



# Discrete and Conservative Factorizations in $\mathbf{Fib}(B)$

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## Abstract

We focus on the transfer of some known orthogonal factorization systems from  $\mathbf{Cat}$  to the 2-category  $\mathbf{Fib}(B)$  of fibrations over a fixed base category  $B$ : the internal version of the *comprehensive factorization*, and the factorization systems given by (sequence of coidentifiers, discrete morphism) and (sequence of coinverters, conservative morphism) respectively. For the class of fibrewise opfibrations in  $\mathbf{Fib}(B)$ , the construction of the latter two simplify to a single coidentifier (respectively coinverter) followed by an internal discrete opfibration (resp. fibrewise opfibration in groupoids). We show how these results follow from their analogues in  $\mathbf{Cat}$ , providing suitable conditions on a 2-category  $\mathcal{C}$ , that allow the transfer of the construction of coinverters and coidentifiers from  $\mathcal{C}$  to  $\mathbf{Fib}_{\mathcal{C}}(B)$ .

**Keywords** Internal fibration · Factorization system · Coidentifier · Coinverter

**Mathematics Subject Classification** 18A32 · 18D05 · 18D30

## 1 Introduction

The crucial point of the work [2] was to recover Yoneda’s Classification Theorem in [17, §3.2] as a result of the factorization of a *regular span*  $S: X \rightarrow B \times A$  through an internal discrete opfibration, in the 2-category  $\mathbf{Fib}(B)$  of fibrations over  $B$ . Yoneda’s Theorem was the base to give a new interpretation of cohomology groups in the additive case. Theorem 3.2

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in [2], which extends the above factorization to any *fibrewise opfibration* (see Definition 4.7) with codomain a split fibration, allows us to enlarge this point of view to non-additive cases, such as cohomology of groups and of associative algebras.

The first aim of the present work is to show that the above mentioned result, obtained in [2] via an ad hoc construction, is actually an application of an orthogonal factorization system in  $\text{Fib}(B)$ , whose right class is given by internal discrete opfibrations. This is explained in Theorem 4.10, which generalizes to  $\text{Fib}(B)$  the well known *comprehensive factorization system* of  $\text{Cat}$  [15]. The above theorem relies on the fact that, for the class of opfibrations, the comprehensive factorization system coincides with the orthogonal factorization system in  $\text{Cat}$  given by (sequence of coidentifiers, discrete functor), whose construction in this case simplifies to a single coidentifier (see Theorem 4.1 and Corollary 4.2). Actually, a similar phenomenon occurs in  $\text{Fib}(B)$ . In fact,  $\text{Fib}(B)$  inherits from  $\text{Cat}$  the latter factorization system (see Theorem 4.5 (i)), which, restricted to fibrewise opfibrations, can be performed by a single coidentifier. This implies that for such class it coincides with the internal comprehensive factorization system (see Corollary 4.11).

As an obvious consequence of the results reported so far, we get a reflection of any morphism in  $\text{Fib}(B)$  onto a discrete morphism. For some purposes, this may cause an undesired loss of information with respect to the initial data. So, one may look for a finer reflection, where fibres are turned into groupoids. This gives a richer structure which is at the base, for example, of the interpretation given in [1] of Schreier–Mac Lane Theorem on the classification of group extension and its further generalizations (see Proposition 2.7 in [1]).

The main goal of the present work is to show first in Theorem 4.5 (ii) that  $\text{Fib}(B)$  inherits such a finer factorization from the one in  $\text{Cat}$  given by (sequence of coinverters, conservative functor) introduced in [7]; second, that for fibrewise opfibrations, it is performed by a single coconverter followed by a fibrewise opfibration whose fibres are groupoids (Theorem 4.12).

The achieved results rely on specific 2-categorical properties of  $\text{Cat}$  and of  $\text{Fib}(B)$  which are studied in detail in Sect. 3. In particular, in Proposition 3.8 we detect a sufficient condition to transfer the construction of coidentifiers and coinverters from a 2-category  $\mathcal{C}$  to the 2-category  $\text{Fib}_{\mathcal{C}}(B)$  of internal fibrations over a fixed object  $B$ . This happens when the 2-monad  $R: \mathcal{C}/B \rightarrow \mathcal{C}/B$ , whose pseudo-algebras define internal fibrations (in the sense of Street [10]), preserves coidentifiers and coinverters of idteetes. In Propositions 3.13 and 3.16 we prove that the property stated above holds when  $\mathcal{C} = \text{Cat}$  and  $\mathcal{C} = \text{Fib}(B)$ , for any  $B$ , thanks to the exponentiability of fibrations and opfibrations in  $\text{Cat}$ .

Throughout the paper, 2-limits and 2-colimits are to be understood in a strict sense.

### 2 Review of Internal Fibrations

Let  $\mathcal{C}$  be a finitely complete 2-category [11]. For a fixed object  $B$  in  $\mathcal{C}$ , we shall denote by  $\mathcal{C}/B$  the slice 2-category over  $B$  and by  $\mathcal{C}//B$  the pseudo-slice 2-category over  $B$ .

We shall denote as follows the (strict) comma objects in  $\mathcal{C}$  of identities, along identities on the left and on the right respectively, and iso-comma along identities:

$$\begin{array}{cccc}
 \begin{array}{ccc} B/B & \xrightarrow{d_1} & B \\ \downarrow d_0 & \nearrow \mu_B & \downarrow 1 \\ B & \xrightarrow{1} & B \end{array} & 
 \begin{array}{ccc} B/f & \xrightarrow{d_1} & A \\ \downarrow Rf & \nearrow \varphi_f & \downarrow f \\ B & \xrightarrow{1} & B \end{array} & 
 \begin{array}{ccc} f/B & \xrightarrow{L_f} & B \\ \downarrow d_0 & \nearrow \psi_f & \downarrow 1 \\ A & \xrightarrow{f} & B \end{array} & 
 \begin{array}{ccc} f/\cong B & \xrightarrow{I_f} & B \\ \downarrow w_f & \nearrow \omega_f & \downarrow 1 \\ A & \xrightarrow{f} & B \end{array} & (1)
 \end{array}$$

Following [10], one can extend the assignment  $f \mapsto Lf$  to 1-cells and 2-cells, yielding a 2-functor  $L: C//B \rightarrow C//B$ . Moreover there exist: a pseudo 2-natural transformation  $u: 1_{C//B} \rightarrow L$ , a (strict) 2-natural transformation  $m: L^2 \rightarrow L$  and a modification  $\lambda: Lu \rightarrow uL$  such that  $(L, u, m, \lambda)$  gives rise to a KZ-doctrine in the sense of Definition 1.1 in [9]. In other words, this structure provides a lax-idempotent 2-monad.

Let us observe that, in fact,  $L: C//B \rightarrow C//B$  factors through the inclusion of  $C/B$  in  $C//B$ , and the above 2-monad on  $C//B$  restricts to a *strict* 2-monad on  $C/B$ , which is also part of a KZ-doctrine by the same  $\lambda$ . We will adopt the same notation for both monads as far as no confusion arises.

