



Crossed Modules of Monoids III. Simplicial Monoids of Moore Length 1

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Abstract

This is the last part of a series of three strongly related papers in which three equivalent structures are studied:

- Internal categories in categories of monoids; defined in terms of pullbacks relative to a chosen class of spans
- Crossed modules of monoids relative to this class of spans
- Simplicial monoids of so-called Moore length 1 relative to this class of spans.

The most important examples of monoids that are covered are small categories (treated as monoids in categories of spans) and bimonoids in symmetric monoidal categories (regarded as monoids in categories of comonoids). In this third part relative simplicial monoids are analyzed. Their Moore length is introduced and the equivalence is proven between relative simplicial monoids of Moore length 1, and relative categories of monoids in Part I. This equivalence is obtained in one direction by truncating a simplicial monoid at the first two degrees; and in the other direction by taking the simplicial nerve of a relative category.

Keywords Simplicial monoid · Moore length · Internal category · Bimonoid

1 Introduction

Whitehead's crossed modules of groups [22] received a lot of attention because of their appearance in many different contexts; see the review articles [17–19] and the references in them. In many of the applications they did not appear in their original form; but in the disguise of the equivalent notion of internal groupoid in the category of groups—also called a strict 2-group—(see [6,7,13] for proofs of the equivalence).

Groups can be regarded as (distinguished) monoids in the cartesian monoidal category of sets. In our antecedent papers [3] and [4] we worked out the notion of crossed module of monoids in more general, not necessarily cartesian monoidal categories, relative to a chosen

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suitable class of spans. The main examples described by the theory are crossed modules of monoids in [16], groupoids in [5] and crossed modules of Hopf monoids in [1, 10–12, 15].

The aim of this article is to extend to this level of generality a third equivalent description of crossed modules of groups: as simplicial groups whose Moore complex has length 1. Recall that for any simplicial group, the Moore complex is the associated chain complex, which is given at each degree by the joint kernel of the face maps of positive label, and the restriction of the zero'th face map as the differential (this is part of the Dold-Kan correspondence). A chain complex—so in particular the Moore complex—is said to have length ℓ if it is trivial for all degrees greater than ℓ .

We consider simplicial monoids; that is, functors from the opposite of the simplicial category Δ to the category of monoids in some monoidal category C . Assuming the existence of certain relative pullbacks (cf. [3]), in Sect. 2 we associate to a simplicial monoid a sequence of morphisms which yields a chain complex whenever the monoidal unit of C is a terminal (hence zero) object in the category of monoids in C . It can be seen as a generalization of the Moore complex of a simplicial group. We also give a meaning to its length, and study the consequences of its having some finite length n .

Section 3 contains some technical material about the invertibility of a certain canonical morphism playing an essential role in the theory.

The main result can be found in Sect. 4 where we prove an equivalence between the category of relative categories in the category of monoids in C (cf. [3]) and the category of relative simplicial monoids in C whose Moore length is 1. The functors establishing the equivalence carry transparent meanings. For obtaining a relative category from a simplicial monoid we truncate it at the first two degrees. In the opposite direction, a simplicial monoid is obtained from a relative category of monoids as the (relative) simplicial nerve.

In the particular monoidal category of spans over a given set, we obtain an equivalence between the categories of certain double categories, and of certain simplicial categories. These equivalent categories contain, as equivalent full subcategories, the category of 2-groupoids on one hand, and the category of crossed modules of groupoids on the other hand.

In the particular monoidal category of comonoids in some symmetric monoidal category M , we obtain an equivalence between the categories of certain category objects in the category of bimonoids in M , and of certain simplicial bimonoids. These equivalent categories contain, as equivalent full subcategories, the category of internal categories in the category of cocommutative Hopf monoids in M on one hand, and the category of simplicial cocommutative Hopf monoids of Moore length 1 on the other hand. This includes, in particular, the equivalence in [8] between the category objects in the category of cocommutative Hopf algebras over a field, and the category of simplicial cocommutative Hopf algebras whose Moore complex has length 1.

Throughout, we freely use definitions, notation and results from [3, 4]. The composition of some morphisms $A \xrightarrow{g} B$ and $B \xrightarrow{f} C$ in an arbitrary category will be denoted by

$A \xrightarrow{f \cdot g} C$. Identity morphisms will be denoted by 1 (without any reference to the (co)domain object if it causes no confusion). In any monoidal category C the monoidal product will be denoted by juxtaposition and the monoidal unit will be I . For the monoidal product of n copies of the same object A also the power notation A^n will be used. For any monoid A in C , the multiplication and the unit morphisms will be denoted by $A^2 \xrightarrow{m} A$ and $I \xrightarrow{u} A$, respectively. If C is also braided, then for the braiding the symbol c will be used.

2 The Moore Length

Recall that a *simplicial object* in an arbitrary category C is a functor S from the opposite of the simplicial indexing category to C . Explicitly, this means objects and morphisms of C in

$$\begin{array}{ccccccc}
 & & & & \xleftarrow{\partial_0} & & \\
 & & & & \xleftarrow{\sigma_0} \xrightarrow{\quad} & & \\
 & & & & \xleftarrow{\partial_1} & & \\
 & & & & \xleftarrow{\sigma_1} \xrightarrow{\quad} & & \\
 & & & & \vdots & & \\
 & & & & \xleftarrow{\sigma_{n-1}} \xrightarrow{\quad} & & \\
 & & & & \xleftarrow{\partial_n} & & \\
 S_0 & \xleftarrow{\partial_0} & S_1 & \xleftarrow{\sigma_0} \xrightarrow{\quad} & S_2 \cdots S_{n-1} & \cdots & S_n \cdots
 \end{array} \tag{1}$$

subject to the following simplicial relations.

$$\begin{aligned}
 \partial_i \cdot \partial_j &= \partial_{j-1} \cdot \partial_i & \text{if } i < j \\
 \sigma_j \cdot \sigma_i &= \sigma_i \cdot \sigma_{j-1} & \text{if } i < j \\
 \partial_i \cdot \sigma_j &= \begin{cases} \sigma_{j-1} \cdot \partial_i & \text{if } i < j \\ 1 & \text{if } i \in \{j, j + 1\} \\ \sigma_j \cdot \partial_{i-1} & \text{if } i > j + 1 \end{cases}
 \end{aligned}$$

In the category of groups (and in more general semi-abelian categories [14]), a chain complex—the so called *Moore complex*—can be associated to any simplicial object. The full subcategory of those simplicial objects whose Moore complex has length 1, turns out to be equivalent to the category of crossed modules and to the category of internal categories in the semi-abelian category in question, see e.g. [19] (in the case of the category of groups) and [9,21] (more generally).

However, as in [3] and [4], here we work in categories (of monoids in some monoidal category C) where the existence of zero objects is not assumed. So the notion of chain complex is not available. We shall see below, however, that when certain relative pullbacks exist, we can still associate to a simplicial object S a sequence of composable morphisms (reproducing the Moore complex whenever the monoidal unit of C is terminal in the category of monoids in C). Although the chain condition can not be formulated at this level of generality, there is a natural way to define the length of this sequence that we call the *Moore length* of the simplicial monoid S .

The construction of this sequence of composable morphisms is based on Assumption 2.1 below.

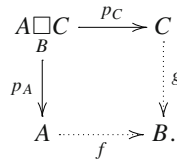
Recall that a class S of spans in an arbitrary category is said to be *admissible* if it satisfies the following two properties in ([3], Definition 2.1).

(POST) If $X \xleftarrow{f} A \xrightarrow{g} Y \in S$ then $X' \xleftarrow{f'} X \xleftarrow{f} A \xrightarrow{g} Y \xrightarrow{g'} Y' \in S$ too, for any morphisms $X \xrightarrow{f'} X'$ and $Y \xrightarrow{g'} Y'$.

(PRE) If $X \xleftarrow{f} A \xrightarrow{g} Y \in S$ then $X \xleftarrow{f} A \xleftarrow{h} B \xrightarrow{h} A \xrightarrow{g} Y \in S$, for any morphism $B \xrightarrow{h} A$.

The *relative pullback* of a cospan $A \xrightarrow{f} B \xleftarrow{g} C$ with respect to such a class S was introduced in ([3], Definition 3.1) as a span $A \xleftarrow{p_A} A \square_B C \xrightarrow{p_C} C$ in S satisfying the following properties.

– *Commutativity* of the diagram



- *Universality*. For any $A \xleftarrow{f'} X \xrightarrow{g'} C \in \mathcal{S}$ such that $f \cdot f' = g \cdot g'$, there is a unique morphism $X \xrightarrow{h} A \square_B C$ which satisfies $p_A \cdot h = f'$ and $p_C \cdot h = g'$.
- *Reflection*. If both

$$A \xleftarrow{p_A} A \square_B C \xleftarrow{k} D \xrightarrow{l} E \quad \text{and} \quad C \xleftarrow{p_C} A \square_B C \xleftarrow{k} D \xrightarrow{l} E$$

belong to \mathcal{S} then also $A \square_B C \xleftarrow{k} D \xrightarrow{l} E$ belongs to \mathcal{S} ; and symmetrically, if

$$E \xleftarrow{l} D \xrightarrow{k} A \square_B C \xrightarrow{p_A} A \quad \text{and} \quad E \xleftarrow{l} D \xrightarrow{k} A \square_B C \xrightarrow{p_C} C$$

belong to \mathcal{S} then also $E \xleftarrow{l} D \xrightarrow{k} A \square_B C$ belongs to \mathcal{S} .

A class \mathcal{S} of spans in a *monoidal* category is said to be *monoidal* if it satisfies the following two conditions in ([3], Definition 2.5).

(UNITAL) For any morphisms f and g whose domain is the monoidal unit I ,

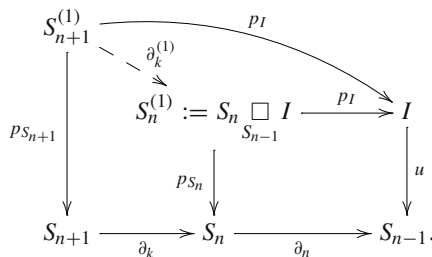
$$X \xleftarrow{f} I \xrightarrow{g} Y \in \mathcal{S}.$$

(MULTIPLICATIVE) If both $X \xleftarrow{f} A \xrightarrow{g} Y \in \mathcal{S}$ and $X' \xleftarrow{f'} A' \xrightarrow{g'} Y' \in \mathcal{S}$ then also

$$XX' \xleftarrow{ff'} AA' \xrightarrow{gg'} YY' \in \mathcal{S}.$$

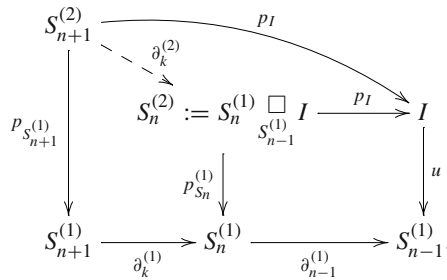
Assumption 2.1 Let \mathcal{S} be a monoidal admissible class of spans—in the sense of ([3], Definitions 2.1 and 2.5)—in some monoidal category \mathcal{C} . For any simplicial object S in the category of monoids in \mathcal{C} let us use the notation of (1) and consider the following successive assumptions.

