories* [3], and many other models [4, 6]. More recent developments have focused on general concepts that incorporate these models, such as  $\infty$ -cosmoi [65].

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Communicated by Nicola Gambino.

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Given the important role of the Yoneda lemma for classical categories, we would expect a similarly important role for the Yoneda lemma for  $(\infty, 1)$ -categories. However, unlike the 1-categorical case, the study of the Yoneda lemma for  $(\infty, 1)$ -categories depends on the particular model we are working with. When we are using the model of Kan enriched categories, we can rely on the extensive literature regarding enriched categories to obtain a Yoneda lemma for Kan enriched categories [63, Lemma 7.3.5], [41, 1.9]. However, other common models of  $(\infty, 1)$ -categories, such as quasi-categories or complete Segal spaces, are not strict categories and so we need to use an alternative approach. Indeed, even trying to define an analogue to the representable functor  $\text{Hom}_{\mathcal{C}}(c, -)$  would require us to choose compositions, which are only defined up to a contractible choice.

Here we can take another look at the historical development of the Yoneda lemma in the setting of classical categories. Starting with ideas of Grothendieck [28], but also important figures such as Gray [27] and Bénabou [68], a certain full subcategory of  $\text{Cat}_{/\mathcal{C}}$ , now known as *discrete Grothendieck opfibrations* over  $\mathcal{C}$ , was recognized as an alternative characterization of the functor category, meaning it is equivalent to the category of set-valued functors out of the category  $\mathcal{C}$ , via the *Grothendieck construction*. In particular, this means there is also a *fibrational Yoneda lemma*, which states that for every object  $c$  there exists a *representable Grothendieck opfibration*  $\mathcal{C}_{c/} \rightarrow \mathcal{C}$  such that for every discrete Grothendieck opfibration  $p : \mathcal{D} \rightarrow \mathcal{C}$  a morphism  $\mathcal{C}_{c/} \rightarrow \mathcal{D}$  over  $\mathcal{C}$  (a “natural transformation”) is uniquely determined by an element in the fiber of  $p$  over  $c$  (the “value of  $c$ ”).

Such discrete Grothendieck opfibrations are characterized via a lifting property and are hence much more amenable to generalizations to the  $(\infty, 1)$ -categorical setting. It was in fact Joyal who first realized that the lifting condition can be generalized to quasi-categories, introducing *left fibrations* of simplicial sets [38, 39]. Since then, many other approaches to left fibrations and the Yoneda lemma have been studied for several models of  $(\infty, 1)$ -categories:

- *Quasi-categories*: After Joyal also studied by Lurie [42]. Has since been reworked using different methods by Heuts–Moerdijk [31, 33], Stevenson [67], Cisinski [16] and Nguyen [49].
- *$\infty$ -Cosmoi*: Introduced and proven by Riehl and Verity [65, Chapter 5].
- *Segal spaces*: Studied by de Brito [19] and Kazhdan–Varshavsky [40].

In this work we want to focus on left fibrations and Yoneda lemma for general simplicial spaces. Concretely we want to tackle the following previously unanswered questions:

- (1) How can we define left fibrations over an arbitrary simplicial space?
- (2) For a given simplicial space  $X$  can we define a homotopy theory of left fibrations over  $X$  (in the form of a model structure)?
- (3) For a given simplicial space  $X$  and a choice of point  $x$  can we construct a *representable left fibration*  $L_x \rightarrow X$  and prove the *Yoneda lemma for simplicial spaces*: A morphism of left fibrations  $L_x \rightarrow L$  over  $X$  is uniquely determined by a choice of element in the fiber of  $p : L \rightarrow X$  over  $x$ ?
- (4) Given an arbitrary simplicial space  $X$ , what is the relation between left fibrations over  $X$  and left fibrations over its free complete Segal space (i.e. the fibrant replacement)?

Answering these questions (and many more) is the goal of this work, the major outcomes of which have been summarized more precisely in Sect. 0.4.

## 0.2 Why Simplicial Spaces?

Given that most results here appeared in one form or another in the language of quasi-categories or  $\infty$ -cosmoi why present the material in the language of simplicial spaces? Here we will list several valuable implications:

*Simplified Arguments:* Using the approach via simplicial spaces permits very straightforward definitions and proofs, a key example being the Grothendieck construction for left fibrations over categories. In the simplicial space approach we can simply use the classical Grothendieck construction level-wise, as we prove in Theorem 4.18, thus giving a computationally much more feasible description. In particular, one aspect of the Grothendieck construction is its naturality. This is stated without proof in [42, Proposition 2.2.1.1], only to be recently proven (using an intricate argument) in [30, Sect. 6]. The simplified construction given here enables us to give a similar naturality proof in a much more straightforward manner.

*Internalization:* The study of left fibrations via simplicial sets are built on the specific combinatorial properties of the category of simplicial sets. In the simplicial space approach, however, we characterize left fibrations as morphisms of simplicial spaces, which are simplicial diagrams in spaces. Hence this approach does not rely on the same combinatorial techniques (beyond possibly what is necessary to construct Kan complexes in the first place). As a result this approach can be effectively generalized to many settings beyond spaces. An elegant example is recent work by Martini, who studies left fibrations (using the definition given here) via simplicial diagrams in an arbitrary Grothendieck  $\infty$ -topos [47].

*Synthetic  $\infty$ -Category Theory: Homotopy Type Theory* is a new foundation for mathematics that is axiomatically homotopy invariant<sup>1</sup> and thus in many ways better suited for homotopical constructions [69]. One long term goal is to use Homotopy Type Theory to introduce a *synthetic* notion of  $\infty$ -categories. A first step in this regard has been taken by Riehl and Shulman [64] who introduced a notion of a *Rezk type* as a way to define  $(\infty, 1)$ -categories in Homotopy Type Theory. The notion of a Rezk type is motivated by complete Segal spaces and so in particular their approach to fibrations and the Yoneda lemma corresponds to fibrations of complete Segal spaces rather than quasi-categories. Recently there has been new development, which takes ideas regarding left fibrations of Segal spaces to study higher category theory (such as limits and adjunctions) in the setting of Homotopy Type Theory [12, 46, 71].

*Completeness:* One defining property of  $(\infty, 1)$ -categories is *completeness*, first introduced by Rezk [58] and then also by Voevodsky in the context of Homotopy Type Theory, where it is introduced as the *univalence axiom* [69]. A more precise comparison between the univalence axiom in a type theory and completeness of the corresponding Segal object can be found in [66].

From a foundational perspective we want to determine which results in  $(\infty, 1)$ -category theory depend on the univalence axiom (i.e. require completeness) and which ones hold in a more general foundation. However, we cannot directly use quasi-categories to address this problem as quasi-categories are automatically complete. Rather it would require us to use technical tools such as *flagged  $\infty$ -categories* [1]. On the other hand complete Segal spaces can be easily generalized to Segal spaces and so give us a direct computational way to study the necessity of completeness: We simply have to check whether a result only holds for complete Segal spaces or can be generalized to Segal spaces.

<sup>1</sup> More precisely, type theoretic constructors are invariant under identities, which by the *univalence axiom* correspond to equivalences.

Let us give two concrete examples to illustrate possible results. In Sect. 5.2 we observe that we can define and study limits and colimits in a Segal space without any reference to the completeness condition. On the other hand, in order to establish the equivalence of several notions of left adjoints, Riehl and Shulman need to explicitly require the completeness condition [64, Theorem 11.23].

*Decomposition Spaces:* *Decomposition spaces*, also known as *2-Segal spaces*, are a generalization of Segal spaces with relevance in algebraic  $K$ -theory [10], combinatorics [23, 24] and representation theory [22]. This has motivated the study of decomposition spaces and their relevant morphisms, known as CULF morphisms. It was proven by Hackney and Kock [29] that the  $\infty$ -category of CULF morphisms over a given simplicial space is equivalent to right fibrations over its edge-wise subdivision (which is generally not a Segal space). Hence, studying right fibrations over general simplicial spaces provides us with effective tools to better understand decomposition spaces and CULF morphisms as well.

*Fibrations of  $(\infty, n)$ -Categories:* The same way that 1-categories have been generalized to  $(\infty, 1)$ -categories, strict  $n$ -categories have been generalized to  $(\infty, n)$ -categories. Similar to the  $(\infty, 1)$ -categorical case there is now a long list of models [5], however, unlike in the  $(\infty, 1)$ -categorical case, many important questions about  $(\infty, n)$ -categories have remained unanswered.

First of all it is not yet proven that the common models of  $(\infty, n)$ -category that appear in the literature are actually equivalent. For example, it is known that  $\Theta_n$ -spaces [60] are equivalent to  *$n$ -fold complete Segal spaces* [2] as proven by Bergner and Rezk [8, 9]. However, it is not known whether they are equivalent to *complicial sets* [70] and both of those are not known to be equivalent to *comical sets* [14]. These are just some of the models that appear in the literature and clearly illustrate the challenges that lie ahead.

On the other hand each model has its own applications in various branches of mathematics. For example,  *$n$ -fold complete Segal spaces* have been the primary model in the study of topological field theories and the cobordism hypothesis [13, 43] and thus merit a theory of fibrations. However, given the difficulties we currently face comparing different models and the fact that the theory of fibrations has not been developed for any of the models, transferring results from one model to another (the way we could for  $(\infty, 1)$ -categories) is currently not possible. It is thus imperative to study fibration of  *$n$ -fold complete Segal spaces* in their own right.

The fact that  *$n$ -fold complete Segal spaces* are a direct generalization of complete Segal spaces thus means that an important first step towards realizing this goal is to study left fibrations of complete Segal spaces.

In all three examples, what is important is not just to know that a certain result holds, but rather how to prove the desired results. For example, the study of fibrations of  *$n$ -fold complete Segal spaces* is expected to be a direct generalization of the results for simplicial spaces proven here.

### 0.3 Relation to Other Work

The idea of working on a Yoneda lemma for simplicial spaces was suggested to me by my advisor Charles Rezk in 2015, as at that time the literature had mostly focused on the study of quasi-categories. Since then however, significant progress has been made:

- De Brito developed left fibrations of Segal spaces in [19] and so several result proven here (independently) already appear in [19], which have been pointed out when appropriate.

- Building on the ideas of this paper, several subsequent papers have been written studying (Cartesian) fibrations of complete Segal spaces by the author [52, 53, 55].
- The key ideas of this paper have also been generalized in [54] to the setting of  $(\infty, n)$ -categories in the particular model of  $n$ -fold complete Segal spaces (and various other models of  $(\infty, n)$ -categories introduced by Bergner and Rezk [9]).
- Independently Nuiten [50] has also studied fibrations of  $n$ -fold complete Segal spaces, which imply certain results proven here, when restricting to  $n = 1$ .

The main benefit of this work consists in uniting different results that have already appeared before, as well as generalizing them to arbitrary simplicial spaces. This in particular includes, but is not limited to, the recognition principle for covariant equivalences (Theorem 4.41) and the invariance of the covariant model structure under CSS equivalences (Theorem 5.1).

### 0.4 Main Results

The paper focuses on the study of *left fibrations*. A left fibration is a Reedy fibration of simplicial spaces  $p : Y \rightarrow X$  such that for all  $n \geq 0$  the commutative square

$$\begin{array}{ccc}
 Y_n & \xrightarrow{p_n} & X_n \\
 \downarrow \langle 0 \rangle^* & & \downarrow \langle 0 \rangle^* \\
 Y_0 & \xrightarrow{p_0} & X_0
 \end{array}$$

is a homotopy pullback square of spaces (Definition 3.2). It generalizes the unique lifting condition of a discrete Grothendieck opfibration for categories (Definition 1.15). It is well-established that discrete Grothendieck opfibrations model covariant functors valued in sets (Proposition 1.10) and by analogy we think of left fibrations as a model for covariant functors valued in spaces, which guides our work throughout this paper.<sup>2</sup>

Unlike Grothendieck opfibrations, the study of left fibrations requires  $(\infty, 1)$ -categorical techniques, which is why we will use the theory of model categories (Appendix A). Concretely, we will show that for each simplicial space  $X$  there is a unique simplicial model structure on the category of simplicial spaces over  $X$ , denoted  $sS_{/X}$  and called the *covariant model structure*, such that the cofibrations are exactly monomorphisms and fibrant objects are precisely the left fibrations (Theorem 3.12).

One important model of an  $(\infty, 1)$ -category is a *complete Segal space*, which is a simplicial space that satisfies the certain lifting conditions that endows it with the structure of a homotopy-coherent generalization of a category (Definition 2.21). As a first step, we thus expect that left fibrations over complete Segal spaces share some of the attributes of discrete Grothendieck opfibrations over categories. We will in fact take a more general step and study left fibrations over *Segal spaces* (Definition 2.15), getting the following generalization of the Yoneda lemma (here  $F(n)$  is the representable simplicial discrete space Sect. 2.3(5)).

**Theorem 3.49** *Let  $W$  be a Segal space and  $x : F(0) \rightarrow W$  be an object. Let*

$$W_{x/} = W^{F(1)^s} \times_W^{\{x\}} F(0) \xrightarrow{t} W$$

*be the under-category projection, which is a left fibration (Theorem 3.44). Then the map*

<sup>2</sup> Left fibrations are a special case of coCartesian fibrations, which themselves can be thought of as  $(\infty, 1)$ -categorical analogues of general Grothendieck opfibrations.

$$\begin{array}{ccc}
 F(0) & \xrightarrow[\simeq]{\{id_x\}} & W_{x/} \\
 & \searrow \{x\} & \swarrow t \\
 & & W
 \end{array}$$

is a covariant equivalence over the Segal space  $W$  and so in particular for every left fibration  $L \rightarrow W$  the induced map

$$\{id_x\}^* : \text{Map}_{/W}(W_{x/}, L) \rightarrow \text{Map}_{/W}(F(0), L)$$

is a Kan equivalence (Corollary 3.50).

If the Segal space is the nerve of a category  $\mathcal{C}$ , then we can give a very precise relationship between left fibrations over  $N^h \mathcal{C}$  (Notation 2.23) and functors out of  $\mathcal{C}$  valued in spaces.

**Theorem 4.18** *Let  $\mathcal{C}$  be a small category. The two simplicially enriched adjunctions*

$$\text{Fun}(\mathcal{C}, \mathcal{S})^{proj} \begin{array}{c} \xrightarrow{s\mathcal{F}_{\mathcal{C}}} \\ \perp \\ \xleftarrow{s\mathcal{H}_{\mathcal{C}}} \end{array} (s\mathcal{S}_{/N^h \mathcal{C}})^{cov} \begin{array}{c} \xleftarrow{s\mathbb{T}_{\mathcal{C}}} \\ \perp \\ \xrightarrow{s\mathbb{I}_{\mathcal{C}}} \end{array} \text{Fun}(\mathcal{C}, \mathcal{S})^{proj}$$

are Quillen equivalences, which are (up to equivalence) natural in  $\mathcal{C}$ . Moreover, the derived counit map  $s\mathcal{F}_{\mathcal{C}} Qs\mathcal{H}_{\mathcal{C}} L \rightarrow L$  is in fact a Reedy equivalence. Here  $\text{Fun}(\mathcal{C}, \mathcal{S})$  has the projective model structure and  $s\mathcal{S}_{/N^h \mathcal{C}}$  has the covariant model structure over  $N^h \mathcal{C}$ .

The result in particular implies (Corollary 4.22) that a morphism of simplicial spaces  $X \rightarrow Y$  over  $N^h \mathcal{C}$  is a covariant equivalence if and only if for all objects  $c$  in  $\mathcal{C}$  the morphism

$$N^h \mathcal{C}_{/c} \times_{N^h \mathcal{C}} X \rightarrow N^h \mathcal{C}_{/c} \times_{N^h \mathcal{C}} Y$$

is a diagonal equivalence. One new and important aspect of this work is that we generalize this last result from nerves of categories to arbitrary simplicial spaces. The key input is the following natural zig-zag of equivalences.

**Theorem 4.39** *Let  $p : Y \rightarrow X$  be a map of simplicial spaces. For every  $\{x\} : F(0) \rightarrow X$ , there is a natural zig-zag of diagonal equivalences (Theorem 2.11)*

$$R_x \times_X Y \xrightarrow{\simeq} R_x \times_X \hat{Y} \xleftarrow{\simeq} F(0) \times_X \hat{Y}.$$

Here  $i : Y \rightarrow \hat{Y}$  is a choice of a left fibrant replacement of  $Y$  over  $X$  and  $R_x \rightarrow X$  is a contravariant fibrant replacement of  $\{x\} : F(0) \rightarrow X$  (Remark 4.25).

Building on this zig-zag we can now establish the recognition principle, classifying general covariant equivalences.

**Theorem 4.41 (Recognition principle)** *For every morphism  $\{x\} : F(0) \rightarrow X$  fix a contravariant fibrant replacement  $R_x \rightarrow X$ . Let  $g : Y \rightarrow Z$  be a morphism over  $X$ . Then  $g : Y \rightarrow Z$  over  $X$  is a covariant equivalence over  $X$  if and only if for every  $\{x\} : F(0) \rightarrow X$*

$$R_x \times_X Y \rightarrow R_x \times_X Z$$

is a diagonal equivalence.

The second is the *invariance property* of the covariant model structure.

**Theorem 5.1** (*Invariance property*) *Let  $f : X \rightarrow Y$  be a CSS equivalence (Theorem 2.22). Then the adjunction*

$$(s\mathcal{S}_{/X})^{cov} \begin{array}{c} \xrightarrow{f_!} \\ \xleftarrow{f^*} \end{array} (s\mathcal{S}_{/Y})^{cov}$$

*is a Quillen equivalence. Here both sides have the covariant model structure.*

Using the invariance property, we can finally establish the *Yoneda lemma for simplicial spaces* (Corollary 5.10), generalizing the Yoneda lemma for Segal spaces (Corollary 3.50). For a simplicial space  $X$  fix a CSS fibrant replacement  $i : X \hookrightarrow \hat{X}$ . Then, for any point  $x : F(0) \rightarrow X$  and left fibration  $L \rightarrow X$ , there is an equivalence of Kan complexes

$$\{s_0(x)\}^* : \text{Map}_{/X}(X_{x/}, L) \xrightarrow{\cong} \text{Map}_{/X}(F(0), L),$$

where  $X_{x/} = X \times_{\hat{X}} \hat{X}^{F(1)} \times_{\hat{X}} F(0)$ . Finally, relying on these results we can establish the following further facts.

**Theorem 5.11** *The covariant model structure is a localization of the CSS model structure on  $s\mathcal{S}_{/X}$ .*

**Theorem 5.15** *Base change by left fibrations preserves CSS equivalences.*

### 0.5 Outline

In Sect. 1 we review the classical Yoneda lemma in Sect. 1.1, the Grothendieck construction in Sect. 1.2 and the fibrational Yoneda lemma for categories in Sect. 1.3 with an eye towards a generalization to simplicial spaces.

Section 2 is a review of necessary background concepts: Joyal–Tierney calculus (Sect. 2.1), spaces (Sect. 2.2), simplicial spaces (Sect. 2.3), the Reedy model structure (Sect. 2.4) and complete Segal spaces (Sect. 2.6).

In Sect. 3 we begin the study of left fibrations. In Sect. 3.1 we introduce left fibrations and give various alternative characterizations. We then move on in Sect. 3.2 to define a model structure for left fibrations, the *covariant model structure*, over arbitrary simplicial spaces (Theorem 3.12). Finally, in Sect. 3.3 we study left fibrations over Segal spaces and in particular prove the *Yoneda lemma for Segal spaces* (Theorem 3.49).

In the next section, Sect. 4, we first take a technical digression in Sect. 4.1 and focus on the covariant model structure over nerves of categories and in particular prove the *Grothendieck construction* in Theorem 4.18. We then use these new technical results in Sect. 4.2 to prove the *recognition principle for covariant equivalences* (Theorem 4.41).

In the final section, Sect. 5, we study the relation between left fibrations and complete Segal spaces. In particular, in Sect. 5.1 we prove the *invariance of the covariant model structure* (Theorem 5.1) and several important implications. Finally, in Sect. 5.2 we apply these result to the study of colimits in Segal spaces.

There are two appendices. In Appendix A we review some key lemmas about model categories. In Appendix B we prove that the covariant model structure for simplicial spaces is Quillen equivalent to the covariant model structure for simplicial sets studied in [42].

### 0.6 Background

The main language here is the language of model categories and complete Segal spaces. So, we assume familiarity with both throughout. Only a few results are explicitly stated here. For a basic introduction to the theory of model categories see [21, 34, 35]. For an introduction to complete Segal spaces see the original source [58].

### 0.7 Notation

We mostly follow the notation as introduced in [58] and will be reviewed in Sect. 2. We use categories with different enrichments and use the following notation to distinguish between them. Fix a category  $\mathcal{C}$  and two objects  $x, y$ .

- We denote the set of maps between them by  $\text{Hom}_{\mathcal{C}}(x, y)$ . For a further object  $z$  and maps  $g : y \rightarrow x, f : z \rightarrow x$ , we will denote the set of maps  $\text{Hom}_{\mathcal{C}/x}(y, z)$  by  $\text{Hom}_{/x}(y, z)$ .
- There is one exception to the previous rule. If  $\mathcal{C}$  is a category of functors, then we denote the set of natural transformations from  $F$  to  $G$  by  $\text{Nat}(F, G)$ , following conventional notation.
- If  $\mathcal{C}$  is enriched over the category of simplicial sets, we denote the mapping simplicial set by  $\text{Map}_{\mathcal{C}}(x, y)$  or, if  $\mathcal{C}$  is clear from the context, by  $\text{Map}(x, y)$ . Similar to the last one we will, instead of  $\text{Map}_{\mathcal{C}/x}(y, z)$ , use  $\text{Map}_{/x}(y, z)$ .
- If  $\mathcal{C}$  is Cartesian closed, we denote the internal mapping object by  $y^x$ .
- There is one exception to the previous rule. For two categories  $\mathcal{C}, \mathcal{D}$ , we denote the category of functors by  $\text{Fun}(\mathcal{C}, \mathcal{D})$ , following conventional notation.
- If  $W$  is a Segal space, then for two objects  $x, y$  in  $W$  there is a mapping space, which we denote by  $\text{map}_W(x, y)$  (Definition 2.17).

For a functor between small categories  $F : \mathcal{C} \rightarrow \mathcal{D}$  and a bicomplete category  $\mathcal{E}$  we use the following notation for the induced functors at the level of functor categories:

$$\begin{array}{ccc}
 & \xrightarrow{F_!} & \\
 \text{Fun}(\mathcal{C}, \mathcal{E}) & \xleftarrow{F^*} & \text{Fun}(\mathcal{D}, \mathcal{E}) \\
 & \xrightarrow{F_*} & 
 \end{array}$$

Here  $F^*$  is defined by precomposition,  $F_!$  is the left Kan extension and  $F_*$  the right Kan extension.

Similarly, for a given morphism  $f : c \rightarrow d$  in a locally Cartesian closed category  $\mathcal{C}$  with small limits and colimits we denote the adjunctions

$$\begin{array}{ccc}
 & \xrightarrow{f_!} & \\
 \mathcal{C}/c & \xleftarrow{f^*} & \mathcal{C}/d \\
 & \xrightarrow{f_*} & 
 \end{array}$$

where  $f_!$  is the postcomposition functor,  $f^*$  the pullback functor and  $f_*$  is the right adjoint to  $f^*$ .

Finally, let  $\mathcal{C}$  be a category with final object  $1$ . Then for a given morphism  $\{y\} : 1 \rightarrow Y$ , we use the notation  $\{y\} : X \rightarrow Y$  for the unique map that factors through the map  $\{y\} : 1 \rightarrow Y$  that picks out the element  $y$  in  $Y$ .



### 1 Another Look at the Yoneda Lemma for Classical Categories

The Yoneda lemma is an important result in classical category theory and is thus well known among practitioners of category theory. A lesser known aspect of the Yoneda lemma is that it can be expressed in several different ways. Concretely we want to review four different faces of the Yoneda lemma, which are summarized in this table:

	<i>Hom</i>	<i>Tensor</i>
<i>Functor</i>	$\text{Nat}(\text{Hom}_{\mathcal{C}}(c, -), F) \xrightarrow{\cong} F(c)$ Lemma 1.1	$\text{Hom}_{\mathcal{C}}(c, -) \otimes_{\mathcal{C}} F \xrightarrow{\cong} F(c)$ Lemma 1.4
<i>Fibration</i>	$\text{Hom}_{/\mathcal{C}}(\mathcal{C}/c, \mathcal{D}) \xrightarrow{\cong} \{c\} \times_{\mathcal{C}} \mathcal{D}$ Lemma 1.21	$\mathcal{C}/c \otimes_{\mathcal{C}} \mathcal{D} \xrightarrow{\cong} \{c\} \otimes_{\mathcal{C}} \mathcal{D}$ Lemma 1.22

Let us start with the most common form of the Yoneda lemma, which can be found in any introductory book on classical category theory. Here is a version that appears in [45, Page 61].

**Lemma 1.1** (Hom version of Yoneda for functors) *If  $F : \mathcal{C} \rightarrow \text{Set}$  is a functor and  $c \in \mathcal{C}$  an object, then the natural map*

$$\begin{aligned} \text{Nat}(\text{Hom}_{\mathcal{C}}(c, -), F) &\xrightarrow{\cong} F(c) \\ [\alpha : \text{Hom}_{\mathcal{C}}(c, -) \rightarrow F] &\longmapsto \alpha_c(\text{id}_c) \end{aligned}$$

is a bijection.

There is, however, a different way this equivalence can be phrased. It relies on the *Hom-Tensor Adjunction*.

#### 1.1 Tensor Product of Functors and Yoneda Lemma

Most of the material in this subsection can be found in greater detail in [44, VII.2]. For this subsection let  $\mathcal{C}$  be a fixed category and  $F : \mathcal{C} \rightarrow \text{Set}$  and  $P : \mathcal{C}^{op} \rightarrow \text{Set}$  be two functors. Then we define the tensor product as the following colimit diagram  $F \otimes_{\mathcal{C}} P$

$$\coprod_{c, c' \in \mathcal{C}} P(c) \times \text{Hom}_{\mathcal{C}}(c, c') \times F(c) \begin{array}{c} \xrightarrow{\varphi} \\ \xrightarrow{\psi} \end{array} \coprod_{c \in \mathcal{C}} P(c) \times F(c) \longrightarrow F \otimes_{\mathcal{C}} P,$$

where  $\varphi(a, f, b) = (P(f)(a), b)$  and  $\psi(a, f, b) = (a, F(f)(b))$ . So the tensor product of two functors is the product of the values quotiented out by the mapping relations. This definition generalizes the tensor product of a right and left module over a ring, which is the motivation for this notation. Similar to the case of rings this definition of a tensor product fits into a *hom-tensor adjunction*.

**Theorem 1.2** *Let  $\mathcal{C}$  be a category and  $F : \mathcal{C} \rightarrow \text{Set}$  a functor. Then we have the adjunction*

$$\text{Fun}(\mathcal{C}^{op}, \text{Set}) \begin{array}{c} \xleftarrow{- \otimes_{\mathcal{C}} F} \\ \xleftarrow{\text{Hom}_{\text{Set}}(F(-), -)} \end{array} \text{Set}$$

where the left adjoint takes  $P$  to  $P \otimes_{\mathcal{C}} F$  and the right adjoint takes a set  $S$  to the functor which takes an object  $C$  to  $\text{Hom}_{\text{Set}}(F(c), S)$ .

**Remark 1.3** Note that we could have made the same construction for the case where  $\text{Set}$  is replaced with any category which has all colimits. However, here we do not need to work at this level of generality. For more details on the general construction see [44, Page 358].

With the tensor product at hand we can state another version of the Yoneda lemma.

**Lemma 1.4** (Tensor version of Yoneda for functors) *If  $P : \mathcal{C}^{op} \rightarrow \text{Set}$  is a functor and  $C \in \mathcal{C}$  an object, then the natural map*

$$\begin{aligned} \text{Nat}(\text{Hom}_{\mathcal{C}}(c, -), F) &\xrightarrow{\cong} F(C) \\ [\alpha : \text{Hom}_{\mathcal{C}}(c, -) \rightarrow F] &\longmapsto \alpha_c(\text{id}_c) \end{aligned}$$

is a bijection.

This version of the Yoneda lemma has the following basic corollaries, which should look quite familiar.

**Corollary 1.5** *Let  $\mathcal{C}$  be a category and  $C, C'$  two objects. Then we have the following isomorphism.*

$$\text{Hom}_{\mathcal{C}}(c, -) \otimes_{\mathcal{C}} \text{Hom}_{\mathcal{C}}(-, c') \cong \text{Hom}_{\mathcal{C}}(c, c')$$

**Corollary 1.6** *Let  $\mathcal{C}$  be a category and  $P, Q : \mathcal{C}^{op} \rightarrow \text{Set}$  be two functors. Then a natural transformation  $\alpha : P \rightarrow Q$  is a natural isomorphism if and only if*

$$\text{Hom}_{\mathcal{C}}(c, -) \otimes_{\mathcal{C}} P \rightarrow \text{Hom}_{\mathcal{C}}(c, -) \otimes_{\mathcal{C}} Q$$

is a bijection for every object  $C \in \mathcal{C}$ .

### 1.2 From Functors to Fibrations: The Grothendieck Construction

We want to now translate the Yoneda lemma from a statement about set-valued functors to a statement about fibrations. This requires us to translate between functors and fibrations which we will do via the Grothendieck construction.

**Remark 1.7** We will need a careful understanding of the Grothendieck construction in the coming sections. Thus the review in this section is self-contained. However, the ideas are in no way new and a more detailed approach can be found in many places, such as [44, I.5], or [36, A1.1.7, B1.3.1].

**Definition 1.8** Let  $\mathcal{C}$  be a category. Define

$$\int_{\mathcal{C}} : \text{Fun}(\mathcal{C}, \text{Set}) \rightarrow \text{Cat}_{/\mathcal{C}}$$

as the functor that takes  $F : \mathcal{C} \rightarrow \text{Set}$  to the category  $\int_{\mathcal{C}} F \rightarrow \mathcal{C}$  with

- Objects: Pairs  $(c, x)$  where  $c$  is an object in  $\mathcal{C}$  and  $x \in F(c)$ .

- **Morphisms:** For two objects  $(c, x), (d, y)$  we have

$$\text{Hom}_{\int_{\mathcal{C}} F}((c, x), (d, y)) = \{f \in \text{Hom}_{\mathcal{C}}(c, d) : F(f)(x) = y\}.$$

It comes with an evident projection map  $\pi_F : \int_{\mathcal{C}} F \rightarrow \mathcal{C}$ . Moreover, for a natural transformation  $\alpha : F \Rightarrow G$ , the functor  $\int_{\mathcal{C}} \alpha : \int_{\mathcal{C}} F \rightarrow \int_{\mathcal{C}} G$  is given as  $(c, x) \mapsto (c, \alpha_c x)$ .

Notice, by construction the fiber of  $\pi_F : \int_{\mathcal{C}} F \rightarrow \mathcal{C}$  over an object  $c$  is the discrete category with object set  $F(c)$ . This functor has a left adjoint and a right adjoint that we want to define in detail.

**Definition 1.9** Let  $\mathcal{C}$  be a category. We define the functor

$$\mathcal{C}_{/-} : \mathcal{C} \rightarrow \text{Cat}_{/\mathcal{C}}$$

that takes an object to the over-category  $\mathcal{C}_{/c}$  and a morphism  $f : c \rightarrow d$  to the post-composition  $f_! : \mathcal{C}_{/c} \rightarrow \mathcal{C}_{/d}$ .

Similarly, define the functor

$$\mathcal{C}_{-/} : \mathcal{C}^{op} \rightarrow \text{Cat}_{/\mathcal{C}}$$

that takes an object to the under-category  $\mathcal{C}_{c/}$  and a morphism  $f : c \rightarrow d$  to the precomposition  $f^* : \mathcal{C}_{d/} \rightarrow \mathcal{C}_{c/}$ .

For a given category over  $\mathcal{C}$ ,  $p : \mathcal{D} \rightarrow \mathcal{C}$  define

$$\mathcal{T}_{\mathcal{C}}(p : \mathcal{D} \rightarrow \mathcal{C}) : \mathcal{C} \rightarrow \text{Set}$$

as the composition

$$\mathcal{C} \xrightarrow{\mathcal{C}_{/-}} \text{Cat}_{/\mathcal{C}} \xrightarrow{- \times_{\mathcal{C}} \mathcal{D}} \text{Cat} \xrightarrow{\pi_0} \text{Set}$$

in other words we have  $\mathcal{T}_{\mathcal{C}}(c) = \pi_0(\mathcal{C}_{/c} \times_{\mathcal{C}} \mathcal{D})$ . Similarly, define

$$\mathcal{H}_{\mathcal{C}}(p : \mathcal{D} \rightarrow \mathcal{C}) : \mathcal{C} \rightarrow \text{Set}$$

as the composition

$$\mathcal{C} \xrightarrow{(\mathcal{C}_{-/})^{op}} (\text{Cat}_{/\mathcal{C}})^{op} \xrightarrow{\text{Hom}_{/\mathcal{C}}(-, \mathcal{D})} \text{Set}$$

meaning we have  $\mathcal{H}_{\mathcal{C}}(c) = \text{Hom}_{/\mathcal{C}}(\mathcal{C}_{c/}, \mathcal{D})$ . We claim that  $\mathcal{T}_{\mathcal{C}}$  is the left adjoint and  $\mathcal{H}_{\mathcal{C}}$  is the right adjoint to  $\int_{\mathcal{C}}$ .

**Proposition 1.10** We have the following diagram of adjunctions

$$\begin{array}{ccc} & \mathcal{T}_{\mathcal{C}} & \\ & \perp & \\ \text{Fun}(\mathcal{C}, \text{Set}) & \xrightarrow{f_{\mathcal{C}}} & \text{Cat}_{/\mathcal{C}} \\ & \perp & \\ & \mathcal{H}_{\mathcal{C}} & \end{array}$$

**Proof** We first prove the right adjoint. First of all notice  $\int_{\mathcal{C}}$  commutes with colimits. Indeed, for a given diagram  $G : I \rightarrow \text{Fun}(\mathcal{C}, \text{Set})$  it is direct computation that the induced cocone  $\int_{\mathcal{C}} G_i \rightarrow \int_{\mathcal{C}} \text{colim}_I G_i$  satisfies the universal property of the universal cocone. Now, by [44, Corollary I.5.4], every colimit preserving functor out of  $\text{Fun}(\mathcal{C}, \text{Set})$  is the left Kan extension of its restriction to  $\mathcal{C}^{op} \rightarrow \text{Fun}(\mathcal{C}, \text{Set})$ , meaning we have the following left Kan extension

$$\begin{array}{ccc}
 \mathcal{C}^{op} & \xrightarrow{\mathcal{C}_{-/}} & \mathcal{C}at_{/\mathcal{C}} \\
 \text{Yon} \downarrow & \nearrow f_{\mathcal{C}} & \\
 \text{Fun}(\mathcal{C}, \text{Set}) & & 
 \end{array}$$

which means it has a right adjoint given by  $\mathcal{H}_{\mathcal{C}}(p : \mathcal{D} \rightarrow \mathcal{C}) = \text{Hom}_{\mathcal{C}}(\mathcal{C}_{-/}, \mathcal{D})$ .

For the left adjoint, we first show that  $\mathcal{T}_{\mathcal{C}}$  commutes with colimits. As colimits are computed point-wise this means we have to prove that for every object  $c$  the functor

$$\pi_0(\mathcal{C}_{/c} \times_{\mathcal{C}} -) : \mathcal{C}at_{/\mathcal{C}} \rightarrow \text{Set}$$

commutes with colimits. This functor is a composition and so we check separately that both are left adjoints:

- (1)  $\mathcal{C}_{/c} \times_{\mathcal{C}} -$  is a left adjoint because  $\mathcal{C}_{/c} \rightarrow \mathcal{C}$  is a Conduché functor [17].
- (2)  $\pi_0$  is the left adjoint of the inclusion functor  $\text{Set} \rightarrow \text{Cat}$ .

Now, we prove that  $\mathcal{T}_{\mathcal{C}}$  is the left adjoint. Recall that  $[n]$  is the category given via the poset structure  $\{0 \leq 1 \leq \dots \leq n\}$  and that every category over  $\mathcal{C}$  is a colimit of functors  $\alpha : [n] \rightarrow \mathcal{C}$  and so the result follows from the following natural isomorphisms:

$$\begin{array}{c}
 \text{Hom}_{/\mathcal{C}}(\alpha : [n] \rightarrow \mathcal{C}, f_{\mathcal{C}} G) \xrightarrow{\cong} \text{Hom}_{/\mathcal{C}}(\alpha \circ \{0\} : [0] \rightarrow \mathcal{C}, f_{\mathcal{C}} G) \xrightarrow{\cong} G(\alpha(0)) \\
 \left. \begin{array}{c} \xrightarrow{\text{Yon}_{\alpha(0)}} \\ \xrightarrow{\cong} \end{array} \right\} \\
 \text{Nat}(\text{Hom}(\alpha(0), -), G) \xrightarrow{\cong} \text{Nat}(\mathcal{T}_{\mathcal{C}}(\alpha : [0] \rightarrow \mathcal{C}), G) \xrightarrow{\cong} \text{Nat}(\mathcal{T}_{\mathcal{C}}(\alpha : [n] \rightarrow \mathcal{C}), G)
 \end{array}$$

Let us explain the various natural isomorphisms that require a justification.

Notice a morphism in  $\int_{\mathcal{C}} G$  is of the form  $f : (c, x) \rightarrow (d, G(f)(x))$ , where  $f : c \rightarrow d$  is a morphism in  $\mathcal{C}$ . Hence, for a given functor  $\alpha : [n] \rightarrow \mathcal{C}$ , which we can depict by a chain of  $n$  morphisms in  $\mathcal{C}$ ,  $c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} c_n$ , a functor  $[n] \rightarrow \int_{\mathcal{C}} G$  over  $\mathcal{C}$  that lifts  $\alpha$  is of the form  $(c_0, x) \xrightarrow{f_0} (c_1, G(f_0)(x)) \xrightarrow{f_1} \dots \xrightarrow{f_n} (c_1, G(f_n \circ \dots \circ f_1 \circ f_0)(x))$ , meaning it is uniquely determined by the value of the object 0 in  $[n]$ .

The third isomorphism is the Yoneda lemma. The fourth isomorphism follows from the fact that  $\mathcal{T}_{\mathcal{C}}(\alpha : [0] \rightarrow \mathcal{C}) = \pi_0(\mathcal{C}_{/c} \times_{\mathcal{C}} [0]) \cong \pi_0(\text{Hom}(\alpha(0), c)) = \text{Hom}(\alpha(0), c)$ . Finally, for the last isomorphism we observe that two objects of the form  $\alpha(0) \rightarrow \alpha(1) \rightarrow \dots \rightarrow \alpha(n) \rightarrow c$  in  $\mathcal{C}_{/c} \times_{\mathcal{C}} [n]$  are in the same path-component if they compose to the same morphism  $\alpha(0) \rightarrow c$  giving us the desired bijection  $\mathcal{T}_{\mathcal{C}}(\alpha \circ \{0\}^* : [0] \rightarrow \mathcal{C}) \cong \mathcal{T}_{\mathcal{C}}(\alpha : [n] \rightarrow \mathcal{C}) = \pi_0(\mathcal{C}_{/c} \times_{\mathcal{C}} [n])$ . □

In fact  $\int_{\mathcal{C}}$  has even more desirable properties.

**Lemma 1.11**  $\int_{\mathcal{C}} : \text{Fun}(\mathcal{C}, \text{Set}) \rightarrow \mathcal{C}at_{/\mathcal{C}}$  is fully faithful.

**Proof** Let  $F, G : \mathcal{C} \rightarrow \text{Set}$  be two functors. We need to prove that the map

$$\text{Nat}(F, G) \rightarrow \text{Hom}_{/\mathcal{C}}\left(\int_{\mathcal{C}} F, \int_{\mathcal{C}} G\right)$$

is a bijection of sets. For that we will construct an inverse. Concretely, we define

$$\mathcal{T}_{\mathcal{C}} : \text{Hom}_{/\mathcal{C}}\left(\int_{\mathcal{C}} F, \int_{\mathcal{C}} G\right) \rightarrow \text{Nat}(F, G)$$

as follows. For a given functor  $H : \int_{\mathcal{C}} F \rightarrow \int_{\mathcal{C}} G$  over  $\mathcal{C}$  we define the natural transformation  $\mathcal{J}_{\mathcal{C}}(H)_c(x) = H(c, x)$ . The functoriality of  $H$  implies that  $\mathcal{J}(H)$  is natural.