Like  $L$ , also the 2-functors  $R$  and  $I$  on  $C//B$ , defined by the corresponding comma squares in (1), can be endowed with a structure of 2-monad, which is colax-idempotent in the case of  $R$  and pseudo-idempotent in the case of  $I$ . In both cases, these structures restrict to strict 2-monads  $(R, v, n, \rho)$  and  $(I, i, l, \iota)$  on  $C/B$ .

One of the most important features of KZ-doctrines is that the corresponding (pseudo-)algebra structures are unique up to isomorphism for each object and they are characterized as right (pseudo-)inverse left adjoint to the unit component of the monad. Applying this observation and its dual to the special cases of the 2-functors  $L, R$  and  $I$  described above, one can characterize (pseudo-)opfibrations (and dually fibrations) and isofibrations in  $C$ .

**Proposition 2.1** *For a morphism  $f: A \rightarrow B$  in  $C$  the following conditions are equivalent and define an internal fibration (respectively pseudo-fibration):*

1. (i) *For all  $X$  in  $C$ ,  $C(X, f): C(X, A) \rightarrow C(X, B)$  is a fibration (respectively pseudo-fibration) in  $\text{Cat}$ ;*  
 (ii) *for all  $g: Y \rightarrow X$ , the commutative square below is a morphism of fibrations (respectively pseudo-fibrations) in  $\text{Cat}$ :*

$$\begin{array}{ccc}
 C(X, A) & \xrightarrow{C(g,A)} & C(Y, A) \\
 C(X,f) \downarrow & & \downarrow C(Y,f) \\
 C(X, B) & \xrightarrow{C(g,B)} & C(Y, B)
 \end{array}$$

2.  *$f$  admits a structure of pseudo-algebra for the 2-monad  $R: C/B \rightarrow C/B$  (respectively  $R: C//B \rightarrow C//B$ );*
3. *The morphism  $v_f: f \rightarrow Rf$  admits a right adjoint in  $C/B$  (respectively  $C//B$ );*
4. *(Chevalley criterion) The morphism  $f_1: A/A \rightarrow B/f$ , determined by the equations*

$$\begin{cases} (Rf)f_1 = fd_0 \\ d_1f_1 = d_1 \\ \varphi_f f_1 = f\mu_A \end{cases}$$

*admits a right adjoint in  $C$  with counit an identity (respectively isomorphism).*

In practice, given an internal fibration according to the above form 2 of Proposition 2.1, it is convenient to fix a corresponding pseudo-algebra structure once and for all (which in  $\text{Cat}$  means to fix a cleavage). Accordingly, throughout the paper,  $\text{Fib}_C(B)$  will denote the 2-category whose objects are pseudo-algebras for the monad  $R: C/B \rightarrow C/B$ , whose 1-cells are strict pseudo-algebra morphisms, and with the obvious 2-cells (we shall write just  $\text{Fib}(B)$  for  $C = \text{Cat}$ ).

**Remark 2.2** The definition of internal fibration (resp. pseudo-fibration) in a representable 2-category appears in the form 2 of Proposition 2.1 in the works of Street [10,12]. The characterizations 1 and 3 in Proposition 2.1 are well-known and already present in the literature (see, for example, [16]). As for the Chevalley criterion, it was first proved by Gray [4] for fibrations in  $\mathbf{Cat}$ , while an internal version of it appears in [10, Proposition 9], asking for the unit to be an isomorphism. In fact, the latter characterizes pseudo-opfibrations (see (3.17) in [12]). This is the reason why we consider the characterization 4 also for internal (strict) fibrations.

**Definition 2.3** An internal fibration is said to be *discrete* if the functor  $\mathcal{C}(X, f)$  in 1. (i) of Proposition 2.1 is a discrete fibration.

The equivalent conditions of Proposition 2.1 may be easily adapted to define *internal opfibrations* (respectively pseudo-opfibrations), replacing the monad  $R$  with the monad  $L$ . For the reader’s convenience, throughout the paper, most of the results are stated in terms of fibrations. Where not explicitly provided, the corresponding results for opfibrations can be obtained by duality.

**Proposition 2.4** For a morphism  $f : A \rightarrow B$  in  $\mathcal{C}$  the following conditions are equivalent and define an internal isofibration:

1. For all  $X$  in  $\mathcal{C}$ ,  $\mathcal{C}(X, f) : \mathcal{C}(X, A) \rightarrow \mathcal{C}(X, B)$  is an isofibration in  $\mathbf{Cat}$ ;
2.  $f$  admits a structure of pseudo-algebra for the 2-monad  $I : \mathcal{C}/B \rightarrow \mathcal{C}/B$ ;
3. the morphism  $i_f : f \rightarrow If$  admits a right adjoint in  $\mathcal{C}/B$ .

### 3 Coinverters and Coidentifiers in $\mathbf{Fib}_{\mathcal{C}}(B)$

From now on, let  $\mathcal{C}$  be a finitely complete 2-category with coidentifiers and coinverters of reflexive 2-cells, whose definition we recall for the sake of completeness (the reader may refer to [8,13] for example).

**Definition 3.1** The coidentifier (respectively coinverter) of a 2-cell  $\alpha$  is a 1-cell  $q$  such that:

1.  $q\alpha$  is an identity (resp. isomorphism);
2. for any other 1-cell  $f$  such that  $f\alpha$  is an identity (resp. isomorphism), there exists a unique 1-cell  $t$  with  $tq = f$ ;
3. for any 2-cell  $\beta : f \rightarrow f'$  such that  $f\alpha$  and  $f'\alpha$  are identities (resp. isomorphisms), there exists a unique 2-cell  $\gamma$  with  $\gamma q = \beta$ ;

In this paper we will consider in particular coidentifiers (coinverters) of idtees (invertees). Given a 1-cell  $f$ , we denote by  $(K, \kappa)$  its idtee, where  $\kappa$  is the 2-universal 2-cell making  $f\kappa$  an identity. We denote by  $(W, \omega)$  the invertee of  $f$ , where  $\omega$  is the 2-universal 2-cell making  $f\omega$  an isomorphism.

Later on, we will take advantage of the following results concerning isofibrations. Recall that a 2-cell is called *f-vertical* if its image under  $f$  is an identity.

**Lemma 3.2** Let  $f$  be an isofibration and  $\alpha$  a 2-cell such that  $f\alpha$  is an isomorphism. Then  $\alpha$  factorizes as  $\alpha = \sigma \cdot \tau$ , where  $\tau$  is *f-vertical* and  $\sigma$  is an isomorphism.

**Proof** Since  $f$  is an isofibration, the isomorphism  $f\alpha$  admits a cartesian lifting  $\sigma$ , which is an isomorphism, at the codomain of  $\alpha$ . Then  $\tau$  is the unique *f-vertical* factorization of  $\alpha$  through  $\sigma$ . □

**Corollary 3.3** *Let  $f$  be an isofibration, then:*

- (i)  $f$  is conservative if and only if its fibres are groupoids;
- (ii) the coinverter of the identee of  $f$  coincides with the coinverter of its invertee.