- (1) Assume that for any positive integer n there exists the \mathcal{S} -relative pullback—in the sense of ([3], Definition 3.1)— $S_n^{(1)}$ in



Since ∂_k is compatible with the units of the monoids S_n and S_{n-1} ; and by the simplicial relation $\partial_n \cdot \partial_k = \partial_k \cdot \partial_{n+1}$ for any $0 \leq k \leq n$, we may apply ([3], Proposition 3.5 (1)) to conclude on the existence of the unique morphism $\partial_k^{(1)} := \partial_k \square 1$ rendering commutative the above diagram.

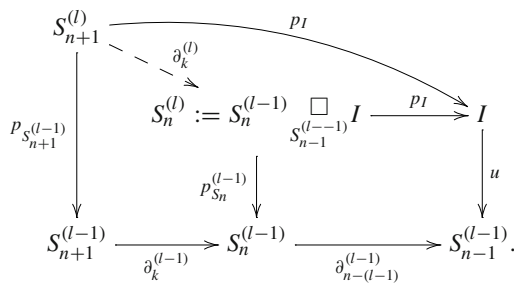
- (2) In addition to the assumption in item (1) above, assume that for all $n \geq 2$ there exists the \mathcal{S} -relative pullback $S_n^{(2)}$ in



Since $\partial_k^{(1)}$ is compatible with the units of the monoids $S_n^{(1)}$ and $S_{n-1}^{(1)}$, and by the simplicial relation $\partial_{n-1} \cdot \partial_k = \partial_k \cdot \partial_n$ for any $0 \leq k \leq n-1$, we may apply ([3], Proposition 3.5 (1)) to conclude on the existence of the unique morphism $\partial_k^{(2)} := \partial_k^{(1)} \square 1$ rendering commutative the above diagram.

⋮

- (l) In addition to the assumptions in all items (1)⋯(l-1) above, assume that for all $n \geq l$ there exists the \mathcal{S} -relative pullback $S_n^{(l)}$ in



Since $\partial_k^{(l-1)}$ is compatible with the units of the monoids $S_n^{(l-1)}$ and $S_{n-1}^{(l-1)}$; and by the simplicial relation $\partial_{n-(l-1)} \cdot \partial_k = \partial_k \cdot \partial_{n+1-(l-1)}$ for any $0 \leq k \leq n-(l-1)$, we may apply ([3], Proposition 3.5 (1)) to conclude on the existence of the unique morphism $\partial_k^{(l)} := \partial_k^{(l-1)} \square 1$ rendering commutative the above diagram.

⋮

Example 2.2 For this example note that in any monoidal category M the monoidal unit I carries a trivial monoid structure which is initial in the category of monoids in M . Symmetrically, I carries a trivial comonoid structure which is terminal in the category of comonoids in M . Whenever M is braided, the trivial monoid and comonoid structures of I combine to a bimonoid which is thus the zero object in the category of bimonoids in M .

If moreover M has equalizers which are preserved by taking the monoidal product with any object, then the category of bimonoids in M has equalizers—see ([3], Example 3.3)—and

thus kernels. The kernel of any bimonoid morphism $A \xrightarrow{f} B$ is computed as the equalizer in M of

$$\widehat{f} := A \xrightarrow{\delta} A^2 \xrightarrow{\delta_1} A^3 \xrightarrow{1f1} ABA \tag{2}$$

and $\widehat{u \cdot \varepsilon} = A \xrightarrow{\delta} A^2 \xrightarrow{1u1} ABA$, see [2].

So let M be a symmetric monoidal category in which equalizers exist and are preserved by taking the monoidal product with any object. Let C be the monoidal category of comonoids in M and consider the monoidal admissible class S of spans in C from ([3], Example 2.3). Since ε is the counit, the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\delta} & A^2 & \xrightarrow{c} & A^2 \\ \delta \downarrow & & \searrow g & & \downarrow g\varepsilon \\ A^2 & \xrightarrow{g\varepsilon} & & & B \end{array}$$

commutes for any bimonoid morphism g proving that

$$B \xleftarrow{g} A \xrightarrow{\varepsilon} I \in S. \tag{3}$$

From Example 3.3 and Proposition 3.7 in [3] we know that all S -relative pullbacks exist in the category of bimonoids in M ; hence any simplicial bimonoid in M —that is, any functor from Δ^{op} to the category of monoids in C —satisfies the successive assumptions of Assumption 2.1 for any positive integer. Still—say, for an easier comparison with [8]—below we present a more explicit description of the objects $S_n^{(k)}$.

For any positive integer n and any $0 < k \leq n$ the desired objects $S_n^{(k)}$ are constructed as the joint kernels of the morphisms $\{\partial_n, \partial_{n-1}, \dots, \partial_{n-k+1}\}$ in the category of bimonoids in M ; that is, as the joint equalizers

$$S_n^{(k)} \xrightarrow{j_n^{(k)}} S_n \begin{array}{c} \xrightarrow{\widehat{\partial}_n} \\ \xrightarrow{\widehat{\partial}_{n-1}} \\ \vdots \\ \xrightarrow{\widehat{\partial}_{n-k+1}} \\ \xrightarrow{1u1 \cdot \delta} \end{array} S_n S_{n-1} S_n$$

in M (where the “hat notation” of (2) is used). By construction they are bimonoids. Using the universality of the equalizer (in M) in the bottom rows, for $k > 1$ we construct bimonoid morphisms in

$$\begin{array}{ccc} \begin{array}{ccc} & S_n^{(k)} & \\ \swarrow P_{S_n^{(k-1)}} & \downarrow j_n^{(k)} & \\ S_n^{(k-1)} & \xrightarrow{j_n^{(k-1)}} & S_n \end{array} & \begin{array}{c} \xrightarrow{\widehat{\partial}_n} \\ \xrightarrow{\widehat{\partial}_{n-1}} \\ \vdots \\ \xrightarrow{\widehat{\partial}_{n-k+2}} \\ \xrightarrow{1u1 \cdot \delta} \end{array} & \begin{array}{ccc} & S_n S_{n-1} S_n & \\ & \downarrow \partial_{n-k+1} \partial_{n-k+1} \partial_{n-k+1} & \\ & S_{n-1} S_{n-2} S_{n-1} & \end{array} \\ \begin{array}{ccc} & S_n^{(k-1)} & \\ \swarrow P_{S_n^{(k-1)}} & \downarrow j_n^{(k-1)} & \\ S_n^{(k-1)} & \xrightarrow{j_n^{(k-1)}} & S_n \end{array} & \begin{array}{c} \xrightarrow{\widehat{\partial}_{n-1}} \\ \xrightarrow{\widehat{\partial}_{n-2}} \\ \vdots \\ \xrightarrow{\widehat{\partial}_{n-k+1}} \\ \xrightarrow{1u1 \cdot \delta} \end{array} & \begin{array}{ccc} & S_n S_{n-1} S_n & \\ & \downarrow \partial_{n-k+1} \partial_{n-k+1} \partial_{n-k+1} & \\ & S_{n-1} S_{n-2} S_{n-1} & \end{array} \end{array}$$

(note the serial commutativity of the right-hand diagram thanks to the simplicial identities) and show that they give rise to the \mathcal{S} -relative pullback

$$\begin{array}{ccc}
 S_n^{(k)} & \xrightarrow{\varepsilon} & I \\
 p_{S_n^{(k-1)}} \downarrow & & \downarrow u \\
 S_n^{(k-1)} & \xrightarrow{\partial_{n-k+1}^{(k-1)}} & S_{n-1}^{(k-1)}
 \end{array} \tag{4}$$

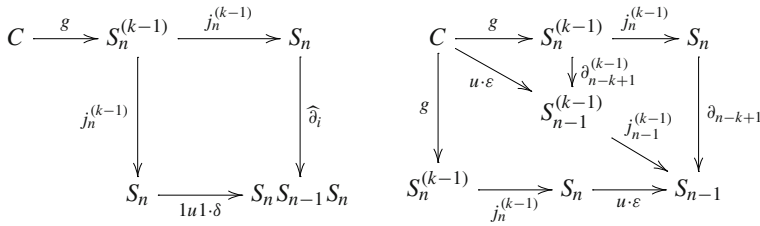
From (3) we infer $S_{n-1}^{(k-1)} \xleftarrow{P_{S_n^{(k-1)}}} S_n^{(k)} \xrightarrow{\varepsilon} I \in \mathcal{S}$. The square of (4) commutes since $j_{n-1}^{(k-1)}$ in the right vertical side of the commutative diagram

$$\begin{array}{ccccc}
 & & S_n^{(k-1)} & \xrightarrow{\partial_{n-k+1}^{(k-1)}} & S_{n-1}^{(k-1)} \\
 & p_{S_n^{(k-1)}} \curvearrowright & \downarrow j_n^{(k-1)} & & \downarrow j_{n-1}^{(k-1)} \\
 S_n^{(k)} & \xrightarrow{j_n^{(k)}} & S_n & \xrightarrow{\partial_{n-k+1}} & S_{n-1} \\
 & \searrow j_n^{(k)} & \downarrow \varepsilon & \nearrow u & \downarrow j_{n-1}^{(k-1)} \\
 & & I & \xrightarrow{u} & S_{n-1}^{(k-1)}
 \end{array}$$