It remains to show these are inverses. For a given natural transformation  $\alpha : F \Rightarrow G$  we have

$$(\mathcal{J}_{\mathcal{C}} \int_{\mathcal{C}} (\alpha))_c(x) = \int_{\mathcal{C}} (\alpha)(c, x) = \alpha_c(x)$$

and on the other side

$$\int_{\mathcal{C}} (\mathcal{J}_{\mathcal{C}}(H))(c, x) = (\mathcal{J}_{\mathcal{C}}(H))_c(x) = H(c, x)$$

finishing the proof. □

This has a direct implication for  $\mathcal{T}_{\mathcal{C}}$  and  $\mathcal{H}_{\mathcal{C}}$ .

**Corollary 1.12**  $\mathcal{T}_{\mathcal{C}}$  is a localization functor and  $\mathcal{H}_{\mathcal{C}}$  is a colocalization functor.

We end this subsection by observing that while  $\int_{\mathcal{C}}$  is fully faithful, it is in fact not essentially surjective.

**Definition 1.13** A functor  $p : \mathcal{D} \rightarrow \mathcal{C}$  is *conservative* if it reflects isomorphisms.

**Lemma 1.14** Let  $F : \mathcal{C} \rightarrow \text{Set}$  be a functor. Then  $\pi_F : \int_{\mathcal{C}} F \rightarrow \mathcal{C}$  is conservative.

**Proof** Let  $f : (c, x) \rightarrow (d, y)$  be a morphism in  $\int_{\mathcal{C}} F$  such that the underlying morphism  $\pi_F(f) : c \rightarrow d$  is an isomorphism. We need to show that  $f$  in  $\int_{\mathcal{C}} F$  is an isomorphism and we will do so by providing an inverse. Let  $f^{-1} : d \rightarrow c$  in  $\mathcal{C}$  be the inverse of  $\pi_F(f)$  in  $\mathcal{C}$ . The inverse is now given by the lift  $f^{-1} : (d, y) \rightarrow (c, x)$ . □

Thus we need to restrict our attention to the essential image of  $\int_{\mathcal{C}}$ , which leads us to discrete Grothendieck fibrations.

### 1.3 Yoneda Lemma for Grothendieck Fibrations

In this subsection we want to use the fully faithful functor  $\int_{\mathcal{C}}$  to translate both versions of the Yoneda lemma from a functorial statement to a fibrational one.

This might not be a major improvement when studying 1-categories, however, in the world of higher categories functors can be difficult to study, because of the homotopy coherence. On the other hand, fibrations can be defined and studied in a straightforward manner. Thus a fibrational approach to the Yoneda lemma is an excellent first step for a generalization to a Yoneda lemma for simplicial spaces.

As we observed in Lemma 1.11,  $\int_{\mathcal{C}}$  is fully faithful, however, it is not essentially surjective!

**Definition 1.15** A functor  $P : \mathcal{D} \rightarrow \mathcal{C}$  is a *discrete Grothendieck opfibration* over  $\mathcal{C}$  if it is in the essential image of  $\int_{\mathcal{C}}$ , meaning there exists a functor  $F : \mathcal{C} \rightarrow \text{Set}$  and isomorphism  $\int_{\mathcal{C}} F \xrightarrow{\cong} \mathcal{D}$  over  $\mathcal{C}$ .

Fortunately, there is also an internal characterization of discrete Grothendieck opfibrations.

**Lemma 1.16** A functor  $P : \mathcal{D} \rightarrow \mathcal{C}$  is a discrete Grothendieck opfibration over  $\mathcal{C}$  if and only if for any map  $f : c \rightarrow c'$  in  $\mathcal{C}$  and object  $d$  in  $\mathcal{D}$  such that  $P(d) = c$ , there exists a unique lift  $\hat{f} : d \rightarrow d'$  such that  $P(\hat{f}) = f$ .

**Proof** Let  $F : \mathcal{C} \rightarrow \text{Set}$  be a functor. Then  $\int_{\mathcal{C}} F \rightarrow \mathcal{C}$  satisfies the lifting condition stated in the lemma. Indeed, for a morphism  $f : c \rightarrow c'$  and a lift  $(c, x)$  where  $x \in F(c)$ , there is a unique lift given by the morphism  $f : (c, x) \rightarrow (c', F(f)(x))$ .

On the other hand let us assume that  $P : \mathcal{D} \rightarrow \mathcal{C}$  satisfies the lifting condition of the lemma. Then we will construct a functor  $F : \mathcal{C} \rightarrow \text{Set}$  such that  $\int_{\mathcal{C}} F \cong \mathcal{D}$  over  $\mathcal{C}$ .

First note that the unique lifting condition implies that the fiber of  $P$  over every given point  $c, P^{-1}(c)$ , is a discrete category i.e. a set. Indeed let  $f$  be a morphism in  $\mathcal{D}$  such that  $P(f) = \text{id}_c$ . Then by the uniqueness assumption  $f = \text{id}$ .

Now define  $F$  as follows:

- *Objects:* For an object  $c$  in  $\mathcal{C}$  define  $F(c) = P^{-1}(c)$
- *Morphisms:* For a morphism  $f : c \rightarrow c'$  define  $F(f) : F(c) \rightarrow F(c')$  as the map that takes  $x \in F(c)$  to the target of the unique lift of  $f$  in  $\mathcal{D}$ .

The standard projection  $\pi : \int_{\mathcal{C}} F \rightarrow \mathcal{C}$  exactly recovers  $P : \mathcal{D} \rightarrow \mathcal{C}$ . □

The previous characterization allows us to give a contravariant version of Grothendieck opfibrations.

**Definition 1.17**  $P : \mathcal{D} \rightarrow \mathcal{C}$  is called a *discrete Grothendieck fibration* if for any map  $f : c \rightarrow c'$  in  $\mathcal{C}$  and object  $d'$  in  $\mathcal{D}$  such that  $P(d') = c'$ , there exists a *unique* lift  $\hat{f} : d \rightarrow d'$  such that  $P(\hat{f}) = f$ .

**Remark 1.18** Note that it is very rare that a functor  $p : \mathcal{D} \rightarrow \mathcal{C}$  is a discrete Grothendieck fibration as well as a discrete Grothendieck opfibration. Concretely it only happens if  $\mathcal{D} \cong \int_{\mathcal{C}} F$  over  $\mathcal{C}$  where  $F : \mathcal{C} \rightarrow \text{Set}$  takes every morphism to an isomorphism.

**Remark 1.19** Discrete Grothendieck fibrations are a special case of more general *Grothendieck fibrations* that correspond to functors valued in categories. See [68] for a readable introduction to general Grothendieck fibrations.

We now want to move on to the Yoneda lemma for discrete Grothendieck fibrations, but for that we need the analogue of representable functors.

**Example 1.20** Let us determine the category  $\int_{\mathcal{C}} \text{Hom}(c, -)$ . Its objects are pairs  $(d, f : c \rightarrow d)$  and a morphism  $(d, f : c \rightarrow d) \rightarrow (d', f' : c \rightarrow d')$  is a morphism  $g : d \rightarrow d'$  such that  $gf = f' : c \rightarrow d'$ . Thus we just rediscovered the projection from the under-category  $\mathcal{C}_{c/} \rightarrow \mathcal{C}$ .

With the previous remarks at hand we can now phrase the first fibered version of the Yoneda Lemma.

**Lemma 1.21** (Hom version of Yoneda for fibered categories) *Let  $P : \mathcal{D} \rightarrow \mathcal{C}$  be a discrete Grothendieck opfibration. Then the natural map between the sets of functors*

$$\begin{aligned} \text{Hom}_{/\mathcal{C}}(\mathcal{C}_{c/}, \mathcal{D}) &\xrightarrow{\cong} \text{Hom}_{/\mathcal{C}}(\{c\}, \mathcal{D}) = \{c\} \times_{\mathcal{C}} \mathcal{D} \\ [F : \mathcal{C}_{c/} \rightarrow \mathcal{D}] &\longmapsto F(\text{id}_c) \end{aligned}$$

*is a bijection. Here  $\text{Hom}_{/\mathcal{C}}$  denotes the hom set in the category  $\text{Cat}_{/\mathcal{C}}$  and  $\text{id}_c : c \rightarrow c$  is seen as an object in  $\mathcal{C}_{c/}$ .*

**Proof** Let  $F : \mathcal{C} \rightarrow \text{Set}$  be a functor such that  $\int_{\mathcal{C}} F \cong \mathcal{D}$  over  $\mathcal{C}$ . We now have the following commutative diagram

$$\begin{array}{ccc}
 \text{Hom}_{/\mathcal{C}}(\mathcal{C}_{c/}, \mathcal{D}) & \xrightarrow{\cong} & \text{Hom}_{/\mathcal{C}}(\{\text{id}_c\}, \mathcal{D}) = \{c\} \times_{\mathcal{C}} \mathcal{D} \\
 \uparrow \cong \int_{\mathcal{C}} & & \uparrow \cong \\
 \text{Nat}(\text{Hom}_{\mathcal{C}}(c, -), F) & \xrightarrow{\text{Yon}} & F(c)
 \end{array}$$

Here the left morphism is a bijection as  $\int_{\mathcal{C}}$  is fully faithful and the bottom morphism is a bijection by the Yoneda lemma. □

Similar to the previous part we also have a tensor version of the Yoneda lemma for fibered categories. First, however, we have to define a notion of tensor product for fibered categories. Recall that the tensor product of two functors  $F : \mathcal{C} \rightarrow \text{Set}$ ,  $P : \mathcal{C}^{op} \rightarrow \text{Set}$  is given by a quotient on the set  $\coprod_{c \in \mathcal{C}} F(c) \times P(c)$ , which reminds us of a fibered product. Hence the fibrational analogue of the tensor product of a discrete Grothendieck opfibration  $\mathcal{D} \rightarrow \mathcal{C}$  and discrete Grothendieck fibration  $\mathcal{E} \rightarrow \mathcal{C}$  is given by

$$\mathcal{D} \otimes_{\mathcal{C}} \mathcal{E} = \pi_0(\mathcal{D} \times_{\mathcal{C}} \mathcal{E})$$

where  $\pi_0$  is the set of connected components. With this definition we can state our last version of the Yoneda lemma.

**Lemma 1.22** (Tensor version of Yoneda for fibered categories) *Let  $P : \mathcal{D} \rightarrow \mathcal{C}$  be a discrete Grothendieck opfibration. Then the natural map*

$$\begin{aligned}
 \mathcal{C}_{/c} \otimes_{\mathcal{C}} \mathcal{D} &\xrightarrow{\cong} \{c\} \times_{\mathcal{C}} \mathcal{D} \\
 [(f : c' \rightarrow c, d')] &\longmapsto \text{Codomain}(\hat{f})
 \end{aligned}$$

*is a bijection (here  $\hat{f}$  is the unique lift of  $f$  with domain  $d'$ ).*

**Proof** Notice  $P : \mathcal{D} \rightarrow \mathcal{C}$  is a discrete Grothendieck opfibration and so the fiber  $\{c\} \times_{\mathcal{C}} \mathcal{D}$  is already a discrete category, and so we can directly establish a bijection of sets. For every arbitrary morphism  $(f : c' \rightarrow c, d')$ , the elements  $(f : c' \rightarrow c, d')$  and  $(\text{id}_c : c \rightarrow c, \text{Codomain}(\hat{f}))$  are in the same equivalence class of  $\mathcal{C}_{/c} \otimes_{\mathcal{C}} \mathcal{D}$ . Moreover,  $(\text{id}_c : c \rightarrow c, d)$  and  $(\text{id}_c : c \rightarrow c, d')$  are in the same equivalence class if and only if  $d = d'$ . This proves that the assignment  $(\text{id}_c, d) \mapsto d$  induces the desired bijection. □

Our goal in the coming sections is to build the necessary machinery to generalize these statements to the setting of simplicial spaces. In particular, we will define the correct analogue to discrete Grothendieck opfibrations, study their properties and prove the Yoneda lemma.

Concretely, we have the following generalizations:

Statement	Category	Higher Category
Grothendieck Construction	Proposition 1.10	Theorem 4.18
Fibration	Definition 1.15/Lemma 1.16	Definition 3.2
Conservativity	Lemma 1.14	Lemma 3.36
Yoneda Lemma (Hom)	Lemma 1.21	Corollary 3.50/Corollary 5.10
Yoneda Lemma (Tensor)	Lemma 1.22	Theorem 4.39/Remark 4.40

## 2 Basics and Conventions

In this section we review some basic concepts that we will need in the coming sections. In particular, we review the Joyal–Tierney calculus (Sect. 2.1) [37] as a powerful notational tool. Moreover, we review notation for simplicial sets (Sect. 2.2), simplicial spaces (Sect. 2.3) and its associated Reedy model structure (Sect. 2.4) along with two localizations of the Reedy model structure (Sect. 2.5). Finally, we use complete Segal spaces as our model of higher categories and thus will end the section with a quick review following [58] (Sect. 2.6).

### 2.1 Joyal–Tierney Calculus

As we primarily work with simplicial spaces it is helpful to first set up some notation that will simplify many statements. The notation introduced here is due to Joyal and Tierney [37, Sect. 7].

**Notation 2.1** For this subsection let  $\mathcal{C}$  be a locally Cartesian closed bicomplete category.

**Definition 2.2** Let  $f : A \rightarrow B$  and  $g : C \rightarrow D$  be two maps in  $\mathcal{C}$ . We define the pushout product as the universal map out of the pushout

$$f \square g : A \times D \coprod_{A \times C} B \times C \rightarrow B \times D$$

induced by the commutative square

$$\begin{array}{ccc}
 A \times C & \xrightarrow{\text{id}_A \times g} & A \times D \\
 \downarrow f \times \text{id}_C & \lrcorner & \downarrow \\
 B \times C & \longrightarrow & A \times D \coprod_{A \times C} B \times C \\
 & \searrow \text{id}_B \times g & \downarrow f \times \text{id}_D \\
 & & B \times D
 \end{array}$$

$f \square g$

Moreover, for two sets of maps  $\mathcal{A}$  and  $\mathcal{B}$  we use the notation

$$\mathcal{A} \square \mathcal{B} = \{f \square g : f \in \mathcal{A}, g \in \mathcal{B}\}.$$

**Definition 2.3** For two maps  $f : A \rightarrow B$  and  $p : Y \rightarrow X$  we define the pullback exponential

$$\exp(f, p) : Y^B \rightarrow Y^A \times_{X^A} X^B$$

induced by the commutative square

$$\begin{array}{ccc}
 Y^B & \xrightarrow{Y^f} & Y^A \\
 \downarrow \exp(f, p) & \lrcorner & \downarrow p^A \\
 Y^A \times_{X^A} X^B & \longrightarrow & Y^A \\
 \downarrow p^f & \lrcorner & \downarrow \\
 X^B & \xrightarrow{X^f} & X^A
 \end{array}$$



Moreover, for two sets of maps  $\mathcal{A}$  and  $\mathcal{X}$  we use the notation

$$\text{exp}(\mathcal{A}, \mathcal{X}) = \{\text{exp}(f, p) : f \in \mathcal{A}, p \in \mathcal{X}\}$$

These two functors give us an adjunction of arrow categories:

$$\mathcal{C}^{[1]} \begin{array}{c} \xrightarrow{-\square f} \\ \perp \\ \xleftarrow{\text{exp}(f, -)} \end{array} \mathcal{C}^{[1]} .$$

The key result about these two constructions is that they can help us better understand lifting conditions.

**Notation 2.4** Let  $\mathcal{L}$  and  $\mathcal{R}$  be two sets of morphisms in  $\mathcal{C}$ . If every morphism in  $\mathcal{L}$  has the left lifting property with respect to morphisms in  $\mathcal{R}$  then we use the notation  $\mathcal{L} \pitchfork \mathcal{R}$ .

**Proposition 2.5** ([37, Proposition 7.6]) *Let  $\mathcal{A}, \mathcal{B}$  and  $\mathcal{X}$  be three sets of morphisms in  $\mathcal{C}$ . Then:*

$$\mathcal{A} \square \mathcal{B} \pitchfork \mathcal{X} \Leftrightarrow \mathcal{A} \pitchfork \text{exp}(\mathcal{B}, \mathcal{X}) \Leftrightarrow \mathcal{B} \pitchfork \text{exp}(\mathcal{A}, \mathcal{X}).$$

### 2.2 Simplicial Sets

$\mathcal{S}$  will denote the category of simplicial sets. Following [58, 2.1] we will also use the terminology *spaces*. We will use the following notation with regard to spaces:

- (1)  $\Delta$  is the indexing category with objects posets  $[n] = \{0, 1, \dots, n\}$  and mappings maps of posets.
- (2) We will denote a morphism  $[n] \rightarrow [m]$  by a sequence of numbers  $\langle a_0, \dots, a_n \rangle$ , where  $a_i$  is the image of  $i \in [n]$ .
- (3)  $\Delta[n]$  denotes the simplicial set representing  $[n]$  i.e.  $\Delta[n]_k = \text{Hom}_\Delta([k], [n])$ .
- (4)  $\partial\Delta[n]$  denotes the boundary of  $\Delta[n]$  i.e. the largest sub-simplicial set which does not include  $id_{[n]} : [n] \rightarrow [n]$ . Similarly  $\Delta[n]_l$  denotes the largest simplicial set in  $\Delta[n]$  which does not include the  $l^{\text{th}}$  face.
- (5) For a simplicial set  $S$  we denote the face maps by  $d_i : S_n \rightarrow S_{n-1}$  and the degeneracy maps by  $s_i : S_n \rightarrow S_{n+1}$ .
- (6) Let  $I[l]$  be the category with  $l$  objects and one unique isomorphism between any two objects. Then we denote the nerve of  $I[l]$  as  $J[l]$ . It is a Kan fibrant replacement of  $\Delta[l]$  and comes with an inclusion  $\Delta[l] \hookrightarrow J[l]$ , which is a Kan equivalence.
- (7) We say a space  $K$  is *discrete* if for each  $n$ ,  $K_n = K_0$  and all simplicial operators are identity maps.

### 2.3 Simplicial Spaces

$s\mathcal{S} = \text{Fun}(\Delta^{op}, \mathcal{S})$  denotes the category of simplicial spaces (bisimplicial sets). We have the following basic notations with regard to simplicial spaces:

- (1) We embed the category of spaces inside the category of simplicial spaces as constant simplicial spaces (i.e. the simplicial spaces  $S$  such that  $S_n = S_0$  for all  $n$  and all simplicial operator maps are identities).
- (2) More generally we say a simplicial space is *homotopically constant* if all simplicial operator maps  $X_n \rightarrow X_m$  are weak equivalences (and in particular  $X_n$  are all equivalent to  $X_0$ ).

- (3) On the other hand we say a simplicial space  $X$  is a simplicial discrete space if for all  $n$ , the space  $X_n$  is discrete.
- (4) For a given simplicial space  $X$  we use the notation  $s$ , the *source map*, for  $d_1 : X_1 \rightarrow X_0$  and  $t$ , the *target map*, for  $d_0 : X_1 \rightarrow X_0$ . This is motivated by thinking of a simplicial diagram as a generalization of a directed graph.
- (5) Denote  $F(n)$  to be the simplicial discrete space defined as

$$F(n)_k = \text{Hom}_\Delta([k], [n]).$$

- (6) Similar to Sect. 2.2(3) we denote a morphism  $F(n) \rightarrow F(m)$  by  $\langle a_0, \dots, a_n \rangle$ .
- (7)  $\partial F[n]$  denotes the boundary of  $F(n)$ . Similarly  $L(n)_l$  denotes the largest simplicial space in  $F(n)$  which lacks the  $l^{\text{th}}$  face.
- (8) The category  $s\mathcal{S}$  is enriched over spaces via

$$\text{Map}_{s\mathcal{S}}(X, Y)_n = \text{Hom}_{s\mathcal{S}}(X \times \Delta[n], Y).$$

- (9) The category  $s\mathcal{S}$  is also enriched over itself via

$$(Y^X)_{kn} = \text{Hom}_{s\mathcal{S}}(X \times F(k) \times \Delta[n], Y).$$

- (10) By the Yoneda lemma, for a simplicial space  $X$  we have an isomorphism of spaces

$$X_n \cong \text{Map}_{s\mathcal{S}}(F(n), X).$$

### 2.4 Reedy Model Structure

The category of simplicial spaces has a Reedy model structure [56], which is defined as follows:

- (F) A map  $f : Y \rightarrow X$  is a (trivial) Reedy fibration if for each  $n \geq 0$  the following map of spaces is a (trivial) Kan fibration

$$\text{Map}_{s\mathcal{S}}(F(n), Y) \rightarrow \text{Map}_{s\mathcal{S}}(\partial F(n), Y) \times_{\text{Map}_{s\mathcal{S}}(\partial F(n), X)} \text{Map}_{s\mathcal{S}}(F(n), X).$$

- (W) A map  $f : Y \rightarrow X$  is a Reedy weak equivalence if it is a level-wise Kan weak equivalence.
- (C) A map  $f : Y \rightarrow X$  is a Reedy cofibration if it is an inclusion.

The Reedy model structure is very helpful as it enjoys many features that can help us while doing computations. In particular, it is *cofibrantly generated, simplicial and proper*. Moreover, it is also *compatible with Cartesian closure*, by which we mean that if  $i : A \rightarrow B$  and  $j : C \rightarrow D$  are cofibrations and  $p : X \rightarrow Y$  is a fibration then the map  $i \square j$  is a cofibration and  $\text{exp}(i, p)$  is a fibration, which are trivial if any of the involved maps are trivial.

These properties in particular imply that we can apply Bousfield localizations to the Reedy model structure. See Appendix A for more details.

### 2.5 Diagonal and Kan Model Structure

In the coming sections we will use various localizations of simplicial spaces with the Reedy model structure that are equivalent to the Kan model structure. Although these results have been studied before, we will make ample use of the notation and thus will do a careful review here.

First we need two adjunctions between spaces and simplicial spaces.

**Notation 2.6** Let

$$\text{Diag} : \Delta \rightarrow \Delta \times \Delta$$

be the diagonal functor that takes an object  $[n]$  to the pair  $([n], [n])$  and let

$$\pi_1, \pi_2 : \Delta \times \Delta \rightarrow \Delta$$

be the projection functors that take an object  $([n], [m])$  to the projections  $[n]$  or  $[m]$ , respectively. Finally, let

$$i_1, i_2 : \Delta \rightarrow \Delta \times \Delta$$

be the inclusion functors that take an object  $[n]$  to  $([n], 0)$  or  $(0, [n])$ , respectively.

These functors give us three adjunctions

$$s\mathcal{S} \begin{array}{c} \xrightarrow{\text{Diag}^*} \\ \xleftarrow{\text{Diag}_*} \end{array} \mathcal{S} \begin{array}{c} \xrightarrow{(\pi_1)^*} \\ \xleftarrow{(\pi_1)_*} \end{array} s\mathcal{S} \begin{array}{c} \xrightarrow{(i_1)^*} \\ \xleftarrow{(i_1)_*} \end{array} \mathcal{S} .$$

**Remark 2.7** These adjunctions  $(\text{Diag}^*, \text{Diag}_*)$ ,  $((\pi_1)^*, (\pi_1)_*)$  and  $((i_1)^*, (i_1)_*)$  are in fact enriched adjunctions, meaning that we have a natural isomorphism of spaces

$$\begin{aligned} \text{Map}_{\mathcal{S}}(\text{Diag}^* X, Y) &\cong \text{Map}_{s\mathcal{S}}(X, \text{Diag}_* Y), \\ \text{Map}_{s\mathcal{S}}((\pi_1)^* X, Y) &\cong \text{Map}_{\mathcal{S}}(X, (\pi_1)_*(Y)), \\ \text{Map}_{\mathcal{S}}((i_1)^* X, Y) &\cong \text{Map}_{s\mathcal{S}}(X, (i_1)_*(Y)). \end{aligned}$$

**Remark 2.8** Our notation convention, Sect. 2.3(1), implies that for a space  $K$  we denote the simplicial space  $(\pi_1)^*(K)$  by  $K$  as well.

**Remark 2.9** Notice we have  $\pi_1 \circ i_1 = \text{id}$  and so  $(i_1)^* \circ (\pi_1)^* = \text{id}$ . More importantly, for every simplicial set  $X$

$$(\pi_1)_*(X)_n \cong \text{Hom}_{\mathcal{S}}(\Delta[n], (\pi_1)_*(X)) \cong \text{Hom}_{s\mathcal{S}}((\pi_1)^* \Delta[n], X) = \text{Hom}_{\mathcal{S}}(\Delta[n], X) \cong X_{0n},$$

hence  $(\pi_1)_* = (i_1)^*$ . Thus, we can also think of  $(i_1)_*$  as the right adjoint to  $(\pi_1)_*$ .

Before we move on we want to give very detailed descriptions of these functors.

For a simplicial space  $X$  we have:

$$\begin{aligned} (\text{Diag}^* X)_n &= X_{nn}, \\ (\pi_1)_*(X)_n &= (i_1)^*(X)_n = X_{0n}. \end{aligned}$$

Also, for a simplicial set  $K$  we have:

$$\begin{aligned} \text{Diag}_*(K)_n &= K^{\Delta[n]}, \\ (\pi_1)^*(K)_n &= K, \\ (i_1)_*(K)_n &= K^{n+1}. \end{aligned}$$

**Remark 2.10** The direct computation above in particular implies that for a given Kan complex  $K$  there is a natural map

$$(\pi_1)^* K \rightarrow \text{Diag}_*(K)$$

that is a Reedy trivial cofibration. Indeed, we have a retract diagram  $K \rightarrow K^{\Delta[n]} \rightarrow K$ , that we can make into a deformation retract via morphism  $\Delta[n] \times \Delta[1] \rightarrow \Delta[n]$  induced by the morphism  $[n] \times [1] \rightarrow [n]$  given by  $(i, 0) \mapsto i$  and  $(i, 1) \mapsto n$ .

We want to show that the category of simplicial spaces has two model structures that makes the three adjunctions above into Quillen equivalences and that will play an important role in the coming sections.

The first one is the *diagonal model structure* on simplicial spaces. Given its prominent role in homotopy theory, it has already been considered in a variety of settings, such as, among others, by Moerdijk [48, Proposition 1.2], Rezk–Schwede–Shipley [61, Lemma 4.3], Dugger [20, Example 5.6], and Cisinski [15, Corollary 3.16].

**Theorem 2.11** *There is a unique, cofibrantly generated, simplicial model structure on  $s\mathcal{S}$ , called the diagonal model structure and denoted by  $s\mathcal{S}^{diag}$ , with the following specifications.*

(W) *A map  $f : X \rightarrow Y$  is a weak equivalence if the diagonal map of spaces*

$$\text{Diag}^*(f) : \text{Diag}^*(X) \rightarrow \text{Diag}^*(Y)$$

*is a Kan equivalence.*

(C) *A map  $f : X \rightarrow Y$  is a cofibration if it is an inclusion.*

(F) *A map  $f : X \rightarrow Y$  is a fibration if it satisfies the right lifting condition for trivial cofibrations.*

*In particular, an object  $W$  is fibrant if and only if it is Reedy fibrant and a homotopically constant simplicial space.*

**Proof** Let  $\mathcal{L}$  be the following set of cofibrations

$$\mathcal{L} = \{ \langle 0 \rangle : F(0) \rightarrow F(n) : n \geq 0 \}.$$

Then by Theorem A.7 there is a localized model structure on  $s\mathcal{S}$  such that cofibrations are inclusions and fibrant objects are Reedy fibrant simplicial spaces  $K$  such that

$$K_n \cong \text{Map}_{s\mathcal{S}}(F(n), K) \rightarrow \text{Map}_{s\mathcal{S}}(F(0), K) \cong K_0$$

is a Kan equivalence.

In order to finish the proof we only need to prove that  $f : X \rightarrow Y$  is a weak equivalence in the localized model structure if and only if  $\text{Diag}^*(f)$  is a Kan equivalence. For that we first observe that for a given fibrant simplicial space  $K$ , we have the following diagram

$$\begin{array}{ccc} (\pi_1)^* K_0 & \xrightarrow{\cong} & K \\ \cong \downarrow & \nearrow \gamma_K & \downarrow \cong \\ \text{Diag}_* K_0 & \longrightarrow & F(0) \end{array} .$$

Here the left hand morphism is the Reedy trivial cofibration described in Remark 2.10, implying the existence of a lift  $\gamma_K : \text{Diag}_* K_0 \rightarrow K$ , which by 2-out-of-3 is a Reedy equivalence as well.

Now, notice also that  $\text{Diag}_*(K_0)$  is Reedy fibrant, as the Reedy morphism is given by the Kan fibration  $(K_0)^{\Delta[n]} \rightarrow (K_0)^{\partial\Delta[n]}$ . Hence, for the fibrant simplicial space  $K$  we have the following diagram

$$\begin{array}{ccc}
 \text{Map}_{s\mathcal{S}}(Y, K) & \longrightarrow & \text{Map}_{s\mathcal{S}}(X, K) \\
 \uparrow (\gamma_K)_* \simeq & & \uparrow \simeq (\gamma_K)_* \\
 \text{Map}_{s\mathcal{S}}(Y, \text{Diag}_*(K_0)) & \longrightarrow & \text{Map}_{s\mathcal{S}}(X, \text{Diag}_*(K_0)) \\
 \downarrow \cong & & \downarrow \cong \\
 \text{Map}_{\mathcal{S}}(\text{Diag}^*Y, K_0) & \longrightarrow & \text{Map}_{\mathcal{S}}(\text{Diag}^*X, K_0)
 \end{array}$$

Here the vertical morphisms in the top square are equivalences, because  $\gamma_K : \text{Diag}_* K_0 \rightarrow K$  is a Reedy equivalence between Reedy fibrant objects and the vertical morphisms in the bottom square are isomorphisms because of the simplicially enriched adjunction  $(\text{Diag}^*, \text{Diag}_*)$ .

Now the map  $f : X \rightarrow Y$  is an equivalence in this localized model structure if and only if the top horizontal map is a Kan equivalence for all fibrant objects  $K$  (as the model structure is simplicial), which by the diagram above is equivalent to the bottom map being a Kan equivalence for all Kan complexes  $K_0$ .

The equivalence above holds in particular when  $K$  is a simplicial space of the form  $\text{Diag}_* L$ , where  $L$  is an arbitrary Kan complex, which, combined with the fact that the Kan model structure is simplicial, implies that this is equivalent to

$$\text{Diag}^* f : \text{Diag}^* X \rightarrow \text{Diag}^* Y$$

being a Kan equivalence, which is exactly the desired statement and finishes the proof.  $\square$

**Theorem 2.12** *There is a unique, cofibrantly generated, simplicial model structure on  $s\mathcal{S}$ , called the Kan model structure and denoted by  $s\mathcal{S}^{Kan}$ , with the following specifications.*

(W) *A map  $f : X \rightarrow Y$  is a weak equivalence if*

$$(\pi_1)_* f : (\pi_1)_* X \rightarrow (\pi_1)_* Y$$

*is a Kan equivalence.*

(C) *A map  $f : X \rightarrow Y$  is a cofibration if it is an inclusion.*

(F) *A map  $f : X \rightarrow Y$  is a fibration if it satisfies the right lifting condition for trivial cofibrations.*

*In particular, an object  $W$  is fibrant if and only if it is Reedy fibrant and the map*

$$\text{Map}_{s\mathcal{S}}(F(n), W) \rightarrow \text{Map}_{s\mathcal{S}}(\partial F(n), W)$$

*is a trivial Kan fibration for  $n > 0$ .*

**Proof** Let  $\mathcal{L}$  be the following set of cofibrations

$$\mathcal{L} = \{\partial F(n) \rightarrow F(n) : n \geq 1\}.$$

Then by Theorem A.7 there is a localized model structure on  $s\mathcal{S}$  such that cofibrations are inclusions and fibrant objects are Reedy fibrant simplicial spaces  $K$  such that

$$\text{Map}_{s\mathcal{S}}(F(n), K) \rightarrow \text{Map}_{s\mathcal{S}}(\partial F(n), K)$$

is a Kan equivalence.

Before we can finish the proof, we need a better understanding of the fibrant objects in this model structure. A Reedy fibrant simplicial space  $X$  is fibrant if and only if the map  $X_n \rightarrow (X_0)^{n+1}$  is an equivalence. Thus a Reedy fibrant simplicial space  $X$  is fibrant in this model structure if and only if the natural map  $X \rightarrow (i_1)_*(X_0)$  is a Reedy equivalence.

In order to finish the proof we only need to prove that  $f : X \rightarrow Y$  is a weak equivalence in this localized model structure if and only if  $(\pi_1)_*(f)$  is a Kan equivalence. Fix a map of simplicial spaces  $X \rightarrow Y$  and a fibrant simplicial space  $Z$ . Then we have the following diagram of spaces:

$$\begin{array}{ccc}
 \text{Map}_{sS}(Y, Z) & \longrightarrow & \text{Map}_{sS}(X, Z) \\
 \downarrow \cong & & \downarrow \cong \\
 \text{Map}_{sS}(Y, (i_1)_*(Z_0)) & \longrightarrow & \text{Map}_{sS}(X, (i_1)_*(Z_0)) \\
 \downarrow \cong & & \downarrow \cong \\
 \text{Map}_S((\pi_1)_*Y, Z_0) & \longrightarrow & \text{Map}_S((\pi_1)_*X, Z_0)
 \end{array}$$

The vertical maps in the top square are Kan equivalences as  $Z$  and  $(i_1)_*(Z_0)$  are both Reedy fibrant, and moreover, by the explanation in the previous paragraph,  $Z \rightarrow (i_1)_*(Z_0)$  is a Reedy equivalence. Indeed,  $(i_1)_*(Z_0)_0 = Z_0$  is Kan fibrant and for  $n > 0$ , the morphism  $\text{Map}_{sS}(F(n), (i_1)_*(Z_0)) \rightarrow \text{Map}_{sS}(\partial F(n), (i_1)_*(Z_0))$  is the identity. The vertical maps in the bottom square are isomorphisms because  $(i_1)_*$  is the right adjoint to  $(\pi_1)_*$  as explained in Remark 2.9.

Now the map  $f : X \rightarrow Y$  is an equivalence in this localized model structure if and only if the top horizontal map is a Kan equivalence for all fibrant objects  $Z$  (as the model structure is simplicial), which by the diagram above is equivalent to the bottom map being a Kan equivalence for all Kan complexes  $Z_0$ .

We can in particular apply this equivalence to the fibrant simplicial space  $(i_1)_*(L)$  for an arbitrary Kan complex  $L$ , meaning the result holds for an arbitrary Kan complex and the Kan model structure is simplicial. Hence this is equivalent to

$$(\pi_1)_*f : (\pi_1)_*X \rightarrow (\pi_1)_*Y$$

being a Kan equivalence of simplicial sets. □

We are finally in a position to prove the existence of the chain of Quillen equivalences.

**Theorem 2.13** *The following three simplicially enriched adjunctions*

$${}_s\text{Diag} \begin{array}{c} \xrightarrow{\text{Diag}^*} \\ \xleftarrow{\text{Diag}_*} \end{array} {}_s\text{Kan} \begin{array}{c} \xleftarrow{(\pi_1)^*} \\ \xrightarrow{(\pi_1)_*} \end{array} {}_s\text{SKan} \begin{array}{c} \xleftarrow{(i_1)^*} \\ \xrightarrow{(i_1)_*} \end{array} {}_s\text{Kan}$$

are Quillen equivalences.

**Proof Quillen Adjunctions:** The fact that these three adjunctions are Quillen adjunctions is similar for all three cases and so we will combine the argument. In all three cases we observe that the left adjoint  $\text{Diag}^*, (\pi_1)^*, (i_1)^*$  preserve inclusions and weak equivalences and thus preserves cofibrations and trivial cofibrations, which imply they are left Quillen functors.

Indeed, the fact that they preserve inclusions is immediate. The fact that they preserve weak equivalences is an immediate observation for  $(\pi_1)^*$  and  $(i_1)^*$  and for  $\text{Diag}^*$  follows from Theorem 2.11.

*Quillen Equivalence:* We move on to prove that these are Quillen equivalences. First we prove that  $(\pi_1)^*$  and  $(i_1)^*$  are derived inverses of each other. Notice that for every simplicial set  $K$ ,  $(i_1)^*(\pi_1)^*(K) = K$ . So, in order to prove they are derived inverses it suffices to prove there is a natural equivalence

$$(\pi_1)^*(i_1)^*(X) \xrightarrow{\cong} X$$

for every simplicial space  $X$  fibrant in the Kan model structure on simplicial spaces. This will then imply that  $((\pi_1)^*, (\pi_1)_*)$  and  $((i_1)^*, (i_1)_*)$  are Quillen equivalences and in fact are inverses of each other.

We have  $(\pi_1)^*(i_1)^*(X)_n = X_0$  and so the natural map  $(\pi_1)^*(i_1)^*(X) \rightarrow X$  is an equivalence in the Kan model structure on simplicial spaces.

Next we move on to prove that  $(\text{Diag}^*, \text{Diag}_*)$  is a Quillen equivalence. Based on Lemma A.4 it suffices to prove that  $\text{Diag}^*$  reflects weak equivalences and the derived counit map (which is the actual counit map as all objects are cofibrant)  $\text{Diag}^*\text{Diag}_*K \rightarrow K$  is a Kan equivalence of simplicial sets, for  $K$  a Kan complex.

The fact that  $\text{Diag}^*$  reflects weak equivalences is part of Theorem 2.11. Before we move on to the second part we first observe that

$$(\text{Diag}^*\text{Diag}_*K)_n = (\text{Diag}_*K)_{nn} = \text{Hom}(\Delta[n] \times \Delta[n], K).$$

This in particular means that  $\text{Diag}^*\text{Diag}_*K$  is a Kan complex. Indeed, we need to witness that for all  $n \geq 0$  and  $0 \leq l \leq n$   $\text{Hom}(\Delta[n], \text{Diag}^*\text{Diag}_*K) \rightarrow \text{Hom}(\Delta[n]_l, \text{Diag}^*\text{Diag}_*K)$  is surjective, which is equivalent to  $\text{Hom}(\Delta[n] \times \Delta[n], K) \rightarrow \text{Hom}(\Delta[n]_l \times \Delta[n]_l, K)$  being surjective, which follows from the fact that  $\Delta[n]_l \times \Delta[n]_l \rightarrow \Delta[n] \times \Delta[n]$  is a trivial cofibration in the Kan model structure [26, Corollary 4.6].

Now, the counit map

$$\Delta^* : \text{Hom}(\Delta[\bullet] \times \Delta[\bullet], K) \rightarrow \text{Hom}(\Delta[\bullet], K)$$

is induced by the diagonal map  $\Delta : \Delta[\bullet] \rightarrow \Delta[\bullet] \times \Delta[\bullet]$ . By 2-out-of-3 it suffices to show that the morphism  $(\pi_1)^* : \text{Hom}(\Delta[\bullet], K) \rightarrow \text{Hom}(\Delta[\bullet] \times \Delta[\bullet], K)$  is a Kan equivalence as  $(\pi_1)^*$  and  $\Delta^*$  compose to the identity.

We will now construct an explicit deformation retract of Kan complexes for

$$(\pi_1)^* : \text{Hom}(\Delta[\bullet], K) \rightarrow \text{Hom}(\Delta[\bullet] \times \Delta[\bullet], K),$$

finishing the proof. As  $\pi_1 \circ i_1$  is the identity, we only need a morphism  $\gamma : \Delta[\bullet] \times \Delta[\bullet] \times \Delta[1] \rightarrow \Delta[\bullet] \times \Delta[\bullet]$ , which satisfies  $\gamma(-, -, 0) = i_1\pi_1$  and  $\gamma(-, -, 1) = \text{id}$ . We can obtain such a morphism, by defining the morphism of posets  $g : [n] \times [n] \times [1] \rightarrow [n] \times [n]$  with  $g(i, j, 0) = g(i, 0, 0)$  and  $g(i, j, 1) = (i, j)$  and then applying nerves.  $\square$

**Remark 2.14** Composing the two Quillen equivalences  $(\text{Diag}^*, \text{Diag}_*)$  and  $((\pi_1)^*, (\pi_1)_*)$  we get a Quillen equivalence

$$s\mathcal{S}^{diag} \begin{array}{c} \xrightarrow{(\pi_1)^*\text{Diag}^*} \\ \xleftrightarrow{\text{Diag}_*(\pi_1)_*} \\ \xrightarrow{\text{Diag}_*(\pi_1)_*} \end{array} s\mathcal{S}^{Kan}$$

however, this Quillen equivalence is not the identity adjunction. Thus the Kan model structure and diagonal model structure on simplicial spaces are Quillen equivalent, but not the same model structures (as their set of weak equivalences and fibrations differ).