**Proof** Let  $(W, \omega)$  and  $(K, \omega c)$  be the invertee and the identee, respectively, of  $f$ :

$$K \xrightarrow{c} W \begin{array}{c} \xrightarrow{w_0} \\ \Downarrow \omega \\ \xrightarrow{w_1} \end{array} A \xrightarrow{f} B.$$

Since  $f\omega$  is an isomorphism by definition, as in Lemma 3.2, we can factorize  $\omega$  as a composite  $\omega = \sigma \cdot \tau$ , where  $\sigma$  is a cartesian lifting of  $f\omega$ , and  $\tau$  the unique  $f$ -vertical comparison 2-cell.  $\tau$  being vertical, there is a unique  $c' : W \rightarrow K$  such that  $\omega c c' = \tau$ . So we have factorized  $\omega$  as in the following diagram:

$$\begin{array}{ccccc} W & \xrightarrow{c'} & K & \xrightarrow{c} & W \\ & \searrow & \swarrow \sigma & & \downarrow \omega \\ & & & & A \end{array} \quad \begin{array}{c} \xrightarrow{w_1} \\ \xleftarrow{w_0} \end{array}$$

It is now easy to see that  $f$  is conservative, i.e. its invertee is an isomorphism, if and only if its fibres are groupoids, i.e. its identee is an isomorphism.

As for the second statement, it suffices to observe that the coinverter of the identee  $\omega c$  coinverts also  $\omega = \sigma \cdot \omega c c'$ . □

Obviously the last result does not hold in general if  $f$  is not an isofibration, as it is witnessed by the non-constant functor from the arrow-category  $\mathbf{2}$  to the groupoid  $\mathbf{I}$  with two objects and two non-trivial arrows.

Identees in  $\mathcal{C}/B$  are computed in  $\mathcal{C}$ , while this is not true for invertees (and 2-limits in general). On the other hand, it is easy to check that the following holds for 2-colimits.

**Lemma 3.4** *The forgetful 2-functor  $\text{dom} : \mathcal{C}/B \rightarrow \mathcal{C}$  creates 2-colimits, and in particular coidentifiers and coinverters of identees.*

We are going to explore the behaviour of the monad  $R$  with respect to these limits and colimits. Analogous results can be proved for the monad  $L$ .

**Remark 3.5** It is worth observing that the functor  $R$  can be described by means of the following construction:

$$\begin{array}{ccccc} & & Rf & & \\ & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\ B/f & \xrightarrow{d_1^* f} & B/B & \xrightarrow{d_0} & B \\ \downarrow d_1 & & \downarrow d_1 & \swarrow \mu_B & \parallel \\ A & \xrightarrow{f} & B & \xrightarrow{\quad} & B \end{array}$$

That is,  $R = (d_0)_! d_1^*$ , i.e. the composite of the change-of-base 2-functor along  $d_1$  with the composition 2-functor with  $d_0$ , which is left adjoint to  $d_0^*$ .

**Lemma 3.6** *The monad  $R : C/B \rightarrow C/B$  preserves identees.*

**Proof** By Remark 3.5, the thesis follows from the fact that  $d_1^*$  preserves limits, being a right adjoint, and  $(d_0)_!$  preserves identees.  $\square$

**Lemma 3.7** *The identee of a morphism  $p : (A, f) \rightarrow (C, g)$  in  $\text{Fib}_C(B)$  can be computed as in  $C$ .*

**Proof** Let

$$\begin{array}{ccccc}
 & & k_0 & & \\
 & & \rightarrow & & \\
 K & \xrightarrow{\quad} & A & \xrightarrow{p} & C \\
 & \Downarrow \kappa & \downarrow f & & \swarrow g \\
 & & B & & \\
 & \nearrow h & & & 
 \end{array} \tag{2}$$

be an identee diagram in  $C/B$  (which means that  $\kappa$  is also the identee of  $p$  in  $C$ ). Since  $p$  is a morphism in  $\text{Fib}_C(B)$  and  $R$  preserves identees by Lemma 3.6, it is straightforward to prove that the adjunctions  $v_f \dashv r_f$  and  $v_g \dashv r_g$  in the diagram

$$\begin{array}{ccccc}
 & & Rk_0 & & \\
 & & \rightarrow & & \\
 B/h & \xrightarrow{\quad} & B/f & \xrightarrow{Rp} & B/g \\
 & \Downarrow R\kappa & \downarrow v_f & & \swarrow v_g \\
 & & A & \xrightarrow{p} & C \\
 & \nearrow Rk_1 & \downarrow r_f & & \swarrow r_g \\
 & & K & & \\
 & \nearrow v_h & \downarrow k_0 & & \\
 & & A & & \\
 & \searrow r_h & \downarrow \kappa & & \\
 & & B & & \\
 & & \downarrow k_1 & & 
 \end{array}$$

induce an adjunction  $v_h \dashv r_h$  by the universal property of the identees.  $\square$

There’s no obvious reason why coinverters and coidentifiers should be preserved by the monad  $R$ , however this happens in some cases of interest which we will explore later on. So, for a given object  $B$  in  $C$ , we shall consider the property

( $\dagger$ ) The monad  $R : C/B \rightarrow C/B$  preserves coidentifiers and coinverters of identees.

**Proposition 3.8** *Let  $B$  be an object in  $C$  satisfying ( $\dagger$ ),  $p : (A, f) \rightarrow (C, g)$  a morphism in  $\text{Fib}_C(B)$  and  $\kappa$  its identee in  $C$ . Then we get a factorization*

$$\begin{array}{ccccc}
 & & p & & \\
 & & \rightarrow & & \\
 A & \xrightarrow{q} & Q & \xrightarrow{s} & C \\
 & \searrow f & \downarrow gs & & \swarrow g \\
 & & B & & 
 \end{array}$$

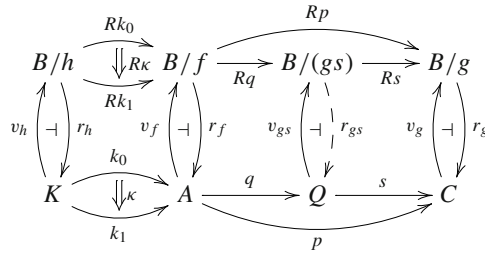
of  $p$  in  $\text{Fib}_C(B)$  in either of the following ways:

- (i) taking as  $q$  the coidentifier of  $\kappa$  in  $C$ ; then  $q : (A, f) \rightarrow (Q, gs)$  is the coidentifier of  $\kappa$  in  $\text{Fib}_C(B)$ ;
- (ii) taking as  $q$  the coinverter of  $\kappa$  in  $C$ ; then  $q : (A, f) \rightarrow (Q, gs)$  is the coinverter of  $\kappa$  in  $\text{Fib}_C(B)$ .

**Proof** We shall only prove (ii), (i) is proved analogously.

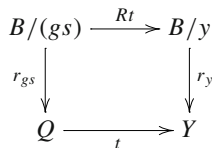
Recall that  $\kappa$  is also the identee of  $p$  in  $\mathcal{C}/B$  and consider the corresponding identee diagram (2) in  $\mathcal{C}/B$ . Since  $p$  (and then  $f$ ) coinverts  $\kappa$ , the morphisms  $s$  and  $g_s$  in the factorization above are uniquely determined by the universal property of  $q$ , and this explains why  $q$  is a coinverter of  $\kappa$  in  $\mathcal{C}/B$ .