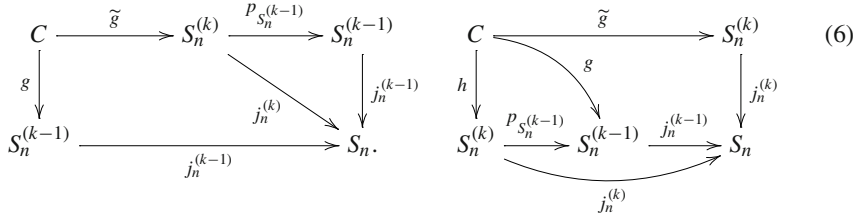
is a monomorphism. In order to check the universality of (4), take a bimonoid morphism $C \xrightarrow{g} S_n^{(k-1)}$ such that the exterior of the left-hand diagram of

$$\begin{array}{ccc}
 C & \xrightarrow{\varepsilon} & I \\
 \downarrow g & \searrow \varepsilon & \downarrow u \\
 S_n^{(k-1)} & \xrightarrow{\partial_{n-k+1}^{(k-1)}} & S_{n-1}^{(k-1)} \\
 \uparrow p_{S_n^{(k-1)}} & & \\
 S_n^{(k)} & \xrightarrow{\varepsilon} & I
 \end{array}
 \qquad
 \begin{array}{ccc}
 C & \xrightarrow{g} & S_n^{(k-1)} \\
 \downarrow \varepsilon & & \downarrow j_n^{(k-1)} \\
 S_n^{(k)} & \xrightarrow{j_n^{(k)}} & S_n \\
 & & \downarrow \widehat{\partial}_n \\
 & & S_n \xrightarrow{\widehat{\partial}_{n-1}} S_{n-1} \\
 & & \vdots \\
 & & S_n \xrightarrow{\widehat{\partial}_{n-k+1}} S_{n-1} \\
 & & \underbrace{1u1 \cdot \delta = u \cdot \varepsilon}
 \end{array} \tag{5}$$

commutes (we know that $S_n^{(k-1)} \xleftarrow{g} C \xrightarrow{\varepsilon} I \in \mathcal{S}$ by (3)). Then a filler \tilde{g} of the left-hand diagram of (5) is constructed using the universality of the equalizer in the bottom row of the right-hand diagram of (5). The occurring morphism $j_n^{(k-1)} \cdot g$ renders commutative the diagrams

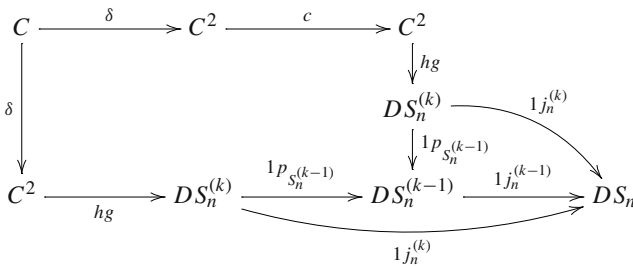


for $n - k + 1 < i \leq n$. Thus since it is a comonoid morphism, it equalizes the parallel morphisms of the right-hand diagram of (5). The so constructed morphism \tilde{g} renders commutative the left-hand diagram of (5) since the right column and the bottom row of the left-hand commutative diagram in



are equal monomorphisms. Finally \tilde{g} is the unique filler of the left-hand diagram of (5); as if also h makes the left-hand diagram of (5) commute then also the right-hand diagram of (6) commutes. Since $j_n^{(k)}$ is a monomorphism, this proves $h = \tilde{g}$. In order to see that the span (4) satisfies the reflection property of ([3], Definition 3.1) on the right, take bimonoid morphisms

$D \xleftarrow{h} C \xrightarrow{g} S_n^{(k)}$ such that $D \xleftarrow{h} C \xrightarrow{g} S_n^{(k)} \xrightarrow{P_{S_n^{(k-1)}}} S_n^{(k-1)} \in \mathcal{S}$; equivalently, the large square on the left of the diagram



commutes. Since $D S_n^{(k-1)} \xrightarrow{1_{j_n^{(k-1)}}} D S_n$ is a monomorphism, this is equivalent to the commutativity of the exterior diagram. Since also $D S_n^{(k)} \xrightarrow{1_{j_n^{(k)}}} D S_n$ is a monomorphism, this is further equivalent to $hg \cdot c \cdot \delta = hg \cdot \delta$; that is, to $D \xleftarrow{h} C \xrightarrow{g} S_n^{(k)} \in \mathcal{S}$. Reflectivity on the left follows symmetrically.

Proposition 2.3 *Let \mathcal{S} be a monoidal admissible class of spans in some monoidal category \mathcal{C} and let S be a simplicial monoid in \mathcal{C} which satisfies the successive conditions in Assumption 2.1 for any positive integer. For any positive integer n , the morphisms*

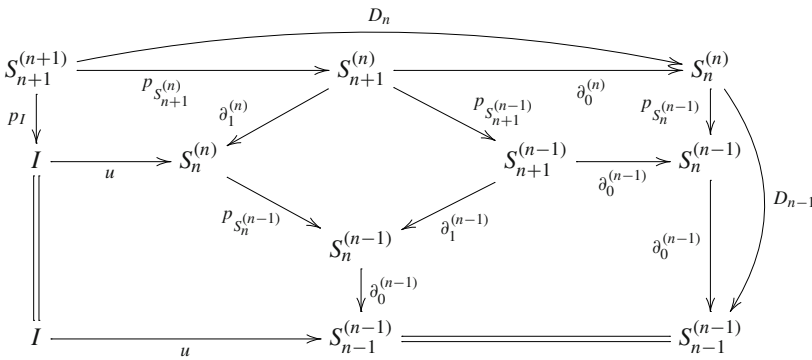
$D_{n-1} := S_n^{(n)} \xrightarrow{P_{S_n^{(n-1)}}} S_n^{(n-1)} \xrightarrow{\partial_0^{(n-1)}} S_{n-1}^{(n-1)}$ (where $S_n^{(0)} := S_n$ and $\partial_0^{(0)} := \partial_0$) render commutative the diagram

$$\begin{array}{ccc} S_{n+1}^{(n+1)} & \xrightarrow{D_n} & S_n^{(n)} \\ p_I \downarrow & & \downarrow D_{n-1} \\ I & \xrightarrow{u} & S_{n-1}^{(n-1)}. \end{array}$$

In particular, whenever the monoidal unit I is a zero object in the category of monoids in \mathcal{C} , there is a chain complex

$$\dots \xrightarrow{D_{n+1}} S_{n+1}^{(n+1)} \xrightarrow{D_n} S_n^{(n)} \xrightarrow{D_{n-1}} \dots \xrightarrow{D_1} S_1^{(1)} \xrightarrow{D_0} S_0. \tag{7}$$

Proof The morphisms D_n are clearly well-defined and they render commutative



□

Example 2.4 The definition of cartesian monoidal category includes the fact that the monoidal unit is a terminal object, which is therefore a zero object in the category of monoids. Thus Proposition 2.3 implies the well-known fact that any simplicial monoid in a cartesian monoidal category admits a Moore complex (7).

Example 2.5 If M is a symmetric monoidal category in which equalizers exist and are preserved by taking the monoidal product with any object, then we know from Example 2.2 that Assumption 2.1 holds for any simplicial bimonoid in M (that is, for any functor from Δ^{op} to the category of monoids in the category of comonoids in M). Since in the category of bimonoids in M the monoidal unit is the zero object, we conclude by Proposition 2.3 that any simplicial bimonoid in M admits a Moore complex (7).

Definition 2.6 Let S be a monoidal admissible class of spans in some monoidal category \mathcal{C} . We say that a simplicial monoid S in \mathcal{C} has Moore length l if the successive conditions in Assumption 2.1 hold for any positive integer and $I \xrightarrow{u} S_n^{(n)}$ and $S_n^{(n)} \xrightarrow{p_I} I$ are mutually inverse isomorphisms for all $n > l$.

Lemma 2.7 Let S be a monoidal admissible class of spans in some monoidal category \mathcal{C} . For a simplicial monoid S in \mathcal{C} of Moore length l , there are mutually inverse isomorphisms $I \xrightarrow{u} S_n^{(n-i)}$ and $S_n^{(n-i)} \xrightarrow{p_I} I$ for any non-negative integer i and any $n > i + l$.

Proof We proceed by induction on i .

For $i = 0$ and $n > l$ the claim holds by definition.

Assume that it holds for some fixed value of i and all $n > l + i$.

If $n > l + i + 1$ then in the \mathcal{S} -relative pullback in the inner square of

$$\begin{array}{ccc}
 S_n^{(n-i-1)} & \xrightarrow{p_I} & I \\
 \downarrow p_{S_n^{(n-i-1)}} & \searrow p_I & \downarrow u \\
 S_n^{(n-i)} & \xrightarrow{p_I} & I \\
 \downarrow p_{S_n^{(n-i-1)}} & \searrow p_I & \downarrow u \\
 S_n^{(n-i-1)} & \xrightarrow{\partial_{i+1}^{(n-i-1)}} & S_{n-1}^{(n-i-1)}
 \end{array} \tag{8}$$

the right column is an isomorphism by the induction hypothesis. We claim that so is then the left vertical of the inner square. By the monoidality of \mathcal{S} , $I = I = I \in \mathcal{S}$, hence by ([3], Lemma 3.4 (2)) also $S_n^{(n-i-1)} = S_n^{(n-i-1)} \xrightarrow{p_I} I \in \mathcal{S}$. The exterior of (8) commutes by the commutativity of

$$\begin{array}{ccc}
 S_n^{(n-i-1)} & \xrightarrow{p_I} & I \\
 \downarrow \partial_{i+1}^{(n-i-1)} & \nearrow p_I & \downarrow u \\
 S_{n-1}^{(n-i-1)} & \xlongequal{\quad} & S_{n-1}^{(n-i-1)}
 \end{array}$$

where the bottom triangle commutes by the induction hypothesis. Then by the universality of the \mathcal{S} -relative pullback in (8) there is a unique morphism $p_{S_n^{(n-i-1)}}^{-1}$ in (8). It is the right inverse

of $p_{S_n^{(n-i-1)}}$ by construction and also the left inverse since $S_n^{(n-i-1)} \xleftarrow{p_{S_n^{(n-i-1)}}^{-1}} S_n^{(n-i)} \xrightarrow{p_I} I$ are jointly monic and the following diagrams commute

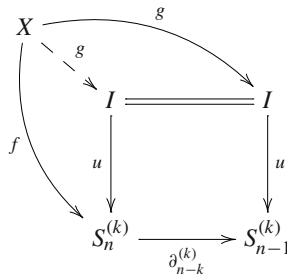
$$\begin{array}{ccc}
 & \xrightarrow{p_{S_n^{(n-i-1)}}^{-1}} & S_n^{(n-i)} \\
 S_n^{(n-i)} & \xrightarrow{p_{S_n^{(n-i-1)}}} & S_n^{(n-i-1)} \xlongequal{\quad} S_n^{(n-i-1)} \\
 & \searrow p_{S_n^{(n-i-1)}} & \downarrow p_{S_n^{(n-i-1)}}
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \xrightarrow{p_{S_n^{(n-i-1)}}^{-1}} & S_n^{(n-i-1)} \xrightarrow{p_{S_n^{(n-i-1)}}^{-1}} S_n^{(n-i)} \\
 S_n^{(n-i)} & \xrightarrow{p_I} & I \\
 & \searrow p_I & \downarrow p_I
 \end{array}$$

Using that also $I \xrightarrow{u} S_n^{(n-i)}$ is an isomorphism by the induction hypothesis, so is the composite morphism $I \xrightarrow{u} S_n^{(n-i)} \xrightarrow{p_{S_n^{(n-i-1)}}^{-1}} S_n^{(n-i-1)} = I \xrightarrow{u} S_n^{(n-i-1)}$, and this completes the proof. \square

Lemma 2.8 *Let \mathcal{S} be a monoidal admissible class of spans in some monoidal category \mathcal{C} and let S be a simplicial monoid in \mathcal{C} for which the successive conditions of Assumption 2.1 hold for any positive integer. If there is some non-negative integer k for which $I \xrightarrow{u} S_n^{(k)}$ and*

$S_n^{(k)} \xrightarrow{PI} I$ are mutually inverse isomorphisms for all $n \geq k$, then also $I \xrightarrow{u} S_n^{(k+1)}$ and $S_n^{(k+1)} \xrightarrow{PI} I$ are mutually inverse isomorphisms for all $n > k$.