### 2.6 Complete Segal Spaces

The Reedy model structure can be localized such that it models  $(\infty, 1)$ -categories [58]. This is done in two steps. First we define *Segal spaces*. For that let  $\alpha_i : [1] \rightarrow [n]$  be the morphism given by  $\alpha_i(0) = i$  and  $\alpha_i(1) = i + 1$ , where  $0 \leq i < n$ .

**Definition 2.15** [58, Page 11] A Reedy fibrant simplicial space  $X$  is called a Segal space if the map

$$(\alpha_0^*, \dots, \alpha_{n-1}^*) : X_n \xrightarrow{\cong} X_1 \times_{X_0} \dots \times_{X_0} X_1$$

is a Kan equivalence for  $n \geq 2$ .

Segal spaces come with a model structure, namely the *Segal space model structure*.

**Theorem 2.16** [58, Theorem 7.1] *There is a simplicial closed model category structure on the category  $sS$  of simplicial spaces called the Segal space model category structure, and denoted  $sS^{Seg}$ , with the following properties.*

- (1) *The cofibrations are precisely the monomorphisms.*
- (2) *The fibrant objects are precisely the Segal spaces.*
- (3) *The weak equivalences are precisely the maps  $f$  such that  $Map_{sS}(f, W)$  is a weak equivalence of spaces for every Segal space  $W$ .*
- (4) *A Reedy weak equivalence between any two objects is a weak equivalence in the Segal space model category structure, and if both objects are themselves Segal spaces then the converse holds.*
- (5) *For two cofibrations  $i$  and  $j$ ,  $i \square j$  is a cofibration, which is trivial if either of  $i$  or  $j$  are.*
- (6) *The model structure is the localization of the Reedy model structure with respect to the maps*

$$G(n) = F(1) \coprod_{F(0)} \dots \coprod_{F(0)} F(1) \rightarrow F(n)$$

for  $n \geq 2$ .

A Segal space already has many characteristics of a category, such as objects and morphisms.

**Definition 2.17** Let  $W$  be a Segal space. Then an *object*  $x$  in  $W$  is a point in  $W_0$ . Moreover for two objects  $x, y$  we define the *mapping space* as the pullback

$$\begin{array}{ccc} \text{map}_W(x, y) & \longrightarrow & W_1 \\ \downarrow & & \downarrow \\ \Delta[0] & \xrightarrow{(\{x\}, \{y\})} & W_0 \times W_0 \end{array}$$

Unlike classical categories, the mapping spaces of a Segal space do not come with strict composition maps. Rather there is a natural zig-zag. For more details see [58, Sect. 5]. On the other hand we do get an actual homotopy category:

**Definition 2.18** Let  $W$  be a Segal space. We define the *homotopy category* of  $W$ , denoted  $\text{Ho}W$ , as the following category:



- (1) Objects of  $\text{Ho}W$  are objects of  $W$ .
- (2) For two objects  $x, y$  we have

$$\text{Hom}_{\text{Ho}W}(x, y) = \pi_0(\text{map}_W(x, y)).$$

This indeed gives us a category [58, 5.5]. Moreover, a morphism  $f$  in  $W$  is a weak equivalence precisely if the morphism  $[f]$  in  $\text{Ho}W$  is an isomorphism.

Segal spaces do not give us a model of  $(\infty, 1)$ -categories. For that we need *complete Segal spaces*.

**Definition 2.19** Let  $J[n]$  be the fibrant replacement of  $\Delta[n]$  in the Kan model structure on simplicial sets (Sect. 2.2(6)). We define the simplicial discrete space  $E(n)$  as  $E(n) = (\pi_2)^*J[n]$ , where  $(\pi_2)^*$  was defined in Notation 2.6. In particular,  $E(1)$  is the free invertible arrow, meaning a morphism of Segal spaces  $E(1) \rightarrow W$  is precisely given by a choice of weak equivalence in  $W$ .

**Definition 2.20** Let  $W$  be a Segal space. We define the space of weak equivalences  $W_{\text{hoequiv}}$  as

$$W_{\text{hoequiv}} = \text{Map}_{sS}(E(1), W).$$

Notice  $W_{\text{hoequiv}} \rightarrow W_1$  is an equivalence when restricted to each path component in  $W_{\text{hoequiv}}$ . Moreover, for any two objects  $x, y$  in  $W$  define  $\text{hoequiv}_W(x, y) = W_{\text{hoequiv}} \times_{W_0 \times W_0} \Delta[0]$  and notice the morphism  $\text{hoequiv}_W(x, y) \rightarrow \text{map}_W(x, y)$  is also an equivalence on each path component of  $\text{hoequiv}_W(x, y)$ .

**Definition 2.21** A Segal space  $W$  is called a *complete Segal space* if it satisfies one of the following equivalent conditions.

- (1) The inclusion map

$$W_0 \hookrightarrow W_{\text{hoequiv}}$$

is a weak equivalence.

- (2) The map

$$\langle 0 \rangle^*: W_{\text{hoequiv}} = \text{Map}(E(1), W) \rightarrow \text{Map}(F(0), W) = W_0$$

is a trivial Kan fibration.

- (3) The map

$$\langle 1 \rangle^*: W_{\text{hoequiv}} = \text{Map}(E(1), W) \rightarrow \text{Map}(F(0), W) = W_0$$

is a trivial Kan fibration.

Complete Segal spaces come with their own model structure, the *complete Segal space model structure*.

**Theorem 2.22** [58, Theorem 7.2] *There is a simplicial closed model category structure on the category  $sS$  of simplicial spaces, called the complete Segal space model category structure, and denoted  $sS^{CSS}$ , with the following properties.*

- (1) *The cofibrations are precisely the monomorphisms.*
- (2) *The fibrant objects are precisely the complete Segal spaces.*
- (3) *The weak equivalences are precisely the maps  $f$  such that  $\text{Map}_{sS}(f, W)$  is a weak equivalence of spaces for every complete Segal space  $W$ .*

- (4) A Reedy weak equivalence between any two objects is a weak equivalence in the complete Segal space model category structure, and if both objects are themselves complete Segal spaces then the converse holds.
- (5) For two cofibrations  $i$  and  $j$ ,  $i \square j$  is a cofibration, which is trivial if either of  $i$  or  $j$  are.
- (6) The model structure is the localization of the Segal space model structure with respect to the map

$$\langle 0 \rangle: F(0) \rightarrow E(1).$$

A complete Segal space is a model for an  $(\infty, 1)$ -category. For a better understanding of complete Segal spaces see [58, 6] and for a comparison with other models see [7, 37].

We end this section with reviewing the relationship between categories and (complete) Segal spaces. For that we first establish the following notational conventions.

**Notation 2.23** For a given category  $\mathcal{C}$ , following our notational convention (Remark 2.8)  $(\pi_1)^*N\mathcal{C} = N\mathcal{C}$ . Hence, we introduce the notation  $N^h\mathcal{C} = (\pi_2)^*N\mathcal{C}$ , called the *horizontal nerve* and note the resulting simplicial space is a Segal space with  $N^h\mathcal{C}_n$  a discrete space.

While  $N^h\mathcal{C}$  is a Segal space (and in fact the Kan equivalences in Definition 2.15 are isomorphisms of discrete simplicial sets), it is usually not complete. That is why there is an alternative construction, the *classifying diagram*, given as follows.

**Definition 2.24** Let  $\mathcal{C}$  be a category. Then the classifying diagram of  $\mathcal{C}$  is the simplicial space defined as  $N(\mathcal{C}, \mathcal{Jso})_{kn} = \text{Fun}([k] \times I[n], \mathcal{C})$ .

We now have the following results in [58, Proposition 6.1].

**Proposition 2.25** *Let  $\mathcal{C}$  be a category, then  $N(\mathcal{C}, \mathcal{Jso})$  is a complete Segal space, moreover, the evident morphism  $N^h\mathcal{C} \rightarrow N(\mathcal{C}, \mathcal{Jso})$  is a complete Segal space equivalence of Segal spaces.*

### 3 Left Fibrations and the Covariant Model Structure

This section is focused on the study of left fibrations. We first focus on various characterizations of left fibrations (Sect. 3.1). Then we show that left fibrations can be seen as fibrant objects in a model structure, the covariant model structure (Sect. 3.2). Finally, we do a careful analysis of left fibrations over Segal spaces (Sect. 3.3).

**Remark 3.1** *Historical note on left fibrations for simplicial spaces:* Left fibrations for complete Segal spaces were first considered by Charles Rezk, motivated by his paper on complete Segal spaces [58], however, he never published those ideas.

The first record of left fibrations for Segal spaces can be found in the work of de Brito [19, 0.1] and Kazhdan-Varshavsky [40, Definition 2.1.1], where both authors (independently) give the same definition of a left fibration for Segal spaces.

The definition of left fibration given here was suggested to the author by Charles Rezk and generalizes those definitions from Segal spaces to an arbitrary simplicial space.

#### 3.1 The Many Faces of Left Fibrations

In this section we first define left fibrations (Definition 3.2) and then prove they can be characterized in several alternative ways: Lemmas 3.5, 3.6, Proposition 3.7, Lemma 3.10.

There is one final characterization of left fibrations, Lemma 3.20, which we relegate to the next section.

We want to generalize the definition of a discrete Grothendieck opfibration (Definition 1.15) to simplicial spaces. The guiding principle towards a working definition is the following idea:

*uniqueness in set theory*  $\Leftrightarrow$  *contractibility in homotopy theory*

Thus we need to find an appropriate contractibility condition. Using our intuition from Segal spaces for a given simplicial space  $X$  we should think of the space  $X_0$  as the *space of objects*,  $X_1$  as the *space of morphisms* and the simplicial map  $s : X_1 \rightarrow X_0$  as map that takes a morphism to its source (Sect. 2.3(4)).

This motivates the following definition:

**Definition 3.2** A map of simplicial spaces  $p : L \rightarrow X$  is called a *left fibration* if it is a Reedy fibration such that the following square is a homotopy pullback square for all  $n \geq 0$

$$\begin{array}{ccc}
 L_n & \xrightarrow{\langle 0 \rangle^*} & L_0 \\
 \downarrow p_n & \lrcorner & \downarrow p_0 \\
 X_n & \xrightarrow{\langle 0 \rangle^*} & X_0
 \end{array} \tag{3.3}$$

where the horizontal maps come from the map  $\langle 0 \rangle : [0] \rightarrow [n]$  taking the point to  $0 \in [n]$ . More generally, a morphism of simplicial spaces is a *left morphism* if it satisfies the pullback condition in 3.3.

**Remark 3.4** As we observed in Lemma 1.16, for a given category  $\mathcal{C}$ , discrete Grothendieck opfibrations over  $\mathcal{C}$  correspond to functors  $\mathcal{C} \rightarrow \text{Set}$ . Given that left fibrations are the homotopical analogue of Grothendieck opfibrations they are expected to model functors valued in spaces. We will in fact prove this statement for the specific case where  $X = \mathbb{N}^h \mathcal{C}$  (Theorem 4.18) and use this idea as a guide towards studying left fibrations over general simplicial spaces.

The proof of the general case can be found for quasi-categories in [42, Theorem 2.2.1.2], [31, Theorem C].

Let us start with a simple, alternative way of characterizing left fibrations.

**Lemma 3.5** *Let  $p : Y \rightarrow X$  be a Reedy fibration. The following are equivalent:*

- (1)  $p$  is a left fibration.
- (2) For each  $n \geq 0$ , the map

$$(p_n, \langle 0 \rangle^*) : Y_n \rightarrow X_n \times_{X_0} Y_0$$

*is a Kan equivalence.*

- (3) For each  $n \geq 0$ , the map

$$(p_n, \langle 0 \rangle^*) : Y_n \rightarrow X_n \times_{X_0} Y_0$$

*is a trivial Kan fibration.*

**Proof** (1  $\Leftrightarrow$  2) This follows from the definition of a homotopy pullback and right-properness of the Kan model structure.

(2  $\Leftrightarrow$  3) The map  $\langle 0 \rangle : F(0) \rightarrow F(n)$  is a cofibration, which implies that

$$\text{Map}_{\mathcal{S}\mathcal{S}}(F(n), Y) \rightarrow \text{Map}_{\mathcal{S}\mathcal{S}}(F(0), Y) \times_{\text{Map}_{\mathcal{S}\mathcal{S}}(F(0), X)} \text{Map}_{\mathcal{S}\mathcal{S}}(F(n), X)$$

or, equivalently, the map

$$Y_n \rightarrow X_n \times_{X_0} Y_0$$

is a Kan fibration. Thus it is a weak equivalence if and only if it is a trivial Kan fibration.  $\square$

Next we give an alternative, inductive characterization of left fibrations.

**Lemma 3.6** *Let  $p : Y \rightarrow X$  be a Reedy fibration. The following two are equivalent:*

(1) *The commutative square*

$$\begin{array}{ccc} Y_n & \xrightarrow{\langle 0 \rangle^*} & Y_0 \\ p_n \downarrow & \lrcorner & \downarrow p_0 \\ X_n & \xrightarrow{\langle 0 \rangle^*} & X_0 \end{array}$$

*is a homotopy pullback square for all  $n \geq 0$ , meaning  $p$  is a left fibration.*

(2) *The commutative square*

$$\begin{array}{ccc} Y_n & \xrightarrow{\langle 0, \dots, n-1 \rangle^*} & Y_{n-1} \\ p_n \downarrow & \lrcorner & \downarrow p_{n-1} \\ X_n & \xrightarrow{\langle 0, \dots, n-1 \rangle^*} & X_{n-1} \end{array}$$

*is a homotopy pullback square for all  $n \geq 1$ .*

**Proof** We have the following diagram:

$$\begin{array}{ccccc} Y_n & \xrightarrow{\langle 0, \dots, n-1 \rangle^*} & Y_{n-1} & \xrightarrow{\langle 0 \rangle^*} & Y_0 \\ p_n \downarrow & & \downarrow p_{n-1} & & \downarrow p_0 \\ X_n & \xrightarrow{\langle 0, \dots, n-1 \rangle^*} & X_{n-1} & \xrightarrow{\langle 0 \rangle^*} & X_0 \end{array} .$$

(1  $\Rightarrow$  2) In this case the rectangle and the right square is a homotopy pullback and therefore the left hand square is also a homotopy pullback.

(2  $\Rightarrow$  1) For this case we use induction. The case  $n = 1$  is clear. If it is true for  $n - 1$  then this means that in the diagram above the right hand square is a homotopy pullback. By assumption the left hand square is a homotopy pullback and so the whole rectangle has to be a homotopy pullback and we are done.  $\square$

Next we give a characterization of left fibrations via lifting conditions.

**Proposition 3.7** *Let  $p : Y \rightarrow X$  be a Reedy fibration. Then the following are equivalent:*

- (1)  $p$  is a left fibration.
- (2)  $p$  satisfies the right lifting property with respect to maps of the form

$$(< 0 > : F(0) \rightarrow F(n)) \square (\partial \Delta[l] \rightarrow \Delta[l]),$$

for all  $n \geq 0$  and  $l \geq 0$ .

- (3) The map of simplicial sets  $(\pi_1)^*(\exp(< 0 > : F(0) \rightarrow F(n), p))$  (where  $\pi_1$  was introduced in Notation 2.6) satisfies the right lifting property with respect to maps of the form

$$\partial \Delta[l] \rightarrow \Delta[l],$$

for all  $n \geq 0$  and  $l \geq 0$ .

**Proof** By Lemma 3.5, the map  $p : Y \rightarrow X$  is a left fibration if and only if

$$(Y_n \rightarrow X_n \times_{X_0} Y_0) = (\pi_1)_*(\exp(< 0 > : F(0) \rightarrow F(n), p : Y \rightarrow X))$$

is a trivial fibration, where  $\pi_1$  was introduced in Notation 2.6. This is equivalent to  $(\pi_1)_*(\exp(< 0 > : F(0) \rightarrow F(n), p : Y \rightarrow X))$  having the right lifting property with respect to the inclusion maps  $\partial \Delta[l] \rightarrow \Delta[l]$ . Using the adjunction  $((\pi_1)^*, (\pi_1)_*)$ , this is equivalent to  $\exp(< 0 > : F(0) \rightarrow F(n), p : Y \rightarrow X)$  having the right lifting property with respect to  $(\pi_1)^*(\partial \Delta[l]) \rightarrow (\pi_1)^*\Delta[l]$ , which by our notation convention (Remark 2.8) we denote by  $\partial \Delta[l] \rightarrow \Delta[l]$ .

Thus,  $p : Y \rightarrow X$  is a left fibration if and only if

$$\partial \Delta[l] \rightarrow \Delta[l] \pitchfork \exp(< 0 > : F(0) \rightarrow F(n), p : Y \rightarrow X).$$

Now using Proposition 2.5 with the set of morphisms:

- $\mathcal{A} = \{< 0 > : F(0) \rightarrow F(n) : n \geq 0\}$
- $\mathcal{B} = \{\partial \Delta[l] \rightarrow \Delta[l] : l \geq 0\}$
- $\mathcal{L} = \{\text{left fibrations}\}$

we have

$$\mathcal{A} \pitchfork \exp(\mathcal{B}, \mathcal{L}) \Leftrightarrow \mathcal{A} \square \mathcal{B} \pitchfork \mathcal{L} \Leftrightarrow \mathcal{B} \pitchfork \exp(\mathcal{A}, \mathcal{L}).$$

Hence we are done. □

The pullback characterization of left fibrations (Lemma 3.5) immediately has the following implication.

**Lemma 3.8** *Let  $f : Y \rightarrow X, g : Z \rightarrow Y$  and  $h : W \rightarrow V$  be three Reedy fibrations.*

- (1) If  $f$  and  $g$  are left fibrations then  $fg$  is also a left fibration.
- (2) If  $f$  and  $fg$  are left fibrations then  $g$  is also a left fibration.
- (3) If  $f$  and  $h$  are weakly equivalent Reedy fibrations, then  $f$  is a left fibration if and only if  $h$  is a left fibration.

The fact that left fibrations are given via a right lifting property has the following formal consequence.

**Lemma 3.9** *The pullback of a left fibration is a left fibration.*

We can use the pullback stability of left fibrations to give several local characterizations of left fibrations.

**Lemma 3.10** *Let  $p : L \rightarrow X$  be a Reedy fibration. Then the following are equivalent:*

- (1)  $p$  is a left fibration.
- (2) For every map  $F(n) \times \Delta[I] \rightarrow X$  the pullback  $L \times_X (F(n) \times \Delta[I]) \rightarrow F(n) \times \Delta[I]$  is a left fibration.
- (3) For every map  $F(n) \rightarrow X$  the pullback  $L \times_X F(n) \rightarrow F(n)$  is a left fibration.

**Proof** (1  $\Rightarrow$  3) This follows from Lemma 3.9.

(2  $\Rightarrow$  1) By Proposition 3.7 it suffices to prove that the map  $p$  has the right lifting property with respect to the maps  $(F(0) \rightarrow F(n)) \square (\partial\Delta[I] \rightarrow \Delta[I])$ , meaning we have to show the following diagram has a lift

$$\begin{array}{ccc}
 F(n) \times \partial\Delta[I] & \coprod_{F(0) \times \partial\Delta[I]} \Delta[I] & \longrightarrow L \\
 \downarrow & & \downarrow p \\
 F(n) \times \Delta[I] & \xrightarrow{f} & X
 \end{array}$$

We can now take a pullback of  $p$  along  $f$  to obtain the following diagram

$$\begin{array}{ccccc}
 F(n) \times \partial\Delta[I] & \coprod_{F(0) \times \partial\Delta[I]} \Delta[I] & \longrightarrow & f^*L & \xrightarrow{p^*f} & L \\
 \downarrow & & \nearrow \hat{f} & \downarrow & & \downarrow p \\
 F(n) \times \Delta[I] & \xrightarrow{\text{id}} & F(n) \times \Delta[I] & \xrightarrow{f} & X
 \end{array}$$

By assumption  $f^*L \rightarrow F(n) \times \Delta[I]$  is a left fibration, which means the lift  $\hat{f}$  exists. Hence,  $p^*f \circ \hat{f}$  is the desired lift for the original diagram.

(3  $\Rightarrow$  2) Fix a map  $f : F(n) \times \Delta[I] \rightarrow X$ . The map  $\langle 0 \rangle : \Delta[0] \rightarrow \Delta[I]$  gives us the following pullback square

$$\begin{array}{ccc}
 (f \circ (\text{id} \times 0))^*(L) & \xrightarrow{\cong} & f^*L \\
 \downarrow & & \downarrow \\
 F(n) & \xrightarrow[\cong]{\text{id} \times \langle 0 \rangle} & F(n) \times \Delta[I]
 \end{array}$$

The bottom map is a Reedy equivalence, which implies the top map is also a Reedy equivalence. The maps  $f^*p$  and  $(f \circ (\text{id} \times 0))^*(p)$  are Reedy fibrations and so, by Lemma 3.8(3),  $f^*p$  is a left fibration if and only if  $(f \circ (\text{id} \times 0))^*(p)$  is a left fibration, which holds by assumption. □

Using the same argument as in the previous proof we can prove the analogous statement about diagonal fibrations that will become useful later on.

**Lemma 3.11** *Let  $p : Y \rightarrow X$  be a Reedy fibration. Then the following are equivalent:*

- (1)  $p$  is a diagonal fibration.
- (2) For every map  $F(n) \times \Delta[I] \rightarrow X$  the pullback  $Y \times_X (F(n) \times \Delta[I]) \rightarrow F(n) \times \Delta[I]$  is a diagonal fibration.
- (3) For every map  $F(n) \rightarrow X$  the pullback  $Y \times_X F(n) \rightarrow F(n)$  is a diagonal fibration.

**Proof** Following [61, Definition 3.3, Lemma 4.3] a morphism  $p : Y \rightarrow X$  is a fibration in the diagonal model structure if and only if  $p$  is a Reedy fibration and for all  $d_i : [n] \rightarrow [n + 1]$  the induced morphism  $(d_i, p_{n+1}) : Y_{n+1} \rightarrow Y_n \times_{X_n} X_{n+1}$  is a weak equivalence, where  $0 \leq i \leq n$ . This means  $p$  is a diagonal fibration if it is a Reedy fibration and satisfies the right lifting property with respect to morphisms  $d_i \square (\partial \Delta[I] \rightarrow \Delta[I])$ . The result now follows from applying the same proof as the one given in Lemma 3.10 with the set of morphisms  $d_i \square (\partial \Delta[I] \rightarrow \Delta[I])$ , where  $0 \leq i \leq n$ . □

### 3.2 The Covariant Model Structure

Let  $X$  be a simplicial space. In this subsection we define a model structure on the over-category  $s\mathcal{S}/X$ , the *covariant model structure*, which has fibrant objects precisely the left fibrations over  $X$  (Theorem 3.12). We end the subsection by giving a useful criterion for determining covariant equivalences by generalizing deformation retracts from classical homotopy theory (Theorem 3.27).

**Theorem 3.12** *Let  $X$  be a simplicial space. There is a unique simplicial left proper model structure on the over-category  $s\mathcal{S}/X$ , called the covariant model structure and denoted by  $(s\mathcal{S}/X)^{cov}$ , which satisfies the following conditions:*

- (1) *The fibrant objects are the left fibrations over  $X$ .*
- (2) *Cofibrations are monomorphisms.*
- (3) *A map  $f : A \rightarrow B$  over  $X$  is a covariant weak equivalence if*

$$\text{Map}_{s\mathcal{S}/X}(B, L) \rightarrow \text{Map}_{s\mathcal{S}/X}(A, L)$$

*is a Kan equivalence for every left fibration  $L \rightarrow X$ .*

- (4) *A weak equivalence (covariant fibration) between fibrant objects is a level-wise equivalence (Reedy fibration).*

**Proof** Let  $\mathcal{L}$  be the collection of maps of the following form

$$\mathcal{L} = \{F(0) \xrightarrow{\langle 0 \rangle} F(n) \rightarrow X : n \geq 0\}.$$

Note that  $\mathcal{L}$  is a set of cofibrations in  $s\mathcal{S}/X$  with the Reedy model structure. This allows us to use the theory of Bousfield localizations (Theorem A.7) with respect to  $\mathcal{L}$  on the category  $s\mathcal{S}/X$ . The resulting model structure immediately satisfies all the conditions above and in particular the fibrant objects are precisely the left fibrations by Lemma A.12. □

**Remark 3.13** This model structure is also constructed in [19, Proposition 1.10] for the particular case where the base  $X$  is a Segal space, and so the theorem could be deduced from that result as well.

We can actually say more about the fibrations in the covariant model structure.

**Lemma 3.14** *Let  $p : L \rightarrow X$  and  $q : L' \rightarrow X$  be two left fibrations. A map  $f : L \rightarrow L'$  over  $X$  is a fibration in the covariant model structure if and only if it is a left fibration.*

**Proof** As  $p$  and  $q$  are fibrant,  $f$  is a fibration if and only if it is a Reedy fibration (Theorem 3.12(4)). The statement now follows from Lemma 3.8 as  $qf = p$ .  $\square$

Note the covariant model structure behaves well with respect to base change.

**Theorem 3.15** *Let  $f : X \rightarrow Y$  be map of simplicial spaces. Then the following adjunction*

$$(s\mathcal{S}/X)^{cov} \begin{array}{c} \xrightarrow{f_!} \\ \perp \\ \xleftarrow{f^*} \end{array} (s\mathcal{S}/Y)^{cov}$$

is a Quillen adjunction, which is a Quillen equivalence if  $f$  is a Reedy equivalence. Here  $f_!$  is the composition map and  $f^*$  is the pullback map.

**Proof** This is the special case of Lemma A.9 when  $\mathcal{L} = \{< 0 > : F(0) \rightarrow F(n) : n \geq 0\}$ .  $\square$

**Remark 3.16** Later we will prove a much stronger result, namely if  $f$  is an equivalence in the CSS model structure then the Quillen adjunction is actually a Quillen equivalence (Theorem 5.1).

**Theorem 3.17** *The following is a Quillen adjunction*

$$(s\mathcal{S}/X)^{cov} \begin{array}{c} \xrightarrow{id} \\ \perp \\ \xleftarrow{id} \end{array} (s\mathcal{S}/X)^{diag}$$

which is a Quillen equivalence if  $X$  is a homotopically constant simplicial space. Here the left side has the covariant model structure and the right side has the induced diagonal model structure (Proposition A.5). This implies that the diagonal model structure over  $X$  is a localization of the covariant model structure over  $X$ .

**Proof** If we localize the Reedy model structure on  $s\mathcal{S}/X$  with respect to maps of the form  $F(0) \rightarrow F(n) \rightarrow X$  we get the covariant model structure (Theorem 3.12) whereas if we localize the Reedy model structure on  $s\mathcal{S}$  with respect to maps  $< 0 > : F(0) \rightarrow F(n)$  we get the diagonal model structure (Theorem 2.11). This means we can apply Theorem A.13 to deduce that this a Quillen adjunction.

Now let us assume  $X$  is homotopically constant. Let  $j : X \rightarrow \hat{X}$  be a Reedy fibrant replacement of  $X$ . Then  $\hat{X}$  is also homotopically constant, which means it is fibrant in the diagonal model structure (Theorem 2.11). We now have the following diagram of Quillen adjunctions

$$\begin{array}{ccc} (s\mathcal{S}/X)^{cov} & \begin{array}{c} \xrightarrow{id} \\ \perp \\ \xleftarrow{id} \end{array} & (s\mathcal{S}/X)^{diag} \\ \uparrow j^* \quad \downarrow j_! & & \uparrow j^* \quad \downarrow j_! \\ (s\mathcal{S}/\hat{X})^{cov} & \begin{array}{c} \xrightarrow{id} \\ \perp \\ \xleftarrow{id} \end{array} & (s\mathcal{S}/\hat{X})^{diag} \end{array} .$$

The left hand vertical adjunction is a Quillen equivalence as  $j$  is a Reedy equivalence and by Lemma A.9. The right hand Quillen adjunction is a Quillen equivalence as the diagonal model structure is right proper and [59, Proposition 2.5]. The bottom horizontal adjunction is a Quillen equivalence because  $\hat{X}$  is fibrant in the diagonal model structure and Theorem A.13. Thus, 2-out-of-3 implies that the top adjunction is a Quillen equivalence as well.  $\square$



**Remark 3.18** The theorem implies that every covariant equivalence is a diagonal equivalence, whereas the opposite direction is obviously not true. On the other hand, in Sect. 4.2 we will prove that we can determine whether a map is a covariant equivalence by checking whether a certain collection of maps consists of diagonal equivalences (Theorem 4.41).

**Example 3.19** One very important instance is the case  $X = F(0)$ . The theorem shows that  $s\mathcal{S}^{cov}$  is the same as  $s\mathcal{S}^{diag}$ .

Using the covariant model structure we can add one final alternative characterization of left fibrations.

**Lemma 3.20** *Let  $p : L \rightarrow X$  be a Reedy fibration. Then the following are equivalent:*

- (1)  $p$  is a left fibration.
- (2) For all  $n \geq 0$

$$\begin{array}{ccc}
 \text{Map}_{s\mathcal{S}}(F(n) \times F(1), L) & \longrightarrow & \text{Map}_{s\mathcal{S}}(F(n) \times F(1), X) \\
 \downarrow & & \downarrow \\
 \text{Map}_{s\mathcal{S}}(F(n), L) & \longrightarrow & \text{Map}_{s\mathcal{S}}(F(n), X)
 \end{array} \tag{3.21}$$

is a homotopy pullback square.

- (3)  $\exp\langle 0 \rangle : F(0) \rightarrow F(1), p$  is a trivial Reedy fibration.

**Proof** (1  $\Rightarrow$  2) First, let us assume  $p$  is a left fibration. Fix a morphism  $p : F(n) \times F(1) \rightarrow X$ . We can write  $F(n) \times F(1)$  as a colimit of a diagram

$$F(n + 1) \leftarrow F(n) \rightarrow \dots \leftarrow F(n) \rightarrow F(n + 1) \tag{3.22}$$

over  $X$  such that all maps  $F(n) \rightarrow F(n + 1)$  take 0 to 0 (for a more detailed description of this diagram see [58, Diagram 10.4]). Thus, by applying 2-out-of-3 to the diagram  $F(0) \rightarrow F(n) \rightarrow F(n+1)$  over  $X$ , the maps in 3.22 are covariant equivalences over  $X$ . As the covariant model structure is left proper (Theorem 3.12) this implies that  $F(n) \times \{0\} \rightarrow F(n) \times F(1)$  is a covariant equivalence over  $X$ . As the covariant model structure is simplicial, this implies that the morphism  $\text{Map}_{/X}(F(n) \times F(1), L) \rightarrow \text{Map}_{/X}(F(n), L)$  is a Kan equivalence. As the morphism is precisely the fiber of the diagram of Kan fibrations

$$\begin{array}{ccc}
 \text{Map}(F(n) \times F(1), L) & \longrightarrow & \text{Map}(F(n), L) \times_{\text{Map}(F(n), X)} \text{Map}(F(n) \times F(1), X) \\
 & \searrow & \swarrow \\
 & \text{Map}(F(n) \times F(1), X) &
 \end{array}$$

over the point  $p : F(n) \times F(1) \rightarrow X$ , this proves, by Corollary A.2, that the square 3.21 is a homotopy pullback square for all  $n \geq 0$ .

(2  $\Rightarrow$  1) Let 3.21 be a homotopy pullback square for all  $n \geq 0$ , which means

$$\text{Map}_{s\mathcal{S}}(F(n) \times F(1), L) \rightarrow \text{Map}_{s\mathcal{S}}(F(n), L) \times_{\text{Map}_{s\mathcal{S}}(F(n), X)} \text{Map}_{s\mathcal{S}}(F(n) \times F(1), X) \tag{3.23}$$

a trivial Kan fibration.

Let  $r : [n] \times [1] \rightarrow [n + 1]$  be the functor given by  $r(i, 0) = i, r(i, 1) = n + 1$ . Then notice the following is a retract diagram

$$\begin{array}{ccccc}
 F(n) & \xrightarrow{\text{id}} & F(n) & \xrightarrow{\text{id}} & F(n) \\
 \downarrow \langle 0, \dots, n \rangle & & \downarrow \text{id} \times \langle 0, \dots, 0 \rangle & & \downarrow \langle 0, \dots, n \rangle \cdot \\
 F(n+1) & \xrightarrow{\langle (0,0), \dots, (n,0), (n,1) \rangle} & F(n) \times F(1) & \xrightarrow{Nr} & F(n+1)
 \end{array}$$

Let us denote  $i = \langle (0, 0), \dots, (n, 0), (n, 1) \rangle : F(n+1) \rightarrow F(n) \times F(1)$ . The diagram gives us the following retract diagram of morphisms

$$\begin{array}{ccc}
 \text{Map}_{sS}(F(n+1), L) & \longrightarrow & \text{Map}_{sS}(F(n), L) \times_{\text{Map}_{sS}(F(n), X)} \text{Map}_{sS}(F(n+1), X) \\
 \downarrow (Nr)^* & & \downarrow \text{id} \times (Nr)^* \\
 \text{Map}_{sS}(F(n) \times F(1), L) & \xrightarrow{\simeq} & \text{Map}_{sS}(F(n), L) \times_{\text{Map}_{sS}(F(n), X)} \text{Map}_{sS}(F(n) \times F(1), X) \\
 \downarrow i^* & & \downarrow \text{id} \times i^* \\
 \text{Map}_{sS}(F(n+1), L) & \longrightarrow & \text{Map}_{sS}(F(n), L) \times_{\text{Map}_{sS}(F(n), X)} \text{Map}_{sS}(F(n+1), X)
 \end{array}$$

As the middle horizontal diagram is a weak equivalence, by 3.23, it follows that the top morphism is a weak equivalence. By Lemma 3.6, this implies that  $p : L \rightarrow X$  is a left fibration.

(2  $\Leftrightarrow$  3) The map  $\text{exp}(\langle 0 \rangle : F(0) \rightarrow F(1), p)$  is a trivial Reedy fibration if and only if it is a level-wise weak equivalence, meaning the maps 3.21 are Kan equivalences for all  $n \geq 0$ . □

**Remark 3.24** One interesting implication of this lemma is that we can get the covariant model structure on  $sS_{/X}$  (Theorem 3.12) also by localizing with respect to the maps

$$\{ \langle (0, 0) \rangle : F(0) \rightarrow F(1) \times F(n) \rightarrow X : n \geq 0 \}.$$

Indeed, Lemma 3.20 implies that the left Bousfield localization with respect to the two sets of morphisms  $\{ \langle (0, 0) \rangle : F(0) \rightarrow F(1) \times F(n) \rightarrow X : n \geq 0 \}$ . and  $\{ \langle 0 \rangle : F(0) \rightarrow F(n) \rightarrow X : n \geq 0 \}$  have the same cofibrations (monomorphisms) and fibrant objects (left fibrations) and hence are the same model structure (the covariant model structure over  $X$ ).

We can apply this argument inductively to conclude that we can also obtain the covariant model structure by localizing with respect to

$$\{ \langle (0, \dots, 0) \rangle : F(0) \rightarrow F(1)^n \rightarrow X : n \geq 0 \}.$$

One important goal in the coming sections is to give a recognition principle for covariant equivalences. This will be done in Theorem 4.41 and needs us to first discuss the simplicial Grothendieck construction (Theorem 4.18). However, there are certain instances, motivated by classical homotopy theory, where recognizing covariant equivalences is quite easy. First we need to prove an important lemma about closure properties of some covariant equivalences.

**Lemma 3.25** *Let  $\mathcal{L}$  be the class of monomorphisms of simplicial spaces that have the left lifting property with respect to all left fibrations.*

- (1) *If  $i : A \rightarrow B \in \mathcal{L}$  and  $j$  is a cofibration then  $i \square j$  is in  $\mathcal{L}$ .*
- (2) *If  $p$  is a left fibration and  $j$  a cofibration then  $\text{exp}(j, p)$  is a left fibration.*
- (3) *If  $i : A \rightarrow B \in \mathcal{L}$  and  $B \rightarrow X$  is an arbitrary morphism, then  $i$  over  $X$  is a covariant equivalence.*

**Proof** (1), (2) By Proposition 2.5 and the fact that left fibrations are described via right lifting property (Proposition 3.7), the first two statements are equivalent and so it suffices to prove the second one. Following Lemma 3.20, we only have to prove that  $\exp(\langle 0 \rangle : F(0) \rightarrow F(1), \exp(j, p))$  is a trivial fibration. Now, we have the following chain of isomorphisms

$$\begin{aligned} \exp(\langle 0 \rangle : F(0) \rightarrow F(1), \exp(j, p)) &\cong \exp(\langle \langle 0 \rangle : F(0) \rightarrow F(1) \rangle \square j, p) \\ &\cong \exp(j, \exp(\langle 0 \rangle, p)). \end{aligned}$$

By Lemma 3.20,  $\exp(\langle 0 \rangle : F(0) \rightarrow F(1), p)$  is a trivial fibration. Moreover, as the Reedy model structure is Cartesian closed (Sect. 2.4),  $\exp(j, \exp(\langle 0 \rangle, p))$  is also a trivial fibration. Hence,  $\exp(\langle 0 \rangle : F(0) \rightarrow F(1), \exp(j, p))$  is a trivial fibration and we are done.

(3) We need to prove that for every  $g : B \rightarrow X$  and left fibration  $p : L \rightarrow X$ , the induced morphism  $\text{Map}_{/X}(B, L) \rightarrow \text{Map}_{/X}(A, L)$  is a Kan equivalence. This morphism is obtained by considering the morphism  $t : \text{Map}(B, L) \rightarrow \text{Map}(B, X) \times_{\text{Map}(A, X)} \text{Map}(A, L)$  over  $\text{Map}(B, X)$  and then taking the fiber over  $g$ . Hence, by Corollary A.2, it suffices to show that  $t$  is a Kan equivalence.

By Sect. 2.3(9) and the definition of pullback exponentials (Definition 2.3), we have  $t = \exp(i, p)_0$ , meaning in order to complete this proof it suffices to show that  $\exp(i, p)$  is a Reedy weak equivalence. We will in fact prove it is a trivial fibration, by establishing it has the right lifting property with respect to all cofibrations  $j$ . By Proposition 2.5, this claim is equivalent to  $i \square j$  having the left lifting property with respect to  $p$ , which holds by (1).  $\square$

This lemma has a useful corollary.

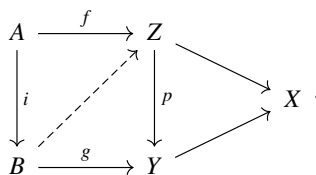
**Corollary 3.26** *Let  $L \rightarrow X$  be a left fibration and  $Y$  a simplicial space, then  $L^Y \rightarrow X^Y$  is a left fibration.*

**Proof** This follows from the previous result if we use the cofibration  $\emptyset \rightarrow Y$ .  $\square$

With the technical lemma at hand, we can give a helpful characterization of covariant equivalences.

**Theorem 3.27** *Let  $i : A \rightarrow B$  be a monomorphism over  $X$ . Then  $i$  is a trivial cofibration in the covariant model structure on  $sS_{/X}$  if there exists a retraction  $r : B \rightarrow A$  (not necessarily over  $X$ ) and a  $H : B \times F(1) \rightarrow B$  (relative to  $A$ ) such that  $H(-, 0) = ir$  and  $H(-, 1) = \text{id}_B$ .*

**Proof** For the proof we simply adapt the argument in [33, Lemma 2.1] to simplicial spaces. Let  $Z \xrightarrow{p} Y \rightarrow X$  be a fibration in the covariant model structure over  $X$ . We need to prove that a lift of the following diagram exists:



Let  $j = (\langle 0 \rangle : F(0) \rightarrow F(1)) \square (i : A \rightarrow B)$ . Using the homotopy  $H$  and the map  $j$ , we can factor this diagram as follows:

$$\begin{array}{ccccc}
 A & \longrightarrow & F(1) \times A & \coprod_{F(0) \times A} & F(0) \times B & \xrightarrow{f\pi_A \amalg fr} & Z \\
 \downarrow i & & & \downarrow j & & \nearrow & \downarrow p \\
 B & \xleftarrow{\text{id}_B \times \langle 1 \rangle} & B \times F(1) & \xrightarrow{gH} & Y & & X
 \end{array}$$

It suffices to show the right hand square has a lift. For that we need to show that  $j$  is a trivial cofibration in the covariant model structure over  $X$ . However, this follows immediately from the fact that  $\langle 0 \rangle : F(0) \rightarrow F(1)$  has the left lifting property with respect to all left fibrations (Proposition 3.7) and so, by Lemma 3.25,  $j$  is a covariant equivalence over  $X$ .  $\square$

**Remark 3.28** As already stated in the proof, this theorem was stated and proven for simplicial sets by Heuts and Moerdijk [33, Lemma 2.1].