Since  $f$  is a fibration, the unit component  $v_f: (A, f) \rightarrow (B/f, Rf)$  admits a right adjoint  $r_f$  in  $\mathcal{C}/B$ . We call  $\eta_f$  and  $\epsilon_f$  the corresponding unit and counit. Likewise,  $v_g$  has a right adjoint  $r_g$ , and  $pr_f = r_g(Rp)$  since  $p$  is a morphism in  $\text{Fib}_{\mathcal{C}}(B)$ . Let us consider the following diagram:



Now,  $pr_f(R\kappa) = r_g(Rp)(R\kappa) = 1$ , hence  $r_f(R\kappa)$  factors through  $\kappa$  and  $qr_f(R\kappa)$  is an isomorphism. By the assumption ( $\dagger$ ),  $Rq$  is the coinverter of  $R\kappa$ , so there exists a unique  $r_{gs}: B/(gs) \rightarrow Q$  such that  $r_{gs}(Rq) = qr_f$ . By the 2-dimensional universal property of the coinverters  $q$  and  $Rq$ , one can prove that a unit  $\eta_{gs}$  and a counit  $\epsilon_{gs}$  are induced by  $\eta_f$  and  $\epsilon_f$  respectively, making  $v_{gs} \dashv r_{gs}$  an adjoint pair in  $\mathcal{C}/B$ , so that  $g_s$  is a fibration. As a consequence of this construction,  $q$  turns out to be a morphism of fibrations over  $B$ .

It remains to show that for each  $c: (A, f) \rightarrow (Y, y)$  in  $\text{Fib}_{\mathcal{C}}(B)$  such that  $c\kappa$  is an isomorphism, the unique comparison morphism  $t$  in  $\mathcal{C}/B$ , induced by the coinverter  $q$  and such that  $tq = c$ , is actually a morphism in  $\text{Fib}_{\mathcal{C}}(B)$ . Let us denote by  $r_y$  the  $R$ -pseudo-algebra structure on  $y$ , i.e. the right adjoint to  $v_y$ , and observe that the diagram



commutes since  $Rq$  is a coinverter, then epimorphic, and precomposition with  $Rq$  gives the commutative square presenting  $c$  as a morphism of  $R$ -pseudo-algebras.  $\square$

**Corollary 3.9** *Let  $B$  be an object in  $\mathcal{C}$  satisfying ( $\dagger$ ),  $f: A \rightarrow B$  a fibration in  $\mathcal{C}$ ,  $\kappa$  its identee in  $\mathcal{C}$ , and  $q: A \rightarrow Q$  its coinverter (respectively coidentifier) in  $\mathcal{C}$ . Then the unique comparison morphism  $s: Q \rightarrow B$ , such that  $sq = f$ , is a fibration and  $q$  is the coinverter (respectively coidentifier) of  $\kappa$  in  $\text{Fib}_{\mathcal{C}}(B)$ .*

**Proof** Apply Proposition 3.8 to the morphism  $f: (A, f) \rightarrow (B, 1_B)$  in  $\text{Fib}_{\mathcal{C}}(B)$ .  $\square$

In the cases we are interested in, which will be studied in Sect. 3.1, the property ( $\dagger$ ) relies upon the exponentiability of split opfibrations in  $\mathcal{C}$ . In this context, exponentiability is to be understood in a 2-categorical sense: a 1-cell  $f$  is exponentiable if the change-of-base 2-functor along  $f$  has a right adjoint.

**Lemma 3.10** *If for an object  $B$  in  $\mathcal{C}$ , the comma projection  $d_1$  in the diagram*

$$\begin{array}{ccc}
 B/B & \xrightarrow{d_0} & B \\
 d_1 \downarrow & \swarrow \mu_B & \parallel \\
 B & \xlongequal{\quad} & B
 \end{array}$$

*is exponentiable, then the functor*

$$R: \mathcal{C}/B \rightarrow \mathcal{C}/B$$

*has a right adjoint, hence  $B$  satisfies the condition  $(\dagger)$ . In particular, this holds for any  $B$  when split opfibrations in  $\mathcal{C}$  are exponentiable.*

**Proof** By Remark 3.5,  $R = (d_0)_! d_1^*$ . Hence  $R$  is left adjoint to  $(d_1)_* d_0^*$ , where  $(d_1)_*$  denotes the right adjoint to  $d_1^*$ , which exists by assumption.  $\square$

**Proposition 3.11** *Under the assumptions of Lemma 3.10, the functor  $\text{dom}: \text{Fib}_{\mathcal{C}}(B) \rightarrow \mathcal{C}$  creates 2-colimits.*

**Proof** One can repeat the same arguments of the proof of Proposition 3.8, using the fact that now  $R$  preserves any 2-colimit, and that  $\text{dom}: \mathcal{C}/B \rightarrow \mathcal{C}$  creates 2-colimits.  $\square$

**Remark 3.12** If instead of  $(\dagger)$  we ask for

$(\dagger')$  The monad  $L: \mathcal{C}/B \rightarrow \mathcal{C}/B$  preserves coidentifiers and coinverters of idtees,

then the results of Proposition 3.8 hold with  $\text{Fib}_{\mathcal{C}}(B)$  replaced by  $\text{OpFib}_{\mathcal{C}}(B)$ . Accordingly, if  $d_0$  is exponentiable, and in particular when split fibrations are exponentiable in  $\mathcal{C}$ , then  $L$  admits a right adjoint,  $(\dagger')$  holds for  $B$ , and  $\text{dom}: \text{OpFib}_{\mathcal{C}}(B) \rightarrow \mathcal{C}$  creates colimits.

### 3.1 Case Study: Cat and Fib(B)

It is well-known that fibrations in  $\text{Cat}$  are exponentiable [3] in the classical 1-categorical sense. As observed by Johnstone [6], this property holds also in the 2-categorical sense recalled above. As a consequence, by Lemma 3.10 and Remark 3.12, we have:

**Proposition 3.13** *In the 2-category  $\text{Cat}$ , each object  $B$  satisfies the conditions  $(\dagger)$  and  $(\dagger')$ .*

We will see in Proposition 3.16 that one can extend the last property from  $\text{Cat}$  to  $\text{Fib}(B)$  for each  $B$ , by means of the pseudo-functorial interpretation of fibrations in  $\text{Cat}$ .

**Proposition 3.14** *Let  $B$  be a category. Then*

- (i) *2-colimits, in particular coidentifiers and coinverters of idtees, exist in  $\text{Fib}(B)$ ;*
- (ii) *their construction can be performed fibrewise.*

**Proof** (i) The existence is guaranteed by Proposition 3.11, since  $\text{Cat}$  is 2-cocomplete. We shall prove (ii) just for coidentifiers, the general case can be treated likewise.