Proof We need to show that under the standing assumptions the inner square of



is an \mathcal{S} -relative pullback for all $n > k$. Commutativity of this square is immediate by the unitality of the morphism in the bottom row and $S_n^{(k)} \xleftarrow{u} I = I \in \mathcal{S}$ by the unitality of \mathcal{S} . For the universality observe that since the verticals are isomorphisms by assumption, the exterior of our diagram commutes if and only if g is equal to $X \xrightarrow{f} S_n^{(k)} \xrightarrow{\partial_{n-k}^{(k)}} S_{n-1}^{(k)} \xrightarrow{PI} I = X \xrightarrow{f} S_n^{(k)} \xrightarrow{PI} I$; if and only if f is equal to $X \xrightarrow{g} I \xrightarrow{u} S_n^{(k)}$. Hence the (obviously unique) filler is g . The top row of the inner square is the identity morphism. Therefore the reflection property in ([3], Definition 3.1) of relative pullback trivially holds. \square

Corollary 2.9 Let \mathcal{S} be a monoidal admissible class of spans in some monoidal category \mathcal{C} . For a simplicial monoid S in \mathcal{C} which satisfies the successive conditions of Assumption 2.1 for any positive integer, the following assertions are equivalent.

- (i) S has Moore length l .
- (ii) $I \xrightarrow{u} S_n^{(l+1)}$ and $S_n^{(l+1)} \xrightarrow{PI} I$ are mutually inverse isomorphisms for all $n > l$.

Proof Statement (i) implies (ii) by Lemma 2.7, and (ii) implies (i) by Lemma 2.8. \square

3 Invertibility of Some Canonical Morphisms

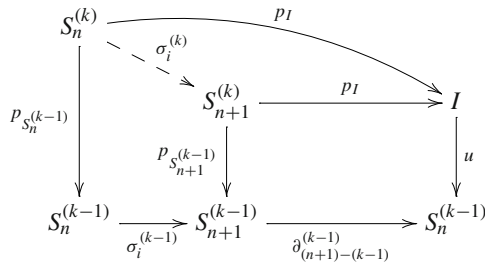
As in ([4], Theorems 1.1, 2.1 and 3.10), also in the forthcoming equivalence between relative categories and certain simplicial monoids a crucial role is played by the invertibility of some canonical morphisms discussed in this section.

Consider a monoidal admissible class of spans in an arbitrary monoidal category \mathcal{C} . Take a simplicial monoid S as in (1) in \mathcal{C} for which the successive conditions in Assumption 2.1 hold for any positive integer. For any non-negative integer n , for $0 \leq i \leq n$ and for $0 \leq k \leq n - i$,

we define a morphism $S_n^{(k)} \xrightarrow{\sigma_i^{(k)}} S_{n+1}^{(k)}$ iteratively as follows.

- $\sigma_i^{(0)} := \sigma_i$.

- For $0 < k \leq n - i$ we define $\sigma_i^{(k)}$ as the unique filler in

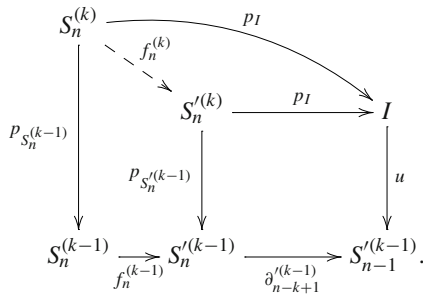


It is well-defined by the simplicial identity $\partial_{(n+1)-(k-1)}\sigma_i = \sigma_i\partial_{n-(k-1)}$ and the unitality of $\sigma_i^{(k-1)}$, see ([3], Proposition 3.5).

With these morphisms $\sigma_i^{(k)}$ at hand, we introduce for any positive integer n and $0 \leq k < n$ the morphisms

$$\gamma_{(n,k)} := S_n^{(k+1)} S_{n-1}^{(k)} \xrightarrow{p_{S_n^{(k)}} \sigma_{n-1-k}^{(k)}} (S_n^{(k)})^2 \xrightarrow{m} S_n^{(k)}. \tag{9}$$

Note that they are natural in the following sense. For a simplicial monoid morphism $\{ S_n \xrightarrow{f_n} S'_n \}_{n \geq 0}$, let us define inductively the morphisms $f_n^{(0)} := f_n$ and for $0 < k \leq n$ the unique morphism $f_n^{(k)}$ which renders commutative



(This definition makes sense—see ([3], Proposition 3.5)—since $\{f_n\}_{n \geq 0}$ is a morphism of simplicial monoids by assumption, hence so is $\{f_n^{(k-1)}\}_{n \geq 0}$ and therefore

$$\begin{array}{ccccc} S_n^{(k-1)} & \xrightarrow{\partial_{n-k+1}^{(k-1)}} & S_{n-1}^{(k-1)} & \xleftarrow{u} & I \\ f_n^{(k-1)} \downarrow & & f_{n-1}^{(k-1)} \downarrow & & \parallel \\ S'_n{}^{(k-1)} & \xrightarrow{\partial_{n-k+1}^{(k-1)}} & S'_{n-1}{}^{(k-1)} & \xleftarrow{u} & I \end{array}$$

commutes.) These morphisms and $y_{(n,k)}$ fit into the commutative diagram

$$\begin{array}{ccc}
 S_n^{(k+1)} S_{n-1}^{(k)} & \xrightarrow{y_{(n,k)}} & (S_n^{(k)})^2 \xrightarrow{m} S_n^{(k)} \\
 \downarrow f_n^{(k+1)} f_{n-1}^{(k)} & \searrow P_{S_n^{(k)}} \sigma_{n-k-1}^{(k)} & \downarrow f_n^{(k)} f_{n-1}^{(k)} \\
 S_n^{(k+1)} S_{n-1}^{(k)} & \xrightarrow{P_{S_n^{(k)}} \sigma_{n-k-1}^{(k)}} & (S_n^{(k)})^2 \xrightarrow{m} S_n^{(k)} \\
 & \searrow y'_{(n,k)} & \downarrow f_n^{(k)} \\
 S_n^{(k+1)} S_{n-1}^{(k)} & \xrightarrow{P_{S_n^{(k)}} \sigma_{n-k-1}^{(k)}} & (S_n^{(k)})^2 \xrightarrow{m} S_n^{(k)}
 \end{array} \tag{10}$$

Lemma 3.1 Consider a monoidal admissible class of spans in an arbitrary monoidal category C . Take a simplicial monoid S as in (1) in C for which the successive conditions in Assumption 2.1 hold for any positive integer. If for some positive integer n and some $0 \leq k < n - 1$ the morphism $y_{(n,k)}$ of (9) is invertible then also $y_{(n-1,k)}$ is invertible.

Proof It follows by the commutativity of both diagrams

$$\begin{array}{ccc}
 S_n^{(k+1)} S_{n-1}^{(k)} & \xrightarrow{y_{(n,k)}} & (S_n^{(k)})^2 \xrightarrow{m} S_n^{(k)} \\
 \downarrow \partial_i^{(k+1)} \partial_i^{(k)} & \searrow P_{S_n^{(k)}} \sigma_{n-1-k}^{(k)} & \downarrow \partial_i^{(k)} \partial_i^{(k)} \\
 S_{n-1}^{(k+1)} S_{n-2}^{(k)} & \xrightarrow{P_{S_{n-1}^{(k)}} \sigma_{n-2-k}^{(k)}} & (S_{n-1}^{(k)})^2 \xrightarrow{m} S_{n-1}^{(k)} \\
 & \searrow y_{(n-1,k)} & \downarrow \partial_i^{(k)} \\
 S_{n-1}^{(k+1)} S_{n-2}^{(k)} & \xrightarrow{P_{S_{n-1}^{(k)}} \sigma_{n-2-k}^{(k)}} & (S_{n-1}^{(k)})^2 \xrightarrow{m} S_{n-1}^{(k)}
 \end{array} \tag{11}$$

for any $0 \leq i < n - k - 1$ together with the simplicial identity $\partial_i \cdot \sigma_i = 1$ that $y_{(n-1,k)}$ has the inverse

$$S_{n-1}^{(k)} \xrightarrow{\sigma_i^{(k)}} S_n^{(k)} \xrightarrow{y_{(n,k)}^{-1}} S_n^{(k+1)} S_{n-1}^{(k)} \xrightarrow{\partial_i^{(k+1)} \partial_i^{(k)}} S_{n-1}^{(k+1)} S_{n-2}^{(k)}.$$

□

Note that in addition to the diagrams of (11), $y_{(n,k)}$ renders commutative also

$$\begin{array}{ccc}
 S_n^{(k+1)} S_{n-1}^{(k)} & \xrightarrow{y_{(n,k)}} & (S_n^{(k)})^2 \xrightarrow{m} S_n^{(k)} \\
 \downarrow \partial_{n-k-1}^{(k+1)} 1 & \searrow \partial_{n-k-1}^{(k)} 1 & \downarrow \partial_{n-k-1}^{(k)} \partial_{n-k-1}^{(k)} \\
 S_{n-1}^{(k+1)} S_{n-1}^{(k)} & \xrightarrow{P_{S_{n-1}^{(k)}} 1} & (S_{n-1}^{(k)})^2 \xrightarrow{m} S_{n-1}^{(k)}
 \end{array} \tag{12}$$

and

$$\begin{array}{ccc}
 S_n^{(k+1)} S_{n-1}^{(k)} & \xrightarrow{y_{(n,k)}} & (S_n^{(k)})^2 \xrightarrow{m} S_n^{(k)} \\
 \downarrow p_i 1 & \searrow \partial_{n-k}^{(k)} 1 & \downarrow \partial_{n-k}^{(k)} \partial_{n-k}^{(k)} \\
 S_{n-1}^{(k)} & \xrightarrow{u 1} & (S_{n-1}^{(k)})^2 \xrightarrow{m} S_{n-1}^{(k)}
 \end{array}$$