We will use this result in the next subsection to study left fibrations of Segal spaces.

### 3.3 Yoneda Lemma for Segal Spaces

In this subsection we want to use the results we have proven until now to study left fibrations over Segal spaces. In particular, we prove the Yoneda lemma for Segal spaces (Theorem 3.49).

We will start by simplifying the definition of a left fibration over Segal spaces.

**Lemma 3.29** *Let  $p : Y \rightarrow X$  be a Reedy fibration and  $X$  a Segal space. The following are equivalent:*

(1)  $Y$  is a Segal space and the following is a homotopy pullback square:

$$\begin{array}{ccc}
 Y_1 & \xrightarrow{\langle 0 \rangle^*} & Y_0 \\
 \downarrow f_1 & \lrcorner & \downarrow f_0 \\
 X_1 & \xrightarrow{\langle 0 \rangle^*} & X_0
 \end{array}$$

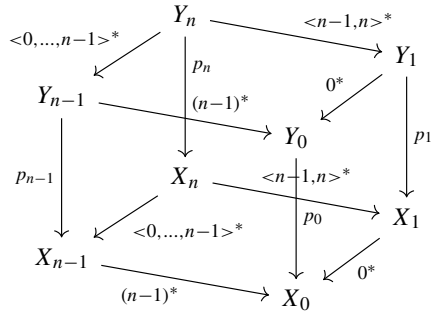
(2)  $p$  is a fibration in the Segal space model structure and the following is a homotopy pullback square:

$$\begin{array}{ccc}
 Y_1 & \xrightarrow{\langle 0 \rangle^*} & Y_0 \\
 \downarrow f_1 & \lrcorner & \downarrow f_0 \\
 X_1 & \xrightarrow{\langle 0 \rangle^*} & X_0
 \end{array}$$

(3)  $p$  is a left fibration.

**Proof** (1  $\Leftrightarrow$  2) This follows immediately from the fact that a Reedy fibration between Segal spaces is a Segal fibration (Theorem 2.16).

(3  $\Leftrightarrow$  1) For all  $n \geq 2$  we have the following diagram:



We have the following facts about this diagram:

- The bottom square is a homotopy pullback for all  $n \geq 1$  if and only if  $X$  is a Segal space.
- Similarly, the top square is a homotopy pullback for all  $n \geq 1$  if and only if  $Y$  is a Segal space.
- The four squares around the cube are homotopy pullback squares if and only if  $p$  is a left fibration (Lemma 3.6).

The result now follows from checking homotopy pullback squares. The bottom square is always a homotopy pullback square.

If we assume (2) then the top square and the right hand square are homotopy pullback squares, which directly implies that left hand square is a homotopy pullback square (by the cancellation property of homotopy pullbacks) and so, by Lemma 3.6,  $p$  is a left fibration.

If we assume (3) then all four squares around are homotopy pullback squares which implies that the top square is a homotopy pullback square as well.  $\square$

**Remark 3.30** Note that [19] and [40] use the characterization in Lemma 3.29 as a definition of left fibrations rather than the definition we have given here (Definition 3.2). Hence this lemma proves that our definition agrees with theirs when the base is a Segal space and thus is a proper generalization of their definition.

Note that a left fibration of Segal spaces generalizes a discrete Grothendieck opfibration between categories.

**Lemma 3.31** *Let  $W$  be a Segal space and  $p : L \rightarrow W$  be a left fibration. Then the induced functor on homotopy categories  $\text{Ho}(p) : \text{Ho}(L) \rightarrow \text{Ho}(W)$  is a discrete Grothendieck opfibration.*

**Proof** By Lemma 3.29  $L$  is a Segal space. Thus  $\text{Ho}(p) : \text{Ho}(L) \rightarrow \text{Ho}(W)$  is a functor. We want to prove it is a discrete Grothendieck opfibration. Let  $[f] : x \rightarrow y$  be a morphism in  $\text{Ho}(W)$  and  $\hat{x}$  be a lift of  $x$  in  $\text{Ho}(L)$ . We need to prove there is a unique lift  $[\hat{f}]$  of  $f$  such that  $\text{Ho}(p)([\hat{f}]) = [f]$ .

Let  $f$  in  $W_1$  be a representative for the class  $[f] \in \pi_0(W_1)$ . Then  $(\hat{x}, f)$  is a point in  $L_0 \times_{W_0} W_1$ . The fact that  $p : L \rightarrow W$  is a left fibration implies that

$$L_1 \rightarrow L_0 \times_{W_0} W_1$$

is a trivial Kan fibration and so the fiber over  $(\hat{x}, f)$ , which we denote by  $F_f$ , is contractible. This means  $\pi_0(F_f) = \{[\hat{f}]\}$  has a single element, which is precisely the unique lift.  $\square$

There is an inverse argument to Lemma 3.31.

**Lemma 3.32** *Let  $p : \mathcal{D} \rightarrow \mathcal{C}$  be a Grothendieck opfibration. Then  $N^h(p) : N^h\mathcal{D} \rightarrow N^h\mathcal{C}$  is a left fibration.*

**Proof** Notice  $N^h\mathcal{D}$  and  $N^h\mathcal{C}$  are simplicial discrete spaces, which means  $N^h(p)$  is a Reedy fibration. Moreover,  $N^h\mathcal{D}$  and  $N^h\mathcal{C}$  are nerves of categories and hence Segal spaces. Thus by Lemma 3.29 we have to show that the square

$$\begin{array}{ccc} N^h\mathcal{D}_1 & \xrightarrow{s} & N^h\mathcal{D}_0 \\ \downarrow & & \downarrow \\ N^h\mathcal{C}_1 & \xrightarrow{s} & N^h\mathcal{C}_0 \end{array}$$

is a pullback square. However, this is precisely the lifting condition of a Grothendieck opfibration (Lemma 1.16). □

We will see later (Proposition 5.20) that this result also holds when we replace the nerve with the classifying diagram (Definition 2.24). We can use the connection between left fibrations and Grothendieck opfibrations to study conservativity of left fibrations.

**Definition 3.33** A map of Segal spaces  $p : V \rightarrow W$  is *conservative* if the square

$$\begin{array}{ccc} V_{hoequiv} & \longrightarrow & V_1 \\ \downarrow & & \downarrow \\ W_{hoequiv} & \longrightarrow & W_1 \end{array} \tag{3.34}$$

is a homotopy pullback square.

We can characterize conservativity via the homotopy category.

**Lemma 3.35** *Let  $p : V \rightarrow W$  be a morphism of Segal spaces. Then the following are equivalent:*

- (1)  $p$  is conservative.
- (2) For all objects  $x, y$  in  $V$ , the square

$$\begin{array}{ccc} \text{hoequiv}_V(x, y) & \longrightarrow & \text{map}_V(x, y) \\ \downarrow & & \downarrow \\ \text{hoequiv}_W(px, py) & \longrightarrow & \text{map}_W(px, py) \end{array}$$

is a homotopy pullback square.

- (3) The functor of categories  $\text{Ho}(p) : \text{Ho}(\mathcal{C}) \rightarrow \text{Ho}(\mathcal{D})$  is conservative.

**Proof** (1)  $\Leftrightarrow$  (2) The square 3.34 is a homotopy pullback square if and only if the morphism  $V_{hoequiv} \rightarrow W_{hoequiv} \times_{W_1} V_1$  is a weak equivalence, which is equivalent to being a weak equivalence over  $W_0 \times W_0$  in the following diagram

$$\begin{array}{ccc}
 V_{\text{hoequiv}} & \longrightarrow & W_{\text{hoequiv}} \times_{W_1} V_1 \\
 & \searrow & \swarrow \\
 & W_0 \times W_0 &
 \end{array}$$

Now, as both legs of the triangle are Kan fibrations, by Corollary A.2, the top morphism is an equivalence if and only if it is a fiber-wise equivalence. However, by Definition 2.17 and Definition 2.20 for a given point  $(x, y) : \Delta[0] \rightarrow W_0 \times W_0$ , the fiber is precisely given by the morphism

$$\text{hoequiv}_V(x, y) \rightarrow \text{hoequiv}_W(px, py) \times_{\text{map}_W(px, py)} \text{map}_V(x, y)$$

finishing the proof.

(2)  $\Leftrightarrow$  (3) Fix two arbitrary objects  $x, y$  in  $W$ . By Definition 2.20, the morphism  $\text{hoequiv}_W(x, y) \rightarrow \text{map}_W(x, y)$  is an equivalence when restricted to each path-component and so

$$\text{hoequiv}_V(x, y) \rightarrow \text{hoequiv}_W(px, py) \times_{\text{map}_W(px, py)} \text{map}_V(x, y)$$

is a weak equivalence if and only if

$$\pi_0 \text{hoequiv}_V(x, y) \rightarrow \pi_0 \text{hoequiv}_W(px, py) \times_{\pi_0 \text{map}_W(px, py)} \pi_0 \text{map}_V(x, y)$$

is a bijection of sets. However, by definition  $\pi_0 \text{map}_W(x, y) = \text{Hom}_{\text{Ho}W}(x, y)$   $\pi_0 \text{hoequiv}_W(x, y) = \mathcal{J}_{\text{SoHo}W}(x, y)$ , meaning this is equivalent to  $\mathcal{J}_{\text{SoHo}(V)}(px, py) \times_{\text{Hom}_{\text{Ho}(W)}(px, py)} \text{Hom}_{\text{Ho}(V)}(x, y)$  being a bijection, which by Definition 1.13, is equivalent to  $\text{Ho}(p)$  being conservative.  $\square$

We can finally relate conservativity and left fibrations.

**Lemma 3.36** *Let  $W$  be a Segal space and  $p : L \rightarrow W$  a left fibration. Then  $p$  is conservative.*

**Proof** By Lemma 3.29,  $L$  is a Segal space and so  $p$  is a map of Segal spaces. Thus, by Lemma 3.35,  $p$  is conservative if and only if the functor  $\text{Ho}p : \text{Ho}W \rightarrow \text{Ho}V$  is conservative. However, by Lemma 3.31,  $p$  is a discrete Grothendieck opfibration and thus is conservative.  $\square$

**Remark 3.37** This result was proven for quasi-categories in [39, Proposition 4.9], which could give us the analogous argument for complete Segal spaces. However, this proof generalizes the result to arbitrary Segal spaces.

We can use conservativity to characterize left fibrations over complete Segal spaces.

**Lemma 3.38** *Let  $W$  be a complete Segal space and  $p : V \rightarrow W$  be a left fibration. Then  $V$  is a complete Segal space.*

**Proof** We have the diagram

$$\begin{array}{ccccc}
 V_{\text{hoequiv}} & \longrightarrow & V_1 & \longrightarrow & V_0 \\
 \downarrow & & \downarrow & & \downarrow \\
 W_{\text{hoequiv}} & \longrightarrow & W_1 & \longrightarrow & W_0 \\
 & & \searrow & \curvearrowright & \\
 & & & \simeq &
 \end{array}$$

The left hand square is a homotopy pullback square as  $p$  is a left fibration and thus conservative (Lemma 3.36). The right hand square is a pullback square because of  $p$  is a left fibration. Hence the whole rectangle is a homotopy pullback.

Now completeness of  $W$  implies that the bottom map is an equivalence (Definition 2.21) and, as the square is a homotopy pullback, this means  $V_{hoequiv} \rightarrow V_0$  is an equivalence. Hence, again by Definition 2.21, this means that  $V$  is a complete Segal space.  $\square$

We can combine Lemmas 3.29 and 3.38 into the following very useful result.

**Proposition 3.39** *Let  $W$  be a Segal space. Then the adjunction*

$$(s\mathcal{S}/_W)^{Seg} \begin{array}{c} \xrightarrow{\text{id}} \\ \xleftarrow{\perp} \\ \xleftarrow{\text{id}} \end{array} (s\mathcal{S}/_W)^{cov}$$

is a Quillen adjunction, where the left hand side has the induced Segal space model structure and the right hand side has the covariant model structure.

Moreover, if  $W$  is also complete then the adjunction

$$(s\mathcal{S}/_W)^{CSS} \begin{array}{c} \xrightarrow{\text{id}} \\ \xleftarrow{\perp} \\ \xleftarrow{\text{id}} \end{array} (s\mathcal{S}/_W)^{cov}$$

is a Quillen adjunction, where the left hand side has the CSS model structure and the right hand side has the covariant model structure.

**Proof** Let us focus on the case for Segal spaces first. As, following Theorem A.13, the induced and the localized Segal space model structures coincide when  $W$  is a Segal space, we can apply Corollary A.10. This means it suffices to prove that the left adjoint preserves monomorphisms and the right adjoint preserves Reedy fibrations and fibrant objects. It is evident that the identity functor preserves monomorphisms and Reedy fibrations. For the last part, observe that fibrant objects in the induced Segal space model structure are precisely Segal fibrations over  $W$ . Thus we only have to prove that if  $W$  is a Segal space and  $p : V \rightarrow W$  is a left fibration then  $p$  is a Segal fibration, which is precisely the statement of Lemma 3.29.

The case for complete Segal spaces is identical except in the last step we use the fact that a left fibration over a complete Segal space is a complete Segal space fibration as shown in Lemma 3.38.  $\square$

**Remark 3.40** The assumption that the base simplicial space is fibrant is not necessary and this proposition can be generalized to arbitrary simplicial spaces as we will do in Theorem 5.11. However, before we can do that we need to understand invariance properties of the covariant model structure (Theorem 5.1).

Having a better understanding of left fibrations over Segal spaces, we can move on to prove the Yoneda lemma for Segal spaces.

**Definition 3.41** Let  $W$  be a Segal space and  $x$  an object in  $W$ . Then we define the under-Segal space  $W_{x/}$  as

$$W_{x/} = W^{F(1)^s} \underset{W}{\times}^{\{x\}} F(0)$$

**Remark 3.42** In the particular case when  $W$  is a complete Segal space, it comes with an underlying quasi-category, denoted  $i_1^*W$  [37] (see Appendix B for more details regarding  $i_1^*$ ). This suggests the natural question how  $i_1^*(W_{x/})$  compares with  $(i_1^*W)_{x/}$ .



The common definition of an under-quasi-category (for example the one that can be found in [42]) relies on the join of simplicial sets. Hence,  $i_1^*(W_{x/})$  (as constructed in Definition 3.41) would not be isomorphic as a simplicial set to  $(i_1^*W)_{x/}$ , as constructed in [42, Proposition 1.2.9.2] via the join. However, in Appendix B we prove that  $i_1^*$  in fact induces a Quillen equivalence of covariant model structures, which in particular implies that the quasi-categories  $(i_1^*W)_{x/}, i_1^*(W_{x/})$  are equivalent left fibrations over  $i_1^*W$ .

An alternative argument that shows that the constructions of under-categories via joins and pullbacks are equivalent (in a general  $\infty$ -cosmos) can be found in [62, Corollary 4.2.8].

Notice the under-Segal space is in fact a Segal space.

**Lemma 3.43** *For a Segal space  $W$  and an object  $x$ , the projection  $W_{x/} \rightarrow W$  is a Segal space fibration and so the under-Segal space  $W_{x/}$  is a Segal space.*

**Proof** We have the following pullback diagram

$$\begin{array}{ccc}
 W_{x/} & \longrightarrow & W^{F(1)} \\
 \downarrow & & \downarrow (s,t) \\
 F(0) \times W & \xrightarrow{\{x\} \times \text{id}_W} & W \times W
 \end{array}$$

The right hand map is a fibration in the Segal space model structure as it is the pullback exponential of a cofibration and a fibrant object (Theorem 2.16). Thus the pullback is a Segal fibration. The result now follows from the fact that  $W$  itself is a Segal space.  $\square$

We have shown that the projection map  $W_{x/} \rightarrow W$  that takes each morphism to its target is a fibration in the Segal space model structure. We want to show that it is actually a left fibration.

**Theorem 3.44** *Let  $W$  be a Segal space and  $x$  an object in  $W$ . Then the projection map  $W_{x/} \rightarrow W$  is a left fibration.*

**Proof** In order to simplify notation we will denote the four vertices  $F(1) \times F(1)$  by  $\{00, 01, 10, 11\}$ .

By Lemma 3.29 it suffices to prove that the map

$$\pi : (W_{x/})_1 \rightarrow (W_{x/})_0 \times_{W_0} W_1$$

is a trivial Kan fibration.

Notice

$$(W_{x/})_0 \times_{W_0} W_1 \cong \Delta[0] \times_{W_0} W_1 \times_{W_0} W_1 \cong \Delta[0] \times_{W_0} \text{Map}_{sS}(G(2), W)$$

and

$$\begin{aligned}
 (W_{x/})_1 &= \Delta[0] \times_{W_1} (W^{F(1)})_1 \cong \Delta[0] \times_{W_0} W_0 \times_{W_1} (W^{F(1)})_1 \\
 &\cong \Delta[0] \times_{W_0} \text{Map}_{sS}(F(0) \coprod_{F(1)} \langle^{10,11} \rangle (F(1) \times F(1)), W).
 \end{aligned}$$

Thus, (recalling the definition of  $G(2)$  in Theorem 2.16) it suffices to prove the map

$$\text{Map}_{sS}(F(0) \coprod_{F(1)}^{<10,11>} (F(1) \times F(1)), W) \rightarrow \text{Map}_{sS}(G(2), W)$$

is a trivial Kan fibration, or, equivalently, the map

$$\langle 00, 01 \rangle \coprod_{\langle 01 \rangle} \langle 01, 11 \rangle : G(2) \hookrightarrow F(0) \coprod_{F(1)} (F(1) \times F(1)) \tag{3.45}$$

is a trivial cofibration in the Segal space model structure (as  $W$  is a Segal space).

This map factors as

$$G(2) \hookrightarrow F(2) \xrightarrow{\langle 00,01,11 \rangle} F(0) \coprod_{F(1)} (F(1) \times F(1))$$

and so we only need to show the second map is a Segal equivalence. Using the isomorphism

$$F(1) \times F(1) \cong F(2) \coprod_{F(1)}^{<0,2>} F(2),$$

the map  $\langle 00, 01, 11 \rangle$  is the pushout of the following diagram:

$$\begin{array}{ccccc} F(1) & \xleftarrow{\text{id}} & F(1) & \xrightarrow{\langle 0,2 \rangle} & F(2) \\ \downarrow & & \downarrow = & & \downarrow = \\ F(0) \coprod_{F(1)} F(2) & \xleftarrow{\langle 0,2 \rangle} & F(1) & \xrightarrow{\langle 0,2 \rangle} & F(2) \end{array}$$

So, the result follows from knowing that the left hand map is a trivial cofibration, as the Segal space model structure is left proper by Theorem 2.16. The left hand map itself is the pushout of the following diagram:

$$\begin{array}{ccccc} F(0) & \xleftarrow{\quad} & F(1) & \xrightarrow{\langle 1,2 \rangle} & G(2) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ F(0) & \xleftarrow{\quad} & F(1) & \xrightarrow{\langle 1,2 \rangle} & F(2) \end{array}$$

As all vertical arrows are equivalences in the Segal space model structure and the Segal space model structure is left proper, the pushout is a Segal space equivalence as well. Hence we are done. □

**Remark 3.46** In order to better understand the proof it might be helpful to visualize the map 3.45 as:

$$\begin{array}{ccc} \langle 00, 01 \rangle \coprod_{\langle 01 \rangle} \langle 01, 11 \rangle : & & \\ \begin{array}{ccc} 0 & \longrightarrow & 1 \\ & & \downarrow \\ & & 2 \end{array} & \longleftarrow & \begin{array}{ccc} 00 & \longrightarrow & 01 \\ & \searrow & \downarrow \\ & & 11 \end{array} \end{array}$$

**Remark 3.47** The fact that the left fibration  $W_{x/} \rightarrow W$  happened to be a Segal fibration is not a coincidence. We will later see that every left fibration is indeed a Segal fibration (Theorem 5.11).

**Remark 3.48** The Segal space condition in Theorem 3.44 is in fact a key condition and the theorem does not hold for general Reedy fibrant simplicial spaces, as we will show with  $G(2)$ . We want to prove that  $G(2)_{0/} \rightarrow G(2)$  is not a left fibration. For that it suffices to observe that the map

$$\pi : (G(2)_{0/})_1 \rightarrow (G(2)_{0/})_0 \times_{G(2)_0} G(2)_1$$

is not a trivial Kan fibration.

Notice that

$$\begin{aligned} (G(2)_{0/})_0 \times_{G(2)_0} G(2)_1 &= \{00, 01\} \times_{\{0,1,2\}} \{00, 01, 11, 12, 22\} \\ &= \{(00, 00), (00, 01), (01, 11), (01, 12)\}. \end{aligned}$$

On the other hand we have

$$(G(2)_{0/})_1 = \{\alpha : F(1) \times F(1) \rightarrow G(2) : \alpha(0, 0) = 0\}$$

and the map  $\pi$  simply restricts  $\alpha : F(1) \times F(1) \rightarrow G(2)$  to the pair  $(\alpha \circ <0, 0>, (0, 1) >, \alpha \circ <0, 1>, (1, 1) >)$ . Hence, the point  $(01, 12)$  has no lift along  $\pi$  as any choice of lift  $\alpha$  necessarily satisfies  $\alpha(1, 1) = 2$ , which is impossible.

We are now at the point where we can prove the Yoneda lemma for Segal spaces.

**Theorem 3.49** *Let  $W$  be a Segal space and  $x$  an object. Then the map  $\{id_x\} : F(0) \rightarrow W_{x/}$  is a covariant equivalence over  $W$ .*

**Proof** Let  $mul : F(1) \times F(1) \rightarrow F(1)$  be the map defined by  $mul(i, j) = ij$  where  $i, j = 0, 1$ . Moreover, let  $mul^* : W^{F(1)} \rightarrow \widehat{W^{F(1) \times F(1)}}$ . Using the adjunction between product and exponential, we get a morphism  $mul^* : F(1) \times \widehat{W^{F(1)}} \rightarrow W^{F(1)}$ , which fits into the following diagram:

$$\begin{array}{ccc} F(0) \times W^{F(1)} & & \\ \downarrow \langle 0 \rangle \times id & \searrow^{s_0 s} & \\ F(1) \times W^{F(1)} & \xrightarrow{\widehat{mul^*}} & W^{F(1)} \\ \uparrow \langle 1 \rangle \times id & \nearrow_{id} & \\ F(0) \times W^{F(1)} & & \end{array}$$

Now, notice we have the following diagram

$$\begin{array}{ccc} W_{x/} & \xrightarrow{mul^*} & (W_{x/})^{F(1)} \\ \downarrow & & \downarrow \\ W^{F(1)} & \xrightarrow{mul^*} & (W^{F(1)})^{F(1)} \end{array},$$

as  $\text{mul}^*$  takes a morphism  $f : x \rightarrow y$  to the square

$$\begin{array}{ccc} x & \xrightarrow{=} & x \\ \downarrow = & & \downarrow f \\ x & \xrightarrow{f} & y \end{array}$$

This means we can restrict the diagram via the inclusion  $W_{x/} \hookrightarrow W^{F(1)}$  to get the following diagram

$$\begin{array}{ccc} F(0) \times W_{x/} & & \\ \downarrow \langle 0 \rangle \times \text{id} & \searrow \{\text{id}_x\} & \\ F(1) \times W_{x/} & \xrightarrow{\widehat{\text{mul}}^*} & W_{x/} \\ \uparrow \langle 1 \rangle \times \text{id} & \nearrow \text{id} & \\ F(0) \times W_{x/} & & \end{array}$$

Notice, by definition of  $\widehat{\text{mul}}^*$  we have the commutative diagram

$$\begin{array}{ccc} F(1) \times \{\text{id}_x\} & \hookrightarrow & F(1) \times W_{x/} \\ \pi_2 \downarrow & & \downarrow \widehat{\text{mul}}^* \\ \{\text{id}_x\} & \hookrightarrow & W_{x/} \end{array}$$

meaning  $\widehat{\text{mul}}^*$  is a homotopy between  $\text{id}$  and  $\{\text{id}_x\} : W_{x/} \rightarrow W_{x/}$  relative to  $\{\text{id}_x\}$ . Hence, by Theorem 3.27, the map  $\{\text{id}_x\} : F(0) \rightarrow W_{x/}$  over  $W$  is a covariant equivalence.  $\square$

Why do we call this the Yoneda lemma? The next corollary makes the connection more clear:

**Corollary 3.50** *Let  $W$  be a Segal space and  $L \rightarrow W$  be a left fibration. Then the map of spaces*

$$\{\text{id}_x\}^* : \text{Map}_{/W}(W_{x/}, L) \rightarrow \text{Map}_{/W}(F(0), L)$$

*is a trivial Kan fibration.*

**Proof** Follows from Theorem 3.49 and the fact that the covariant model structure is simplicial (Theorem 3.12).  $\square$

**Remark 3.51** This Yoneda lemma for Segal spaces has also been proven (independently) by Boavida [19, Lemma 1.31]. The analogous version for quasi-categories, meaning a Kan equivalence  $\text{Map}_{/S}(S_{x/}, L) \rightarrow \text{Map}_{/S}(\Delta[0], L)$ , where  $L \rightarrow S$  is a left fibration of quasi-categories and  $S_{x/} \rightarrow S$  is the under-quasi-category projection, has also been established in the literature. It was proven directly by Joyal [39, Chapter 11], but is also an implication of the straightening construction by Lurie [42, Theorem 2.2.1.2]. Finally, there is an analogous statement in an arbitrary  $\infty$ -cosmos, meaning a Kan equivalence  $\text{Map}_{/S}(S_{x/}, L) \rightarrow \text{Map}_{/S}(\Delta[0], L)$ , where  $S$  is an  $\infty$ -category in an  $\infty$ -cosmos  $\mathcal{K}$ ,  $L \rightarrow S$  a left fibration (there called discrete coCartesian fibration [62, Definition 5.5.3]) and  $S_{x/} \rightarrow S$  the under- $\infty$ -category [62, Theorem 5.7.3].

We can use this result to study the relation between initial objects and representable functors. Let  $W$  be a Segal space. Then for two objects  $x, y$  we can define the mapping space  $\text{map}_W(x, y)$  (Definition 2.17). We would hope that this choice is functorial, meaning we get a functor  $\text{map}_W(x, -)$ . This would require an actual composition map

$$f_* : \text{map}_W(x, y) \rightarrow \text{map}_W(x, z)$$

for any map  $f : y \rightarrow z$  in  $W$ . However, composition of morphisms in a Segal space is only defined up to contractible ambiguity. For more details on composition in Segal spaces see [58, Sect. 5]

We will thus take a fibrational approach. We observed in Example 1.20 that the Grothendieck opfibration associated to the representable functor  $\text{Hom}(a, -) : \mathcal{C} \rightarrow \text{Set}$  is the under-category  $\mathcal{C}_{a/} \rightarrow \mathcal{C}$ . This motivates the following definition:

**Definition 3.52** A left fibration  $p : V \rightarrow W$  is called *representable* if there exists an object  $x$  in  $W$  and a Reedy equivalence  $f : W_{x/} \rightarrow V$  over  $X$ .

Using covariant equivalences we can relate representable left fibrations with the concept of *initiality*.

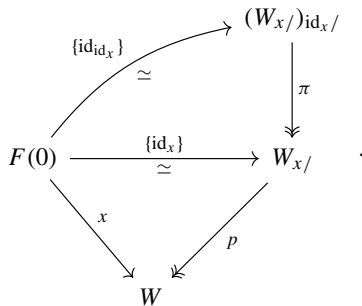
**Definition 3.53** Let  $W$  be a Segal space. An object  $x$  in  $W$  is called *initial* if the map  $\{x\} : F(0) \rightarrow W$  is a covariant equivalence over  $W$ .

**Remark 3.54** Initial objects are a special kind of colimit as we shall see in Sect. 5.2. Initial objects were thus studied in the context of colimits in quasi-categories [42, 1.2.12], [38, 10.1].

**Theorem 3.55** Let  $p : L \rightarrow W$  be a left fibration. Then the following are equivalent:

- (1)  $p$  is representable.
- (2)  $L$  has an initial object

**Proof** (1)  $\Rightarrow$  (2) If  $p$  is representable then  $L$  is Reedy equivalent to  $W_{x/}$  for some object  $x$  in  $W$ . Thus it suffices to prove  $W_{x/}$  has an initial object. We have the following diagram.



$W_{x/}$  is a left fibration over  $W$ . By Lemma 3.8  $(W_{x/})_{id_x/}$  is also a left fibration over  $W$  as the composition of left fibrations is a left fibration. By Theorem 3.49, the map  $\{id_x\}$  is a covariant equivalence over  $W$ . By the same argument the map  $\{id_{id_x}\}$  is a covariant equivalence over  $W_{x/}$ , which implies it is also a covariant equivalence over  $W$  (Theorem 3.15). By 2-out-of-3, we get that  $\pi$  is a covariant equivalence over  $W$ . But  $\pi$  is a map between left fibrations over  $W$  and thus must be a trivial Reedy fibration (Theorem 3.12).

(2)  $\Rightarrow$  (1) Let  $\{x\} : F(0) \rightarrow L$  be a covariant equivalence over  $L$ . Then, by Theorem 3.15,  $\{l\} : F(0) \rightarrow L$  is a covariant equivalence over  $W$ . By Theorem 3.49,  $F(0) \rightarrow W_{p(l)l}$  is a trivial covariant cofibration over  $W$  and, by assumption,  $L \rightarrow W$  is a left fibration and thus a covariant fibration over  $W$  and so we can lift the diagram below

$$\begin{array}{ccc}
 F(0) & \xrightarrow[\simeq]{\{x\}} & L \\
 \downarrow \{id_{p(l)}\} \simeq & & \downarrow p \\
 W_{p(l)l} & \longrightarrow & W
 \end{array}$$

As the top map is a covariant equivalence over  $W$ , by 2-out-of-3, the lift  $W_{p(l)l} \rightarrow L$  is a covariant equivalence over  $W$ . As both are left fibrations, by Theorem 3.12, this map is a Reedy equivalence.  $\square$

**Remark 3.56** Notice the second condition only depends on  $L$ . Thus representability of a left fibration  $p : L \rightarrow W$  is independent of the map  $p$  and base  $W$ .

### 4 From the Grothendieck Construction to the Yoneda Lemma

In Sect. 3.3 we studied many important features of the covariant model structure over Segal spaces. The goal is to generalize all those results to the covariant model structure over an arbitrary simplicial space. An important step is to have a precise characterization of left fibrations over  $F(n)$  and a computationally feasible way for characterizing covariant equivalences over an arbitrary simplicial space. The goal of this section is to address both these concerns.

In Sect. 4.1 we prove the *simplicial Grothendieck construction* for categories (Theorem 4.18), which in particular gives us a characterization of left fibrations over  $F(n)$ . In Sect. 4.2, we will then use this characterization to prove the *recognition principle* for covariant equivalences (Theorem 4.41).

**Notation 4.1** For this section, recall that for a given category  $\mathcal{C}$ ,  $N^h\mathcal{C}$  is the simplicial space given as  $(\pi_2)^*N\mathcal{C}$ , which is a (generally non-complete) Segal space with  $(\pi_2)^*N\mathcal{C}_n$  being a discrete simplicial set given by the set  $\text{Fun}([n], \mathcal{C})$  (Notation 2.23).

#### 4.1 Grothendieck Construction over Categories

In Proposition 1.10 we constructed an adjunction between set-valued functors out of  $\mathcal{C}$  and functors over  $\mathcal{C}$ , which gives us an equivalence when we restrict to discrete Grothendieck opfibrations.

In this subsection we generalize this result and prove two Quillen equivalences between a model category of space valued functors out of  $\mathcal{C}$  and a model category of left fibrations over  $N^h\mathcal{C}$ . We will then use this result to give a precise characterization of left fibrations over  $F(n)$ .

**Definition 4.2** Let  $\mathcal{C}$  be a small category. We define the *projective* model structure on the functor category  $\text{Fun}(\mathcal{C}, \mathcal{S})$  as follows.

- (F) A natural transformation  $\alpha : G \rightarrow H$  is a projective fibration if and only if for every object  $c$  in  $\mathcal{C}$  the map  $\alpha_c : G(c) \rightarrow H(c)$  is a Kan fibration.

- (W) A natural transformation  $\alpha : G \rightarrow H$  is a projective equivalence if and only if for every object  $c$  in  $\mathcal{C}$  the map  $\alpha_c : G(c) \rightarrow H(c)$  is a Kan equivalence.
- (C) A natural transformation is a projective cofibration if it satisfies the left lifting property with respect to all trivial projective fibrations.

The projective model structure on  $\text{Fun}(\mathcal{C}, \mathcal{S})$  exists [42, Proposition A.2.8.2]. Recall that for a given simplicial set  $S$  we denote the constant functor as  $\{S\} : \mathcal{C} \rightarrow \mathcal{S}$  (Sect. 0.7).

**Remark 4.3** The projective model structure has many desirable properties.

- (1) It is proper.
- (2) It is combinatorial.
- (3) It is a simplicial model category, with simplicial enrichment given by

$$\text{Map}(F, G)_n = \text{Nat}(F \times \{\Delta[n]\}, G).$$

- (4) It is compatible with Cartesian closure of the underlying category: If  $A \rightarrow B$  and  $C \rightarrow D$  are cofibrations then  $(A \rightarrow B) \square (C \rightarrow D)$  is a cofibration, which is trivial if either is trivial.

**Remark 4.4** Using the isomorphism of functor categories

$$\text{Fun}(\mathcal{C}, \mathcal{S}) \cong \text{Fun}(\Delta^{op}, \text{Fun}(\mathcal{C}, \text{Set}))$$

we can think of a space valued functor  $G : \mathcal{C} \rightarrow \mathcal{S}$  as a simplicial object in set valued functors  $G_\bullet : \mathcal{C} \rightarrow \text{Set}$ . Thus we will often switch between those when required.

Our first step is to generalize the adjunction from Proposition 1.10.

**Definition 4.5** Let

$$s \int_{\mathcal{C}} : \text{Fun}(\mathcal{C}, \mathcal{S}) \rightarrow s\mathcal{S}_{/N^h\mathcal{C}}$$

be the functor that applies  $\int_{\mathcal{C}}$  level-wise to the functor  $G : \mathcal{C} \rightarrow \mathcal{S}$ , meaning  $(s \int_{\mathcal{C}} G)_{nl} = (N \int_{\mathcal{C}}(G_l))_n$ .

**Remark 4.6** By direct computation, the simplicial space  $s \int_{\mathcal{C}} G$  is level-wise equal to

$$\left( s \int_{\mathcal{C}} G \right)_n = \coprod_{c_0 \rightarrow \dots \rightarrow c_n} G(c_0)$$

with projection  $(s \int_{\mathcal{C}} G)_n \rightarrow N^h\mathcal{C}_n$  taking an element  $(c_0 \rightarrow \dots \rightarrow c_n, x)$  to  $c_0 \rightarrow \dots \rightarrow c_n$ .

**Definition 4.7** Let

$$s\mathcal{T}_{\mathcal{C}} : s\mathcal{S}_{/N^h\mathcal{C}} \rightarrow \text{Fun}(\mathcal{C}, \mathcal{S})$$

be the functor defined as the left Kan extension of the functor

$$s\mathcal{T}_{\mathcal{C}}(p : F(n) \times \Delta[l] \rightarrow N^h\mathcal{C}) = \text{Hom}(p(0, 0), -) \times \Delta[l]$$

**Definition 4.8** Let

$$s\mathcal{H}_{\mathcal{C}} : s\mathcal{S}_{/N^h\mathcal{C}} \rightarrow \text{Fun}(\mathcal{C}, \mathcal{S})$$

be the functor that takes a map  $p : Y \rightarrow N^h\mathcal{C}$  to the functor

$$\mathcal{C} \xrightarrow{e_{-/}} \mathcal{C}at_{/\mathcal{C}} \xrightarrow{N^h} s\mathcal{S}_{/N^h\mathcal{C}} \xrightarrow{\text{Map}_{/N^h\mathcal{C}}(-, Y)} \mathcal{S}$$

meaning that for an object  $c$  in  $\mathcal{C}$  the value is given by

$$s\mathcal{H}_{\mathcal{C}}(p : Y \rightarrow N^h\mathcal{C})(c) = \text{Map}_{/N^h\mathcal{C}}(N^h\mathcal{C}_{c/}, Y).$$

**Lemma 4.9** *The functors  $s\mathcal{T}_{\mathcal{C}}, s\mathcal{f}_{\mathcal{C}}, s\mathcal{H}_{\mathcal{C}}$  give us two adjunctions  $(s\mathcal{T}_{\mathcal{C}} \dashv s\mathcal{f}_{\mathcal{C}}), (s\mathcal{f}_{\mathcal{C}} \dashv s\mathcal{H}_{\mathcal{C}})$*

$$\begin{array}{ccc} & s\mathcal{T}_{\mathcal{C}} & \\ & \downarrow & \\ \text{Fun}(\mathcal{C}, \mathcal{S}) & \xrightarrow{s\mathcal{f}_{\mathcal{C}}} & s\mathcal{S}_{/N^h\mathcal{C}} \\ & \uparrow & \\ & s\mathcal{H}_{\mathcal{C}} & \end{array}$$

Moreover, the adjunction  $(s\mathcal{f}_{\mathcal{C}} \dashv s\mathcal{H}_{\mathcal{C}})$  is simplicially enriched.

**Proof** By definition  $s\mathcal{T}_{\mathcal{C}}$  commutes with colimits. Hence it suffices to observe that we have the following natural bijections

$$\begin{aligned} \text{Nat}(s\mathcal{T}_{\mathcal{C}}(p : F(n) \times \Delta[l] \rightarrow N^h\mathcal{C}), G) &\cong \text{Nat}(\text{Hom}(p(0, 0), -) \times \Delta[l], G) \cong G_l(p(0, 0)) \cong \\ &\text{Hom}_{/N^h\mathcal{C}}(p : F(n) \times \Delta[l] \rightarrow N^h\mathcal{C}, s\int_{\mathcal{C}} G) \end{aligned}$$

which establishes the adjunction  $s\mathcal{T}_{\mathcal{C}} \dashv s\mathcal{f}_{\mathcal{C}}$  (here  $G_l : \mathcal{C} \rightarrow \text{Set}$  is as described in Remark 4.4).

On the other hand we have

$$\begin{aligned} \text{Hom}_{/N^h\mathcal{C}}(s\int_{\mathcal{C}}(\text{Hom}(c, -) \times \Delta[l]) \rightarrow N^h\mathcal{C}, p : Y \rightarrow N^h\mathcal{C}) &\cong \\ \text{Hom}_{/N^h\mathcal{C}}(N^h\mathcal{C}_{c/} \times \Delta[l] \rightarrow N^h\mathcal{C}, p : Y \rightarrow N^h\mathcal{C}) &= s\mathcal{H}_{\mathcal{C}}(p : Y \rightarrow N^h\mathcal{C})_l(c) \cong \\ \text{Nat}(\text{Hom}(c, -) \times \Delta[l], s\mathcal{H}_{\mathcal{C}}(p : Y \rightarrow N^h\mathcal{C})) & \end{aligned}$$

which establishes the adjunction  $s\mathcal{f}_{\mathcal{C}} \dashv s\mathcal{H}_{\mathcal{C}}$ .

Following [63, Proposition 3.7.10], in order to show the adjunction is simplicially enriched it suffices to show that the adjunction preserves the simplicial tensor. Following the explicit description of the tensor given in Sect. 2.3 and Remark 4.3, this means we need to prove  $s\mathcal{f}_{\mathcal{C}}(F \times K) \cong (s\mathcal{f}_{\mathcal{C}} F) \times K \xrightarrow{\pi_1} s\mathcal{f}_{\mathcal{C}} F \rightarrow N^h\mathcal{C}$ . However, it is an immediate computation that  $s\mathcal{f}_{\mathcal{C}}\{K\} = K \times N^h\mathcal{C} \xrightarrow{\pi_2} N^h\mathcal{C}$ . Moreover,  $s\mathcal{f}_{\mathcal{C}}$  is a right adjoint and hence commutes with products and so we have  $s\mathcal{f}_{\mathcal{C}}(F \times K) \cong s\mathcal{f}_{\mathcal{C}} F \times_{N^h\mathcal{C}} K \times N^h\mathcal{C} \cong s\mathcal{f}_{\mathcal{C}} F \times K$  over  $N^h\mathcal{C}$  and hence we are done.  $\square$

We would like to prove that if a functor  $G : \mathcal{C} \rightarrow \mathcal{S}$  is valued in Kan complexes then the map of simplicial spaces  $s\mathcal{f}_{\mathcal{C}} G \rightarrow N^h\mathcal{C}$  is a left fibration. However, this does not hold in general and so we instead have the following lemma.