Let us consider a morphism  $p: (A, f) \rightarrow (C, g)$  in  $\text{Fib}(B)$  and focus our attention on its restriction  $p_b$  to a single fibre over some  $b$  in  $B$ . We can consider the coidentifier  $q_b: A_b \rightarrow Q_b$  of its idtee  $(K_b, \kappa_b)$ . Since  $f$  and  $g$  are fibrations, the assignments  $b \mapsto A_b$  and  $b \mapsto C_b$  are pseudo-functorial and the collection of the functors  $p_b$  gives rise to a



natural transformation of pseudo-functors from  $B^{op}$  to  $\text{Cat}$ . By the universal property of the coidentifiers  $q_b$  for each  $b$ , the assignment  $b \mapsto Q_b$  is also pseudo-functorial and the  $q_b$ 's organize in a natural transformation. Let us briefly show how this can be proved.

In fact, the assignment  $b \mapsto K_b$  is also pseudo-functorial and together with the collection of the  $\kappa_b$ 's, it determines the idantee  $(K, \kappa)$  of the cartesian functor  $p$ . Given an arrow  $\beta: b' \rightarrow b$  in  $B$ , we always denote by  $\beta^*$  its associated change of base functor for any chosen fibration over  $B$ . Since  $q_{b'}\beta^*\kappa_b = q_b\kappa_{b'}\beta^* = 1$ , by the universal property of the coidentifier  $q_b$  there is a unique functor  $\beta^*: Q_b \rightarrow Q_{b'}$  such that  $\beta^*q_b = q_{b'}\beta^*$ :

$$\begin{array}{ccccc}
 K_b & \xrightarrow{\quad} & A_b & \xrightarrow{q_b} & Q_b \\
 \beta^* \downarrow & \curvearrowright & \Downarrow \kappa_b & \downarrow \beta^* & \downarrow \beta^* \\
 K_{b'} & \xrightarrow{\quad} & A_{b'} & \xrightarrow{q_{b'}} & Q_{b'}
 \end{array}$$

Given a composable pair of arrows

$$b'' \xrightarrow{\beta'} b' \xrightarrow{\beta} b$$

in  $B$ , let  $\phi_{\beta, \beta'}: (\beta')^*\beta^* \rightarrow (\beta\beta')^*$  be the corresponding coherence isomorphism induced by the fibration  $f$ . Since  $q_{b''}\phi_{\beta, \beta'}$  is a 2-cell between  $q_{b''}(\beta')^*\beta^* = (\beta')^*\beta^*q_b$  and  $q_{b''}(\beta\beta')^* = (\beta\beta')^*q_b$ , then by the universal property of the coidentifier  $q_b$  there exists a unique invertible 2-cell  $\psi_{\beta, \beta'}$  such that  $\psi_{\beta, \beta'}q_b = q_{b''}\phi_{\beta, \beta'}$ .

$$\begin{array}{ccc}
 A_b & \xrightarrow{q_b} & Q_b \\
 \downarrow \beta^* & & \downarrow \beta^* \\
 A_{b'} & \xrightarrow{q_{b'}} & Q_{b'} \\
 \downarrow (\beta')^* & & \downarrow (\beta')^* \\
 A_{b''} & \xrightarrow{q_{b''}} & Q_{b''}
 \end{array}$$

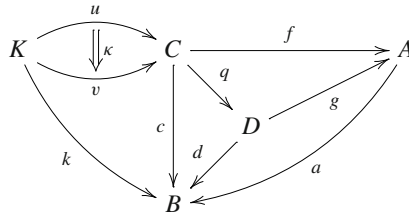
$\phi_{\beta, \beta'}: (\beta')^*\beta^* \rightarrow (\beta\beta')^*$  (left 2-cell),  $\psi_{\beta, \beta'}: (\beta\beta')^* \rightarrow (\beta')^*\beta^*$  (right 2-cell),  $(\beta\beta')^*$  (left arrow),  $(\beta')^*$  (right arrow)

Finally, the coherence conditions on the  $\psi$ 's making the assignment  $b \mapsto Q_b$  into a pseudo-functor can be deduced once again by the universal property of the  $q_b$ 's. Since for a morphism  $t: (A, f) \rightarrow (Y, y)$  in  $\text{Fib}(B)$ ,  $t\kappa = 1$  if and only if  $t_b\kappa_b = 1$  for each  $b$  in  $B$ ,  $q$  is actually the coidentifier of  $\kappa$  in  $\text{Fib}(B)$ . □

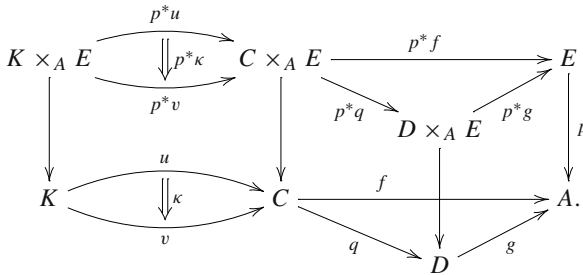
**Lemma 3.15** *For each internal fibration (resp. opfibration)  $p: (E, e) \rightarrow (A, a)$  in  $\text{Fib}(B)$ , the change of base 2-functor  $p^*: \text{Fib}(B)/(A, a) \rightarrow \text{Fib}(B)/(E, e)$  preserves coidentifiers and coinverters of idantees.*

**Proof** We will prove the result concerning internal fibrations and coinverters, the variations involving opfibrations and coidentifiers are obtained analogously.

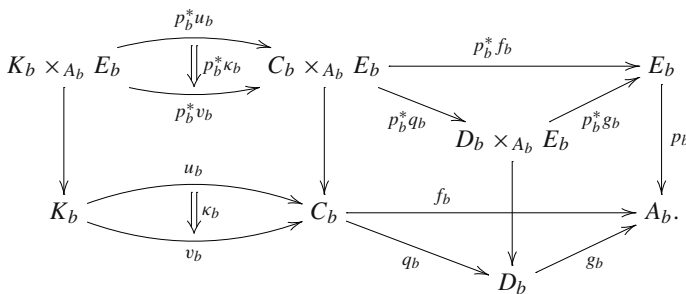
Let the arrow  $q : ((C, c), f) \rightarrow ((D, d), g)$  in the diagram



be the coiverter in  $\text{Fib}(B)/(A, a)$  of an identee  $\kappa$ , and consider its image under the change of base 2-functor  $p^*$ , i.e. the upper part of the next diagram (we omit all arrows over  $B$ , all pullbacks provide in fact fibrations over  $B$ ):



We would like to show that  $p^*q$  is the coiverter of the identee  $p^*\kappa$  in  $\text{Fib}(B)/(E, e)$ . To this end, we consider the restriction of the above diagram to the fibres over any object  $b$  in  $B$ . By limit commutation, the latter is the same as the corresponding change of base diagram in the fibres over  $b$ :