Proposition 3.2 *Let \mathcal{S} be a monoidal admissible class of spans in some monoidal category \mathcal{C} . For a simplicial monoid S in \mathcal{C} of Moore length l , the following assertions hold.*

- (1) *The morphism $y_{(n,k)}$ of (9) is invertible for all $l < k < n$.*
- (2) *For any $n > l$ the following are equivalent.*
 - (i) *$y_{(n,l)}$ of (9) is invertible.*
 - (ii) *$S_{n-1}^{(l)} \xrightarrow{\sigma_{n-1-l}^{(l)}} S_n^{(l)}$ is invertible.*
 - (iii) *The morphisms $S_{n-1}^{(l)} \xrightarrow{\sigma_0^{(l)}} S_n^{(l)}$, $S_{n-1}^{(l)} \xrightarrow{\sigma_1^{(l)}} S_n^{(l)}$, \dots , $S_{n-1}^{(l)} \xrightarrow{\sigma_{n-1-l}^{(l)}} S_n^{(l)}$ are equal isomorphisms. Their inverse is equal to $\partial_0^{(l)} = \partial_1^{(l)} = \dots = \partial_{n-l}^{(l)}$.*

Proof (1) In the unitality diagram

$$\begin{array}{ccc}
 I & \xlongequal{\quad} & I \\
 uu \downarrow & & \downarrow u \\
 S_n^{(k+1)} & S_{n-1}^{(k)} \xrightarrow{y_{(n,k)}} & S_n^{(k)}
 \end{array}$$

the vertical arrows are isomorphisms by Lemma 2.7. Then the bottom row is an isomorphism too.

(2) The equivalence of (i) and (ii) follows by the commutativity of the diagram

$$\begin{array}{ccc}
 S_{n-1}^{(l)} & \xrightarrow{\sigma_{n-1-l}^{(l)}} & S_n^{(l)} \\
 u1 \downarrow & & \downarrow u1 \\
 S_n^{(l+1)} & S_{n-1}^{(l)} \xrightarrow{p_{S_n^{(l)}} \sigma_{n-1-l}^{(l)}} & (S_n^{(l)})^2 \xrightarrow{m} S_n^{(l)} \\
 & \searrow y_{(n,l)} & \nearrow
 \end{array}$$

whose left vertical arrow is an isomorphism by Lemma 2.7.

Assertion (iii) trivially implies (ii). Conversely, if $\sigma_{n-1-l}^{(l)}$ is invertible then by the simplicial relations its inverse is $\partial_{n-1-l}^{(l)} = \partial_{n-l}^{(l)}$ which is then invertible too. Again by the simplicial relations the inverse of $\partial_{n-1-l}^{(l)} = \partial_{n-l}^{(l)}$ is $\sigma_{n-1-l}^{(l)} = \sigma_{n-2-l}^{(l)}$. Iterating this reasoning we conclude that (iii) holds. □

Example 3.3 For a simplicial category \mathcal{S} (that is, a functor \mathcal{S} from Δ^{op} to the category of monoids in the category of spans over a fixed set in [[4], Example 1.2]), $\mathcal{S}_n^{(k+1)}$ is the subcategory of those morphisms in $\mathcal{S}_n^{(k)}$ which are taken by the functor $\partial_{n-k}^{(k)}$ to an identity morphism. Hence for a simplicial *groupoid* \mathcal{S} , the category $\mathcal{S}_n^{(k)}$ is a groupoid for all $0 \leq k \leq n$. In this case all morphisms $\{y_{(n,k)}\}_{0 \leq k < n}$ of (9) are invertible by the same argument applied to the morphism (1.3) in ([4], Example 1.2).

Example 3.4 Let \mathcal{M} be a symmetric monoidal category in which equalizers exist and are preserved by taking the monoidal product with any object. Let \mathcal{C} be the monoidal category of comonoids in \mathcal{M} and let \mathcal{S} be the monoidal admissible class of spans in \mathcal{C} from ([3], Example 2.3). Take a simplicial monoid S in \mathcal{C} (that is, a simplicial bimonoid in \mathcal{M}) such that for all

non-negative integers n , S_n is a cocommutative Hopf monoid in M . Then $y_{(n,k)}$ of (9) is invertible for all $0 \leq k < n$. This can be seen as follows.

Recall from Example 2.2 that $S_n^{(k)}$ is now a joint kernel in the category of bimonoids in M , hence it is a sub bimonoid of the cocommutative bimonoid S_n . Thus $S_n^{(k)}$ is a cocommutative bimonoid. Moreover, by the cocommutativity of S_n , its antipode is a comonoid morphism

$S_n \xrightarrow{z_n} S_n$ for all $n \geq 0$. So we can use the universality of the equalizer in C in the bottom row of

(13)

to define the antipode $z_n^{(k)}$ for all $n > 0$ and $0 \leq k \leq n$ as the restriction of z_n . The diagram of (13) is serially commutative since any bimonoid morphism between Hopf monoids is compatible with the antipodes, thus so are in particular the parallel morphisms of the rows. This makes $S_n^{(k)}$ a cocommutative Hopf monoid for all $0 \leq k \leq n$. By construction

the morphisms $S_n^{(k)} \xrightarrow{\partial_{n-k}^{(k)}} S_{n-1}^{(k)}$ and $S_{n-1}^{(k)} \xrightarrow{\sigma_{n-k-1}^{(k)}} S_n^{(k)}$ are morphisms of bimonoids and therefore of Hopf monoids.

Summarizing, $S_{n-1}^{(k)} \begin{matrix} \xrightarrow{\sigma_{n-k-1}^{(k)}} \\ \xleftarrow{\partial_{n-k}^{(k)}} \end{matrix} S_n^{(k)}$ is a split epimorphism of bimonoids in M for which

conditions (1.a) and (1.b) of ([4], Proposition 1.5) hold; hence by ([4], Proposition 1.5) (see also [20]) the corresponding morphism $y_{(n,k)}$ in ([4], Theorem 1.1 (1.b)) is invertible.

The above assumption about the cocommutativity of each S_n may look quite strong. Note however, that for the application of ([4], Proposition 1.5) we need the assumption that

$S_n^{(k)} = S_n^{(k)} \xrightarrow{\partial_{n-k}^{(k)}} S_{n-1}^{(k)}$ belongs to the class \mathcal{S} . By ([3], Lemma 2.4)¹ this is equivalent to the cocommutativity of $S_{n-1}^{(k)}$. This should hold for all $n > 0$ and $0 \leq k < n$; so in particular for $k = 0$.

4 Equivalence of Relative Categories and Simplicial Monoids of Moore Length 1

Theorem 4.1 Consider a monoidal admissible class \mathcal{S} of spans in a monoidal category C such that there exist the \mathcal{S} -relative pullbacks of those cospans in C whose legs are in \mathcal{S} (cf. ([3], Assumption 4.1)). The equivalent categories $CatMon_{\mathcal{S}}(C)$ and $Xmod_{\mathcal{S}}(C)$ of ([4], Theorem 3.10) are equivalent also to the category $Simp^1 Mon_{\mathcal{S}}(C)$ whose

¹ Apologies about a regrettable typo in the first line of ([3], Lemma 2.4), interchanging the symbols A and B .

objects are simplicial monoids in \mathcal{C} (that, is functors S from Δ^{op} to the category of monoids in \mathcal{C}) such that the following conditions hold.

- (a) S has Moore length l .
- (b) Using the notation from (1), $S_0 \xleftarrow{\partial_1} S_1 = S_1 \xrightarrow{\partial_0} S_0$ belong to \mathcal{S} .
- (c) The morphisms $y_{(n,k)}$ of (9) and the morphisms

$$q_n := S_1^{(1)} S_1^{\square n-1} \xrightarrow{p_{S_1} 1} S_1 S_1^{\square n-1} \xrightarrow{(1 \square \sigma_0 \square \dots \square \sigma_0)(\sigma_0 \square 1)} (S_1^{\square n})^2 \xrightarrow{m} S_1^{\square n}$$

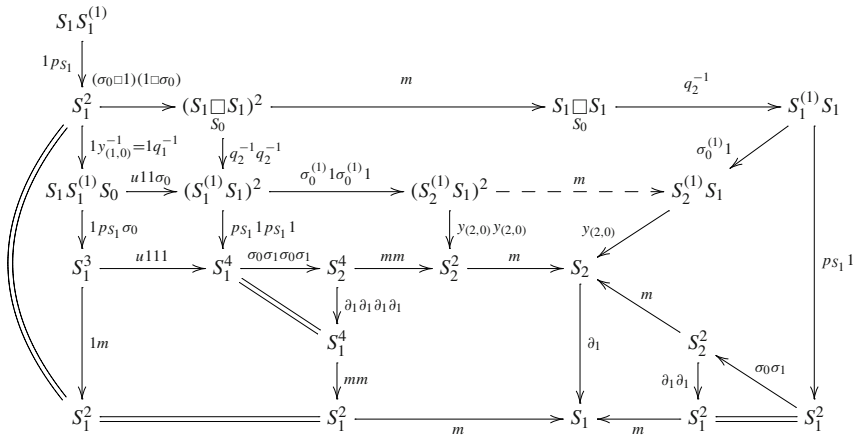
are invertible for all $n > 0$ and $0 \leq k < n$.

morphisms are simplicial monoid morphisms (that is, natural transformations between the functors from Δ^{op} to the category of monoids in \mathcal{C}).

Proof The proof consists of the construction of mutually inverse equivalence functors between $\text{Simp}^1 \text{Mon}_{\mathcal{S}}(\mathcal{C})$ and $\text{CatMon}_{\mathcal{S}}(\mathcal{C})$.

The functor $\text{Simp}^1 \text{Mon}_{\mathcal{S}}(\mathcal{C}) \rightarrow \text{CatMon}_{\mathcal{S}}(\mathcal{C})$ sends an object S in (1) to the \mathcal{S} -relative category whose underlying reflexive graph is $S_0 \begin{matrix} \xleftarrow{\partial_0} \\ \xleftarrow{\sigma_0} \rightarrow \\ \xrightarrow{\partial_1} \end{matrix} S_1$. By construction this is an

object of the category $\text{ReflGraphMon}_{\mathcal{S}}(\mathcal{C})$ in ([4], Theorem 2.1) for which the morphisms q_n of ([4], (3.3)) are invertible. By ([4], Proposition 3.8) it extends uniquely to an object of $\text{CatMon}_{\mathcal{S}}(\mathcal{C})$ since the following diagram commutes.