**Lemma 4.10** *Let  $G : \mathcal{C} \rightarrow \mathcal{S}$  be a functor. Then  $s\mathcal{f}_{\mathcal{C}} G \rightarrow N^h\mathcal{C}$  is a left morphism. Moreover, let  $R s\mathcal{f}_{\mathcal{C}} G \rightarrow N^h\mathcal{C}$  be a Reedy fibrant replacement of  $s\mathcal{f}_{\mathcal{C}} G \rightarrow N^h\mathcal{C}$ . Then  $R s\mathcal{f}_{\mathcal{C}} G \rightarrow N^h\mathcal{C}$  is a left fibration.*

**Proof** First we observe that  $s\mathcal{f}_{\mathcal{C}} G \rightarrow N^h\mathcal{C}$  is a left morphism. Notice the square



$$\begin{array}{ccc}
 \coprod_{c_0 \rightarrow \dots \rightarrow c_n} G(c_0) & \longrightarrow & \coprod_{c_0} G(c_0) \\
 \downarrow & & \downarrow \\
 N^h \mathcal{C}_n & \longrightarrow & N^h \mathcal{C}_0
 \end{array}$$

is a pullback as an element in  $\coprod_{c_0 \rightarrow \dots \rightarrow c_n} G(c_0)_k$  is precisely a choice of  $c_0 \rightarrow \dots \rightarrow c_n$  in  $N^h \mathcal{C}_n$  along with a choice of element in  $G(c_0)_k$ . Moreover, the diagram is already a homotopy pullback square as  $N^h \mathcal{C}_n \rightarrow N^h \mathcal{C}_0$  is a Kan fibration.

Now, let  $Rsf_{\mathcal{C}} G \rightarrow N^h \mathcal{C}$  be a Reedy fibrant replacement. Notice, it is a Reedy fibration by definition and so we only need to verify the locality condition. As  $N^h \mathcal{C}$  is a discrete simplicial space, the map  $N^h \mathcal{C}_n \rightarrow N^h \mathcal{C}_0$  is a Kan fibration for all  $n$ , and so by right properness of the Kan model structure the induced map on pullbacks  $(sf_{\mathcal{C}} G)_0 \times_{N^h \mathcal{C}_0} N^h \mathcal{C}_n \rightarrow (Rsf_{\mathcal{C}} G)_0 \times_{N^h \mathcal{C}_0} N^h \mathcal{C}_n$  is a Kan equivalence. Hence, for every  $n \geq 1$ , we have the following commutative diagram where three sides are weak equivalences

$$\begin{array}{ccc}
 (sf_{\mathcal{C}} G)_n & \xrightarrow{\simeq} & (Rsf_{\mathcal{C}} G)_n \\
 \downarrow \cong & & \downarrow \\
 (sf_{\mathcal{C}} G)_0 \times_{N^h \mathcal{C}_0} N^h \mathcal{C}_n & \xrightarrow{\simeq} & (Rsf_{\mathcal{C}} G)_0 \times_{N^h \mathcal{C}_0} N^h \mathcal{C}_n
 \end{array}$$

and so the desired result follows from 2-out-of-3. □

We can use this observation to determine when  $sf_{\mathcal{C}} \alpha$  is a covariant equivalence.

**Lemma 4.11** *Let  $\alpha : G \rightarrow H$  be a natural transformation. Then  $\alpha$  is a projective equivalence if and only if  $sf_{\mathcal{C}} \alpha$  is a covariant equivalence.*

**Proof** Let  $Rsf_{\mathcal{C}} \alpha : Rsf_{\mathcal{C}} G \rightarrow Rsf_{\mathcal{C}} H$  be a Reedy fibrant replacement of  $sf_{\mathcal{C}} \alpha$ . By the previous lemma  $Rsf_{\mathcal{C}} G \rightarrow N^h \mathcal{C}$ ,  $Rsf_{\mathcal{C}} H \rightarrow N^h \mathcal{C}$  are left fibrations and so  $sf_{\mathcal{C}} \alpha$  is an equivalence if and only if  $(Rsf_{\mathcal{C}} G)_0 \rightarrow (Rsf_{\mathcal{C}} H)_0$  is a Kan equivalence. We now have the following diagram

$$\begin{array}{ccc}
 \coprod_{c_0} G(c_0) & \longrightarrow & \coprod_{c_0} H(c_0) \\
 \downarrow \simeq & & \downarrow \simeq \\
 (Rsf_{\mathcal{C}} G)_0 & \longrightarrow & (Rsf_{\mathcal{C}} H)_0
 \end{array}$$

The vertical maps are Kan equivalences as Reedy equivalences are level-wise Kan equivalences. Hence the top map is an equivalence (which is equivalent to  $\alpha$  being a projective equivalence) if and only if the bottom map is an equivalence (which is equivalent for  $sf_{\mathcal{C}} \alpha$  to be a covariant equivalence). □

Although  $sf_{\mathcal{C}} G \rightarrow N^h \mathcal{C}$  has many desirable properties it is generally not a left fibration, because it is not a Reedy fibration. Hence, we need to define an alternative, yet equivalent, functor that takes projectively fibrant functors to left fibrations. The following remark can help guide us towards a working definition.

**Remark 4.12** In Proposition 1.10 the left adjoint of  $\int_{\mathcal{C}}$ , denoted  $\mathcal{T}_{\mathcal{C}}$ , was defined as  $\mathcal{T}_{\mathcal{C}}(p : \mathcal{D} \rightarrow \mathcal{C})(c) = \pi_0(\mathcal{C}_{/c} \times_{\mathcal{C}} \mathcal{D})$ . This exactly coincides with the left Kan extension of the functor that takes the functor  $p : [n] \rightarrow \mathcal{C}$  to the representable functor  $\text{Hom}(p(0), -)$ .

This would suggest that the correct simplicial generalization should take  $N^h p : N^h \mathcal{D} \rightarrow N^h \mathcal{C}$  to the functor which takes  $c$  to the simplicial set  $N(\mathcal{C}_{/c} \times_{\mathcal{C}} \mathcal{D})$ , as in that case we would have an isomorphism of set valued functors  $\pi_0(N(\mathcal{C}_{/c} \times_{\mathcal{C}} \mathcal{D})) \cong \mathcal{T}_{\mathcal{C}}(p : \mathcal{D} \rightarrow \mathcal{C})$ . Indeed for any category  $\mathcal{C}$  we have a bijection  $\pi_0 N \mathcal{C} \cong \pi_0 \mathcal{C}$ .

However, this is clearly not the case. Indeed, as we proved in Lemma 4.9, the left adjoint is given by the functor  $s\mathcal{T}_{\mathcal{C}}$ , which, following Definition 4.7, satisfies  $s\mathcal{T}_{\mathcal{C}}(p : F(n) \rightarrow N^h \mathcal{C}) = \text{Hom}_{\mathcal{C}}(p(0), -)$ , which is clearly not the same as  $N(\mathcal{C}_{/c} \times_{\mathcal{C}} [n])$ .

Building on this remark, we want to define an appropriate analogue of  $s\mathcal{T}_{\mathcal{C}}$  with the correct values, which we will label  $s\mathbb{T}_{\mathcal{C}}$ .

**Definition 4.13** Let

$$s\mathbb{T}_{\mathcal{C}} : s\mathcal{S}_{/N^h \mathcal{C}} \rightarrow \text{Fun}(\mathcal{C}, \mathcal{S})$$

be the functor that takes a map  $Y \rightarrow N^h \mathcal{C}$  to the functor

$$\mathcal{C} \xrightarrow{\mathcal{C}_{/-}} \text{Cat}_{/\mathcal{C}} \xrightarrow{N^h} s\mathcal{S}_{/N^h \mathcal{C}} \xrightarrow{- \times_{N^h \mathcal{C}} Y} s\mathcal{S} \xrightarrow{\text{Diag}^*} \mathcal{S},$$

concretely meaning the values are given as follows

$$s\mathbb{T}_{\mathcal{C}}(Y \rightarrow N^h \mathcal{C})(c) = \text{Diag}^*(N^h \mathcal{C}_{/c} \times_{N^h \mathcal{C}} Y) = N\mathcal{C}_{/c} \times_{N\mathcal{C}} \text{Diag}^*(Y).$$

For the next definition we use the fact that a map  $F(n) \times \Delta[l] \rightarrow N^h \mathcal{C}$  corresponds to a functor  $[n] \rightarrow \mathcal{C}$ .

For the next definition recall that there is an isomorphism of categories  $\text{Fun}(((\Delta \times \Delta)_{/N^h \mathcal{C}})^{op}, \text{Set}) \cong s\mathcal{S}_{/N^h \mathcal{C}}$  [44, Page 157]. Here  $(\Delta \times \Delta)_{/N^h \mathcal{C}}$  is the full subcategory of  $s\mathcal{S}_{/N^h \mathcal{C}}$  with objects  $F(n) \times \Delta[l] \rightarrow N^h \mathcal{C}$ .

**Definition 4.14** Define

$$s\mathbb{I}_{\mathcal{C}} : \text{Fun}(\mathcal{C}, \mathcal{S}) \rightarrow s\mathcal{S}_{/N^h \mathcal{C}}$$

as the functor that via the adjunction between products and exponentials corresponds to

$$\text{Nat}(s\mathbb{T}_{\mathcal{C}} \text{Yon}(-), -) : ((\Delta \times \Delta)_{/N^h \mathcal{C}})^{op} \times \text{Fun}(\mathcal{C}, \text{Set}) \rightarrow \text{Set},$$

using the isomorphism  $\text{Fun}(((\Delta \times \Delta)_{/N^h \mathcal{C}})^{op}, \text{Set}) \cong s\mathcal{S}_{/N^h \mathcal{C}}$ . Unwinding this definition, it takes  $G$  to the simplicial space  $s\mathbb{I}_{\mathcal{C}}(G) \rightarrow N^h \mathcal{C}$ , whose morphisms from  $p : F(n) \times \Delta[l] \rightarrow N^h \mathcal{C}$  over  $N^h \mathcal{C}$  are given by

$$\text{Hom}_{/N^h \mathcal{C}}(F(n) \times \Delta[l], s\mathbb{I}_{\mathcal{C}}(G)) \cong \text{Nat}(N([n] \times_{\mathcal{C}} \mathcal{C}_{/-} \times \{\Delta[l]\}, G).$$

Here  $[n] \rightarrow \mathcal{C}$  is the functor that corresponds to the map  $p : F(n) \times \Delta[l] \rightarrow N^h \mathcal{C}$ .

It will follow from Theorem 4.18 that for a projectively fibrant functor  $G$ ,  $s\mathbb{I}_{\mathcal{C}}(G)$  is in fact a left fibration and equivalent to  $s\int_{\mathcal{C}} G$ , giving us a fibrant replacement.

We now establish that these functors are in fact adjoints.

**Proposition 4.15** *The functors  $s\mathbb{T}_{\mathcal{C}}$ ,  $s\mathbb{I}_{\mathcal{C}}$  form a simplicially enriched adjunction*

$$s\mathcal{S}_{/N^h\mathcal{C}} \begin{array}{c} \xrightarrow{s\mathbb{T}_{\mathcal{C}}} \\ \perp \\ \xleftarrow{s\mathbb{I}_{\mathcal{C}}} \end{array} \text{Fun}(\mathcal{C}, \mathcal{S})$$

that is natural up to isomorphism in  $\mathcal{C}$ .

**Proof** First we show the functor  $s\mathbb{T}_{\mathcal{C}}$  commutes with colimits. As colimits in  $\text{Fun}(\mathcal{C}, \mathcal{S})$  are evaluated point-wise, we only need to confirm that for every object  $c$  in  $\mathcal{C}$ , the functor  $N\mathcal{C}_{/c} \times_{N\mathcal{C}} \text{Diag}^*(-) : s\mathcal{S}_{/N^h\mathcal{C}} \rightarrow \mathcal{S}$  preserves colimits. This follows immediately from the fact that  $\text{Diag}^*$  preserves colimits (it is a left adjoint) and that  $N^h\mathcal{C}_{/c} \times_{N^h\mathcal{C}} -$  preserves colimits ( $\mathcal{S}$  is locally Cartesian closed). Now, following [44, Corollary I.5.4], every colimit preserving functor out of  $s\mathcal{S}_{/N^h\mathcal{C}}$  is uniquely determined by its restriction to  $\Delta \times \Delta_{/N^h\mathcal{C}}$  and has a right adjoint, which is by definition given by  $s\mathbb{I}_{\mathcal{C}}$ .

We now move on to show the adjunction is enriched. Again, by [63, Proposition 3.7.10], we need to prove that  $s\mathbb{T}_{\mathcal{C}}$  preserves the simplicial tensor. Based on Sect. 2.3 and Remark 4.3 we need to prove that for a given simplicial set  $K$ ,  $s\mathbb{T}_{\mathcal{C}}(Y \times K \xrightarrow{\pi_1} Y \rightarrow N^h\mathcal{C}) \cong s\mathbb{T}_{\mathcal{C}}(Y \rightarrow N^h\mathcal{C}) \times \{K\}$ . This is a direct computation as for every object  $c$  in  $\mathcal{C}$  we have

$$s\mathbb{T}_{\mathcal{C}}(Y \times K \xrightarrow{\pi_1} Y \rightarrow N^h\mathcal{C})(c) = (\text{Diag}^*(Y) \times_{N\mathcal{C}} N\mathcal{C}_{/c}) \times K.$$

Hence, giving us the desired isomorphism of functors  $s\mathbb{T}_{\mathcal{C}}(Y \times K \xrightarrow{\pi_1} Y \rightarrow N^h\mathcal{C}) \cong s\mathbb{T}_{\mathcal{C}}(Y \rightarrow N^h\mathcal{C}) \times \{K\}$ .

Finally, we move on to show the adjunction is natural. Fix a functor  $\alpha : \mathcal{C} \rightarrow \mathcal{D}$ . We want to show the diagram of adjunctions

$$\begin{array}{ccc} s\mathcal{S}_{/N^h\mathcal{C}} & \begin{array}{c} \xrightarrow{s\mathbb{T}_{\mathcal{C}}} \\ \perp \\ \xleftarrow{s\mathbb{I}_{\mathcal{C}}} \end{array} & \text{Fun}(\mathcal{C}, \mathcal{S}) \\ \alpha^* \uparrow \downarrow \alpha_! & & (\alpha)^* \uparrow \downarrow \alpha_! \\ s\mathcal{S}_{/N^h\mathcal{D}} & \begin{array}{c} \xrightarrow{s\mathbb{T}_{\mathcal{D}}} \\ \perp \\ \xleftarrow{s\mathbb{I}_{\mathcal{D}}} \end{array} & \text{Fun}(\mathcal{D}, \mathcal{S}) \end{array}$$

commutes up to isomorphism. By uniqueness of right adjoints it suffices to show that the diagram of left adjoints commute. As left adjoints commute with colimits it suffices to prove that  $\alpha_! \circ s\mathbb{T}_{\mathcal{C}}(F(n) \times \Delta[I] \rightarrow N^h\mathcal{C}) \cong s\mathbb{T}_{\mathcal{D}} \circ \alpha_!(F(n) \times \Delta[I] \rightarrow N^h\mathcal{C})$ . We have shown that  $s\mathbb{T}_{\mathcal{C}}, s\mathbb{T}_{\mathcal{D}}$  are simplicially enriched. Moreover  $\alpha_! : \text{Fun}(\mathcal{C}, \mathcal{S}) \rightarrow \text{Fun}(\mathcal{D}, \mathcal{S})$  is also simplicially enriched [42, Proposition A.3.3.6]. Hence, we can reduce the computation to showing that  $\alpha_! \circ s\mathbb{T}_{\mathcal{C}}(F(n) \rightarrow N^h\mathcal{C}) \cong s\mathbb{T}_{\mathcal{D}} \circ \alpha_!(F(n) \rightarrow N^h\mathcal{C})$ .

We will start with the case  $n = 0$ . In that case a morphism  $\{c\} : F(0) \rightarrow N^h\mathcal{C}$  is given by a choice of object in  $\mathcal{C}$  and by direct computation (Definition 4.13)  $s\mathbb{T}_{\mathcal{C}}(F(0) \rightarrow N^h\mathcal{C}) = \text{Hom}_{\mathcal{C}}(c, -)$  and  $s\mathbb{T}_{\mathcal{D}}(F(0) \rightarrow N^h\mathcal{C} \rightarrow N^h\mathcal{D}) = \text{Hom}_{\mathcal{D}}(\alpha(c), -)$  thought of as a discrete simplicial sets. The desired result  $\alpha_! \text{Hom}_{\mathcal{C}}(c, -) \cong \text{Hom}_{\mathcal{D}}(\alpha(c), -)$  is now a direct computation (see also [45, Page 236]).

We move on to the case  $n > 0$ . Fix a morphism  $F(n) \rightarrow N^h\mathcal{C}$  given by a chain  $c_0 \rightarrow \dots \rightarrow c_n$ . Notice for  $k \geq 0$ , we have

$$s\mathbb{T}_{\mathcal{C}}(F(n) \rightarrow N^h\mathcal{C})_k = \coprod_{\gamma: [k] \rightarrow [n]} \text{Hom}_{\mathcal{C}}(c_{\gamma(k)}, -).$$

Hence, we can repeat the argument of the previous paragraph level-wise (using the fact that left adjoints commute with coproducts) to deduce the desired isomorphism  $\alpha_! \circ s\mathbb{T}_{\mathcal{C}}(F(n) \rightarrow N^h\mathcal{C}) \cong s\mathbb{T}_{\mathcal{D}} \circ \alpha_!(F(n) \rightarrow N^h\mathcal{C})$ , finishing the proof.  $\square$

We now want to prove that these adjunctions are Quillen adjunctions. For that we first need to show that they interact well with the model structures.

**Lemma 4.16** *The functor  $s\mathbb{f}_{\mathcal{C}}$  takes (trivial) projective cofibrations to (trivial) covariant cofibrations.*

**Proof** It suffices to check  $s\mathbb{f}_{\mathcal{C}}$  preserves the generating cofibrations and trivial cofibrations, which, following [42, Remark A.2.8.5], are given by the cofibrations  $\partial\Delta[n] \times \text{Hom}_{\mathcal{C}}(c, -) \rightarrow \Delta[n] \times \text{Hom}_{\mathcal{C}}(c, -)$  and the trivial cofibrations  $\Lambda[n]_i \times \text{Hom}_{\mathcal{C}}(c, -) \rightarrow \Delta[n] \times \text{Hom}_{\mathcal{C}}(c, -)$ .

Observe that  $s\mathbb{f}_{\mathcal{C}}$  takes the generating cofibrations

$$\partial\Delta[n] \times \text{Hom}_{\mathcal{C}}(c, -) \rightarrow \Delta[n] \times \text{Hom}_{\mathcal{C}}(c, -)$$

to the cofibrations

$$\partial\Delta[n] \times N(\mathcal{C}_{c/}) \rightarrow \Delta[n] \times N(\mathcal{C}_{c/})$$

and similarly the generating trivial cofibrations

$$\Lambda[n]_i \times \text{Hom}_{\mathcal{C}}(c, -) \rightarrow \Delta[n] \times \text{Hom}_{\mathcal{C}}(c, -)$$

to the morphisms

$$\Lambda[n]_i \times N(\mathcal{C}_{c/}) \rightarrow \Delta[n] \times N(\mathcal{C}_{c/}),$$

which is a trivial cofibration in the Reedy model structure over  $N^h\mathcal{C}$  and hence also a covariant equivalence.  $\square$

**Lemma 4.17** *The functor  $s\mathbb{I}_{\mathcal{C}}$  takes (trivial) projective fibrations to (trivial) fibrations in the covariant model structure over  $N^h\mathcal{C}$ .*

**Proof** Let  $\alpha : G \rightarrow H$  be a projective fibration. We need to prove that  $s\mathbb{I}_{\mathcal{C}}(\alpha)$  has the right lifting property with respect to maps

- $(\partial F(n) \rightarrow F(n)) \square (\Lambda[I]_i \rightarrow \Delta[I]) \rightarrow N^h\mathcal{C}$ ,
- $(F(0) \rightarrow F(n)) \square (\partial\Delta[I] \rightarrow \Delta[I]) \rightarrow N^h\mathcal{C}$ .

Using the adjunction  $(s\mathbb{T}_{\mathcal{C}} \dashv s\mathbb{I}_{\mathcal{C}})$  this is equivalent to proving that

- $s\mathbb{T}_{\mathcal{C}}((\partial F(n) \rightarrow F(n)) \square (\Lambda[I]_i \rightarrow \Delta[I]) \rightarrow N^h\mathcal{C})$ ,
- $s\mathbb{T}_{\mathcal{C}}((F(0) \rightarrow F(n)) \square (\partial\Delta[I] \rightarrow \Delta[I]) \rightarrow N^h\mathcal{C})$

are trivial projective cofibrations in  $\text{Fun}(\mathcal{C}, \mathcal{S})$ .

By direct computation these are equal to

- $s\mathbb{T}_{\mathcal{C}}(\partial F(n) \rightarrow F(n) \rightarrow N^h\mathcal{C}) \square \{\Lambda[I]_i\} \rightarrow \{\Delta[I]\}$ ,
- $s\mathbb{T}_{\mathcal{C}}(F(0) \rightarrow F(n) \rightarrow N^h\mathcal{C}) \square \{\partial\Delta[I]\} \rightarrow \{\Delta[I]\}$ .

The map  $\{\Lambda[I]_i\} \rightarrow \{\Delta[I]\}$  is a trivial projective cofibration and  $\{\partial\Delta[I]\} \rightarrow \{\Delta[I]\}$  is a projective cofibration. Hence, by Remark 4.3, it suffices to prove that  $s\mathbb{T}_{\mathcal{C}}(\partial F(n) \rightarrow F(n) \rightarrow N^h\mathcal{C})$  is a projective cofibration and  $s\mathbb{T}_{\mathcal{C}}(F(0) \rightarrow F(n) \rightarrow N^h\mathcal{C})$  is a projective trivial cofibration.

Fix a morphism  $\alpha : F(n) \rightarrow N^h\mathcal{C}$  and notice it comes from a functor  $[n] \rightarrow \mathcal{C}$ , which we also denote by  $\alpha$ . By Proposition 4.15, we now have the following commutative diagram of adjunctions

$$\begin{array}{ccc}
 (s\mathcal{S}_{/F(n)})^{cov} & \xrightleftharpoons[s\mathbb{I}_{[n]}]{s\mathbb{T}_{[n]}} & \text{Fun}([n], \mathcal{S})^{proj} \\
 \alpha^* \uparrow \downarrow \alpha! & & (\alpha)^* \uparrow \downarrow \alpha! \\
 (s\mathcal{S}_{/N^h\mathcal{C}})^{cov} & \xrightleftharpoons[s\mathbb{I}_{\mathcal{C}}]{s\mathbb{T}_{\mathcal{C}}} & \text{Fun}(\mathcal{C}, \mathcal{S})^{proj}
 \end{array}$$

The right hand adjunction is a Quillen adjunction of projective model structures [42, Proposition A.2.8.7], meaning  $\alpha_!$  preserves (trivial) cofibrations. Hence it suffices to prove  $s\mathbb{T}_{[n]}(\partial F(n) \rightarrow F(n) \rightarrow F(n))$  is a cofibration and  $s\mathbb{T}_{[n]}(F(0) \rightarrow F(n) \rightarrow F(n))$  is a trivial cofibration. By direct computation for  $i < n$  we have

$$\begin{aligned}
 s\mathbb{T}_{[n]}(\partial F(n) \rightarrow F(n))(i) &= \text{Diag}^*(\partial F(n) \times_{F(n)} F(i)) = \text{Diag}^*(F(n) \times_{F(n)} F(i)) \\
 &= s\mathbb{T}_{[n]}(F(n) \rightarrow F(n))(i)
 \end{aligned}$$

and for  $i = n$  we have

$$s\mathbb{T}_{[n]}(\partial F(n) \rightarrow F(n))(n) = \text{Diag}^*(\partial F(n) \times_{F(n)} F(n)) = \partial \Delta[n].$$

This means we have the following pushout square in  $\text{Fun}(\mathcal{C}, \mathcal{S})$

$$\begin{array}{ccc}
 \text{Hom}(n, -) \times \partial \Delta[n] & \longrightarrow & s\mathbb{T}_{[n]}(\partial F(n) \rightarrow F(n)) \\
 \downarrow & & \downarrow \\
 \text{Hom}(n, -) \times \Delta[n] & \longrightarrow & s\mathbb{T}_{[n]}(F(n) \rightarrow F(n))
 \end{array}$$

proving the desired morphism is a projective cofibration.

Finally, by direct computation

$$\begin{aligned}
 s\mathbb{T}_{[n]}(F(0) \rightarrow F(n))(i) &= \text{Diag}^*(F(0) \times_{F(n)} F(i)) = \Delta[0] \xrightarrow{\cong} \Delta[i] \\
 &= \text{Diag}^*(F(n) \times_{F(n)} F(i)) = s\mathbb{T}_{[n]}(F(n) \rightarrow F(n))(i),
 \end{aligned}$$

giving us the desired equivalence.

Now, let us assume that  $\alpha : G \rightarrow H$  is a trivial projective fibration. We need to prove that  $s\mathbb{I}_{\mathcal{C}}(\alpha)$  has the right lifting property with respect to maps of the form  $(\partial F(n) \rightarrow F(n)) \square (\partial \Delta[l] \rightarrow \Delta[l]) \rightarrow N^h\mathcal{C}$ , which is equivalent to establishing that

$$s\mathbb{T}_{\mathcal{C}}((\partial F(n) \rightarrow F(n)) \square (\partial \Delta[l] \rightarrow \Delta[l])) \cong s\mathbb{T}_{\mathcal{C}}(\partial F(n) \rightarrow F(n) \rightarrow N^h\mathcal{C}) \square \{\partial \Delta[l] \rightarrow \Delta[l]\}$$

is a projective cofibration. However, we already showed above that  $s\mathbb{T}_{\mathcal{C}}(\partial F(n) \rightarrow F(n) \rightarrow N^h\mathcal{C})$  is a projective cofibration and so the desired result follows from the fact that the projective model structure is simplicial.  $\square$

**Theorem 4.18** *Let  $\mathcal{C}$  be a small category. The two simplicially enriched adjunctions*

$$\text{Fun}(\mathcal{C}, \mathcal{S})^{proj} \xrightleftharpoons[s\mathcal{H}_{\mathcal{C}}]{s\mathcal{f}_{\mathcal{C}}} (s\mathcal{S}_{/N^h\mathcal{C}})^{cov} \xrightleftharpoons[s\mathbb{I}_{\mathcal{C}}]{s\mathbb{T}_{\mathcal{C}}} \text{Fun}(\mathcal{C}, \mathcal{S})^{proj}$$

are Quillen equivalences, which are (up to equivalence) natural in  $\mathcal{C}$ . Moreover, the derived counit map  $s\mathcal{f}_{\mathcal{C}} Qs\mathcal{H}_{\mathcal{C}}L \rightarrow L$  is in fact a Reedy equivalence. Here  $\text{Fun}(\mathcal{C}, \mathcal{S})$  has the projective model structure and  $s\mathcal{S}_{/N^h\mathcal{C}}$  has the covariant model structure over  $N^h\mathcal{C}$ .

**Proof** First we show both are Quillen adjunctions. By Lemma 4.16,  $s\mathcal{f}_{\mathcal{C}}$  preserves cofibrations and trivial cofibrations and so is a left Quillen functor. On the other hand, by Lemma 3.14, a fibration between fibrant objects in the covariant model structure is a left fibration. By Lemma 4.17,  $s\mathbb{L}_{\mathcal{C}}$  takes projective fibrations to left fibrations, which means it takes projective fibrations between fibrant objects to covariant fibrations. By the same lemma,  $s\mathbb{L}_{\mathcal{C}}$  takes trivial projective fibrations to trivial covariant fibrations. Hence, by Lemma A.3, it is a right Quillen functor.

We move on to prove they are Quillen equivalences. Notice, the composition functor  $s\mathbb{T}_{\mathcal{C}} \circ s\mathcal{f}_{\mathcal{C}} : \text{Fun}(\mathcal{C}, \mathcal{S}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{S})$  is a colimit preserving functor that takes  $\text{Hom}(c, -) \times \{\Delta[I]\}$  to the functor  $N(\mathcal{C}_{c/} \times_{\mathcal{C}} \mathcal{C}_{-/}) \times \{\Delta[I]\}$ , which is naturally equivalent to  $\text{Hom}(c, -) \times \{\Delta[I]\}$ . Hence, the composition functor is naturally weakly equivalent to the identity and so a Quillen equivalence. Thus in order to prove both adjunctions are Quillen equivalences by 2-out-of-3 it suffices to prove  $s\mathcal{f}_{\mathcal{C}} \dashv s\mathcal{H}_{\mathcal{C}}$  is a Quillen equivalence.

By Lemma A.4, it suffices to prove that the derived counit map is an equivalence and  $s\mathcal{f}_{\mathcal{C}}$  reflects weak equivalences. We already proved that  $s\mathcal{f}_{\mathcal{C}}$  reflects weak equivalences in Lemma 4.11. Let  $L \rightarrow N^h\mathcal{C}$  be a left fibration. We want to prove that  $s\mathcal{f}_{\mathcal{C}} Qs\mathcal{H}_{\mathcal{C}}L \rightarrow L$  is a covariant equivalence, where  $Qs\mathcal{H}_{\mathcal{C}}L \rightarrow s\mathcal{H}_{\mathcal{C}}L$  is a cofibrant replacement of  $s\mathcal{H}_{\mathcal{C}}L$  in the projective model structure on  $\text{Fun}(\mathcal{C}, \mathcal{S})$ . We will in fact prove the derived counit is a Reedy equivalence, hence also proving the second statement of the theorem and so finishing the proof. It suffices to do so fiber-wise.

Fix a map  $F(n) \rightarrow N^h\mathcal{C}$  that we can represent by a diagram  $c_0 \rightarrow \dots \rightarrow c_n$ . As  $L$  is a left fibration we have an equivalence of spaces

$$\text{Map}_{/N^h\mathcal{C}}(F(n), L) \rightarrow \text{Map}_{/N^h\mathcal{C}}(\{c_0\}, L),$$

that restricts along the inclusion  $\langle 0 \rangle : F(0) \rightarrow F(n)$ . Moreover, by Remark 4.6,

$$\text{Map}_{/N^h\mathcal{C}}(F(n), s\int_{\mathcal{C}} Qs\mathcal{H}_{\mathcal{C}}L) \cong (Qs\mathcal{H}_{\mathcal{C}}L)(c_0) \xrightarrow{\cong} (s\mathcal{H}_{\mathcal{C}}L)(c_0) = \text{Map}_{/N^h\mathcal{C}}(N^h\mathcal{C}_{c_0/}, L),$$

which also restricts a morphism  $F(n) \rightarrow s\mathcal{f}_{\mathcal{C}}s\mathcal{H}_{\mathcal{C}}L$  along  $\langle 0 \rangle : F(0) \rightarrow F(n)$  to an element in  $s\mathcal{H}_{\mathcal{C}}L(c_0)$ . Hence, in order to establish the Quillen equivalence, we only have to show that the composition map

$$\begin{aligned} \text{Map}_{/N^h\mathcal{C}}(N^h\mathcal{C}_{c_0/}, L) &\simeq \text{Map}_{/N^h\mathcal{C}}(F(n), s\int_{\mathcal{C}} Qs\mathcal{H}_{\mathcal{C}}L) \rightarrow \text{Map}_{/N^h\mathcal{C}}(F(n), L) \\ &\rightarrow \text{Map}_{/N^h\mathcal{C}}(\{c_0\}, L) \end{aligned}$$

is a Kan equivalence. However, tracing through these morphisms, this morphism is induced by restricting along  $\{\text{id}_c\} \rightarrow \mathcal{C}_{c/}$ . Hence, this is an equivalence precisely by the statement of the Yoneda lemma for Segal spaces (Theorem 3.49).

Let  $\alpha : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. In order to finish the proof, we need to show that the diagram of adjunctions

$$\begin{array}{ccccc} \text{Fun}(\mathcal{C}, \mathcal{S})^{proj} & \xrightleftharpoons[s\mathcal{H}_{\mathcal{C}}]{s\mathcal{f}_{\mathcal{C}}} & (s\mathcal{S}_{/N^h\mathcal{C}})^{cov} & \xrightleftharpoons[s\mathbb{L}_{\mathcal{C}}]{s\mathbb{T}_{\mathcal{C}}} & \text{Fun}(\mathcal{C}, \mathcal{S})^{proj} \\ \alpha^* \uparrow \downarrow \alpha! & & \alpha^* \uparrow \downarrow \alpha! & & \alpha^* \uparrow \downarrow \alpha! \\ \text{Fun}(\mathcal{D}, \mathcal{S})^{proj} & \xrightleftharpoons[s\mathcal{H}_{\mathcal{D}}]{s\mathcal{f}_{\mathcal{D}}} & (s\mathcal{S}_{/N^h\mathcal{D}})^{cov} & \xrightleftharpoons[s\mathbb{L}_{\mathcal{D}}]{s\mathbb{T}_{\mathcal{D}}} & \text{Fun}(\mathcal{D}, \mathcal{S})^{proj} \end{array}$$

commutes up to natural equivalence. We have shown in Proposition 4.15 that the right hand square commutes up to natural isomorphism, so we only need to prove the left hand square commutes up to weak equivalence. We will prove that the right Quillen functors  $s\mathcal{H}_{\mathcal{C}} \circ \alpha^*$  and  $\alpha^* \circ s\mathcal{H}_{\mathcal{D}}$  are naturally equivalent for fibrant objects.

Fix a left fibration  $p : L \rightarrow N^h\mathcal{D}$ . Then

$$s\mathcal{H}_{\mathcal{C}}\alpha^*L(c) = \text{Map}_{/N^h\mathcal{C}}(N^h\mathcal{C}_{/c}, \alpha^*L) \cong \text{Map}_{/N^h\mathcal{D}}(N^h\mathcal{C}_{/c}, L).$$

Hence, in order to establish a natural projective equivalence  $s\mathcal{H}_{\mathcal{C}} \circ \alpha^*L \simeq \alpha^* \circ s\mathcal{H}_{\mathcal{D}}L$  it suffices to prove that the natural morphism  $\alpha^* : \text{Map}_{/N^h\mathcal{D}}(N^h\mathcal{C}_{/c}, L) \rightarrow \text{Map}_{/N^h\mathcal{D}}(N^h\mathcal{D}_{/\alpha(c)}, L)$  is a Kan equivalence. However, this again follows from the Yoneda lemma for Segal spaces (Theorem 3.49) combined with 2-out-of-3, as the morphism  $F(0) \xrightarrow{\{id_c\}} \mathcal{C}_{/c} \xrightarrow{\alpha} \mathcal{D}_{/\alpha(c)}$  induces a commutative diagram

$$\begin{array}{ccc}
 & & \text{Map}_{/N^h\mathcal{C}}(N^h\mathcal{C}_{/c}, L) \\
 & \nearrow^{\{id_c\}^*} & \downarrow^{\alpha^*} \\
 \text{Map}_{/N^h\mathcal{D}}(F(0), L) & \xrightarrow{\simeq} & \\
 & \searrow_{\{id_{\alpha(c)}\}^*} & \downarrow^{\alpha^*} \\
 & & \text{Map}_{/N^h\mathcal{D}}(N^h\mathcal{D}_{/\alpha(c)}, L)
 \end{array}$$

where the two left hand morphisms are Kan equivalences. □

**Remark 4.19** It is interesting to note how this result compares to a similar result in [33, Theorem C]. There the authors study a functor very similar to  $s\int_{\mathcal{C}}$  using quasi-categories, however, as they are using simplicial sets, their functor  $h_1$  is the diagonal of the level-wise Grothendieck construction. Thus, they cannot simply take a Reedy fibrant replacement (as we did in Lemma 4.11) to get a left fibration and thus have to apply more complicated techniques.

**Remark 4.20** The left side of the two Quillen equivalences has also been proven to be an equivalence by de Brito [19, Theorem A]. Moreover, both equivalences have been generalized to the  $(\infty, n)$ -categorical setting, which in particular would restrict to the equivalences given here [54, Theorem 5.50]

The Quillen equivalence can help us find fibrant replacements.

**Corollary 4.21** *Let  $Y \rightarrow N^h\mathcal{C}$  be a map of simplicial spaces. Then the derived unit map  $Y \rightarrow s\mathbb{I}_{\mathcal{C}}Rs\mathbb{T}_{\mathcal{C}}Y$  is the covariant fibrant replacement of  $Y \rightarrow N^h\mathcal{C}$ .*

There is one key example which we want to consider more explicitly. Let  $\mathcal{C} = [n]$ . Then  $N^h\mathcal{C} = F(n)$  (using Notation 4.1), and so the result implies that we have Quillen equivalences

$$\text{Fun}([n], \mathcal{S})^{proj} \begin{array}{c} \xrightarrow{s\int_{[n]} \\ \perp \\ \xleftarrow{s\mathcal{H}_{[n]}} \end{array} (s\mathcal{S}_{/F(n)})^{cov} \begin{array}{c} \xrightarrow{s\mathbb{T}_{[n]} \\ \perp \\ \xleftarrow{s\mathbb{I}_{[n]}} \end{array} \text{Fun}([n], \mathcal{S})^{proj} .$$

The main result (Theorem 4.18) and this diagram has important corollaries that we will use extensively.

**Corollary 4.22** *A morphism of simplicial spaces  $X \rightarrow Y$  over  $N^h\mathcal{C}$  is a covariant equivalence if and only if for all objects  $c$  in  $\mathcal{C}$  the morphism*

$$N^h\mathcal{C}_{/c} \times_{N^h\mathcal{C}} X \rightarrow N^h\mathcal{C}_{/c} \times_{N^h\mathcal{C}} Y$$

is a diagonal equivalence. In particular, a map of simplicial spaces  $X \rightarrow Y$  over  $F(n)$  is a covariant equivalence if and only if for all maps  $\langle 0, \dots, i \rangle : F(i) \rightarrow F(n)$  the induced map

$$F(i) \times_{F(n)} X \rightarrow F(i) \times_{F(n)} Y$$

is a diagonal equivalence for all  $0 \leq i \leq n$ .

**Proof** By Theorem 4.18,  $s\mathbb{T}_{\mathcal{C}}$  reflects weak equivalences meaning a morphism  $X \rightarrow Y$  over  $N^h\mathcal{C}$  is a weak equivalence if and only if  $s\mathbb{T}_{\mathcal{C}}(X \rightarrow N^h\mathcal{C}) \rightarrow s\mathbb{T}_{\mathcal{C}}(Y \rightarrow N^h\mathcal{C})$  is a projective equivalence. By definition of projective weak equivalences (Definition 4.2) this is equivalent to  $s\mathbb{T}_{\mathcal{C}}(X \rightarrow N^h\mathcal{C})(c) \rightarrow s\mathbb{T}_{\mathcal{C}}(Y \rightarrow N^h\mathcal{C})(c)$  being a Kan equivalence for all objects  $c$  in  $\mathcal{C}$ , which is the same as  $\text{Diag}^*(N^h\mathcal{C}_{/c} \times_{N^h\mathcal{C}} X) \rightarrow \text{Diag}^*(N^h\mathcal{C}_{/c} \times_{N^h\mathcal{C}} Y)$  being a Kan equivalence, which, by Theorem 2.11, means  $N^h\mathcal{C}_{/c} \times_{N^h\mathcal{C}} X \rightarrow N^h\mathcal{C}_{/c} \times_{N^h\mathcal{C}} Y$  is a diagonal equivalence for all objects  $c$  in  $\mathcal{C}$ .

Finally, by direct computation  $N^h([n]_{/i}) \rightarrow N^h([n])$  is exactly the map  $\langle 0, \dots, i \rangle : F(i) \rightarrow F(n)$ . This means for a given morphism  $X \rightarrow F(n)$ , we have  $s\mathbb{T}_{[n]}(X \rightarrow F(n))(i) = \text{Diag}^*(F(i) \times_{F(n)} X)$  and so the result follows from the previous paragraph.  $\square$

**Corollary 4.23** Every left fibration  $L \rightarrow F(n)$  is Reedy equivalent to a colimit (and hence a homotopy colimit) of left morphisms  $\langle 0, \dots, i \rangle \circ \pi_1 : F(i) \times \Delta[i] \rightarrow F(n)$ .