Now observe that, by Proposition 3.14 (ii),  $q_b$  is the coiverter of  $\kappa_b$ . Moreover, since  $p_b$  is a fibration in  $\text{Cat}$  by assumption, it is exponentiable, hence  $p_b^*$  is a left adjoint and  $p_b^*q_b$  is the coiverter of  $p_b^*\kappa_b$ . Finally, again by Proposition 3.14,  $p^*q$  is the coiverter of  $p^*\kappa$  in  $\text{Fib}(B)$ , and hence in  $\text{Fib}(B)/(A, a)$ .  $\square$

**Proposition 3.16** *In the 2-category  $\text{Fib}(B)$ , each object satisfies the conditions  $(\dagger)$  and  $(\dagger')$ .*

**Proof** Let  $a: A \rightarrow B$  be a fibration of categories, then the projections  $d_0$  and  $d_1$  of the comma square in  $\text{Fib}(B)$

$$\begin{array}{ccc}
 (A, a)/(A, a) & \xrightarrow{d_0} & (A, a) \\
 d_1 \downarrow & \swarrow \mu_{(A,a)} & \parallel \\
 (A, a) & \xlongequal{\quad} & (A, a)
 \end{array}$$

are an internal fibration and opfibration respectively (see Theorem 14 in [10]). As a consequence, by Lemma 3.15, the corresponding change of base 2-functors  $d_0^*$  and  $d_1^*$  preserve coidentifiers and coinverters of idteets. Now likewise in the proof of Lemma 3.10, the thesis follows from the fact that  $R = (d_0)_! d_1^*$  and  $(d_0)_!$  is a left adjoint (and similarly for  $L$ ).  $\square$

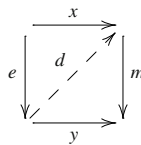
**Proposition 3.17** *Let  $p: (A, f) \rightarrow (C, g)$  be an internal fibration (resp. opfibration) in  $\text{Fib}(B)$ . Then the morphism  $s$  in the factorization of Proposition 3.8 (i) is an internal fibration (resp. opfibration) in  $\text{Fib}(B)$ .*

**Proof** By Proposition 3.16, we can apply Corollary 3.9 to the fibration  $p$  in  $\text{Fib}(B)$ .  $\square$

### 4 Three Factorization Systems in $\text{Fib}(B)$

Let us recall that an (orthogonal) factorization system on a 1-category  $C$  is given by a pair  $(\mathcal{E}, \mathcal{M})$  of classes of morphisms in  $C$  such that:

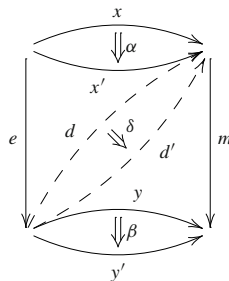
- (i)  $\mathcal{E}$  and  $\mathcal{M}$  are closed under composition and contain isomorphisms;
- (ii) each morphism  $f$  in  $C$  admits a factorization  $f = m \cdot e$  with  $m$  in  $\mathcal{M}$  and  $e$  in  $\mathcal{E}$ ;
- (iii) for each commutative square



with  $m$  in  $\mathcal{M}$  and  $e$  in  $\mathcal{E}$ , there exists a unique morphism  $d$  such that  $md = y$  and  $de = x$ .

When  $C$  is a 2-category such factorization system  $(\mathcal{E}, \mathcal{M})$  becomes a strict 2-categorical (or a Cat-enriched) factorization system if the following additional property holds:

- (iv) for each diagram

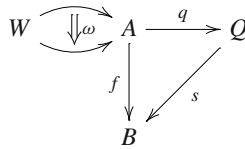


where  $m\alpha = \beta e$  and  $d$  and  $d'$  are determined by  $(x, y)$  and  $(x', y')$  respectively, there exists a unique  $\delta$  such that  $m\delta = \beta$  and  $\delta e = \alpha$ .

### 4.1 Three Factorization Systems in Cat

The *comprehensive factorization system* introduced in [15] and given by (initial functor, discrete opfibration) is a well known example of factorization system in  $\text{Cat}$ , which is actually strict 2-categorical. We will call by the same name the dual factorization given by (final functor, discrete fibration).

Another example of factorization system in  $\text{Cat}$ , which is 2-categorical as well, and has conservative functors as right class, was introduced in [7, Theorem C.0.31] relying on the notion of coinverter. Let us illustrate how this is obtained. For a functor  $f$

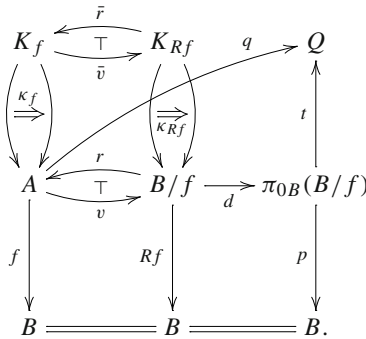


the factorization  $s$  of  $f$  through the coinverter  $q$  of the invertree  $\omega$  of  $f$  is not conservative in general. One has to repeat this “invertree–coinverter” procedure possibly infinitely many times in order to get a conservative comparison, and an actual factorization system. A similar phenomenon occurs when taking the “identree–coidentifier” analogue of the previous procedure, which allows to factor any functor as a (possibly infinite) sequence of coidentifiers followed by a discrete functor, i.e. a functor whose fibres are discrete, yielding another (strict 2-categorical) factorization system in  $\text{Cat}$ .

We are going to show that if we restrict ourselves to fibrations, the latter two transfinite procedures above reduce to a single step. We start with the second one and we show also that, for fibrations, it coincides with the comprehensive factorization system. This actually holds not only in  $\text{Cat}$ , but in any 2-category  $\text{Cat}(\mathcal{E})$  of internal categories where the construction of the comprehensive factorization of any functor provided in [14] is still valid, as, for example, when  $\mathcal{E}$  is a finitely cocomplete locally cartesian closed category, like any topos  $\mathcal{E}$ .

**Theorem 4.1** *Let  $f : A \rightarrow B$  be a fibration in  $\text{Cat}(\mathcal{E})$  as above. The coidentifier of the identree of  $f$  factorizes  $f$  into a final functor followed by a discrete fibration, giving then the comprehensive factorization of  $f$ . Starting with  $f$  opfibration, the same procedure yields the dual factorization of  $f$  given by an initial functor followed by a discrete opfibration.*

**Proof** We consider just the case of fibrations. Following the approach of Section 3 in [14], we perform the comprehensive factorization of  $f$  by taking the free  $R$ -algebra  $Rf$ , which is a split fibration, and then reflecting it into a discrete fibration  $p$ :



By construction of the above reflection,  $d$  is the coidentifier of the idantee  $\kappa_{Rf}$  of  $Rf$ . Considering the adjunction  $v \dashv r$  provided by the fact that  $f$  is a fibration, we get  $dv\kappa_f = d\kappa_{Rf}\bar{v} = 1$ , where  $\kappa_f$  is the idantee of  $f$ . Let now  $q$  be a functor such that  $q\kappa_f = 1$ , and consider the unit  $\eta: 1 \rightarrow rv$  of the adjunction  $v \dashv r$  in  $\text{Cat}(\mathcal{E})/B$ . Then  $f\eta = 1$  and  $\eta$  is contained in  $\kappa_f$ , so that  $q\eta = 1$  as well, and  $qrv = q$ . On the other hand, the counit  $\epsilon: vr \rightarrow 1$  is such that  $(Rf)\epsilon = 1$ , so it is contained in  $\kappa_{Rf}$  and hence  $d\epsilon = 1$  and  $dvr = d$ .