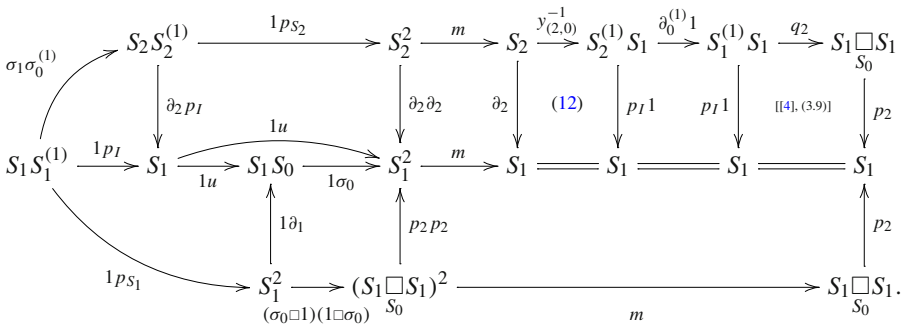
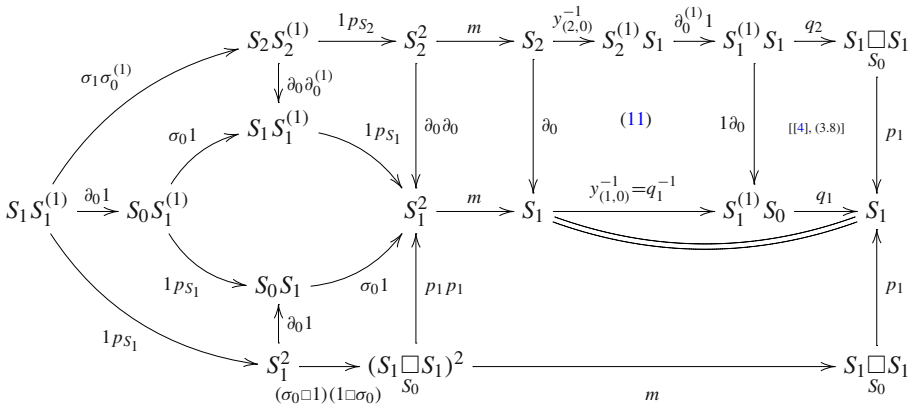
Recall from Proposition 3.2 that $S_1^{(1)} \xrightarrow{\sigma_0^{(1)}} S_2^{(1)}$ and $S_2^{(1)} \xrightarrow{\partial_0^{(1)}} S_1^{(1)}$ are mutually inverse isomorphisms. Hence the regions of the above diagram sharing the dashed arrow commute because both $q_2 \cdot \partial_0^{(1)} 1$ and $y_{(2,0)}$ are multiplicative by [[3], Lemma 1.5], with respect to the multiplications induced by the respective distributive laws

$$S_1 S_2^{(1)} \xrightarrow{1_{\partial_0^{(1)}}} S_1 S_1^{(1)} \xrightarrow{1_{p_{S_1}}} S_1^2 \xrightarrow{(\sigma_0 \square 1)(1 \square \sigma_0)} (S_1 \square_{S_0} S_1)^2 \xrightarrow{m} S_1 \square_{S_0} S_1 \xrightarrow{q_2^{-1}} S_1^{(1)} S_1 \xrightarrow{\sigma_0^{(1)} 1} S_2^{(1)} S_1$$

and

$$S_1 S_2^{(1)} \xrightarrow{\sigma_1 p_{S_2}} S_2^2 \xrightarrow{m} S_2 \xrightarrow{y_{(2,0)}^{-1}} S_2^{(1)} S_1$$

whose equality follows by the commutativity of the diagrams



A morphism $S \xrightarrow{F} S'$ in $\text{Simp}^1 \text{Mon}_{\mathcal{S}}(\mathcal{C})$ is sent to the morphism of reflexive graphs $(S_0 \xrightarrow{F_0} S'_0, S_1 \xrightarrow{F_1} S'_1)$; it is an \mathcal{S} -relative functor by ([4], Proposition 3.9).

In the opposite direction $\text{CatMon}_{\mathcal{S}}(\mathcal{C}) \rightarrow \text{Simp}^1 \text{Mon}_{\mathcal{S}}(\mathcal{C})$, we send an \mathcal{S} -relative category

$$B \begin{array}{c} \xleftarrow{s} \\ \xleftarrow{i} \\ \xleftarrow{t} \end{array} A \xleftarrow{d} A \square_B A \text{ to its 'S-relative nerve'}$$

$$B \begin{array}{c} \xleftarrow{t} \\ \xleftarrow{i} \\ \xleftarrow{s} \end{array} A \begin{array}{c} \xleftarrow{p_1} \\ \xleftarrow{1 \square i} \\ \xleftarrow{d} \\ \xleftarrow{i \square 1} \\ \xleftarrow{p_2} \end{array} A \square_B A \dots A \square_B^{n-1} \begin{array}{c} \xleftarrow{\partial_0} \\ \xleftarrow{\sigma_0} \\ \xleftarrow{\partial_1} \\ \vdots \\ \xleftarrow{\sigma_{n-1}} \\ \xleftarrow{\partial_n} \end{array} A \square_B^n \dots$$

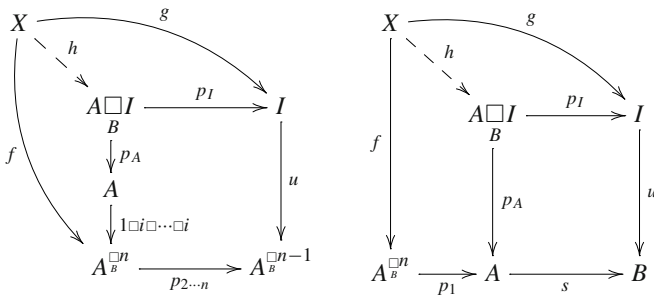
where for any positive integer n we put

$$\begin{aligned} \sigma_k &:= 1 \square^{n-k-1} \square i \square 1 \square^k && \text{for } 0 \leq k < n \\ \partial_0 &:= 1 \square^{n-1} \square t = p_1 \dots p_{n-1} \\ \partial_k &:= 1 \square^{n-k-1} \square d \square 1 \square^{k-1} && \text{for } 0 < k < n \\ \partial_n &:= s \square 1 \square^{n-1} = p_2 \dots p_n. \end{aligned}$$

By the functoriality of \square —cf. ([3], Proposition 3.5)—they constitute a simplicial monoid which obeys property (b) by construction and for which the morphisms q_n of part (c) are invertible. In order to see that it has Moore length 1, note first that $A^{(1)} = A \square_B I$ exists; see ([4], Theorem 1.1). We claim that also for any $n > 0$ there is an \mathcal{S} -relative pullback

$$\begin{array}{ccc}
 A \square_B I & \xrightarrow{p_I} & I \\
 p_A \downarrow & & u \downarrow \\
 A & \xrightarrow{s} & B \\
 1 \square i \square \dots \square i \downarrow & & i \square \dots \square i \downarrow \\
 A_B^{\square n} & \xrightarrow{p_{2 \dots n}} & A_B^{\square n-1}
 \end{array}
 \quad \begin{array}{c}
 \curvearrowright \\
 u \\
 \curvearrowleft
 \end{array}$$

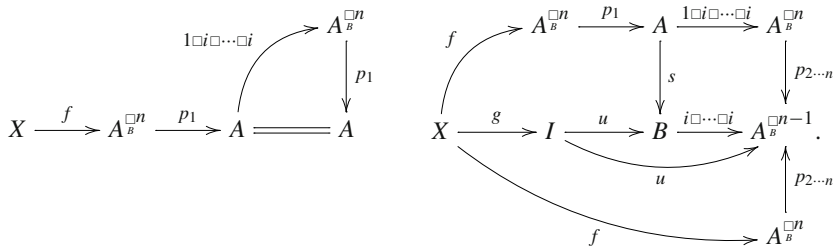
determining $(A_B^{\square n})^{(1)}$. By construction $A \xleftarrow{p_A} A \square_B I \xrightarrow{p_I} I \in \mathcal{S}$ hence by (POST) also $A_B^{\square n} \xleftarrow{1 \square i \square \dots \square i} A \xleftarrow{p_A} A \square_B I \xrightarrow{p_I} I \in \mathcal{S}$. If some morphisms f and g render commutative the left-hand diagram of



then they make commute the right-hand diagram as well by the commutativity of

$$\begin{array}{ccccc}
 X & \xrightarrow{g} & I & & \\
 f \downarrow & & u \swarrow & u \downarrow & \\
 A_B^{\square n} & \xrightarrow{p_{2 \dots n}} & A_B^{\square n-1} & \xrightarrow{p_1} & A \\
 p_1 \downarrow & & & & i \downarrow \\
 A & \xrightarrow{s} & B & & \\
 & & & & \curvearrowright \\
 & & & & u \\
 & & & & \curvearrowleft
 \end{array}$$

Whenever $A_B^{\square n} \xleftarrow{f} X \xrightarrow{g} I \in \mathcal{S}$ also $A \xleftarrow{p_1} A_B^{\square n} \xleftarrow{f} X \xrightarrow{g} I \in \mathcal{S}$ by (POST). Hence by the universality of the \mathcal{S} -relative pullback in the right-hand diagram, it has a unique filler h . But the same morphism h is a filler also for the left-hand diagram by the commutativity of both diagrams



The unique filler of the right-hand diagram is the unique filler for the left-hand diagram since any filler h of the left-hand diagram is clearly a filler for the right-hand diagram as well. For the reflection property let us use again that $1 \square i \square \dots \square i$ is a monomorphism split by p_1 . Then if $Y \xleftarrow{f} X \xrightarrow{g} A \square_B I \xrightarrow{p_A} A \xrightarrow{1 \square i \square \dots \square i} A_B^{\square n} \in \mathcal{S}$ then by (POST) also

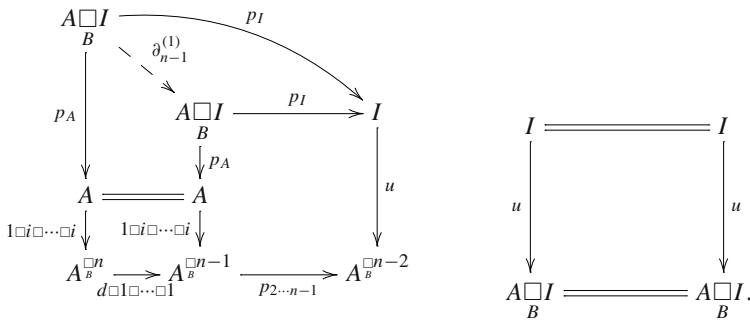
$$Y \xleftarrow{f} X \xrightarrow{g} A \square_B I \xrightarrow{p_A} A \in \mathcal{S} . \text{ So, whenever also } Y \xleftarrow{f} X \xrightarrow{g} A \square_B I \xrightarrow{p_I} I \in \mathcal{S}, \text{ it fol-}$$

lows from the reflection property of the relative pullback span $A \xleftarrow{p_A} A \square_B I \xrightarrow{p_I} I$ (in the

sense of ([3], Definition 3.1)) that $Y \xleftarrow{f} X \xrightarrow{g} A \square_B I \in \mathcal{S}$. The reflection property on the left is proven symmetrically.