**Proof** Let  $L \rightarrow F(n)$  be a left fibration. Then by Theorem 4.18 there exists a functor  $G : [n] \rightarrow \mathcal{S}$  and a Reedy equivalence  $L \simeq s\int_{[n]} G$  over  $F(n)$  (concretely we can take  $G = s\mathcal{H}_{\mathcal{C}}(L)$ ). But  $G$  is a simplicial presheaf and so there is an isomorphism  $G \cong \text{colim}(\text{Hom}([i], -) \times \Delta[i])$  and so

$$L \simeq s\int_{[n]} G \cong \text{colim}(s\int_{[n]} \text{Hom}([i], -) \times \Delta[i]) \cong \text{colim}(\langle \langle 0, \dots, i \rangle \circ \pi_1 : F(i) \times \Delta[i] \rightarrow F(n) \rangle)$$

giving us the desired result.  $\square$

**Corollary 4.24** Let  $L \rightarrow F(n) \times \Delta[i]$  be a left fibration. Then there is a Reedy equivalence

$$L \simeq \text{colim}(\langle \langle 0, \dots, i \rangle \times \text{id}_{\Delta[i]} \rangle \circ \pi_1 : (F(i) \times \Delta[i]) \times \Delta[j] \rightarrow F(n) \times \Delta[i])$$

over  $F(n) \times \Delta[i]$ .

**Proof** The projection map  $\pi_1 : F(n) \times \Delta[i] \rightarrow F(n)$  is a Reedy equivalence and so by Theorem 3.15 gives us a Quillen equivalence

$$(s\mathcal{S}_{/F(n) \times \Delta[i]})^{cov} \xrightleftharpoons[\langle \pi_1 \rangle^*]{\langle \pi_1 \rangle!} (s\mathcal{S}_{/F(n)})^{cov}$$

which implies that the derived unit map  $L \rightarrow \hat{L} \times \Delta[i]$  is a Reedy equivalence of left fibrations over  $F(n) \times \Delta[i]$ . Here  $\hat{L}$  is given as the left fibrant replacement of the morphism  $L \rightarrow F(n) \times \Delta[i] \rightarrow F(n)$ . Now, by the previous corollary we have a Reedy equivalence  $\hat{L} \simeq \text{colim}(\langle \langle 0, \dots, i \rangle \circ \pi_1 \rangle : F(i) \times \Delta[i] \rightarrow F(n))$  over  $F(n)$ . We can pull back this equivalence along  $F(n) \times \Delta[i] \rightarrow F(n)$  to get an equivalence  $\hat{L} \times \Delta[i] \simeq \text{colim}(\langle \langle 0, \dots, i \rangle \circ \pi_1 \rangle : F(i) \times \Delta[i] \rightarrow F(n)) \times \Delta[i]$ , using the fact that  $- \times \Delta[i] : \mathcal{S} \rightarrow \mathcal{S}$



preserves weak equivalences between non-fibrant simplicial sets. Finally, the desired result follows from the fact that colimits commute with products, giving us

$$L \simeq \hat{L} \times \Delta[l] \simeq [\text{colim}(\langle 0, \dots, i \rangle \circ \pi_1) : F(i) \times \Delta[j] \rightarrow F(n)] \times \Delta[l] \cong \text{colim}(\langle 0, \dots, i \rangle \times \text{id}_{\Delta[l]}) \circ \pi_1 : (F(i) \times \Delta[l]) \times \Delta[j] \rightarrow F(n) \times \Delta[l]$$

over  $F(n) \times \Delta[l]$ . □

In the coming sections we will need the contravariant version of fibrations, *right fibrations*.

**Remark 4.25** Until now we have focused on the covariant approach to fibrations. However, there is also a contravariant analogue. A morphism of simplicial spaces  $p : R \rightarrow X$  is a *right fibration* if the morphism  $p^{op} : R^{op} \rightarrow X^{op}$  is a left fibration. Explicitly this means a right fibration  $p : R \rightarrow X$  is a Reedy fibration such that for all  $n \geq 0$ , the induced morphism  $(p_n, \langle n \rangle) : Y_n \xrightarrow{\simeq} X_n \times_{X_0} Y_0$  is a Kan equivalence.

We can now repeat all the results in Sects. 3 and 4.1 using the fact that we can define right fibrations in terms of left fibrations. However, instead of repeating all theorems for right fibrations we introduce the following table, which simply states the terminology relevant to right fibrations.

Left Fibration (Morphism)	Right Fibration (Morphism)
$(p_n, \langle 0 \rangle^*) : Y_n \xrightarrow{\simeq} X_n \times_{X_0} Y_0$	$(p_n, \langle n \rangle^*) : Y_n \xrightarrow{\simeq} X_n \times_{X_0} Y_0$
Covariant Model Structure	Contravariant Model Structure
Under-Segal Space	Over-Segal Space
$W_{X/} = F(0)^{\{x\}} \times_W^s W^{F(1)}$	$W_{/X} = W^{F(1)t} \times^{\{x\}} F(0)$
initial object	final object
$s\mathcal{J}_{\mathcal{C}} : \text{Fun}(\mathcal{C}, \mathcal{S})^{proj} \rightarrow (s\mathcal{S}_{/N^h\mathcal{C}})^{cov}$	$s\mathcal{J}_{\mathcal{C}}^{op} : \text{Fun}(\mathcal{C}^{op}, \mathcal{S})^{proj} \rightarrow (s\mathcal{S}_{/N^h\mathcal{C}})^{contra}$

Having defined left and right fibrations, we can use our previous results to generalize Remark 1.18 from categories to simplicial spaces.

**Theorem 4.26** *Let  $p : L \rightarrow X$  be a left fibration. Then the following are equivalent:*

- (1)  $p$  is a right fibration.
- (2) For every map  $f : F(1) \rightarrow X$  the map  $f^*L \rightarrow F(1)$  is a right fibration.
- (3)  $p$  is a diagonal fibration.
- (4) For every map  $f : F(1) \rightarrow X$  the map  $f^*L \rightarrow F(1)$  is a diagonal fibration.

**Proof** By Lemmas 3.10, 3.11  $p$  is a left, right or diagonal fibration if and only if  $f^*p : f^*L \rightarrow F(n)$  is such a fibration for every map  $F(n) \rightarrow X$ . We will hence assume that  $X = F(n)$ .

(1  $\Leftrightarrow$  2) One side is a special case. For the other side, as  $F(n)$  is a Segal space, by Lemma 3.29, it suffices to show that  $(p_1, \langle 1 \rangle^*) : L_1 \rightarrow L_0 \times_{F(n)_0} \times F(n)_1$  is an equivalence over  $F(n)_1$ . By Corollary A.2, this is equivalent to a fiber-wise equivalence, meaning we need to show that for every  $\alpha$  in  $F(n)_1$  the induced map on fibers

$$L_1 \times_{F(n)_1} \Delta[0] \rightarrow L_0 \times_{F(n)_0} \times F(n)_1 \times_{F(n)_1} \Delta[0] \cong L_0 \times_{F(n)_0} \Delta[0]$$

is a Kan equivalence. Fix one  $\alpha$  in  $F(n)_1$  and notice this corresponds to a morphism of simplicial spaces  $\{\alpha\} : F(1) \rightarrow F(n)$ , meaning  $\alpha$  is precisely the image of the identity  $\{\alpha\}_1 : F(1)_1 \rightarrow F(n)_1$ . Hence, by the pasting property of pullbacks, the map above is isomorphic to the map

$$(\{\alpha\}^*L)_1 \times_{F(1)_1} \Delta[0] \rightarrow (\{\alpha\}^*L)_0 \times_{F(1)_0} \Delta[0],$$

which is an equivalence as  $\{\alpha\}^*L \rightarrow F(1)$  is a right fibration.

( $I \Leftrightarrow 3$ ) Every diagonal fibration is a left fibration and right fibration. Before we prove the opposite direction, we make the following observation regarding diagonal fibrations. Let  $K$  be a Kan complex, then, by Remark 2.10,  $\text{Diag}_* K$  gives us a Reedy fibrant replacement of  $K$  and so  $F(n) \times \text{Diag}_* K \rightarrow F(n)$  is a Reedy fibration that is in fact a diagonal fibration, as it satisfies the conditions given in [61, Definition 3.3], which precisely describe the diagonal fibrations by [61, Lemma 4.3]. Hence, in order to prove a Reedy fibration  $L \rightarrow F(n)$  is a diagonal fibration, it suffices to show it is Reedy equivalent to a morphism of the form  $F(n) \times K \rightarrow F(n)$  for some Kan complex  $K$ .

Now, let  $L \rightarrow F(n)$  be a right and left fibration. As it is a left fibration, by Theorem 4.18, there exists a functor  $G : [n] \rightarrow \mathcal{S}$  such that  $s_{f[n]} G$  is Reedy equivalent to  $L$  over  $F(n)$ . We can assume without loss of generality that  $G$  takes value in Kan complexes (by fibrantly replacing in the projective model structure).

Following the argument in the first paragraph, in order to finish the proof we only need to prove there is a Reedy weak equivalence  $F(n) \times G(0) \xrightarrow{\simeq} s_{f[n]} G$  over  $F(n)$ . Let  $\{G(0)\} : [n] \rightarrow \mathcal{S}$  be the constant functor and notice it comes with an evident natural transformation  $\alpha : \{G(0)\} \Rightarrow G$ . Moreover,  $s_{f[n]} \{G(0)\} = G(0) \times F(n)$ , hence we need to establish that  $s_{f[n]} \alpha$  is a Reedy weak equivalence. By Remark 4.6, at level  $k$  the morphism is given by  $\coprod_{c_0 \rightarrow \dots \rightarrow c_k} G(0) \rightarrow \coprod_{c_0 \rightarrow \dots \rightarrow c_k} G(c_0)$  and Kan equivalences are closed under coproducts, meaning we only have to show that the morphisms  $G(0) \rightarrow G(i)$  are equivalences for all  $0 \leq i \leq n$ .

Let  $f : 0 \rightarrow i$  be the unique morphism in  $[n]$  from 0 to  $i$ . Taking fiber of the commutative diagram

$$\begin{array}{ccc} (s_{f[n]} G)_1 & \xrightarrow{\langle 1 \rangle^*} & (s_{f[n]} G)_0 \\ \downarrow \simeq & & \downarrow \simeq \\ L_1 & \xrightarrow{\langle 1 \rangle^*} & L_0 \end{array}$$

over  $f$  in  $F(n)_1$  gives us the diagram

$$\begin{array}{ccc} G(0) & \xrightarrow{G(f)} & G(i) \\ \downarrow \simeq & & \downarrow \simeq \\ L_1 \times_{F(n)_1} \Delta[0] & \xrightarrow{\langle 1 \rangle^*} & L_0 \times_{F(n)_0} \Delta[0] \end{array} .$$

Here the vertical morphisms are equivalences because  $s_{f[n]} G \rightarrow L$  is a Reedy equivalence and the bottom vertical morphism is a weak equivalence because  $L \rightarrow F(n)$  is a right fibration, it hence follows that the top morphism is also a weak equivalence.

( $2 \Leftrightarrow 4$ ) We can use the same argument as in the previous part. □

### 4.2 The Yoneda Lemma

We are finally in a position to prove the recognition principle for covariant equivalences. The proof has three main steps:

- (1) Study how right and left fibrations interact: Theorem 4.29.
- (2) Characterize covariant equivalences between left fibrations: Theorem 4.35.
- (3) Prove the recognition principle for covariant equivalences: Theorem 4.41.

However, before we can start we need one technical lemma.

**Lemma 4.27** *Let  $\mathcal{L} = \{A_j \rightarrow N^h(\mathcal{C}_j)\}_{j \in J}$  be a set of monomorphisms in  $s\mathcal{S}$ . For every simplicial space  $X$  denote by  $(s\mathcal{S}/X)^{\mathcal{M}_X}$  the left Bousfield localization of the induced Reedy model structure with respect to the set of monomorphisms  $\{A_j \rightarrow N^h(\mathcal{C}_j) \rightarrow X : j \in J\}$  in  $s\mathcal{S}/X$ . Then the following are equivalent:*

- (1) *For every simplicial space  $X$  and every right fibration  $p : R \rightarrow X$  over  $X$ , the adjunction*

$$(s\mathcal{S}/X)^{\mathcal{M}_X} \begin{array}{c} \xrightarrow{p_!p^*} \\ \perp \\ \xleftarrow{p_*p^*} \end{array} (s\mathcal{S}/X)^{\mathcal{M}_X}$$

*is a Quillen adjunction.*

- (2) *For every  $j \in J$ , every object  $c$  in  $\mathcal{C}_j$  and map  $i : A_j \rightarrow N^h(\mathcal{C}_j)$  in  $\mathcal{L}$  the pullback map*

$$N^h(\pi_c)^*(i) : N^h(\pi_c)^*(A_j) \rightarrow N^h((\mathcal{C}_j)_{/c})$$

*is a trivial cofibration in  $(s\mathcal{S}/N^h\mathcal{C}_j)^{\mathcal{M}_{N^h\mathcal{C}_j}}$ . Here  $\pi_c : (\mathcal{C}_j)_{/c} \rightarrow \mathcal{C}_j$  is the projection map.*

**Proof** (1  $\Rightarrow$  2) This is just the special case of (1) applied to the right fibration  $N^h(\mathcal{C}_j)_{/c} \rightarrow N^h\mathcal{C}_j$ .

(2  $\Rightarrow$  1) The proof consists of several reduction steps.

(I) *Reduce to Fibrant Objects:* First, by Corollary A.10 it suffices to show  $p_!p^*$  preserves cofibrations,  $p_*p^*$  preserves Reedy fibrations and fibrant objects  $Y \rightarrow X$ . The fact that  $p_!p^*$  preserves cofibrations and that  $p_*p^*$  preserves Reedy fibrations follows from the fact that  $(p_!p^*, p_*p^*)$  is a Quillen adjunction when both sides just have the induced Reedy model structure, as the Reedy model structure is right proper (Sect. 2.4). So, we only have to prove that for every fibrant object  $Y \rightarrow X$ ,  $p_*p^*(Y) \rightarrow X$  is also fibrant.

(II) *Reduce to Local Objects:* Next notice that  $p_*p^*(Y) \rightarrow X$  is fibrant if and only if it is Reedy fibrant and local with respect to maps  $A_j \rightarrow N^h(\mathcal{C}_j) \rightarrow X$ , where  $i : A_j \rightarrow N^h(\mathcal{C}_j)$  is in  $\mathcal{L}$ . Again the Reedy fibrancy follows from the previous paragraph and so it suffices to prove that  $p_*p^*(Y) \rightarrow X$  is local. Thus we need to prove that

$$i^* : \text{Map}_X(N^h(\mathcal{C}_j), p_*p^*(Y)) \rightarrow \text{Map}_X(A_j, p_*p^*(Y))$$

is a Kan equivalence for all  $j$  and all morphisms  $N^h\mathcal{C}_j \rightarrow X$ .

(III) *Reduce to Local Trivial Cofibration:* Using the fact that  $p_*p^*$  has a left adjoint this is equivalent to

$$(p_!p^*i)^* : \text{Map}_X(p_!p^*N^h(\mathcal{C}_j), Y) \rightarrow \text{Map}_X(p_!p^*A_j, Y)$$

being a Kan equivalence. As the model structure  $\mathcal{M}_X$  is simplicial and  $Y \rightarrow X$  is an arbitrary fibrant object, this is equivalent to

$$p_!p^*i : p_!p^*A_j \rightarrow p_!p^*N^h(\mathcal{C}_j)$$

being a weak equivalence in  $(s\mathcal{S}/X)^{\mathcal{M}_X}$  for all  $j \in J$  and  $N^h\mathcal{C}_j \rightarrow X$ . Here the morphism  $p_!p^*N^h\mathcal{C}_j \rightarrow X$  is given as  $p_!p^*N^h\mathcal{C}_j \rightarrow N^h\mathcal{C}_j \rightarrow X$ , where the first morphism is the counit.

(IV) *Reduce to Categorical base:* Fix a  $j \in J$  and morphism  $A_j \xrightarrow{i} N^h\mathcal{C}_j \xrightarrow{m} X$ . Then we have the following pullback square

$$\begin{array}{ccc} S & \xrightarrow{k} & R \\ \downarrow q & & \downarrow p \\ N^h\mathcal{C}_j & \xrightarrow{m} & X \end{array}$$

Now the Beck-Chevalley condition for pullback squares [25, 1.2] implies that we have an isomorphism  $p^*m_!(i) \cong k_!q^*(i)$ . However, notice  $k_!$  preserves weak equivalences. Thus to prove that  $p_!p^*i$  is a weak equivalence over  $X$  it suffices to prove that  $q^*i : q^*A_j \rightarrow q^*N^h\mathcal{C}_j$  is a weak equivalence over  $N^h\mathcal{C}_j$  in the  $\mathcal{M}_{N^h\mathcal{C}_j}$  model structure.

(V) *Reduce to Representable Right Fibrations:* By Lemma 3.9, right fibrations are stable under pullback and so  $q : S \rightarrow N^h\mathcal{C}_j$  is also a right fibration. However, by the contravariant analogue of Corollary 4.23, every right fibration over  $N^h\mathcal{C}_j$  is Reedy equivalent to  $\text{colim}(N^h(\mathcal{C}_j)_{/c} \times \Delta[I])$  (and so a homotopy colimit). Thus we can reduce the argument to proving that for all  $j \in J$  and object  $c$  in  $\mathcal{C}_j$

$$A_j \times_{N^h\mathcal{C}_j} N^h(\mathcal{C}_j)_{/c} \times \Delta[I] \rightarrow N^h(\mathcal{C}_j)_{/c} \times \Delta[I]$$

is a weak equivalence in  $s\mathcal{S}_{/N^h\mathcal{C}_j}$ .

(VI) *Reduce to the desired condition:* Finally, using the fact that  $(s\mathcal{S}_{/N^h\mathcal{C}_j})^{\mathcal{M}_{N^h\mathcal{C}_j}}$  is a simplicial model structure with all objects cofibrant, for every object  $X \rightarrow N^h\mathcal{C}_j$ , the induced morphism  $\text{id}_X \times < 0 > : X \times \Delta[0] \rightarrow X \times \Delta[n]$  is a weak equivalence over  $N^h\mathcal{C}_j$  in the  $\mathcal{M}_{N^h\mathcal{C}_j}$  model structure. Hence the previous condition is equivalent to

$$N^h(\pi_c)^*(A) = A \times_{N^h\mathcal{C}_j} N^h(\mathcal{C}_j)_{/c} \rightarrow N^h(\mathcal{C}_j)_{/c}$$

being a trivial cofibration in  $s\mathcal{S}_{/N^h\mathcal{C}_j}$  in the  $\mathcal{M}_{N^h\mathcal{C}_j}$  model structure, for all  $j \in J$  and objects  $c$  in  $\mathcal{C}_j$ . □

**Remark 4.28** We can use the same argument to prove an analogous result for pulling back along left fibrations. Concretely,

$$(s\mathcal{S}/X)^{\mathcal{M}_X} \begin{array}{c} \xrightarrow{p_!p^*} \\ \perp \\ \xleftarrow{p^*p^*} \end{array} (s\mathcal{S}/X)^{\mathcal{M}_X}$$

is a Quillen adjunction for every left fibration  $p : L \rightarrow X$  if and only if

$$N^h(\pi_c)^*(i) : N^h(\pi_c)^*(A_j) \rightarrow N^h((\mathcal{C}_j)_{/c})$$

is a trivial cofibration in  $(s\mathcal{S}_{/N^h\mathcal{C}_j})^{\mathcal{M}_{N^h\mathcal{C}_j}}$  for every  $j \in J$  and object  $c$  in  $\mathcal{C}_j$ .

We can now use this result to give the desired connection between right fibrations and covariant equivalences.

**Theorem 4.29** *Let  $p : R \rightarrow X$  be a right fibration. Then the adjunction*

$$(s\mathcal{S}/X)^{cov} \begin{array}{c} \xrightarrow{p_!p^*} \\ \perp \\ \xleftarrow{p^*p^*} \end{array} (s\mathcal{S}/X)^{cov}$$

is a Quillen adjunction where both sides have the covariant model structure.

**Proof** The covariant model structure is given by localization with respect to maps  $F(0) \xrightarrow{\langle 0 \rangle} F(n) \rightarrow X$ , where  $n \geq 1$ . The over-category  $[n]_i \rightarrow [n]$  is given by the map of simplicial spaces  $\langle 0, \dots, i \rangle: F(i) \rightarrow F(n)$ . Thus, by the previous lemma, we only need to prove that the pullback map  $F(0) = F(0) \times_{F(n)} F(i) \rightarrow F(i)$  is a covariant equivalence over  $F(n)$ . However, that is true by definition.  $\square$

**Remark 4.30** By Remark 4.28 and the analogous argument to Theorem 4.29, for every left fibration  $p : L \rightarrow X$ , we get a Quillen adjunction  $(p_!p^*, p_*p^*)$  between contravariant model structures.

**Remark 4.31** The analogous result for quasi-categories (namely that pulling back along right fibrations of simplicial sets preserves equivalences in the model structure for quasi-categories) was proven independently by Lurie [42, Proposition 4.1.2.15], Joyal [39, Theorem 11.9], and Nguyen [49, Proposition 4.12].

We can also use this lemma to prove a relationship between right fibrations and complete Segal spaces.

**Theorem 4.32** Let  $W$  be a Segal space and  $p : R \rightarrow W$  be a right or left fibration. Then the adjunction

$$(s\mathcal{S}/W)^{Seg} \begin{matrix} \xrightarrow{p^*} \\ \perp \\ \xleftarrow{p_*} \end{matrix} (s\mathcal{S}/R)^{Seg}$$

is a Quillen adjunction where both sides have the induced Segal space model structure (Proposition A.5).

If  $W$  is also complete, then the same statement holds for the adjunction

$$(s\mathcal{S}/W)^{CSS} \begin{matrix} \xrightarrow{p^*} \\ \perp \\ \xleftarrow{p_*} \end{matrix} (s\mathcal{S}/R)^{CSS}$$

where now both sides have the induced complete Segal space model structure (Proposition A.5).

**Proof** We will assume  $p$  is a right fibration. The argument for left fibrations follows similarly, using the adjustment in Remark 4.28.

Let  $W$  be a Segal space. We can extend the adjunction above as follows

$$(s\mathcal{S}/W)^{Seg} \begin{matrix} \xrightarrow{p^*} \\ \perp \\ \xleftarrow{p_*} \end{matrix} (s\mathcal{S}/R)^{Seg} \begin{matrix} \xrightarrow{p_!} \\ \perp \\ \xleftarrow{p^*} \end{matrix} (s\mathcal{S}/W)^{Seg} .$$

In order to show that  $(p^*, p_*)$  is a Quillen adjunction, we have to prove  $p^*$  preserves cofibrations and trivial cofibrations in the Segal space model structure. It is evident that  $p^*$  preserves cofibrations, as they are just monomorphisms. Moreover, by Proposition A.5,  $p_!$  preserves and reflects trivial Segal cofibrations. Hence  $p^*$  preserves trivial cofibrations if and only if  $p_!p^*$  preserves trivial cofibrations, which is equivalent to proving that

$$(s\mathcal{S}/W)^{Seg} \begin{matrix} \xrightarrow{p_!p^*} \\ \perp \\ \xleftarrow{p_*p^*} \end{matrix} (s\mathcal{S}/W)^{Seg}$$

is a Quillen adjunction, where both sides have the Segal space model structure. We can thus apply Lemma 4.27.

By Theorem A.13 the induced Segal space model structure over a Segal space is just given by localizing with respect to the maps  $G(n) \rightarrow F(n) \rightarrow W$ , where  $n \geq 2$ . Thus we only need to check the map  $G(n) \rightarrow F(n)$  satisfies the desired condition in Lemma 4.27. We know the over-category over  $i$  is given by  $\langle 0, \dots, i \rangle: F(i) \rightarrow F(n)$ . Thus we only need to show that  $G(i) = G(n) \times_{F(n)} F(i) \rightarrow F(i)$  is an equivalence in the Segal space model structure, which is true by definition.

Now let us assume also in addition that  $W$  is complete. By the explanation given at the beginning of the proof, it suffices to show that

$$(s\mathcal{S}/W)^{CSS} \begin{array}{c} \xrightarrow{p_! p^*} \\ \perp \\ \xleftarrow{p_* p^*} \end{array} (s\mathcal{S}/W)^{CSS}$$

is a Quillen adjunction, where both sides have the complete Segal space model structure. This means we can again use Lemma 4.27.

By Theorem A.13 the induced complete Segal space model structure is given by localizing with respect to maps  $G(n) \rightarrow F(n) \rightarrow W$  and  $F(0) \rightarrow E(1) \rightarrow W$ . We already observed that  $G(n) \rightarrow F(n)$  satisfies the condition of Lemma 4.27 so we only need to prove the same statement for the map  $F(0) \rightarrow E(1)$ .

However,  $E(1) = N^h(I[1])$ , where  $I[1]$  is the category with two objects and one unique isomorphism (Definition 2.19). By direct computation  $I[1]_{/0} = I[1]_{/1} = I[1]$  and so the projection map from the over-category is just the identity map. Hence we are done.  $\square$

This theorem has the following useful corollary.

**Corollary 4.33** *Let the following diagram be given*

$$\begin{array}{ccc} p^*X & \xrightarrow{p^*f} & L \\ \downarrow & & \downarrow p \\ X & \xrightarrow[\simeq]{f} & W \end{array}$$

where  $W$  is a complete Segal space,  $p : L \rightarrow W$  is a left or right fibration and  $f$  is a complete Segal space weak equivalence. Then  $p^*f : p^*X \rightarrow L$  is a complete Segal space weak equivalence.

**Remark 4.34** The assumptions in the previous theorem seem too strong, the result should also hold if the base simplicial space  $W$  is not a Segal space. This is in fact correct and we will prove this in Theorem 5.15 / Theorem 5.16. However, before we can do that we need to understand the invariance of left fibrations with respect to complete Segal space equivalences, which is the goal of Theorem 5.1.

We can now move on to the second step and characterize covariant equivalences between left fibrations.

**Theorem 4.35** *Let  $L \rightarrow X$  and  $L' \rightarrow X$  be left fibrations and  $f : L \rightarrow L'$  a map over  $X$ . The following are equivalent:*

- (1)  $f$  is a covariant equivalence.

- (2)  $f$  is a Reedy equivalence.
- (3)  $f$  is a Kan equivalence.
- (4)  $f$  is a fiber-wise Reedy equivalence ( $f \times_X F(0) : L \times_X F(0) \rightarrow L' \times_X F(0)$  is a Reedy equivalence for every map  $F(0) \rightarrow X$ ).
- (5)  $f$  is a fiberwise Kan equivalence ( $f \times_X F(0) : L \times_X F(0) \rightarrow L' \times_X F(0)$  is a Kan equivalence for every map  $F(0) \rightarrow X$ ).
- (6)  $f$  is a fiberwise diagonal equivalence ( $f \times_X F(0) : L \times_X F(0) \rightarrow L' \times_X F(0)$  is a diagonal equivalence for every map  $F(0) \rightarrow X$ ).

**Remark 4.36** By Theorem 4.18, a left fibration  $L \rightarrow N^h\mathcal{C}$  is Reedy equivalent to  $s\int_{\mathcal{C}} G \rightarrow N^h\mathcal{C}$  for some functor  $G : \mathcal{C} \rightarrow \mathcal{S}$ . Thus a map of left fibrations  $L \rightarrow L'$  over  $N^h\mathcal{C}$  is an equivalence if and only if the corresponding natural transformation  $G \rightarrow G'$  is an equivalence.

Theorem 4.35 can thus be seen as a generalization of this observation to an arbitrary simplicial space  $X$ : We are comparing two left fibrations over  $X$ , by comparing their fibers, which we should think of as their “values”.

**Proof** (1  $\Leftrightarrow$  2) Follows from the definition of localization as left fibrations are the fibrant objects in the covariant model structure (Theorem 3.12).

(2  $\Leftrightarrow$  3) Clearly (2) implies (3). For the other side let  $f$  be a Kan equivalence, then  $f_0 : L_0 \rightarrow L'_0$  is a Kan equivalence of spaces. This implies that in the diagram

$$\begin{array}{ccc}
 L_n & \xrightarrow{f_n} & L'_n \\
 \downarrow \simeq & & \downarrow \simeq \\
 L_0 \times_{X_0} X_n & \xrightarrow[\simeq]{(f_0, id)} & L'_0 \times_{X_0} X_n
 \end{array}$$

the two vertical maps and the bottom horizontal map are Kan equivalences. Thus  $f_n : Y_n \rightarrow Z_n$  is a Kan equivalence as well, which implies that  $f$  is a Reedy equivalence.

(3  $\Leftrightarrow$  5) This is precisely the statement of Corollary A.2.

(4  $\Leftrightarrow$  5) By Example 3.19, left fibrations over  $F(0)$  are diagonal fibrations and hence homotopically constant, which means  $f \times_X F(0) : L \times_X F(0) \rightarrow L' \times_X F(0)$  is a Kan equivalence if and only if it is a Reedy equivalence.

(4  $\Leftrightarrow$  6) By Lemma 3.9,  $F(0) \times_X L \rightarrow F(0)$  is a left fibration, which by Example 3.19 means that  $L \times_X F(0)$  is diagonally fibrant. Thus  $f \times_X F(0) : L \times_X F(0) \rightarrow L' \times_X F(0)$  is a Reedy equivalence if and only if it is a diagonal equivalence (Theorem 2.11).  $\square$

**Remark 4.37** Note we can reduce condition (5) in Theorem 4.35 to proving that there exists  $x : F(0) \rightarrow X$  such that  $f \times_X F(0)$  is a fiberwise Kan equivalence for every path component of  $X_0$ . Indeed, if  $x$  and  $y$  are in the same path-component then we have an equivalence of fibers  $f_0 \times_{X_0}^x \Delta[0] \simeq f_0 \times_{X_0}^y \Delta[0]$ .

**Remark 4.38** Covariant equivalences of left fibrations have also been studied by de Brito, Moerdijk and Heuts [19, Proposition 1.10], [18, Lemma 4.3], [32, Proposition 13.8].

We can now move on to the general case.

**Theorem 4.39** *Let  $p : Y \rightarrow X$  be a map of simplicial spaces. For every  $\{x\} : F(0) \rightarrow X$ , there is a natural zig-zag of diagonal equivalences*

$$R_x \times_X Y \xrightarrow{\simeq} R_x \times_X \hat{Y} \xleftarrow{\simeq} F(0) \times_X \hat{Y}$$

Here  $i : Y \rightarrow \hat{Y}$  is a choice of a left fibrant replacement of  $Y$  over  $X$  and  $R_x \rightarrow X$  is a contravariant fibrant replacement of  $\{x\} : F(0) \rightarrow X$ .

**Proof** Fix a covariant fibrant replacement  $i : Y \rightarrow \hat{Y}$  over  $X$ . Then we have the following zig-zag of equivalences

$$Y \times_X R_x \xrightarrow{cov \simeq} \hat{Y} \times_X R_x \xleftarrow{contra \simeq} \hat{Y} \times_X F(0)$$

By Theorem 4.29 the first map is a covariant equivalence because  $R_x \rightarrow X$  is a right fibration. By the covariant version of the same lemma the second map is a contravariant equivalence because  $\hat{Y} \rightarrow X$  is a left fibration. So, by Theorem 3.17, both are diagonal equivalences.  $\square$

In the case  $X$  is a Segal space, we can replace the zig-zag of equivalences with an actual map.

**Remark 4.40** Let  $X$  be a Segal space. Let  $p : Y \rightarrow X$  be a map of simplicial spaces and  $Y \xrightarrow{i} \hat{Y} \xrightarrow{\hat{p}} X$  its fibrant replacement in the covariant model structure. Then, by Theorem 3.49,  $\{id_x\} : F(0) \rightarrow F(0) \times_X X^{F(1)}$  is a covariant fibrant replacement of  $\{x\} : F(0) \rightarrow X$  over  $X$ . Now we have the following diagram:

$$\begin{array}{ccc}
 Y \times_X^p (X^{F(1)} \times_X^t \times_X^{\{x\}} F(0)) & \xrightarrow[\simeq]{i \times_X X^{F(1)} \times_X F(0)} & \hat{Y} \times_X^{\hat{p}} (X^{F(1)} \times_X^t \times_X^{\{x\}} F(0)) \xleftarrow[\simeq]{\hat{Y} \times_X \{id_x\}} \hat{Y} \times_X^{\hat{p}} \times_X^{\{x\}} F(0) \\
 & & \uparrow \text{sec} \simeq (s, \hat{p}^{F(1)}) \times_X F(0) \\
 & & \hat{Y}^{F(1)} \times_X^{t(\hat{p}^{F(1)})} \times_X^{\{x\}} F(0) \xrightarrow{t} \hat{Y} \times_X^{\hat{p}} \times_X^{\{x\}} F(0)
 \end{array}$$

By Lemma 3.20, the map  $\hat{Y}^{F(1)} \rightarrow \hat{Y} \times_X X^{F(1)}$  is a trivial Reedy fibration and so we can pick a section

$$sec : \hat{Y} \times_X^{\hat{p}} \times_X^s X^{F(1)} \times_X^t \times_X^{\{x\}} F(0) \rightarrow \hat{Y}^{F(1)} \times_X^{t(\hat{p}^{F(1)})} \times_X^{\{x\}} F(0).$$

By Theorem 4.39,  $\hat{Y} \times_X \{id_x\}$  is a diagonal equivalence and so, by 2-out-of-3,  $s$  is a diagonal equivalence. Hence

$$s \circ sec \circ (i \times_X X_{/x}) : Y \times_X X_{/x} \rightarrow \hat{Y} \times_X F(0)$$

is the desired diagonal equivalence.

We can finally prove the recognition principle for covariant equivalences.

**Theorem 4.41 (Recognition principle)** *For every morphism  $\{x\} : F(0) \rightarrow X$  fix a contravariant fibrant replacement  $R_x \rightarrow X$ . Let  $g : Y \rightarrow Z$  be a morphism over  $X$ . Then  $g : Y \rightarrow Z$  over  $X$  is a covariant equivalence over  $X$  if and only if for every  $\{x\} : F(0) \rightarrow X$*

$$R_x \times_X Y \rightarrow R_x \times_X Z$$

is a diagonal equivalence.



**Proof** Let the diagram

$$\begin{array}{ccc} Y & \xrightarrow{i} & \hat{Y} \\ \downarrow g & \simeq & \downarrow \hat{g} \\ Z & \xrightarrow{j} & \hat{Z} \end{array}$$

be a left fibrant replacement of  $g$  over  $X$ . By Theorem 3.12,  $g : Y \rightarrow Z$  is a covariant equivalence if and only if  $\hat{g} : \hat{Y} \rightarrow \hat{Z}$  is a Reedy equivalence. We now have the following diagram:

$$\begin{array}{ccc} Y \times_X R_x & \xrightarrow{g \times id} & Z \times_X R_x \\ \downarrow i \times id \simeq & & \downarrow j \times id \simeq \\ \hat{Y} \times_X R_x & \xrightarrow{\hat{g} \times id} & \hat{Z} \times_X R_x \\ \uparrow \simeq & & \uparrow \simeq \\ \hat{Y} \times_X F(0) & \xrightarrow{\hat{g} \times id} & \hat{Z} \times_X F(0) \end{array}$$

By Theorem 4.39, all vertical maps are diagonal equivalences and so the top horizontal map is a diagonal equivalence if and only if the bottom horizontal map is one. But the bottom map is a diagonal equivalence for every  $x : F(0) \rightarrow X$  if and only if  $\hat{Y} \rightarrow \hat{Z}$  is a Reedy equivalence (Theorem 4.35). Hence, we are done.  $\square$

In the case of Segal spaces the equivalence takes on a very simple form.

**Corollary 4.42** *Let  $X$  be a Segal space and  $f : Y \rightarrow Z$  a map over  $X$ . Then  $f$  is a covariant equivalence if and only if*

$$Y \times_X X_{/x} \rightarrow Z \times_X X_{/x}$$

*is a diagonal equivalence for every object  $x$ .*

**Proof** By Theorem 3.49,  $X_{/x} \rightarrow X$  is the contravariant fibrant replacement of  $\{x\} : F(0) \rightarrow X$ . The result now follows from Theorem 4.41.  $\square$

**Remark 4.43** A similar result has been established by Heuts and Moerdijk [33, Proposition G] using quasi-categories, to characterize covariant equivalences of simplicial sets over a given quasi-category. Notice, we cannot use the Quillen equivalence between complete Segal spaces and quasi-categories to prove Corollary 4.42 using [33, Proposition G] as the proof here holds for all Segal spaces.

**Remark 4.44** Corollary 4.42 should very much remind us of the behavior of  $s\mathbb{T}_c$ , which takes a map  $Y \rightarrow N^h\mathcal{C}$  to the functor  $s\mathbb{T}_c(Y)$  with value  $s\mathbb{T}_c(Y)(c) = \text{Diag}^*(Y \times_{N^h\mathcal{C}} N^h\mathcal{C}_{/c})$ . The functor  $s\mathbb{T}_c$  was only defined over nerves of categories, but Corollary 4.42 suggests that a map of Segal spaces  $Y \rightarrow X$  should via the covariant model structure correspond to a functor with value  $\text{Diag}^*(Y \times_X X_{/x})$ .

## 5 Complete Segal Spaces and Covariant Model Structure

In this section we want to study the relation between complete Segal spaces and left fibrations. Concretely, while the covariant model structure is invariant under Reedy equivalences, meaning a Reedy equivalence induces a Quillen equivalence of covariant model structures, we will prove (the very non-trivial fact) that the covariant model structure is invariant under CSS equivalences as well (Theorem 5.1). We further witness that (although not initially defined this way) the covariant model structure is a Bousfield localization of the complete Segal space model structure (Theorem 5.11), which in particular implies that every left fibration is a complete Segal space fibration (Corollary 5.13). We will use this connection to generalize results regarding Segal spaces proven in Sect. 3.3 to arbitrary simplicial spaces, and in particular establish the *Yoneda lemma for simplicial spaces* (Corollary 5.10). Finally, we apply our new understanding of left fibrations to study colimits in Segal spaces (Definition 5.26) and establish Quillen’s Theorem A (Theorem 5.40) in this setting in Sect. 5.2.

### 5.1 Invariance of the Covariant Model Structure

Until now we have seen several results that suggest a deep connection between complete Segal spaces and left fibrations, in particular over a Segal space. For example the left fibration  $W_{/x} \rightarrow W$  is in fact a Segal fibration (Lemma 3.43). Or pulling back a right fibration of complete Segal spaces  $R \rightarrow X$  preserves CSS equivalences (Theorem 4.32). In this subsection we want to prove that these types of results generalize to an arbitrary simplicial space. The key input is the *invariance theorem* for the covariant model structure, which proves that the covariant model structure is invariant under equivalences in the complete Segal space model structure.

**Theorem 5.1** (*Invariance property*) *Let  $f : X \rightarrow Y$  be a CSS equivalence. Then the adjunction*

$$(s\mathcal{S}/X)^{cov} \begin{array}{c} \xrightarrow{f_!} \\ \perp \\ \xleftarrow{f^*} \end{array} (s\mathcal{S}/Y)^{cov}$$

*is a Quillen equivalence. Here both sides have the covariant model structure.*

**Remark 5.2** As the proof is quite long here is an overview of the essential steps:

- (1) By the diagram in 5.3, we can reduce the proof to a fibrant replacement morphism obtained via the small object argument  $i : X \rightarrow \hat{X}$ .
- (2) We first prove the derived counit map is an equivalence in 5.4.
- (3) We then move on to the derived unit map. By the small object argument the proof reduces to checking for a generating set of trivial cofibrations for the Reedy model structure (1), the morphisms characterizing Segal spaces (2), and the morphisms giving us the completeness condition (1), of which only the case of morphisms for the Segal condition (2) require a longer argument.
- (4) By direct computation we can reduce the case for Segal maps to proving that  $F(0) \rightarrow G(n)$  is a covariant equivalence over  $G(n)$  (5.5).
- (5) We prove this by induction in (5.6).

**Proof** As the first step of the proof we fix a fibrant replacement for  $X$  in the complete Segal space model structure. By Theorem 2.22 a complete Segal space is characterized via right

lifting property with respect to three sets of morphisms, which we explicitly name as the Reedy, Segal and completeness maps:

- (1) *Reedy*:  $(\partial F(n) \rightarrow F(n)) \square (\Delta[l]_i \rightarrow \Delta[l])$
- (2) *Segal*:  $(G(n) \rightarrow F(n)) \square (\partial \Delta[l] \rightarrow \Delta[l])$
- (3) *Completeness*:  $(F(0) \rightarrow E(1)) \square (\partial \Delta[l] \rightarrow \Delta[l])$

Hence, applying the small object argument [34, Proposition 10.5.16] to the morphism  $X \rightarrow F(0)$  in  $s\mathcal{S}$  gives us a factorization  $X \xrightarrow{i} \hat{X} \rightarrow F(0)$  that satisfies the following properties:

- $i : X \rightarrow \hat{X}$  is a cell complex, meaning it is transfinite composition of pushouts of coproducts of the three sets of morphisms.
- The morphism  $\hat{X} \rightarrow F(0)$  satisfies the right lifting property with respect to the three classes of morphisms, which means  $\hat{X}$  is a complete Segal space.