Now,  $qr\kappa_{Rf} = q\kappa_f\bar{r} = 1$ , so by the universal property of the coidentifier  $d$  there exists a unique  $t$  such that  $td = qr$ . Hence  $q = qrv = tdv$ , and  $t$  is unique with this last property. Indeed, if  $t'dv = q$  for some  $t'$ , then  $t'd = t'dvr = qr = td$  and hence  $t' = t$  since  $d$  is epimorphic. This proves that  $dv$  is the coidentifier of  $\kappa_f$ , and then it is final [14].  $\square$

**Corollary 4.2** *For a fibration (resp. opfibration)  $f$ , the factorization of  $f$  given by (sequence of coidentifiers, discrete functor) reduces to a single coidentifier and coincides with the comprehensive factorization.*

In the special case of  $\text{Cat}$ , Theorem 4.1 can be proved directly by means of the pseudo-functorial interpretation of fibrations. This indeed is what we are going to do in order to obtain the analogous result, where coidentifiers are replaced by coinverters and discrete fibrations are replaced by fibrations in groupoids (i.e. fibrations whose idantee is an isomorphism).

**Theorem 4.3** *Each fibration (respectively opfibration)  $f: A \rightarrow B$  in  $\text{Cat}$  admits a factorization given by the coinverter of the idantee of  $f$  followed by a fibration (resp. opfibration) in groupoids. This factorization of  $f$  coincides with the one given by (sequence of coinverters, conservative functor).*

**Proof** Let us denote by

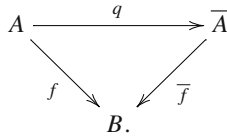
$$\text{Cat} \begin{array}{c} \xrightarrow{\pi} \\ \perp \\ \xleftarrow{i} \end{array} \text{Gpd}$$

the reflection of categories in groupoids, where the left adjoint  $\pi$  can be obtained by taking as unit component, for each category  $A$ , the coinverter  $\eta_A$  of the 2-cell  $\mu_A$  associated with the comma category  $A/A$ :

$$A/A \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \mu_A \\ \xrightarrow{\quad} \end{array} A \xrightarrow{\eta_A} \pi(A).$$

Consider now a fibration  $f: A \rightarrow B$  and denote by  $[f]: B^{\text{op}} \rightarrow \text{Cat}$  the corresponding pseudo-functor. The composite  $\pi[f]: B^{\text{op}} \rightarrow \text{Gpd}$  gives rise to a fibration in groupoids

$\bar{f}: \bar{A} \rightarrow B$ . On the other hand,  $\eta[f]$  corresponds to a morphism  $q: (A, f) \rightarrow (\bar{A}, \bar{f})$  in  $\text{Fib}(B)$ :



The component  $q_b$  of  $\eta[f]$  at an object  $b$  of  $B$  is actually the coinverter  $\eta_{A_b}$  of  $\mu_{A_b}$ :

$$A_b/A_b \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \mu_{A_b} \\ \xrightarrow{\quad} \end{array} A_b \xrightarrow{\eta_{A_b}} \pi(A_b).$$

It is not difficult to see that the pair  $(A_b/A_b, \mu_{A_b})$  coincides with the restriction  $(K_b, \kappa_b)$  of the identee  $(K, \kappa)$  of  $f$  to the fibre over  $b$ . Hence, as explained in Proposition 3.14,  $q$  turns out to be the coinverter of  $\kappa$  in  $\text{Fib}(B)$ . Thanks to Corollary 3.3,  $\bar{f}$  is conservative and we get the desired factorization of  $f$ . □

### 4.2 From Cat to Fib(B)

We are going to use now the results of the previous sections to produce analogous factorization systems in  $\text{Fib}(B)$ . First, we need a preliminary result.

**Lemma 4.4** *For a morphism  $p: (A, f) \rightarrow (C, g)$  in  $\text{Fib}(B)$ , the coinverter  $q$  of the invertee of  $p$  in  $\text{Fib}(B)$  is also the coinverter of the invertee of  $p$  in  $\text{Cat}$ .*

**Proof** The only non-trivial property to prove is that for an arrow  $\alpha$  in  $A$ ,  $q\alpha$  is an isomorphism as soon as  $p\alpha$  is an isomorphism. But if  $p\alpha$  is an isomorphism, then  $\alpha$  factors as  $\alpha = \kappa \cdot \nu$  where  $\kappa$  is an  $f$ -cartesian lifting of  $f\alpha$ , hence an isomorphism, and  $\nu$  is  $f$ -vertical with  $p\nu$  isomorphism. So  $\nu$ , and hence  $\alpha$ , is inverted by  $q$ . □

**Theorem 4.5** *For each category  $B$ ,  $\text{Fib}(B)$  inherits from  $\text{Cat}$  two factorization systems given by*

- (i) (sequence of coidentifiers, discrete cartesian functors);
- (ii) (sequence of coinverters, conservative cartesian functors).

**Proof** Identees in  $\text{Fib}(B)$  are computed in  $\text{Cat}$  by Lemma 3.7. Even if invertees in  $\text{Fib}(B)$  may differ from the corresponding invertees computed in  $\text{Cat}$ , their coinverters coincide by Lemma 4.4. As a consequence, thanks to Proposition 3.11, such factorization systems are just performed in  $\text{Cat}$ . □

Thanks to a result due to Bénabou, proving that a morphism in  $\text{Fib}(B)$  is an internal fibration if and only if it is a fibration in  $\text{Cat}$  (see, for example, Theorem 4.16 in [5]), we easily get the following result.

**Proposition 4.6**  *$\text{Fib}(B)$  inherits from  $\text{Cat}$  the comprehensive factorization system having internal discrete fibrations as right class.*

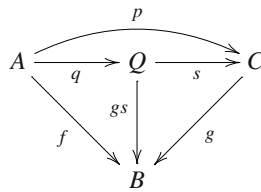
**Proof** Let  $p: (A, f) \rightarrow (C, g)$  be any morphism in  $\text{Fib}(B)$ . Take its factorization  $(q, s)$  in  $\text{Cat}$ , with  $q$  a final functor and  $s$  a discrete fibration. Then  $gs$  is a fibration and  $s$  is an internal fibration. It is easy to see that, since  $p$  is cartesian and  $s$  is discrete,  $q$  is cartesian as well. □

Our next goal is to show that  $\text{Fib}(B)$  admits also a comprehensive factorization system having internal discrete *opfibrations* as right class. We cannot repeat the above argument because internal opfibrations in  $\text{Fib}(B)$  are not opfibrations in  $\text{Cat}$ . Let us recall from [2] the following definition.

**Definition 4.7** (see [2, Definition 2.1]) We say that a morphism  $p: (A, f) \rightarrow (C, g)$  in  $\text{Fib}(B)$  is a *fibrewise (discrete) opfibration* if, for every object  $b$  of  $B$ , the restriction  $p_b: A_b \rightarrow C_b$  of  $p$  to the  $b$ -fibres is a (discrete) opfibration.