With this we proved that $(A_B^{\square n})^{(1)}$ exists for any $n > 0$ and it is isomorphic to $A^{(1)} = A \square_B I$.

The morphism $\partial_{n-1}^{(1)}$ is the identity morphism, being defined as the unique morphism fitting the left-hand commutative diagram of



Since $A \square_B I \xleftarrow{u} I = I \in \mathcal{S}$ by the unitality of \mathcal{S} , the right-hand diagram is obviously an \mathcal{S} -relative pullback. This proves that for any $n > 1$

$$(A_B^{\square n})^{(2)} = (A_B^{\square n})^{(1)} \square_{(A_B^{\square n-1})^{(1)}} I \cong (A \square_B I) \square_{A \square_B I} I$$

exists and it is isomorphic to I . Then by Corollary 2.9 \mathcal{S} has Moore length 1. Above we

proved that $(A_B^{\square n})^{(1)} \xrightarrow{\partial_{n-1}^{(1)}} (A_B^{\square n-1})^{(1)}$ is invertible; then so is its inverse $\sigma_{n-2}^{(1)}$. Therefore by Proposition 3.2 the morphism $y_{(n,k)}$ of (9) is invertible for all $0 < k < n$. For any $n > 0$ the morphism $y_{(n,0)}$ takes now the form

$$(A_B^{\square n})^{(1)} A_B^{\square n-1} \cong (A \square I) A_B^{\square n-1} \xrightarrow{pA^1} AA_B^{\square n-1} \xrightarrow{(1 \square i \square \dots \square i)(i \square 1)} (A_B^{\square n})^2 \xrightarrow{m} A_B^{\square n}$$

in which we recognize the invertible morphism q_n . This proves that the nerve of an object of $\text{CatMon}_{\mathcal{S}}(\mathcal{C})$ is indeed an object of $\text{Simp}^1 \text{Mon}_{\mathcal{S}}(\mathcal{C})$. A morphism

$$(B \begin{matrix} \xleftarrow{s} \\ \xrightarrow{i} \\ \xleftarrow{t} \end{matrix} A \xleftarrow{d} A \square_B A) \xrightarrow{(b,a)} (B' \begin{matrix} \xleftarrow{s'} \\ \xrightarrow{i'} \\ \xleftarrow{t'} \end{matrix} A' \xleftarrow{d'} A' \square_{B'} A')$$

is sent to the family $(B \xrightarrow{b} B', \{A_B^{\square n} \xrightarrow{a^{\square n}} A_{B'}^{\square n}\}_{n>0})$ which is clearly a morphism of simplicial monoids.

It remains to see that the above constructed functors are mutually inverse equivalences. Sending an \mathcal{S} -relative category (and \mathcal{S} -relative functor, respectively) to its nerve and then truncating it as above, we clearly re-obtain the \mathcal{S} -relative category (and \mathcal{S} -relative functor, respectively) that we started with. Composing the functors in the opposite order, an object S of $\text{Simp}^1 \text{Mon}_{\mathcal{S}}(\mathcal{C})$ is sent to

$$S_0 \begin{matrix} \xleftarrow{\partial_0} \\ \xrightarrow{\sigma_0} \\ \xleftarrow{\partial_1} \end{matrix} S_1 \begin{matrix} \xleftarrow{p_1} \\ \xrightarrow{1 \square \sigma_0} \\ \xleftarrow{d} \\ \xrightarrow{\sigma_0 \square 1} \\ \xleftarrow{p_2} \end{matrix} S_1 \square_{S_0} S_1 \dots S_1^{\square n-1} \begin{matrix} \xleftarrow{\tilde{\partial}_0} \\ \xrightarrow{\tilde{\sigma}_0} \\ \xleftarrow{\tilde{\partial}_1} \\ \vdots \\ \xrightarrow{\tilde{\sigma}_{n-1}} \\ \xleftarrow{\tilde{\partial}_n} \end{matrix} S_1^{\square n} \dots \quad (14)$$

where $d = S_1 \square_{S_0} S_1 \xrightarrow{q_2^{-1}} (S_1 \square I) S_1 \xrightarrow{p_{S_1}^1} S_1^2 \xrightarrow{m} S_1$ (see ([4], Proposition 3.8)) and for any positive integer n ,

$$\begin{aligned} \tilde{\sigma}_k &:= 1^{\square n-k-1} \square \sigma_0 \square 1^{\square k} && \text{for } 0 \leq k < n \\ \tilde{\partial}_0 &:= 1^{\square n-1} \square \partial_0 = p_1 \dots n-1 \\ \tilde{\partial}_k &:= 1^{\square n-k-1} \square d \square 1^{\square k-1} && \text{for } 0 < k < n \\ \tilde{\partial}_n &:= \partial_1 \square 1^{\square n-1} = p_2 \dots n. \end{aligned}$$

Note that together with the family of morphisms $\{q_n\}_{n>0}$ they render commutative the following diagrams. For all $0 \leq i < n$,

$$\begin{array}{ccccccc} & & & & q_n & & \\ & & & & \curvearrowright & & \\ S_1^{(1)} S_1^{\square n-1} & \xrightarrow{p_{S_1} \tilde{\sigma}_{n-1}} & S_1 S_1^{\square n} & \xrightarrow{\tilde{\sigma}_0 1} \dots \xrightarrow{\tilde{\sigma}_0 1} & S_1^{\square n-i} S_1^{\square n} & \xrightarrow{\tilde{\sigma}_0 1} \dots \xrightarrow{\tilde{\sigma}_0 1} & (S_1^{\square n})^2 \xrightarrow{m} S_1^{\square n} \\ \downarrow 1 \tilde{\sigma}_i & & \downarrow 1 \tilde{\sigma}_i & & \downarrow \tilde{\sigma}_0 \tilde{\sigma}_i & & \downarrow \tilde{\sigma}_i \\ S_1^{(1)} S_1^{\square n} & \xrightarrow{p_{S_1} \tilde{\sigma}_n} & S_1 S_1^{\square n+1} & \xrightarrow{\tilde{\sigma}_0 1} \dots \xrightarrow{\tilde{\sigma}_0 1} & S_1^{\square n+1-i} S_1^{\square n+1} & \xrightarrow{\tilde{\sigma}_0 1} \dots \xrightarrow{\tilde{\sigma}_0 1} & (S_1^{\square n+1})^2 \xrightarrow{m} S_1^{\square n+1} \\ & & & & q_{n+1} & & \\ & & & & \curvearrowright & & \end{array} \quad (15)$$

For $1 < i < n$,

$$\begin{array}{c}
 \xrightarrow{q_{n+1}} \\
 S_1^{(1)} S_1^{\square n} \xrightarrow{p_{S_1} \tilde{\sigma}_n} S_1 S_1^{\square n+1} \xrightarrow{\tilde{\sigma}_0 1} \dots \xrightarrow{\tilde{\sigma}_0 1} S_1^{\square n-i} S_1^{\square n+1} \xrightarrow{\tilde{\sigma}_0 1} S_1^{\square n+1-i} S_1^{\square n+1} \xrightarrow{\tilde{\sigma}_0 1} \dots \xrightarrow{\tilde{\sigma}_0 1} (S_1^{\square n+1})^2 \xrightarrow{m} S_1^{\square n+1} \\
 \downarrow 1\tilde{\partial}_i \quad \downarrow 1\tilde{\partial}_i \quad \downarrow 1\tilde{\partial}_i \quad \downarrow \tilde{\partial}_0 \tilde{\partial}_i \quad \downarrow \tilde{\partial}_i \tilde{\partial}_i \quad \downarrow \tilde{\partial}_i \\
 S_1^{(1)} S_1^{\square n-1} \xrightarrow{p_{S_1} \tilde{\sigma}_{n-1}} S_1 S_1^{\square n} \xrightarrow{\tilde{\sigma}_0 1} \dots \xrightarrow{\tilde{\sigma}_0 1} S_1^{\square n-i} S_1^{\square n} \xrightarrow{\tilde{\sigma}_0 1} \dots \xrightarrow{\tilde{\sigma}_0 1} (S_1^{\square n})^2 \xrightarrow{m} S_1^{\square n} \\
 \xrightarrow{q_n}
 \end{array}
 \tag{16}$$

and for $0 \leq i \leq 1 < n$ the analogous one

$$\begin{array}{c}
 \xrightarrow{q_{n+1}} \\
 S_1^{(1)} S_1^{\square n} \xrightarrow{p_{S_1} \tilde{\sigma}_n} S_1 S_1^{\square n+1} \xrightarrow{\tilde{\sigma}_0 1} \dots \xrightarrow{\tilde{\sigma}_0 1} S_1^{\square n} S_1^{\square n+1} \xrightarrow{\tilde{\sigma}_0 1} (S_1^{\square n+1})^2 \xrightarrow{m} S_1^{\square n+1} \\
 \downarrow 1\tilde{\partial}_i \quad \downarrow 1\tilde{\partial}_i \quad \downarrow 1\tilde{\partial}_i \quad \downarrow \tilde{\partial}_i \tilde{\partial}_i \quad \downarrow \tilde{\partial}_i \\
 S_1^{(1)} S_1^{\square n-1} \xrightarrow{p_{S_1} \tilde{\sigma}_{n-1}} S_1 S_1^{\square n} \xrightarrow{\tilde{\sigma}_0 1} \dots \xrightarrow{\tilde{\sigma}_0 1} (S_1^{\square n})^2 \xrightarrow{m} S_1^{\square n} \\
 \xrightarrow{q_n}
 \end{array}
 \tag{17}$$