Use the same argument on  $Y$  to obtain a morphism  $i' : Y \rightarrow \hat{Y}$  with the same properties. For the remainder of the proof we will use the fixed  $i : X \rightarrow \hat{X}$  and  $i' : Y \rightarrow \hat{Y}$  obtained via this argument.

Now, pick a morphism  $\hat{f} : \hat{X} \rightarrow \hat{Y}$  that makes the following diagram commute (using the lifting property of  $i$  against  $\hat{Y}$ )

$$\begin{array}{ccc}
 X & \xrightarrow[f \simeq_{CSS}]{} & Y \\
 \downarrow i \simeq_{CSS} & & \downarrow i' \simeq_{CSS} \\
 \hat{X} & \xrightarrow[\hat{f} \simeq_{Ree}]{} & \hat{Y}
 \end{array}$$

Then all maps in the diagram are equivalences in the CSS model structure and so the bottom horizontal map is a Reedy equivalence as  $\hat{X}$  and  $\hat{Y}$  are themselves complete Segal spaces (Theorem 2.22).

This diagram gives us the following diagram of adjunctions:

$$\begin{array}{ccc}
 (s\mathcal{S}/X)^{cov} & \xrightleftharpoons[f^*]{f_!} & (s\mathcal{S}/Y)^{cov} \\
 \uparrow i^* \downarrow i_! & & \uparrow (i')^* \downarrow (i')_! \\
 (s\mathcal{S}/\hat{X})^{cov} & \xrightleftharpoons[\hat{f}^*]{\hat{f}_!} & (s\mathcal{S}/\hat{Y})^{cov}
 \end{array} \tag{5.3}$$

By Theorem 3.15, all four are Quillen adjunctions and the bottom horizontal Quillen adjunction is a Quillen equivalence. So, if we proved that the two vertical Quillen adjunctions are Quillen equivalences, it would then follow from 2-out-of-3 that the top horizontal adjunction is a Quillen equivalence as well. As both vertical Quillen adjunctions are given by a fibrant replacement map, it suffices to prove the left vertical Quillen adjunction  $(i, i^*)$  is a Quillen equivalence.

We will prove that the derived unit and derived counit maps are weak equivalences. First we prove that the derived counit map  $i_! i^* L \rightarrow L$  is a covariant equivalence for every left fibration  $p : L \rightarrow \hat{X}$  (notice again that the derived counit map is the actual counit map as all objects are cofibrant).

The counit map comes from the diagram

$$\begin{array}{ccc}
 i^*L & \longrightarrow & L \\
 \downarrow & & \downarrow p \\
 X & \xrightarrow{i} & \hat{X}
 \end{array} \tag{5.4}$$

As  $p$  is a left fibration over a complete Segal space and  $i$  is a CSS equivalence, it follows from Theorem 4.32 that  $i^*L \rightarrow L$  is a CSS equivalence. Finally, by Proposition 3.39, a CSS equivalence over the CSS  $\hat{X}$  is also a covariant equivalence over  $\hat{X}$  finishing this part of the proof.

We move on to prove that the derived unit map is an equivalence. As explained in the beginning of the proof  $i$  is given as the transfinite composition of pushouts of coproducts of the three sets of morphisms, the Reedy, Segal and completeness morphisms, specified in the beginning of the proof. We claim it suffices to check these three cases separately to deduce that  $(i, i^*)$  is a Quillen equivalence. Indeed, let  $\mathcal{P}$  be the collection of cofibrations  $f : A \rightarrow B$  such that  $(f_!, f^*)$  is a Quillen equivalence between covariant model structures. It suffices to show that  $\mathcal{P}$  is closed under transfinite composition, coproducts and pushouts.

- Let  $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \dots$  be a chain of morphisms such that  $f_i \in \mathcal{P}$  for all  $i$ . Then, the transfinite composition  $f_\infty : A_0 \rightarrow A = \text{colim}(A_0 \rightarrow A_1 \rightarrow \dots)$  is in  $\mathcal{P}$  as the  $(f_\infty)^*$  is given as the projection from the limit of the tower  $s\mathcal{S}/A = \lim(\dots \rightarrow s\mathcal{S}/A_1 \xrightarrow{(f_0)^*} s\mathcal{S}/A_0) \rightarrow s\mathcal{S}/A_0$  and so if every functor in the diagram is a right Quillen equivalence,  $(f_\infty)^*$  is a Quillen equivalence as well. Hence,  $\mathcal{P}$  is closed under transfinite composition.
- For a collection of morphisms  $\{f_j : A_j \rightarrow B_j\}_{j \in J}$  we have an isomorphism between  $(\coprod_j f_j)^* : s\mathcal{S}/\coprod_j B_j \rightarrow s\mathcal{S}/\coprod_j A_j$  and  $\prod_j ((f_j)^* : s\mathcal{S}/B_j \rightarrow s\mathcal{S}/A_j)$  and so if all  $(f_j)^*$  are right Quillen equivalences then so is  $(\coprod_j f_j)^*$ . Hence,  $\mathcal{P}$  is closed under arbitrary coproducts.
- If  $k : A \rightarrow B$  is a monomorphism of simplicial spaces and  $f : A \rightarrow C$  a morphism, then we have a pullback square

$$\begin{array}{ccc}
 s\mathcal{S}/B \amalg_A C & \xrightarrow{(t_1)^*} & s\mathcal{S}/B \\
 \downarrow (t_2)^* & & \downarrow k^* \\
 s\mathcal{S}/C & \xrightarrow{f^*} & s\mathcal{S}/A
 \end{array}$$

So, if  $k^*$  is a right Quillen equivalence, then so is the induced functor  $(t_2)^*$ . Hence,  $\mathcal{P}$  is closed under pushouts.

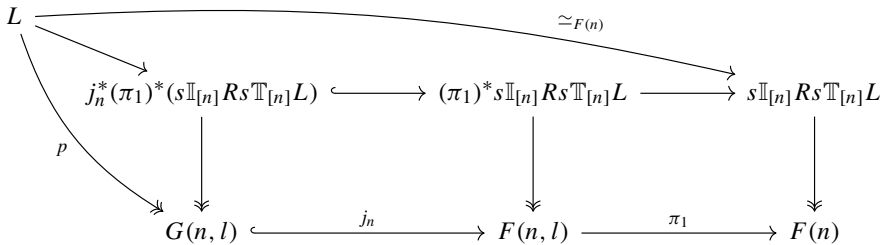
We now prove the three types of maps give us Quillen equivalences. The case for Reedy maps follows from Theorem 3.15. The case for the completeness maps follows from the fact that in the diagram of Quillen adjunctions induced by the functor  $[0] \rightarrow I[1]$

$$\begin{array}{ccc}
 \text{Fun}([0], \mathcal{S})^{proj} & \xrightleftharpoons[s\mathcal{H}_{[0]}]{s f_{[0]}} & s\mathcal{S}^{cov} \\
 \uparrow \downarrow & & \uparrow \downarrow \\
 \{0\}^* \dashv \{0\}_! & & \{0\}^* \dashv \{0\}_! \\
 \text{Fun}(I[1], \mathcal{S})^{proj} & \xrightleftharpoons[s\mathcal{H}_{I[1]}]{s f_{I[1]}} & (s\mathcal{S}/E(1))^{cov}
 \end{array}$$

the horizontal maps are Quillen equivalences (Theorem 4.18) and the left hand side is also a Quillen equivalence ( $\{0\} : [0] \rightarrow I[1]$  is an equivalence of categories) and so by 2-out-of-3 the right hand vertical adjunction is also a Quillen equivalence. Hence we only need to focus on the case of Segal maps.

To simplify notation we denote the map  $(G(n) \rightarrow F(n)) \square (\partial \Delta[l] \rightarrow \Delta[l])$  by  $j_n : G(n, l) \rightarrow F(n, l)$ . Let  $p : L \rightarrow G(n, l)$  be a left fibration. We want to prove the derived unit map  $L \rightarrow j_n^* R(j_n)_! L$  is a covariant equivalence over  $G(n, l)$ . Here  $L \rightarrow R(j_n)_! L$  is a fibrant replacement in the covariant model structure over  $F(n, l)$ . The key idea towards the proof is the appropriate fibrant replacement. The morphism  $\pi_1 : F(n) \times \Delta[l] \rightarrow F(n)$  is a Reedy equivalence and so, by Theorem 3.15, we have an equivalence  $R(j_n)_! L \simeq (\pi_1)^* R(\pi_1 j_n)_! L$ , meaning we can obtain a fibrant replacement of  $L$  over  $F(n, l)$  by taking the fibrant replacement over  $F(n)$  and pulling back the replacement via  $\pi_1 : F(n, l) \rightarrow F(n)$ .

Now, by Corollary 4.21, a fibrant replacement of  $(\pi_1 j_n)_! p : L \rightarrow F(n)$  is given by the derived unit map  $L \rightarrow s\mathbb{I}_{[n]} R s\mathbb{T}_{[n]} L$ , where  $R s\mathbb{T}_{[n]} L$  is the fibrant replacement of the functor  $s\mathbb{T}_{[n]} L : [n] \rightarrow \mathbb{S}$  in the projective model structure. Hence, the derived unit map of  $p : L \rightarrow G(n, l)$  is given by the morphism  $L \rightarrow j_n^* (\pi_1)^* (s\mathbb{I}_{[n]} R s\mathbb{T}_{[n]} L)$  over  $G(n, l)$ . We can summarize this construction in the following diagram:



By Theorem 4.35, it suffices to prove that  $L \rightarrow j_n^* (\pi_1)^* (s\mathbb{I}_{[n]} R s\mathbb{T}_{[n]} L)$  is fiber-wise a Kan equivalence. Fix an object  $m : F(0) \rightarrow G(n, l)$ . On the one hand, as  $L \rightarrow G(n, l)$  is a left fibration, the fiber over  $m : F(0) \rightarrow G(n, l)$  is homotopically constant (Example 3.19) and so  $(L \times_{G(n, l)} F(0))_0 \rightarrow \text{Diag}^*(L \times_{G(n, l)} F(0))$  is a Kan equivalence. On the other hand, we have the following chain of equivalences:

$$\begin{aligned}
 ((\pi_1 j_n)^* (s\mathbb{I}_{[n]} R s\mathbb{T}_{[n]} L) \times_{G(n, l)} F(0))_0 &\cong (s\mathbb{I}_{[n]} R s\mathbb{T}_{[n]} L \times_{F(n)} F(0))_0 && \text{definition of pullback} \\
 &\cong \text{Map}(N([0]^{[m]} \times_{[n]} [n]_{/-}), R s\mathbb{T}_{[n]} L) && \text{Definition 4.14} \\
 &\cong \text{Map}(\text{Hom}_{[n]}(m, -), R s\mathbb{T}_{[n]} L) && \text{computation} \\
 &\cong R s\mathbb{T}_{[n]} L(m) && \text{Lemma 1.1} \\
 &\xrightarrow{\simeq} s\mathbb{T}_{[n]} L(m) && \text{Definition 4.2} \\
 &\cong \text{Diag}^*(L \times_{F(n)} F(n)/_m) && \text{Definition 4.13}
 \end{aligned}$$

where Map denotes the mapping space of the functor category, using the fact that the projective model structure is simplicial (Remark 4.3).

Notice, if we compose the chain of morphisms we get an equivalence  $\text{Diag}^*(L \times_{F(n)} F(n)/_m) \rightarrow ((\pi_1 j_n)^* (s\mathbb{I}_{[n]} R s\mathbb{T}_{[n]} L) \times_{G(n, l)} F(0))_0$ , which takes an element  $\sigma \in \text{Diag}^*(L \times_{F(n)} F(n)/_m)_l$  to the natural transformation

$$N([0]^{[m]} \times_{[n]} [n]_{/-}) \xrightarrow{\cong} \text{Hom}_{[n]}(m, -) \rightarrow (s\mathbb{T}_{[n]} L)_l \rightarrow (R s\mathbb{T}_{[n]} L)_l,$$

where the middle natural transformation is uniquely determined via the Yoneda lemma (Lemma 1.1) by the element  $\sigma \in (s\mathbb{T}_{[n]} L)(m)_l = \text{Diag}^*(L \times_{F(n)} F(n)/_m)_l$ .

Now, by Proposition 4.15, we have that  $s\mathbb{T}_{[n]}(\{m\} : \Delta[l] \rightarrow F(n)) = N([0]^{[m]} \times_{[n]} [n]_{/-}) \times \{\Delta[l]\}$ . This means the second line in the chain of equivalences above is precisely given via the adjunction between  $s\mathbb{T}_{[n]}$  and  $s\mathbb{I}_{[n]}$ . Hence, by definition of the unit of an adjunction, the unit map  $(L \times_{G(n,l)} F(0))_0 \rightarrow (s\mathbb{I}_{[n]}Rs\mathbb{T}_{[n]}L)_0$  at the level of  $l$ -simplices is also given via the morphism

$$\text{Hom}_{/F(n)}(\Delta[l], L) \rightarrow \text{Hom}_{/F(n)}(\Delta[l], s\mathbb{I}_{[n]}Rs\mathbb{T}_{[n]}L) \cong \text{Nat}(N([0]^{[m]} \times_{[n]} [n]_{/-}) \times \{\Delta[l]\}, Rs\mathbb{T}_{[n]}L),$$

that takes an element  $\sigma$  to the natural transformation  $N([0]^{[m]} \times_{[n]} [n]_{/-}) \times \{\Delta[l]\} \rightarrow Rs\mathbb{T}_{[n]}L$  described above. This means if we restrict the chain of equivalences above via the evident inclusion  $\iota : (L \times_{G(n,l)} F(0))_0 \rightarrow \text{Diag}^*(L \times_{F(n)} F(n)_{/m})$  it becomes equal to the unit map. In other words, we have established that the following diagram commutes, where the top map is induced by the unit map

$$\begin{array}{ccc} (L \times_{G(n,l)} F(0))_0 & \longrightarrow & ((\pi_1 j_n)^*(s\mathbb{I}_{[n]}Rs\mathbb{T}_{[n]}L) \times_{G(n,l)} F(0))_0 \\ \downarrow \simeq & \searrow \iota & \simeq \uparrow \\ \text{Diag}^*(L \times_{G(n,l)} F(0)) & \longrightarrow & \text{Diag}^*(L \times_{F(n)} F(n)_{/m}) \end{array} .$$

We already observed above that the two vertical morphisms are equivalences. So, by 2-out-of-3, in order to show that the top morphism is a Kan equivalence it suffices to show that the bottom morphism is a Kan equivalence, or in other words, the map of fibers given as

$$L \times_{G(n,l)} F(0) \rightarrow L \times_{F(n)} F(n)_{/m} \cong L \times_{G(n,l)} G(n, l) \times_{F(n)} F(n)_{/m} \tag{5.5}$$

is a diagonal equivalence.

It suffices to prove that  $F(0) \rightarrow G(n, l) \times_{F(n)} F(n)_{/m}$  is a contravariant equivalence over  $G(n, l)$ . Indeed, in that case, by Theorem 4.29, the map 5.5 is also a contravariant equivalence (as  $L \rightarrow G(n, l)$  is a left fibration) and hence a diagonal equivalence by Theorem 3.17.

By direct computation  $F(n)_{/m} \rightarrow F(n)$  is given by  $\langle 0, \dots, m \rangle : F(m) \rightarrow F(n)$  and so we have an isomorphism

$$G(n, l) \times_{F(n)} F(n)_{/m} \cong G(m, l).$$

Now, the map  $F(0) \rightarrow G(m, l)$  factors as  $F(0) \rightarrow F(0) \times \Delta[l] \rightarrow G(m, l)$ . The first map is a Reedy equivalence (as  $\Delta[l]$  is contractible) and so we need to show that  $F(0) \times \Delta[l] \rightarrow G(m, l)$  is a covariant equivalence over  $G(n, l)$ . The map is the homotopy pushout (in the Reedy model structure) of the diagram

$$\begin{array}{ccccc} F(0) \times \Delta[l] & \longleftarrow & F(0) \times \partial\Delta[l] & \longrightarrow & F(0) \times \partial\Delta[l] \\ \downarrow & & \downarrow & & \downarrow \\ G(m) \times \Delta[l] & \longleftarrow & G(m) \times \partial\Delta[l] & \longrightarrow & F(m) \times \partial\Delta[l] \end{array} .$$

Thus it suffices to prove the vertical maps are contravariant equivalences over  $G(n, l)$  (as the contravariant model structure is left proper by Theorem 3.12).

The contravariant model structure is simplicial (Theorem 3.12) and so we can reduce it to checking that  $\langle 0 \rangle : F(0) \rightarrow G(m)$  is a contravariant equivalence over  $G(n, l)$ . The map is a composition of maps  $g : G(i) \rightarrow G(i + 1)$  and thus it suffices to show that  $g$  is a

contravariant equivalence over  $G(n, l)$ . By Theorem 3.15, we can reduce that to proving that  $g : G(i) \rightarrow G(i + 1)$  is a contravariant equivalence over  $G(i + 1)$ .

Finally, we have the following pushout square:

$$\begin{array}{ccc}
 F(0) & \xrightarrow[\simeq]{\langle 0 \rangle} & F(1) \\
 \downarrow \langle i \rangle & & \downarrow \langle i, i+1 \rangle \\
 G(i) & \xrightarrow[\simeq]{} & G(i + 1)
 \end{array} \quad (5.6)$$

The top horizontal map is a covariant equivalence by definition, which implies that the bottom horizontal map is a contravariant equivalence over  $G(i + 1)$ .  $\square$

**Remark 5.7** This result is also proven by Lurie [42, Remark 2.1.4.11], however, there it relies on translating the problem into the world of simplicial categories and then proving it there, which we managed to avoid. On the other side, it is also proven by Heuts and Moerdijk [33, Proposition F] with simplicial sets, using a very similar approach. In the special case where the domain and codomain are Segal spaces, the result also follows from [19, Proposition 5.5].

**Remark 5.8** Interestingly enough the result does not hold if we replace ‘‘CSS equivalence’’ with covariant or contravariant equivalence. For that it suffices to look at the simple case of  $F(0) \rightarrow F(1)$ , as the covariant model structure over  $F(0)$  is just the diagonal model structure, which is certainly not equivalent to the covariant or contravariant model structure over  $F(1)$ . Note, following [59, Proposition 2.5], this in particular implies that the covariant model structure is not right proper.

**Remark 5.9** Notice, there are maps which are not CSS equivalences, but still induce a Quillen equivalence of covariant model structures, as we shall explain. In Appendix B we prove that the covariant model structure over a simplicial set  $S$  (as defined in [42, Proposition 2.1.4.7]) is equivalent, via a left Quillen equivalence  $p_1^*$ , to the covariant model structure over the simplicial space  $p_1^*S$  (Theorem B.12). Hence, it suffices to witness a morphism of simplicial sets that is not an equivalence in the Joyal model structure (the model structure for quasi-categories), but induces an equivalence of covariant model structures over simplicial sets.

Examples of such maps can be found in [42, Sects. 4.4.5 and 5.1.4]. Indeed, [42, Sect. 5.1.4] studies *idempotent completions of  $\infty$ -categories* and proves in [42, Proposition 5.1.4.9, Corollary 4.4.5.15] that any such idempotent completion induces an equivalence on  $\infty$ -categories of presheaves. Hence, the same argument given in [42, Remark 2.1.4.11] (that relies on the straightening construction [42, Theorem 2.2.1.2]) implies that any non-trivial idempotent completion  $S \rightarrow \hat{S}$  is a map of simplicial sets that is not an equivalence in the Joyal model structure, but induces an equivalence on covariant model structures.

We can now use this theorem to finally establish a *Yoneda lemma for simplicial spaces* generalizing Theorem 3.49.

**Corollary 5.10** *Let  $X$  be a simplicial space and  $i : X \hookrightarrow \hat{X}$  be a chosen CSS fibrant replacement of  $X$ . Then for any point  $x : F(0) \rightarrow X$  the covariant fibrant replacement is given by  $\{s_0(x)\} : F(0) \rightarrow X_{x/} = F(0) \times_{\hat{X}} \hat{X}^{F(1)} \times_{\hat{X}} X$  induced by the composition  $i \circ \{x\} : F(0) \rightarrow \hat{X}$ . In particular, for any left fibration  $L \rightarrow X$  there is an equivalence of Kan complexes*

$$\{s_0(x)\}^* : \text{Map}_{/X}(X_{x/}, L) \xrightarrow{\simeq} \text{Map}_{/X}(F(0), L).$$

**Proof** For the first part, according to Theorem 3.49,  $F(0) \times_{\hat{X}} \hat{X}^{F(1)} \rightarrow \hat{X}$  is the covariant fibrant replacement of  $x : F(0) \rightarrow \hat{X}$  and then by Theorem 5.1 the covariant equivalence is preserved by pulling back along  $i$ . Now, the second part follows from the fact that the covariant model structure is simplicial (Theorem 3.12).  $\square$

In Proposition 3.39 we proved that a left fibration over a complete Segal space is a complete Segal space fibration. Using the invariance theorem, Theorem 5.1, we can now generalize it to left fibrations over every simplicial space.

**Theorem 5.11** *Let  $X$  be a simplicial space. Then the following adjunction*

$$(s\mathcal{S}/X)^{CSS} \begin{array}{c} \xrightarrow{id} \\ \perp \\ \xleftarrow{id} \end{array} (s\mathcal{S}/X)^{cov}$$

is a Quillen adjunction. Here the left hand side has the induced CSS model structure (Definition A.5) and the right hand side has the covariant model structure.

This implies that the covariant model structure over  $X$  is a localization of the induced CSS model structure over  $X$ .

**Proof** We want to prove that the left adjoint preserves cofibrations and trivial cofibrations. Clearly it preserves cofibrations as they are just the monomorphisms. Let  $i : Y \rightarrow Z$  be a trivial CSS cofibration over  $X$ . Then, Theorem 5.1 gives us a Quillen equivalence

$$(s\mathcal{S}/Y)^{cov} \begin{array}{c} \xrightarrow{i_!} \\ \perp \\ \xleftarrow{i^*} \end{array} (s\mathcal{S}/Z)^{cov} .$$

Thus, in particular, the counit map  $i_!i^*Z \rightarrow Z$  is a covariant equivalence over  $Z$ . However, we have  $i_!i^*(id_Z) = i : Y \rightarrow Z$ , which means  $i$  is a covariant equivalence over  $Z$ . Finally, by Theorem 3.15,  $i$  is also a covariant equivalence over  $X$ .  $\square$

**Remark 5.12** Note that we can use the same proofs with the contravariant model structure to show that the contravariant model structure is a localization of the CSS model structure as well.

The result above has the following very important corollary.

**Corollary 5.13** *Every left (and right) fibration is a CSS fibration.*

**Remark 5.14** This result generalizes [19, Sect. 1.4], which proved that left fibrations over Segal spaces are CSS fibrations. Lurie proves the same result over an arbitrary simplicial set [42, Theorem 3.1.5.1], but relies on the straightening construction.

In Theorem 4.32 we proved that pulling back along right and left fibrations over a complete Segal space preserves CSS equivalences. Using the invariance theorem, Theorem 5.1, we can generalize this to arbitrary right and left fibrations.

**Theorem 5.15** *Let  $p : R \rightarrow X$  be a right or left fibration. Then the adjunction*

$$(s\mathcal{S}/X)^{CSS} \begin{array}{c} \xrightarrow{p^*} \\ \perp \\ \xleftarrow{p_*} \end{array} (s\mathcal{S}/R)^{CSS}$$

is a Quillen adjunction. Here both sides have the induced CSS model structure (Proposition A.5).



**Proof** We will assume that  $p$  is a right fibration, the case for left fibrations follows similarly.

In order to prove  $(p^*, p_*)$  is a Quillen adjunction it suffices to prove  $p^*$  preserves cofibrations and trivial cofibrations. Evidently,  $p^*$  preserves cofibrations as they are just monomorphisms. Hence we are left with proving that for a given CSS equivalence  $Y \rightarrow Z$  over  $X$ , the map  $Y \times_X R \rightarrow Z \times_X R$  is a CSS equivalence over  $R$ .

First of all, observe that this property is invariant under Reedy equivalences, meaning if  $g : R \xrightarrow{\simeq} R'$  are Reedy equivalent fibrations over  $X$ , and pulling back along  $R$  preserves CSS equivalences, then so does pulling back along  $R'$ . Indeed, for an arbitrary CSS equivalence  $f : Y \rightarrow Z$  over  $X$  we have the following diagram

$$\begin{CD} Y \times_X R @>{Y \times_X g}>> Y \times_X R' \\ @V{f \times_X R}VV @VV{f \times_X R'}V \\ Z \times_X R @>{Z \times_X g}>> Z \times_X R' \end{CD}$$

The horizontal morphisms are Reedy equivalences, as the Reedy model structure is right proper and  $R \rightarrow X, R' \rightarrow X$  are fibrations [34, Proposition 13.3.9]. Hence, the left hand morphism is a CSS equivalence if and only if the right hand morphism is a CSS equivalence.

Now, let  $R \rightarrow X$  be a right fibration and fix a CSS fibrant replacement  $i : X \rightarrow \hat{X}$ . Then, by Theorem 5.1, there is a Reedy equivalence of right fibrations  $R \rightarrow i^*i_!\hat{R}$  over  $X$ , where  $i_!\hat{R}$  is the contravariant fibrant replacement of  $i_!R \rightarrow \hat{X}$ . So by the previous paragraph it suffices to show that pulling back along  $i^*Ri_!\hat{R} \rightarrow X$  preserves CSS equivalences.

In order to simplify notation we denote  $i_!R \rightarrow \hat{X}$  by  $\hat{R} \rightarrow \hat{X}$  and  $i^*Ri_!\hat{R} \rightarrow X$  by  $R \rightarrow X$  and notice we now have the following pullback square

$$\begin{CD} R @>{j}>> \hat{R} \\ @V{p}VV @VV{\hat{p}}V \\ X @>{i}>> \hat{X} \end{CD}$$

Now, by the Beck-Chevalley condition for pullback squares, we have a natural isomorphism  $j_!p^* \cong \hat{p}^*i_!$  [25, 1.2]. By Proposition A.5,  $j_!$  reflects weak equivalences and so  $p^*$  preserves weak equivalences if and only if  $j_!p^*$  does, which, by the natural isomorphism is equivalent to  $\hat{p}^*i_!$  preserving weak equivalences. However, this follows directly from the fact that  $i_!$  always preserves weak equivalences and  $\hat{p}^*$  does so by Theorem 4.32.  $\square$

The exact same proof can be used to prove the following theorem.

**Theorem 5.16** *Let  $p : R \rightarrow X$  be a right or left fibration. Then the adjunction*

$$(s\mathcal{S}/X)^{Seg} \begin{array}{c} \xrightarrow{p^*} \\ \xleftarrow{\perp} \\ \xrightarrow{p_*} \end{array} (s\mathcal{S}/R)^{Seg}$$

*is a Quillen adjunction. Here both sides have the induced Segal space model structure (A.5).*

**Remark 5.17** This same result is stated in [39, Remark 11.10] in the language of quasi-categories, however without a proof.

The theorem has the following helpful corollary.

**Corollary 5.18** *Let  $X \rightarrow Y$  be a CSS equivalence and  $F \rightarrow Y$  either a right or left fibration over  $Y$ . Then the map  $X \times_Y F \rightarrow F$  is also a CSS equivalence.*

This result is indeed helpful, as the CSS model structure is *not* right proper i.e. generally weak equivalences are not preserved by pullbacks along fibrations in the complete Segal space model structure. We can easily see this in the following example.

**Example 5.19** The map  $G(2) \rightarrow F(2)$  is a Segal equivalence. Let  $F(1) \rightarrow F(2)$  be the unique map that takes 0 to 0 and 1 to 2. Note that this map is a CSS fibration but neither a left fibration nor a right fibration. Now the pullback

$$F(1) \times_{F(2)} G(2) \rightarrow F(1)$$

is clearly not a Segal equivalence as the left hand side is just  $F(0) \coprod F(0)$ .

We will end this subsection with a generalization of Lemma 3.32 using Theorem 5.1. For that recall the classifying diagram  $N(\mathcal{C}, \mathcal{J}so)$  defined in Definition 2.24.

**Proposition 5.20** *Let  $p : \mathcal{D} \rightarrow \mathcal{C}$  be a discrete Grothendieck opfibration. Then  $N(\mathcal{D}, \mathcal{J}so) \rightarrow N(\mathcal{C}, \mathcal{J}so)$  is a left fibration of complete Segal spaces.*

**Proof** It follows from the same argument as in [58, Lemma 3.9] that  $N(\mathcal{D}, \mathcal{J}so) \rightarrow N(\mathcal{C}, \mathcal{J}so)$  is a Reedy fibration. We now have a pullback diagram

$$\begin{array}{ccc} N^h \mathcal{D} & \xrightarrow{\simeq} & N(\mathcal{D}, \mathcal{J}so) \\ \downarrow & \ulcorner & \downarrow \\ N^h \mathcal{C} & \xrightarrow{\simeq} & N(\mathcal{C}, \mathcal{J}so) \end{array}$$

The horizontal morphisms are equivalences in the complete Segal space model structure (Proposition 2.25). Hence, by Theorem 5.1, the left hand morphism is a left fibration if and only if the right hand map is a left fibration. However, we established in Lemma 3.32 that the left hand morphism is a left fibration and so we are done. □

### 5.2 Colimits, Cofinality and Quillen’s Theorem A

One intricate subject in the theory of  $(\infty, 1)$ -categories is the study of limits and colimits. Accordingly, there are now many sources dedicated to the study of colimits of quasi-categories, such as [38, 42]. Thus if we are interested in understanding colimits in a complete Segal space, we can translate those results from quasi-categories to complete Segal spaces using the work of Joyal and Tierney [37], or specialize the model independent approach to colimits via  $\infty$ -cosmoi to the case of complete Segal spaces [65].

In Sect. 0.2 we discussed how we want to understand whether results about  $(\infty, 1)$ -categories, such as various results about their colimits, still hold if we drop the completeness condition. This cannot be directly translated from the corresponding results about quasi-categories or  $\infty$ -cosmoi and needs to be proven directly.

In this final section we will apply our knowledge about left fibrations of Segal spaces to prove several classical results about colimits in the context of Segal spaces. In particular, we prove that a cocone out of a diagram corresponds to a map out of its colimit (Theorem 5.29), that final maps give us equivalent colimits (Theorem 5.37), and Quillen’s theorem A for simplicial spaces (Theorem 5.40) and in particular Segal spaces (Corollary 5.41).

**Remark 5.21** In this section we focus on using left fibrations to study colimits. We can analogously study limits via right fibrations.

**Definition 5.22** Let  $X$  be a Segal space and  $p : K \rightarrow X$  be a map of simplicial spaces. We define the Segal space of *cocones under  $K$* , denoted by  $X_{p/}$ , as the pullback

$$\begin{array}{ccc}
 X_{p/} & \longrightarrow & (X^K)^{F(1)} \\
 \downarrow \pi & & \downarrow (s,t) \\
 X \cong F(0) \times X & \xrightarrow{\{p\} \times \Delta} & X^K \times X^K
 \end{array}$$

where  $\Delta : X \rightarrow X^K$  is the map induced by the final map  $K \rightarrow F(0)$ .

**Remark 5.23** Recall that if  $X$  is a complete Segal space, then the underlying quasi-category of the under-complete Segal space  $i_1^*(W_{x/})$  is equivalent to the under-quasi-category  $i_1^*(W)_{x/}$  (as defined in [42, Proposition 1.2.9.2]), but not isomorphic (Remark 3.42). Again, by [42, Proposition 1.2.9.2], the same applies to the Segal space of cocones. Concretely, this means that if  $X$  is a complete Segal space and  $p : K \rightarrow X$  a morphism of simplicial spaces, then the equivalence of covariant model structures established in Appendix B implies an equivalence of left fibrations  $i_1^*(W_{p/}) \simeq i_1^*(W)_{i_1^*(p)/}$  although the constructions will not be isomorphic.

**Example 5.24** If  $X$  is a Segal space and  $K = F(0)$  then  $p$  is determined by a choice of point  $x$  in  $X$  and we have  $X_{p/} = X_{x/}$ , the Segal space of objects under  $x$ , as defined in Definition 3.41.

**Lemma 5.25** Let  $X$  be a Segal space and  $p : K \rightarrow X$  be a map of simplicial spaces. The projection map

$$\pi : X_{p/} \rightarrow X$$

is a left fibration.

**Proof** In the following pullback diagram

$$\begin{array}{ccc}
 X_{p/} & \longrightarrow & F(0) \times_{X^K} (X^K)^{F(1)} = (X^K)_{p/} \\
 \downarrow \dashv & \lrcorner & \downarrow \dashv \\
 X & \longrightarrow & X^K
 \end{array}$$

the right vertical map is a left fibration, by Theorem 3.44, and so the left vertical map must be a left fibration as well, as left fibrations are closed under pullbacks (Lemma 3.9).  $\square$

**Definition 5.26** Let  $X$  be a Segal space and  $p : K \rightarrow X$  a map of simplicial spaces. We say  $p$  has a colimit if the Segal space  $X_{p/}$  has an initial object (Definition 3.53).

**Remark 5.27** This approach to colimits has already been studied for quasi-categories [42, Definition 1.2.13.4].

The next lemma can help us better understand the definition of a colimit.

**Lemma 5.28** Let  $X$  be a Segal space and  $p : K \rightarrow X$  be a map of simplicial spaces. Then the following are equivalent:

- (1)  $p$  has a colimit.
- (2) There is a covariant equivalence  $\{\sigma\} : F(0) \rightarrow X_{p/}$  over  $X$ .
- (3) There is a Reedy equivalence

$$X_{v/} \rightarrow X_{p/}$$

where  $v$  is an object in  $X$ .

**Proof** (1  $\Rightarrow$  2) If  $p$  is a colimit, then  $X_{p/}$  has an initial object  $\sigma$  which, by Definition 3.53, means  $\{\sigma\} : F(0) \rightarrow X_{p/}$  is a covariant equivalence over  $X$ .

(2  $\Rightarrow$  3) Assume we have a covariant equivalence  $\{\sigma\} : F(0) \rightarrow X_{p/}$  over  $X$  and let  $\{v\} = \pi \circ \{\sigma\} : F(0) \rightarrow X$ . Then the commutative square

$$\begin{array}{ccc} F(0) & \xrightarrow{\cong} & X_{p/} \\ \cong \downarrow & \nearrow \text{dashed} & \downarrow \\ X_{v/} & \longrightarrow & X \end{array}$$

has a lift. Indeed,  $X_{p/} \rightarrow X$  is a left fibration (Lemma 5.25) and so a fibration in the covariant model structure and  $F(0) \rightarrow X_{v/}$  is a covariant equivalence over  $X$  (Theorem 3.49). As the top and left hand maps are covariant equivalences over  $X$ , the map  $X_{v/} \rightarrow X_{p/}$  is also a covariant equivalence over  $X$ . Finally, as both  $X_{v/}$  and  $X_{p/}$  are left fibrations over  $X$ , the covariant equivalence is in fact a Reedy equivalence (Theorem 3.12).

(3  $\Rightarrow$  1) By Theorem 3.55,  $X_{v/}$  has an initial object and so by the Reedy equivalence  $X_{p/}$  also has an initial object. □

We call the object  $\sigma$  in  $X_{p/}$  the *universal cocone* and, by abuse of language, the object  $v$  in  $X$  the *colimit*.

For the remainder of this section we want to use our knowledge of left fibrations to study colimits of Segal spaces. First, we prove the Segal space analogue to the fact that maps out of a colimit are equivalent to maps of cocones.

**Theorem 5.29** *Let  $X$  be a Segal space and  $p : K \rightarrow X$  be a map of simplicial spaces and assume it has universal cocone  $\sigma : F(0) \rightarrow X_{p/}$  with colimit  $v$ . Then for any object  $y$  in  $X$  we have a natural equivalence*

$$\text{comp} : \text{map}_X(v, y) \xrightarrow{\cong} \text{map}_{X^K}(p, \{y\}).$$

**Proof** By Lemma 5.28, we have a Reedy equivalence  $X_{v/} \rightarrow X_{p/}$  over  $X$ . Fix an object  $y : F(0) \rightarrow X$ , which gives us a point  $y : \Delta[0] \rightarrow X_0$ . Then, using Definition 2.17, we get a Kan equivalence

$$\text{comp} : \text{map}_X(v, y) = \text{Map}_{/X}(F(0), X_{v/}) \rightarrow \text{Map}_{/X}(F(0), X_{p/}) = \text{map}_{X^K}(p, \{y\})$$

and hence we are done. □

We now move on to study computational aspects of colimits in Segal spaces. In particular, we introduce final maps and prove they give us equivalent colimits.

**Definition 5.30** A map  $f : X \rightarrow Y$  is called *final* if  $f$  is a contravariant equivalence over  $Y$ . Similarly,  $f : X \rightarrow Y$  is called *initial* if  $f$  is a covariant equivalence over  $Y$ .

**Remark 5.31** The notion of final maps of quasi-categories was first studied in [38, 8.11]. They also have been studied by Lurie [42, Definition 4.1.1.1] where they are called cofinal.

Initial and final maps are a generalization of initial and final objects.

**Example 5.32** A map  $\{x\} : F(0) \rightarrow X$  is initial in the sense of Definition 5.30 if and only if the object  $x$  is initial in the sense of Definition 3.53.

**Lemma 5.33** Let  $f : X \rightarrow Y$  be a map of simplicial spaces. The following are equivalent:

- (1)  $f$  is a final map.
- (2) For any map  $g : Y \rightarrow Z$  the map  $f$  is a contravariant equivalence over  $Z$ .
- (3) For any right fibration  $R \rightarrow Y$  the induced map

$$\text{Map}_{/Y}(Y, R) \rightarrow \text{Map}_{/Y}(X, R)$$

is a Kan equivalence.

**Proof** (1  $\Rightarrow$  2) As  $f : X \rightarrow Y$  is a covariant equivalence over  $Y$ ,  $g_!(X \rightarrow Y)$  is a covariant equivalence over  $Z$  (Theorem 3.15).

(2  $\Rightarrow$  1) This is just a special case where  $g = \text{id}_Y$ .

(1  $\Leftrightarrow$  3) Follows from the definition of contravariant equivalence (Theorem 3.12). □

**Remark 5.34** Although final maps are defined as certain contravariant equivalences, they do not always behave similar to weak equivalences in the model categorical sense. In particular, they do not satisfy the 2-out-of-3 property. For example, in the chain

$$F(0) \xrightarrow{\langle 0 \rangle} F(1) \xrightarrow{\langle 0, 0 \rangle} F(0)$$

the map  $\langle 0, 0 \rangle$  and the composition  $\langle 0 \rangle$  are final, but  $\langle 0 \rangle : F(0) \rightarrow F(1)$  is not.

**Corollary 5.35** If  $f : X \rightarrow Y$  is a CSS equivalence then it is final.

**Proof** This follows directly from Theorem 5.11. □

Before we move on let us note that this gives us one exception to Remark 5.34.

**Lemma 5.36** Let  $X \xrightarrow{f} Y \xrightarrow{g} Z$  be a chain of maps such that  $g$  is a CSS equivalence. Then  $f$  is a final map if and only if  $gf$  is a final map.

**Proof** First note that  $g$  is a final map. So, if  $f$  is final then it is a contravariant equivalence over  $Z$  (Lemma 5.33) and so the composition  $gf$  is also a covariant equivalence over  $Z$ , which by definition means it is final.

On the other side, let us assume  $gf = g_!(f)$  is a final map. Then, by Theorem 5.1, the following adjunction is a Quillen equivalence:

$$(s\mathcal{S}_{/Y})^{contra} \begin{array}{c} \xrightarrow{g_!} \\ \perp \\ \xleftarrow{g^*} \end{array} (s\mathcal{S}_{/Z})^{contra}$$

which implies that  $f : X \rightarrow Y$  is a contravariant equivalence over  $Y$  if and only if  $g_!(f) : X \rightarrow Z$  is a contravariant equivalence over  $Z$ , and so  $f$  is a final map as well. □

Having discussed final maps we can now show how it allows us to simplify colimit diagrams.