From Theorem 2.8 in [2] it follows that every internal opfibration in  $\text{Fib}(B)$  is a fibrewise opfibration, while the latter is exactly a morphism in  $\text{Fib}(B)$  which is an internal opfibration in  $\text{Cat}/B$  (see Propositions 2.5 and 2.7 in [2]). By Corollary 2.9 in [2], the two notions coincide in the discrete case. Recall also from [2] that Yoneda’s *regular spans* and *two-sided fibrations* are instances, respectively, of fibrewise opfibrations and internal opfibrations in  $\text{Fib}(B)$ .

**Proposition 4.8** Let  $p: (A, f) \rightarrow (C, g)$  be a fibrewise (resp. internal) opfibration in  $\text{Fib}(B)$ . Then we get a factorization

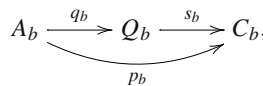


of  $p$  in  $\text{Fib}(B)$  in either of the following ways:

- (i) taking as  $q$  the coidentifier of the identee of  $p$ ; then  $s$  is a discrete opfibration in  $\text{Fib}(B)$ ;
- (ii) taking as  $q$  the coinverter of the identee of  $p$ ; then  $s$  is a fibrewise (resp. internal) opfibration in groupoids in  $\text{Fib}(B)$ .

**Proof** Let us consider a fibrewise opfibration  $p: (A, f) \rightarrow (C, g)$  in  $\text{Fib}(B)$ .

(i) Take the factorization  $(q, s)$  of  $p$  as in Proposition 3.8 (i), with  $q$  the coidentifier of the identee of  $p$ . If we restrict to a single fibre over some  $b$  in  $B$ , we get a factorization



where, thanks to Proposition 3.14 (ii),  $q_b$  is the coidentifier of the identee of  $p_b$ . Since the latter is an opfibration, by Theorem 4.1  $s_b$  is a discrete opfibration, hence  $s$  is an internal discrete opfibration in  $\text{Fib}(B)$ .

(ii) Take instead the factorization  $(q, s)$  of  $p$  as in Proposition 3.8 (ii). Likewise in (i), for each  $b$  in  $B$ , thanks to Theorem 4.3, we get a factorization  $(q_b, s_b)$  of  $p_b$  into a coinverter followed by an opfibration in groupoids.

If moreover  $p$  is an internal opfibration, then  $s$  is also an internal opfibration by Corollary 3.9 applied to  $\text{Fib}(B)$ , thanks to Proposition 3.16. □

The result in Proposition 4.8 (i) (with  $g$  a split fibration) was obtained in Section 3.3 of [2], by providing an explicit construction of the discrete opfibration  $s$  together with an ad hoc definition of  $q$ , which was later on proved to be the coidentifier of the identee of  $p$ .

As a consequence of Proposition 4.8 (i), we get an internal comprehensive factorization system for  $\text{Fib}(B)$ . In the next results, by *initial* we mean a morphism which is left orthogonal

to internal discrete opfibrations. First we need the following lemma, whose dual version is in Proposition 3.5 in [14], for a particular case. For completeness, we formulate it here in a general finitely complete 2-category.

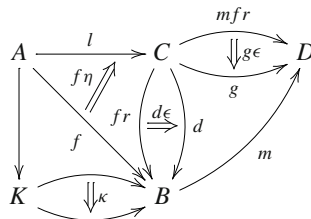
**Lemma 4.9** *In a finitely complete 2-category, left adjoints are initial.*

**Proof** Consider the commutative square

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 l \downarrow & & \downarrow m \\
 C & \xrightarrow{g} & D
 \end{array}$$

where  $m$  is an internal discrete opfibration and  $l$  is left adjoint to some  $r$ , with unit  $\eta$  and counit  $\epsilon$ , so that  $\epsilon l \cdot l \eta = 1_l$  and  $r \epsilon \cdot \eta r = 1_r$ .

Let  $\gamma : fr \rightarrow d$  denote the opcartesian  $m$ -lifting of  $g\epsilon$  at  $fr$ , so that  $md = g$ . The 2-cell  $d\epsilon$  being also an  $m$ -lifting of  $g\epsilon$ , it coincides with  $\gamma$  since  $m$  is a discrete opfibration. Then  $m(d\epsilon \cdot f\eta) = g\epsilon l \cdot m f \eta = g(\epsilon l \cdot l \eta) = 1_{gl}$ .



Hence  $d\epsilon \cdot f\eta$  factors through the identity  $\kappa$  of  $m$ , which is discrete, so  $\kappa$ , and hence  $d\epsilon \cdot f\eta$ , is an identity. Consequently,  $dl = f$ .

Suppose  $d' : C \rightarrow B$  is another morphism such that  $md' = g$  and  $d'l = f$ . Then  $d'\epsilon$  is another  $m$ -lifting of  $g\epsilon$ , so  $d'\epsilon = d\epsilon$  and  $d = d'$ .  $\square$

**Theorem 4.10** *In  $\text{Fib}(B)$  there exists a comprehensive factorization system given by (initial morphism, internal discrete opfibration).*

**Proof** Consider a morphism  $p : (A, f) \rightarrow (C, g)$  in  $\text{Fib}(B)$ , and the monad  $L : \text{Fib}(B)/(C, g) \rightarrow \text{Fib}(B)/(C, g)$ .  $Lp$  is an internal opfibration, so we can factorize it in  $\text{Fib}(B)$  as a coidentifier  $q$  followed by an internal discrete opfibration  $s$ , thanks to Proposition 4.8. Now consider the factorization

$$\begin{array}{ccccc}
 (A, f) & \xrightarrow{u_p} & p/(C, g) & \xrightarrow{q} & (Q, gs) & \xrightarrow{s} & (C, g) \\
 & & & & \searrow & \nearrow & \\
 & & & & & & p
 \end{array}$$

of  $p$ , where the unit component  $u_p$  is initial by Lemma 4.9, since it is a left adjoint (see Corollary 6 in [10]). The result follows from Proposition 4.8 as it is easy to see that coidentifiers are initial.  $\square$

**Corollary 4.11** *For each fibrewise opfibration, the factorization of Theorem 4.10 coincides with the one given by (sequence of coidentifiers, discrete cartesian functor).*



As a consequence of Proposition 4.8 (ii), we get an extension of Theorem 4.3.

**Theorem 4.12** *For every fibrewise opfibration  $p: (A, f) \rightarrow (C, g)$  in  $\text{Fib}(B)$ , the factorization of Proposition 4.8 (ii) coincides with the one given by (sequence of coinverters, conservative cartesian functor).*

**Proof** By Corollary 3.3 applied to  $p$ , which is an opfibration, and then an isofibration, in  $\text{Cat}/B$ ,  $p$  is conservative in  $\text{Cat}/B$ , hence in  $\text{Fib}(B)$ . Then the thesis follows from Theorem 4.5 (ii).  $\square$

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