**Theorem 5.37** *Let  $g : A \rightarrow B$  be a final map and  $X$  be a Segal space. Then for any map  $f : B \rightarrow X$  the induced map*

$$X_{f/} \rightarrow X_{fg/}$$

*is a Reedy equivalence.*

**Proof** Fix an object  $x$  in the Segal space  $X$ . Using adjunctions between products and exponentials, we have the following isomorphism

$$\text{Map}(F(1), X^B) \cong \text{Map}(F(1) \times B, X) \cong \text{Map}(B, X^{F(1)}) \tag{5.38}$$

Let  $\text{Map}^{res}(F(1) \times B, X)$  be the full subspace of  $\text{Map}(F(1) \times B, X)$  consisting of maps  $H : F(1) \times B \rightarrow X$  such that  $H|_{\{0\}} = f : B \rightarrow X$  and  $H|_{\{1\}} = \{x\} : B \rightarrow X$ . Then restricting the two isomorphisms in 5.38 gives us the isomorphisms

$$\text{Map}_{/X}(F(0), X_{f/}) \cong \text{Map}^{res}(F(1) \times B, X) \cong \text{Map}_{/X}(B, X_{/x}).$$

This isomorphism is natural in  $B$  and hence the map  $g : A \rightarrow B$  gives us a commutative diagram:

$$\begin{array}{ccc} \text{Map}_{/X}(F(0), X_{f/}) & \xrightarrow{g_*} & \text{Map}_{/X}(F(0), X_{fg/}) \\ \downarrow \cong & & \downarrow \cong \\ \text{Map}_{/X}(B, X_{/x}) & \xrightarrow[\cong]{f^*} & \text{Map}_{/X}(A, X_{/x}) \end{array}$$

By the explanation above the vertical maps are isomorphisms. Moreover,  $X_{/x} \rightarrow X$  is a right fibration (Theorem 3.44) and so, by Definition 5.30, the bottom map is a Kan equivalence. Hence the top map is a Kan equivalence for every map  $x : F(0) \rightarrow X$ . As  $g_* : X_{f/} \rightarrow X_{fg/}$  is a map of left fibrations over  $X$ , this implies that it is a Reedy equivalence (Theorem 4.35). □

Notice, this statement has also been proven in the context of quasi-categories in [42, Proposition 4.1.1.7].

**Corollary 5.39** *Let  $X$  be a Segal space and  $g : A \rightarrow B$  be a final map of simplicial spaces. Then a map  $f : B \rightarrow X$  has a colimit if and only if  $gf : A \rightarrow X$  has a colimit and in that case they are equivalent objects in  $X$ .*

We end this section by giving a useful criterion for classifying final maps of simplicial spaces, motivated by Quillen’s Theorem A [51].

**Theorem 5.40** *Let  $f : X \rightarrow Y$  be a map of simplicial spaces and  $\{L_y \rightarrow Y\}_{\{y\}:F(0) \rightarrow Y}$  a collection of covariant fibrant replacements of  $\{y\} : F(0) \rightarrow Y$ . The following are equivalent:*

- (1)  *$f$  is a final map.*
- (2) *For any  $y : F(0) \rightarrow Y$ , the simplicial space  $L_y \times_Y X$  is diagonally contractible.*

**Proof** By Theorem 4.41,  $f : X \rightarrow Y$  is a contravariant equivalence over  $Y$  if and only if

$$L_y \times_Y f : L_y \times_Y X \rightarrow L_y \times_Y Y \cong L_y$$

is a diagonal equivalence for all  $y : F(0) \rightarrow Y$ . By assumption  $F(0) \rightarrow L_y$  is a covariant equivalence over  $Y$  and thus, by Theorem 3.17, a diagonal equivalence. Hence,  $L_y \times_Y f$  is a diagonal equivalence if and only if  $L_y \times_Y X$  is diagonally contractible. □

In the case of Segal spaces we can simplify the statement.

**Corollary 5.41** *Let  $Y$  be a Segal space and  $f : X \rightarrow Y$  be a map of simplicial spaces. Then  $f$  is final if and only if for every object  $y$  in  $Y$  the simplicial space  $Y_{y/} \times_Y X$  is diagonally contractible.*

**Remark 5.42** This result was proven for quasi-categories in [42, Theorem 4.1.3.1] where it is attributed to Joyal. Thus, Corollary 5.41 generalizes Quillen's theorem *A* to Segal spaces.

**Acknowledgements** I want to thank my advisor Charles Rezk who has guided me through every step of the work. I want to in particular thank him for many helpful remarks on previous drafts. I also want to thank Matt Ando for many fruitful conversations and suggestions. Moreover, I want to thank the first referee for many helpful comments and suggestions on how to improve and simplify the theorems. Finally, I want to sincerely thank the second referee for several very extensive lists of comments that have significantly improved the results and presentation.

**Funding** Open Access funding enabled and organized by Projekt DEAL. No funding was received to assist with the preparation of this manuscript. The author did not receive support from any organization for the submitted work.

**Availability of Data and Materials** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

## Declarations

**Conflict of interest** The authors have no relevant financial or non-financial interests to disclose.

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## Appendix A: Some Facts about Model Categories

We primarily used the theory of model categories to tackle issues of higher category theory. In this section we will not introduce model categories as there are already several excellent sources. For instance, we refer the reader to [21] for a short introduction to this subject and to [34, 35] for a more detailed discussion. Here we will only state some technical lemmas we have used throughout.

First, we make ample use of the following very classical result about trivial Kan fibrations.

**Lemma A.1** *Let  $p : S \rightarrow T$  be a Kan fibration in  $\mathcal{S}$ . Then  $p$  is a trivial Kan fibration if and only if each fiber of  $p$  is contractible.*

For a readable proof of this statement see [57, Sect. 38]. This lemma has the following important corollary.

**Corollary A.2** *Let  $p : S \rightarrow K$  and  $q : T \rightarrow K$  be two Kan fibrations. A map  $f : S \rightarrow T$  over  $K$  is a Kan equivalence if and only if for each point  $\{k\} : \Delta[0] \rightarrow K$  the map between*

fibers

$$S \times_K \{k\} \rightarrow T \times_K \{k\}$$

is a Kan equivalence.

Next we have two important results that help us study Quillen adjunctions.

**Lemma A.3** ([37, Proposition 7.15], [34, Proposition 8.5.4]) *Let  $\mathcal{M}$  and  $\mathcal{N}$  be two model categories and*

$$\mathcal{M} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} \mathcal{N}$$

be an adjunction of model categories, then the following are equivalent:

- (1)  $(F, G)$  is a Quillen adjunction.
- (2)  $F$  takes cofibrations to cofibrations and  $G$  takes fibrations between fibrant objects to fibrations.
- (3)  $G$  preserves trivial fibrations and takes fibrations between fibrant objects to fibrations.

**Lemma A.4** (Special case of [37, Proposition 7.17, Proposition 7.22]) *Let*

$$\mathcal{M} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} \mathcal{N}$$

be a Quillen adjunction of model categories. Then the following are equivalent:

- (1)  $(F, G)$  is a Quillen equivalence.
- (2)  $F$  reflects weak equivalences between cofibrant objects and the derived counit map  $FLG(n) \rightarrow n$  is an equivalence for every fibrant-cofibrant object  $n \in \mathcal{N}$  (Here  $LG(n)$  is a cofibrant replacement of  $G(n)$  inside  $\mathcal{M}$ ).
- (3)  $G$  reflects weak equivalences between fibrant objects and the derived unit map  $m \rightarrow GRF(m)$  is an equivalence for every fibrant-cofibrant object  $m \in \mathcal{M}$  (Here  $RF(m)$  is a fibrant replacement of  $F(m)$  inside  $\mathcal{N}$ ).

We move on to the main topic of this appendix, namely the existence of two localized model structures on the category of simplicial spaces over a fixed simplicial space and their comparison.

**Proposition A.5** [34, Theorem 7.6.5] *Let  $\mathcal{M}$  be a model structure on  $s\mathcal{S}$ . Let  $X$  be a simplicial space. There is a simplicial model structure on  $s\mathcal{S}/X$ , which we call the induced model structure and denote by  $(s\mathcal{S}/X)^{\mathcal{M}}$ , and which satisfies the following conditions:*

- (F) A map  $f : Y \rightarrow Z$  over  $X$  is a (trivial) fibration if  $Y \rightarrow Z$  is a (trivial) fibration in  $\mathcal{M}$ .
- (W) A map  $f : Y \rightarrow Z$  over  $X$  is a weak equivalence if  $Y \rightarrow Z$  is a weak equivalence in  $\mathcal{M}$ .
- (C) A map  $f : Y \rightarrow Z$  over  $X$  is a (trivial) cofibration if  $Y \rightarrow Z$  is a (trivial) cofibration in  $\mathcal{M}$ .

**Remark A.6** The induced model structure can be defined for any model category and not just for model structures on  $s\mathcal{S}$ , but for our work there is no need for further generality.



**Theorem A.7** *Let  $X$  be a simplicial space and  $\mathcal{L}$  be a set of monomorphisms in  $s\mathcal{S}_{/X}$ . There exists a cofibrantly generated, simplicial model category structure on  $s\mathcal{S}_{/X}$  with the following properties:*

- (1) *The cofibrations are exactly the monomorphisms.*
- (2) *The fibrant objects (called  $\mathcal{L}$ -local objects) are exactly the Reedy fibrations  $W \rightarrow X \in s\mathcal{S}$  such that*

$$\text{Map}_{/X}(B, W) \rightarrow \text{Map}_{/X}(A, W)$$

*is a weak equivalence of spaces for all maps  $f : A \rightarrow B$  over  $X$  in  $\mathcal{L}$ .*

- (3) *The weak equivalences (called  $\mathcal{L}$ -local weak equivalences) are exactly the maps  $g : Y \rightarrow Z$  over  $X$  such that for every  $\mathcal{L}$ -local object  $W \rightarrow X$ , the induced map*

$$\text{Map}_{/X}(Z, W) \rightarrow \text{Map}_{/X}(Y, W)$$

*is a weak equivalence.*

- (4) *Let  $f : Y \rightarrow Z$  be a map over  $X$ .*

- *If  $f$  is a Reedy weak equivalence over  $X$  then it is a  $\mathcal{L}$ -local weak equivalence.*
- *If  $f$  is a  $\mathcal{L}$ -local fibration then it is a Reedy fibration.*

*The opposite of both statements hold if  $Y$  and  $Z$  are  $\mathcal{L}$ -local.*

*We call this model category the localized model structure.*

The model structure is given as the left Bousfield localization of the induced Reedy model structure on  $s\mathcal{S}_{/X}$ . For a careful and detailed proof of the existence of left Bousfield localizations see [34, Theorem 4.1.1] (notice we are using the fact that the induced Reedy model structure on  $s\mathcal{S}_{/X}$  is proper and cellular [34, Proposition 12.1.6]). For a nice summary of this proof that goes over the main steps see [58, Proposition 9.1].

**Remark A.8** Notice, we can in particular take  $X$  to be the final object in which case the theorem gives us a localization model structure of the Reedy model structure on  $s\mathcal{S}$ .

Note any such model structure is invariant under Reedy equivalences.

**Lemma A.9** *Let  $\mathcal{L}$  be a set of monomorphisms in  $s\mathcal{S}$  and let  $f : X \rightarrow Y$  be a map of simplicial spaces. Then the adjunction*

$$(s\mathcal{S}_{/X})^{\mathcal{M}_X} \begin{matrix} \xrightarrow{f_!} \\ \xleftarrow{f^*} \end{matrix} (s\mathcal{S}_{/Y})^{\mathcal{M}_Y}$$

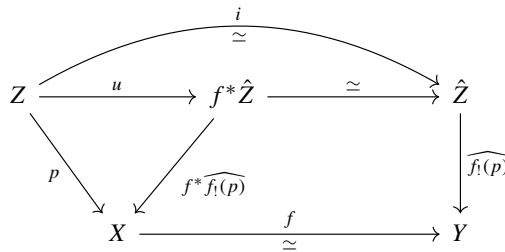
*is a Quillen adjunction which is a Quillen equivalence if  $f$  is a Reedy equivalence. Here the left hand side has the localization model structure with respect to maps  $A \rightarrow B \rightarrow X$  for all maps  $A \rightarrow B$  in  $\mathcal{L}$  and the right hand side has the localization model structure with respect to maps  $A \rightarrow B \rightarrow Y$  for all maps  $A \rightarrow B$  in  $\mathcal{L}$ .*

**Proof** First we use Lemma A.3 to prove it is a Quillen adjunction. Clearly the left adjoint  $f_!$  preserves monomorphisms and thus cofibrations. On the other hand, fibrations between fibrant objects are just Reedy fibrations, which are preserved by  $f^*$ . Thus we only need to prove that  $f^*$  preserves fibrant objects. However, the fibrant objects are just Reedy fibrations which satisfy the right lifting property with respect to maps  $A \rightarrow B$  in  $\mathcal{L}$  and the class of such maps is clearly closed under pullback.

Next we assume that  $f$  is a Reedy equivalence and prove that the adjunction is a Quillen equivalence. We will prove that the derived unit and derived counit maps are equivalences. Before we can analyze the derived unit map we need to better understand fibrant replacements.

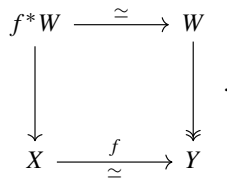
Let  $p : Z \rightarrow X$  be a fibrant object (meaning  $p$  is a local Reedy fibration) and let  $Z \xrightarrow{i} \widehat{Z} \xrightarrow{\widehat{f}_!(p)} Y$  be a Reedy fibrant replacement of  $f_!(p) : Z \rightarrow Y$ . As  $p$  and  $\widehat{f}_!(p)$  are Reedy equivalent and  $p$  is local, it follows that  $\widehat{f}_!(p)$  is a local fibration as well, meaning that  $i : Z \rightarrow \widehat{Z}$  is in fact already a fibrant replacement in the localized Reedy model structure.

Now, the derived unit map,  $f^*Rf_!$ , is given by taking the fibrant replacement in the localized model structure of the map  $Z \rightarrow X \rightarrow Y$  in  $s\mathcal{S}_Y$  and then pulling it back along  $f : X \rightarrow Y$ . This can be depicted as the following diagram:



where (as we explained in the previous paragraph)  $i : Z \rightarrow \widehat{Z}$  is taken to be the Reedy fibrant replacement. The map  $f^*\widehat{Z} \rightarrow \widehat{Z}$  is a Reedy equivalence as Reedy equivalences are preserved by pullbacks along fibrations. Thus, by 2-out-of-3, the derived unit map  $u : Z \rightarrow f^*\widehat{Z}$  is a Reedy weak equivalence.

We move on to the derived counit map. As all objects are cofibrant the derived counit map is simply given by the actual counit map. Let  $W \rightarrow Y$  be a fibrant object (meaning a fibration). Then we have the diagram



As the Reedy model structure is right proper (Sect. 2.4) and thus Reedy weak equivalences are preserved by pullback, the counit map  $f^*W \rightarrow W$  is a Reedy equivalence and so also an equivalence in the localized model structure (Theorem A.7).  $\square$

Our precise understanding of the fibrant objects in the localized model structure allows us to simplify the conditions in Lemma A.3.

**Corollary A.10** *Let  $X$  be a simplicial space and let  $(s\mathcal{S}_X, \mathcal{M})$  and  $(s\mathcal{S}_X, \mathcal{N})$  be two localized model structures of the induced Reedy model structure. Then an adjunction*

$$(s\mathcal{S}_X)^{\mathcal{M}} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow[G]{} \end{array} (s\mathcal{S}_X)^{\mathcal{N}}$$

is a Quillen adjunction if it satisfies the following conditions:

- (1)  $F$  takes cofibrations to cofibrations.

- (2)  $G$  takes fibrant objects to fibrant objects.
- (3)  $G$  takes Reedy fibrations to Reedy fibrations.

**Remark A.11** Let  $X$  be a simplicial space and  $\mathcal{L}$  be a set of monomorphisms over  $X$ . Then we can construct two model structures on  $s\mathcal{S}/X$  using  $\mathcal{L}$ :

- (1) Using Theorem A.7 we can construct a localized model structure on the induced model structure  $s\mathcal{S}/X$ . This is the localized model structure on  $s\mathcal{S}/X$ .
- (2) We can project  $\mathcal{L}$  to  $s\mathcal{S}$  to get a set of monomorphisms in  $s\mathcal{S}$  and then use Theorem A.7 to construct a localized model structure on  $s\mathcal{S}$  and finally take the induced model structure on  $s\mathcal{S}/X$ . We call this the *induced localized model structure* on  $s\mathcal{S}/X$ .

We want to understand how these two model structures compare to each other. Before we can do that we need a precise characterization of the fibrant objects in the localized model structure on  $s\mathcal{S}/X$  in terms of the Reedy model structure on  $s\mathcal{S}$ .

**Lemma A.12** *Let  $X$  be a simplicial space and  $\mathcal{L}$  a set of monomorphisms. An object  $p : Y \rightarrow X$  in  $s\mathcal{S}/X$  is fibrant in the localized model structure if and only if it is a Reedy fibration and for every morphism  $f : A \rightarrow B$  in  $\mathcal{L}$  the commutative square*

$$\begin{array}{ccc}
 \text{Map}(B, Y) & \xrightarrow{f^*} & \text{Map}(A, Y) \\
 \downarrow p_* & & \downarrow p_* \\
 \text{Map}(B, X) & \xrightarrow{f^*} & \text{Map}(A, X)
 \end{array}$$

is a homotopy pullback square of spaces.

**Proof** The square is a homotopy pullback square if and only if the horizontal map below is an equivalence

$$\begin{array}{ccc}
 \text{Map}(B, Y) & \xrightarrow{\quad\quad\quad} & \text{Map}(A, Y) \times_{\text{Map}(A, X)} \text{Map}(B, X) \\
 & \searrow p_* & \swarrow \pi_2 \\
 & \text{Map}(B, X) &
 \end{array}$$

By Corollary A.2, this is equivalent to proving that for each map  $g : B \rightarrow X$  the induced map

$$\text{Map}_{/X}(B, Y) \rightarrow \text{Map}_{/X}(A, Y)$$

is an equivalence, which is exactly the condition of being fibrant in the localized model structure (Theorem A.7). □

**Theorem A.13** *Let  $X$  be a simplicial space and  $\mathcal{L}$  be a set of monomorphisms in  $s\mathcal{S}/X$ . Then the adjunction*

$$(s\mathcal{S}/X)^{loc\mathcal{M}} \begin{array}{c} \xrightarrow{id} \\ \xleftarrow{id} \end{array} (s\mathcal{S}/X)^{\mathcal{M}}$$

is a Quillen adjunction, which is a Quillen equivalence if  $X$  is fibrant in the localized model structure. In fact, in this case the two model structures are isomorphic. Here the left hand side has the localized model structure and the right hand side has the induced localized model structure (Remark A.11).

**Proof** Both sides have the same set of cofibrations. By [42, Corollary A.3.7.2], in order to finish the proof it suffices to show that every fibrant object in the induced localized model structure on  $s\mathcal{S}/X$  is fibrant in the localized model structure on  $s\mathcal{S}/X$  and that the opposite holds if  $X$  is fibrant in the localized model structure.

Let  $p : Y \rightarrow X$  be a Reedy fibration and  $f : A \rightarrow B$  a morphism in  $\mathcal{L}$ . Then we have the following diagram:

$$\begin{array}{ccc} \text{Map}(B, Y) & \xrightarrow{f^*} & \text{Map}(A, Y) \\ \downarrow p_* & & \downarrow p_* \\ \text{Map}(B, X) & \xrightarrow{f^*} & \text{Map}(A, X) \end{array} .$$

If  $p$  is fibrant in the induced localized model structure on  $s\mathcal{S}/X$ , then the square above is a homotopy pullback square as the induced localized model structure on  $s\mathcal{S}/X$  is a simplicial model structure and  $f : A \rightarrow B$  is a trivial cofibration in the localized model structure on  $s\mathcal{S}$ . However, by Lemma A.12, this is equivalent to  $p : Y \rightarrow X$  being fibrant in the localized model structure on  $s\mathcal{S}/X$ . This finishes one side and proves the adjunction above is a Quillen adjunction.

On the other hand, let us assume  $X$  is fibrant in the localized model structure on  $s\mathcal{S}$  and  $p : Y \rightarrow X$  is fibrant in the localized model structure on  $s\mathcal{S}/X$ . The fibrancy of  $p : Y \rightarrow X$  implies, again by Lemma A.12, that the square

$$\begin{array}{ccc} \text{Map}(B, Y) & \xrightarrow[\simeq]{f^*} & \text{Map}(A, Y) \\ \downarrow p_* & & \downarrow p_* \\ \text{Map}(B, X) & \xrightarrow[\simeq]{f^*} & \text{Map}(A, X) \end{array}$$

is a homotopy pullback square for all morphisms  $f : A \rightarrow B$  in  $\mathcal{L}$  and the fibrancy of  $X$  implies the bottom map is a Kan equivalence. Hence the top map is a Kan equivalence as well. This means that  $Y$  is fibrant in the localized model structure on  $s\mathcal{S}$ . Thus, by Theorem A.7, the map  $p : Y \rightarrow X$  is a fibration in the localized model structure on  $s\mathcal{S}$ , as Reedy fibrations between fibrant objects are fibrations in the localized model structure.  $\square$

## Appendix B: Comparison with Quasi-Categories

In this part we confirm that the covariant model structure on simplicial sets coincides with the covariant model structure on simplicial spaces, by proving they are Quillen equivalent, via two different Quillen equivalences.

The trick is to realize that the Quillen equivalences between the model structure for quasi-categories (which we will henceforth call the Joyal model structure) and the model structure for complete Segal spaces constructed by Joyal and Tierney [37] descend to Quillen equivalences between their respective covariant model structures.

We will thus start by giving a quick review of the relevant results in [37] and review the relevant definitions of the covariant model structure on simplicial sets. We will only focus on specific results that we need in this section and refer the reader to the vast literature for any details [33, 42].

**Notation B.1** This section focuses on the interaction between complete Segal spaces and quasi-categories and more specifically relies on [37]. We will hence use different notation in this section alone to match existing literature. Concretely, we use the following notation:

- We use  $\text{sSet}^{Joy}$  for the Joyal model structure on the category of simplicial sets (with fibrant objects quasi-categories) and, for a given simplicial set  $S$ , use the notation  $(\text{sSet}/S)^{cov}$  for the covariant model structure on simplicial sets over  $S$  (defined in Definition B.3).
- Following the notation in [37, Sect. 2] we denote  $p_1 : \Delta \times \Delta \rightarrow \Delta$  defined as  $p_1([n], [m]) = [n]$  and  $i_1 : \Delta \rightarrow \Delta \times \Delta$  defined as  $i_1([n]) = ([n], [0])$ .

**Definition B.2** [42, Definition 2.0.0.3] A map  $f : S \rightarrow T$  of simplicial sets is a *left fibration* if it satisfies the right lifting property with respect to all horn conclusions of the form  $\Lambda_i^n \rightarrow \Delta[n]$ , where  $0 \leq i < n$ .

**Definition B.3** [42, Definition 2.1.4.5, Proposition 2.1.4.7, Proposition 2.1.4.8] Let  $S$  be a simplicial set. There is a left proper, combinatorial, simplicial model structure on  $\text{sSet}/S$ , called the covariant model structure and denoted by  $(\text{sSet}/S)^{cov}$ , which satisfies the following conditions:

- (1) A map  $T \rightarrow U$  over  $S$  is a cofibration if it is a monomorphism.
- (2) The fibrant objects are the left fibrations

We can also characterize the fibrations between fibrant objects.

**Corollary B.4** [42, Corollary 2.2.3.14] *A map between left fibrations is a fibration in the covariant model structure if and only if it is a left fibration.*

There are several important theorems about the covariant model structure we are going to need later on.

**Theorem B.5** [42, Theorem 3.1.5.1] *Let  $S$  be a simplicial set. Then the following adjunction*

$$(\text{sSet}/S)^{Joy} \begin{matrix} \xrightarrow{id} \\ \perp \\ \xleftarrow{id} \end{matrix} (\text{sSet}/S)^{cov}$$

*is a Quillen adjunction, where the left hand side has the Joyal model structure and the right hand side has the covariant model structure. This implies that the covariant model structure is a localization of the Joyal model structure.*

**Theorem B.6** [42, Proposition 2.1.4.10, Remark 2.1.4.11] *Let  $f : S \rightarrow T$  be a map of simplicial sets. Then the adjunction*

$$(\text{sSet}/S)^{cov} \begin{matrix} \xrightarrow{f_!} \\ \perp \\ \xleftarrow{f^*} \end{matrix} (\text{sSet}/T)^{cov}$$

*is a Quillen adjunction, which is a Quillen equivalence if  $f$  is a categorical equivalence. Here both sides have the covariant model structure.*

We now move on to review the main results in [37]. We will hence use the notation given in [37, Sect. 2] instead of the notation we have used before. Concretely,  $p_1 : \Delta \times \Delta \rightarrow \Delta$  is given by  $p_1 * ([n], [m]) = [n]$

**Theorem B.7** [37, Theorem 4.11] *The functors  $p_1, i_1$  (Notation B.1) induce the following adjunction*

$$\mathbf{sSet}^{Joy} \begin{array}{c} \xrightarrow{p_1^*} \\ \perp \\ \xleftarrow{i_1^*} \end{array} \mathbf{sS}^{CSS},$$

which is a Quillen equivalence, where  $\mathbf{sSet}$  has the Joyal model structure and  $\mathbf{sS}$  has the CSS model structure.

**Theorem B.8** [37, Theorem 4.12] *Let  $t_! : \mathbf{sS} \rightarrow \mathbf{sSet}$  be the left Kan extension of the map which is defined on the generators  $F(n) \times \Delta[l]$  as  $t_!(F(n) \times \Delta[l]) = \Delta[n] \times J[l]$ . Let  $t^! : \mathbf{sSet} \rightarrow \mathbf{sS}$  be the right adjoint of this construction, i.e.  $t^!(S)_{nl} = \text{Hom}_{\mathbf{sS}}(\Delta[n] \times J[l], S)$ . Then this defines a Quillen equivalence*

$$(\mathbf{sS})^{CSS} \begin{array}{c} \xrightarrow{t_!} \\ \perp \\ \xleftarrow{t^!} \end{array} (\mathbf{sSet})^{Joy}$$

with  $\mathbf{sS}$  having the CSS model structure and  $\mathbf{sSet}$  having the Joyal model structure.

The two adjunctions  $(p_1^*, i_1^*), (t_!, t^!)$  do interact well with each other.

**Lemma B.9** [37, Proposition 4.10] *Let  $S$  be a quasi-category. Then the natural map*

$$g_S : p_1^* S \rightarrow t^! S$$

is an equivalence in the CSS model structure.

There is an analogous statement for complete Segal spaces.

**Lemma B.10** *Let  $X$  be a complete Segal space, then the natural map*

$$h_X : i_1^* X \rightarrow t_! X$$

induced by applying  $t_!$  to the counit map  $p_1^* i_1^* X \rightarrow X$  and using  $t_! p_1^* = id$  is a categorical equivalence.

We now move on to the main topic of this section: the two Quillen equivalences (Theorems B.12 and B.14).

**Remark B.11** Let  $\mathcal{C}$  be a category with pullbacks and let

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} \mathcal{D}$$

be an adjunction of categories and  $C$  an object in  $\mathcal{C}$ . Then we get an adjunction

$$\mathcal{C}_{/C} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{u^*G} \end{array} \mathcal{D}_{/FC}$$

where the left adjoint takes a map  $f : D \rightarrow C$  to  $Ff : FD \rightarrow FC$  and the right adjoint takes a map  $f : D \rightarrow FC$  to the pullback  $u^*(G(f)) : u^*G(D) \rightarrow C$ , where  $u : C \rightarrow GFC$  is the unit map.

**Theorem B.12** *Let  $S$  be a simplicial set. The adjunction*

$$(s\mathcal{S}et/S)^{cov} \begin{array}{c} \xrightarrow{p_1^*} \\ \perp \\ \xleftarrow{u^*i_1^*} \end{array} (s\mathcal{S}/p_1^*S)^{cov}$$

*is a Quillen equivalence, where both sides have the covariant model structure.*

**Proof** We break down the proof in several steps. In Lemma B.15 we prove the adjunction is a Quillen adjunction. Then, in Remark B.17 we reduce the proof of the Quillen equivalence to the case when  $S$  is a quasi-category. Finally, we show the Quillen adjunction is a Quillen equivalence when  $S$  is a quasi-category in Lemma B.18 and hence we are done.  $\square$

**Remark B.13** In the particular case where the base is of the form  $i_1^*X$ , where  $X$  is a complete Segal space, the adjunction  $(p_1^*, u^*i_1^*)$  has been shown to be a Quillen equivalence independently by de Brito [19, Theorem 1.22].

**Theorem B.14** *Let  $X$  be a simplicial space. The adjunction*

$$(s\mathcal{S}/X)^{cov} \begin{array}{c} \xrightarrow{t_1} \\ \perp \\ \xleftarrow{u^*t_1} \end{array} (s\mathcal{S}et/t_1X)^{cov}$$

*is a Quillen equivalence, where both sides have the covariant model structure.*

**Proof** The proof involves several separate steps. In Lemma B.15 we prove the adjunction is a Quillen adjunction. Then, again in Remark B.17 we reduce the proof of the Quillen equivalence to the case when  $X$  is a complete Segal space. Finally, we show the Quillen adjunction is a Quillen equivalence when  $X$  is a complete Segal space in Lemma B.19 finishing the proof.  $\square$

We start with the Quillen adjunctions.

**Lemma B.15** *Let  $S$  be a simplicial set. The adjunction*

$$(s\mathcal{S}et/S)^{cov} \begin{array}{c} \xrightarrow{p_1^*} \\ \perp \\ \xleftarrow{u^*i_1^*} \end{array} (s\mathcal{S}/p_1^*S)^{cov}$$

*is a Quillen adjunction, where both sides have the covariant model structure.*

**Proof** We use Lemma A.3. Clearly,  $p_1^*$  takes cofibrations to cofibrations as they are just inclusions. So, all that is left is to show that  $u^*i_1^*$  takes fibrations between fibrant objects to fibrations. By Lemma 3.14 a fibration between fibrant objects is just a left fibration. Thus it suffices to prove that  $u^*i_1^*$  preserves left fibrations. The map  $u^*$  just pulls back along the unit, which preserves left fibrations, as it is given by a right lifting property (Definition B.2). So it suffices to prove that  $i_1^*$  preserves left fibrations.

Let  $p : L \rightarrow p_1^*S$  be a left fibration. We have to show that  $i_1^*(p)$  satisfies the right lifting property with respect to horns  $\Delta[n]_i \rightarrow \Delta[n]$  where  $0 \leq i < n$  (Definition B.2). Using the adjunction  $(p_1^*, i_1^*)$  This is equivalent to  $p$  having the right lifting property with to the maps  $j : L(n)_i \rightarrow F(n)$ , which means we have to prove  $j$  is a trivial cofibration in  $s\mathcal{S}$  with respect to the covariant model structure over  $p_1^*S$ .

From Corollary 4.22 and the fact that  $\langle 0, \dots, k \rangle \times_{\text{id}_{F(n)}} j : F(k) \times_{F(n)} L(n)_i \rightarrow F(k)$  is a diagonal equivalence (both sides are diagonally contractible), we deduce that  $j$  is a covariant equivalence over  $F(n)$  and so is a covariant equivalence over  $p_1^*S$  as well (Theorem 3.15).  $\square$

**Lemma B.16** *For a left fibration of simplicial sets  $p : L \rightarrow S$ ,  $t^1L \rightarrow t^1S$  is a left fibration of simplicial spaces. In particular, for every simplicial space  $X$  the adjunction*

$$(s\mathcal{S}/X)^{cov} \begin{array}{c} \xrightarrow{t_1} \\ \perp \\ \xleftarrow{u^*t^1} \end{array} (s\text{Set}/t_1X)^{cov}$$

is a Quillen adjunction where both sides have the covariant model structure.

**Proof** By Theorem B.5,  $p : L \rightarrow S$  is a fibration in the Joyal model structure and so  $t^1(p)$  is a CSS fibration, by Theorem B.8, and so in particular a Reedy fibration. Thus we only have to prove that for every map  $F(n) \rightarrow X$  the induced map

$$\langle 0 \rangle^* : \text{Map}_{/S}(F(n), t^1(p)) \rightarrow \text{Map}_{/S}(F(0), t^1(p))$$

is a Kan equivalence. By adjunction, this is equivalent to proving that  $\langle 0 \rangle : \Delta[0] = t_1(F(0)) \rightarrow t_1(F(n)) = \Delta[n]$  is a covariant equivalence over  $t^1S$ . However, it is a well-established fact that the map  $\langle 0 \rangle : \Delta[0] \rightarrow \Delta[n]$  is a covariant equivalence over  $\Delta[n]$  and so in particular over  $S$  (by Theorem 3.15). For an elegant proof of this fact see [33, Lemma 2.5].

We now move on to the adjunction. We will show the adjunction satisfies the three conditions of Lemma A.3. Clearly,  $t_1$  takes cofibrations to cofibrations as they are just monomorphisms. Thus we only need to prove that  $u^*t^1$  preserves fibrant objects and fibrations between fibrant objects. By Corollary B.4, a fibration between fibrant objects is a left fibration, thus it suffices to prove that  $u^*t^1$  preserves left fibrations. By Lemma 3.9, it suffices to prove that  $t^1$  preserves left fibrations, which we established in the previous paragraph.  $\square$

Next we will reduce the proof to the case of complete Segal spaces and quasi-categories.

**Remark B.17** Let  $S$  be a simplicial set and choose a quasi-category fibrant replacement  $i : S \rightarrow \hat{S}$  and let  $X$  be a simplicial space and choose a complete Segal space fibrant replacement  $j : X \rightarrow \hat{X}$ . Then we have the following diagram of Quillen adjunctions:

$$\begin{array}{ccc} (s\text{Set}/S)^{cov} & \begin{array}{c} \xrightarrow{p_1^*} \\ \perp \\ \xleftarrow{u^*i_1^*} \end{array} & (s\mathcal{S}/p_1^*S)^{cov} \\ \uparrow i^* \downarrow i_! & & (p_1^*(i))^* \downarrow (p_1^*(i))_! \\ (s\text{Set}/\hat{S})^{cov} & \begin{array}{c} \xrightarrow{p_1^*} \\ \perp \\ \xleftarrow{u^*i_1^*} \end{array} & (s\mathcal{S}/p_1^*\hat{S})^{cov} \end{array} \quad \begin{array}{ccc} (s\mathcal{S}/X)^{cov} & \begin{array}{c} \xrightarrow{t_1} \\ \perp \\ \xleftarrow{u^*t^1} \end{array} & (s\text{Set}/t_1X)^{cov} \\ \uparrow j^* \downarrow j_! & & (t_1(j))^* \downarrow (t_1(j))_! \\ (s\mathcal{S}/\hat{X})^{cov} & \begin{array}{c} \xrightarrow{t_1} \\ \perp \\ \xleftarrow{u^*t^1} \end{array} & (s\text{Set}/t_1\hat{X})^{cov} \end{array}$$

All vertical Quillen adjunctions are Quillen equivalences (Theorems B.6, 5.1) thus the top horizontal Quillen adjunctions are Quillen equivalences if and only if the bottom Quillen adjunctions are Quillen equivalences.

We are now ready to move on to the last step.

**Lemma B.18** *Let  $S$  be a quasi-category. Then the adjunction*

$$(s\text{Set}/S)^{cov} \begin{array}{c} \xrightarrow{p_1^*} \\ \perp \\ \xleftarrow{u^*i_1^*} \end{array} (s\mathcal{S}/p_1^*S)^{cov}$$

is a Quillen equivalence, where both sides have the covariant model structure.



**Proof** We prove that the derived unit and counit maps are equivalences.

First we prove the derived counit map is an equivalence. Let  $p : L \rightarrow p_1^*S$  be a left fibration. We need to prove that the map  $p_1^*u^*i_1^*L \rightarrow L$  is a covariant equivalence over  $p_1^*S$ . By Theorem B.7,  $u : S \rightarrow i_1^*p_1^*S$  is the identity map. Hence we only need to prove that  $p_1^*i_1^*L \rightarrow L$  is a covariant equivalence over  $p_1^*S$ . However, this follows immediately from the fact that  $p_1^*i_1^*L \rightarrow L$  is a complete Segal space equivalence (by Theorem B.7) and hence a covariant equivalence over  $p_1^*S$  (Theorem 5.11).

We move on to prove that the derived unit map is an equivalence. Let  $p : L \rightarrow S$  be a left fibration. Then  $p_1^*L \rightarrow p_1^*S$  is generally not a left fibration and so we need to find a left fibrant replacement. We have the following diagram:

$$\begin{array}{ccc}
 p_1^*L & \xrightarrow{g_L} & t^!L \\
 \searrow & \cong & \downarrow \\
 & (g_S)^*t^!L & \xrightarrow{\cong} & t^!L \\
 & \downarrow & & \downarrow \\
 & p_1^*S & \xrightarrow{g_S} & t^!S
 \end{array}$$

By Lemma B.16,  $t^!L \rightarrow t^!S$  is a left fibration and so  $(g_S)^*(t^!L) \rightarrow p_1^*S$  is a left fibration (Lemma 3.9). Moreover,  $(g_S)^*t^!L \rightarrow t^!L$  is a CSS equivalence, as pulling back along left fibrations preserves CSS equivalences (Theorem 5.15) and  $g_L$  is a CSS equivalence by Lemma B.9. Thus  $p_1^*L \rightarrow (g_S)^*t^!L$  is a CSS equivalence over  $p_1^*S$  and so a covariant equivalence over  $p_1^*S$  (by Theorem 5.11). Hence,  $p_1^*L \rightarrow (g_S)^*t^!L$  is a fibrant replacement of  $p_1^*p$  in  $(sS/p_1^*S)^{cov}$ .

Thus, in order to prove the derived unit is an equivalence we only need to show that

$$L \rightarrow i_1^*p_1^*L \rightarrow (i_1^*)(g_S)^*t^!L$$

is a covariant equivalence over  $p_1^*S$ . However, both maps are just the identity map and hence we are done.  $\square$

**Lemma B.19** *Let  $X$  be a complete Segal space. Then the adjunction*

$$(sS/X)^{cov} \begin{array}{c} \xrightarrow{t_!} \\ \xleftarrow{u^*t^!} \end{array} (sSet/t_!X)^{cov}$$

*is a Quillen equivalence, where both sides have the covariant model structure.*

**Proof** We have the following chain of Quillen adjunctions:

$$(sSet/i_1^*X)^{cov} \begin{array}{c} \xrightarrow{p_1^*} \\ \xleftarrow{u^*i_1^*} \end{array} (sS/p_1^*i_1^*X)^{cov} \begin{array}{c} \xrightarrow{c_!} \\ \xleftarrow{c^*} \end{array} (sS/X)^{cov} \begin{array}{c} \xrightarrow{t_!} \\ \xleftarrow{u^*t^!} \end{array} (sSet/t_!X)^{cov} .$$

Here  $c : p_1^*i_1^*X \rightarrow X$  is the counit map of the adjunction.

The first adjunction is a Quillen equivalence by Lemma B.18. The middle one is a Quillen equivalence by Theorem 5.1, as  $c$  is an equivalence of complete Segal spaces (as proven in Theorem B.7). Finally, the composition takes a morphism  $S \xrightarrow{p} i_1^*X \rightarrow t_!p_1^*S \rightarrow t_!p_1^*i_1^*X \rightarrow$

$t_!X$ . By definition of  $h_X$  (Lemma B.10) this morphism is equal to  $S \xrightarrow{p} i_1^*X \xrightarrow{h_X} t_!X$  meaning the composition is precisely the Quillen adjunction

$$(\mathbf{sSet}_{/i_1^*X})^{cov} \begin{matrix} \xrightarrow{(h_X)!} \\ \perp \\ \xleftarrow{(h_X)^*} \end{matrix} (\mathbf{sSet}_{/h_X})^{cov} .$$

Now, by Lemma B.10,  $h_X$  is a categorical equivalence and so  $((h_X)!, (h_X)^*)$  is a Quillen equivalence and so by 2-out-of-3, the Quillen adjunction  $(t_!, u^*t^!)$  is also a Quillen equivalence. □

The Quillen equivalence in Theorem B.12 has an interesting corollary.

**Corollary B.20** *The covariant model structure on  $(\mathbf{sSet}_{/S})^{cov}$  is the localization of the Joyal model structure with respect to the set of maps  $< 0 >: \Delta[0] \rightarrow \Delta[n] \rightarrow S$ .*

**Remark B.21** Essentially we proved that the two Quillen equivalences that Joyal and Tierney introduced remain an equivalence after we localize at both sides. Theoretically, we could have just proven these theorems using the fact that localizing with respect to the "same" maps on both sides preserves Quillen equivalences. However, the issue is that we did not have a good enough understanding of the localization of the Joyal model structure (i.e. it is not clear which maps we are localizing with respect to). It is just after this proof that we get a clear sense of the localizing maps.

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