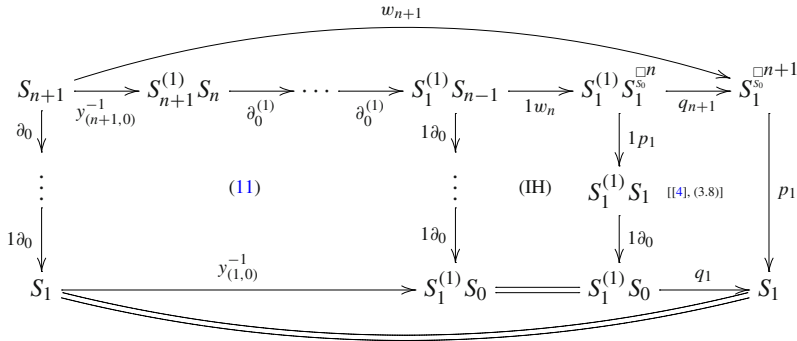
We claim that a natural isomorphism from S to its image in (14) can be constructed iteratively for all $n \geq 0$ as

- $w_0 := 1$
- $w_n := S_n \xrightarrow{y_{(n,0)}^{-1}} S_n^{(1)} S_{n-1} \xrightarrow{\partial_0^{(1)} 1} \dots \xrightarrow{\partial_0^{(1)} 1} S_1^{(1)} S_{n-1} \xrightarrow{1w_{n-1}} S_1^{(1)} S_1^{\square n-1} \xrightarrow{q_n} S_1^{\square n}$. (18)

Note that it gives $w_1 = 1$, but non-trivial higher components. Let us prove by induction on n the equality

$$S_n \xrightarrow{w_n} S_1^{\square n} \xrightarrow{p_1} S_1 = S_n \xrightarrow{\partial_0} \dots \xrightarrow{\partial_0} S_1 \quad \text{for } n > 0. \tag{19}$$

For $n = 1$ both sides yield the identity morphism thus the equality holds. If it holds for some $n > 0$ then the following diagram commutes



proving (19) for all positive n . (The region marked by (IH) commutes by the induction hypothesis.)

The morphisms w_n of (18) are composites of isomorphisms (see Proposition 3.2) hence they are invertible. They are compatible with the units of the domain and codomain monoids by the unitality of the constituent monoid morphisms. Multiplicativity is checked by induction on n again. Trivially, $w_0 = 1$ is multiplicative. If w_{n-1} is multiplicative for some $n > 0$ then by ([3], Corollary 1.7) so are both isomorphisms

$$S_1^{(1)} S_{n-1} \xrightarrow{\sigma_0^{(1)} 1} \dots \xrightarrow{\sigma_0^{(1)} 1} S_n^{(1)} S_{n-1} \xrightarrow{y_{(n,0)}} S_n \text{ and } S_1^{(1)} S_{n-1} \xrightarrow{1w_{n-1}} S_1^{(1)} S_1^{\square n-1} \xrightarrow{q_n} S_1^{\square n} \quad (20)$$

with respect to the multiplications induced by the respective distributive laws

$$S_{n-1} S_1^{(1)} \xrightarrow{1\sigma_0^{(1)}} \dots \xrightarrow{1\sigma_0^{(1)}} S_{n-1} S_n^{(1)} \xrightarrow{\sigma_{n-1} P S_n} S_n^2 \xrightarrow{m} S_n \xrightarrow{y_{(n,0)}^{-1}} S_n^{(1)} S_{n-1} \xrightarrow{\partial_0^{(1)} 1} \dots \xrightarrow{\partial_0^{(1)} 1} S_1^{(1)} S_{n-1}$$

$$S_{n-1} S_1^{(1)} \xrightarrow{w_{n-1} 1} S_1^{\square n-1} S_1^{(1)} \xrightarrow{1p_{S_1}} S_1^{\square n-1} S_1 \xrightarrow{(\sigma_0 \square 1)(1 \square \sigma_0 \square \dots \square \sigma_0)} (S_1^{\square n})^2 \xrightarrow{m} S_1^{\square n} \xrightarrow{q_n^{-1}} S_1^{(1)} S_1^{\square n-1} \xrightarrow{1w_{n-1}^{-1}} S_1^{(1)} S_{n-1}.$$

Their equality follows by the commutativity of the diagrams of Fig. 1, whose right vertical arrows are jointly monomorphic.

Since w_n is the composite of the right-hand morphism of (20) with the inverse of the left-hand one, we conclude that it is multiplicative too.

Next we check by induction on n that $\{w_n\}_{n \geq 0}$ is a simplicial morphism; that is,

$$\begin{array}{ccc}
 S_n & \xrightarrow{w_n} & S_1^{\square n} \\
 \sigma_i \downarrow & & \downarrow \tilde{\sigma}_i \\
 S_{n+1} & \xrightarrow{w_{n+1}} & S_1^{\square n+1}
 \end{array}
 \quad
 \begin{array}{ccc}
 S_{n+1} & \xrightarrow{w_{n+1}} & S_1^{\square n+1} \\
 \partial_j \downarrow & & \downarrow \tilde{\partial}_j \\
 S_n & \xrightarrow{w_n} & S_1^{\square n}
 \end{array}
 \quad (21)$$

commute for all $n \geq 0$ and $0 \leq i \leq n$ and $0 \leq j \leq n + 1$. Note that the induction must be started with $n = 1$ because the left-hand diagram of (24) below only makes sense for $n > 0$. For $n = 0$ the diagrams of (21) commute because w_0 and w_1 are the identity morphisms and the equalities $S_0 \xrightarrow{\tilde{\sigma}_0} S_1 = S_0 \xrightarrow{\sigma_0} S_1$ and $S_1 \xrightarrow{\tilde{\partial}_i} S_0 = S_1 \xrightarrow{\partial_i} S_0$ hold for $i = 0, 1$ by

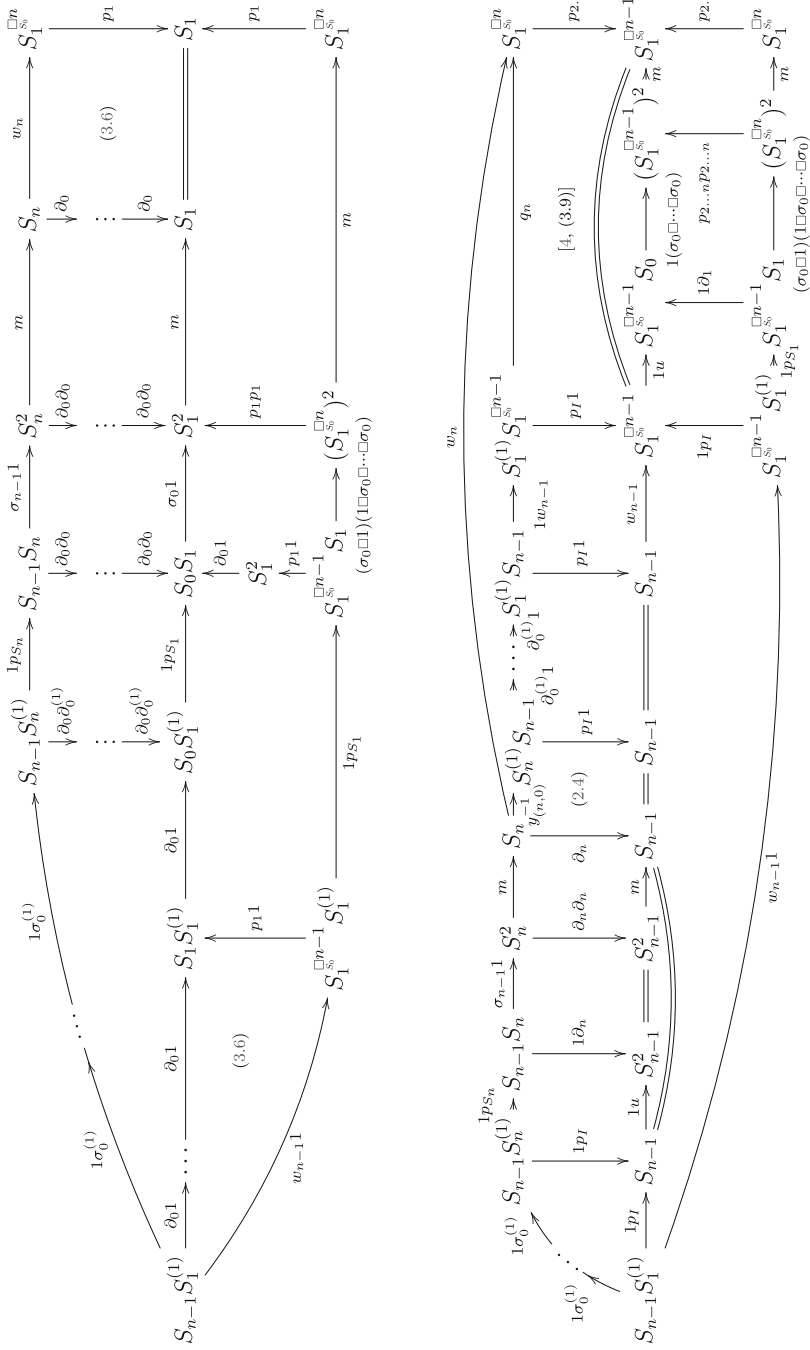
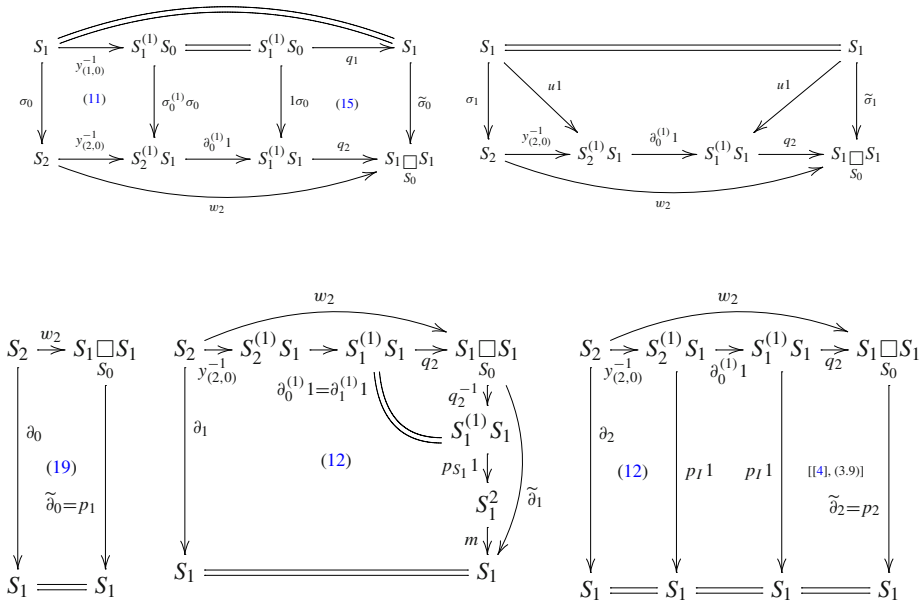
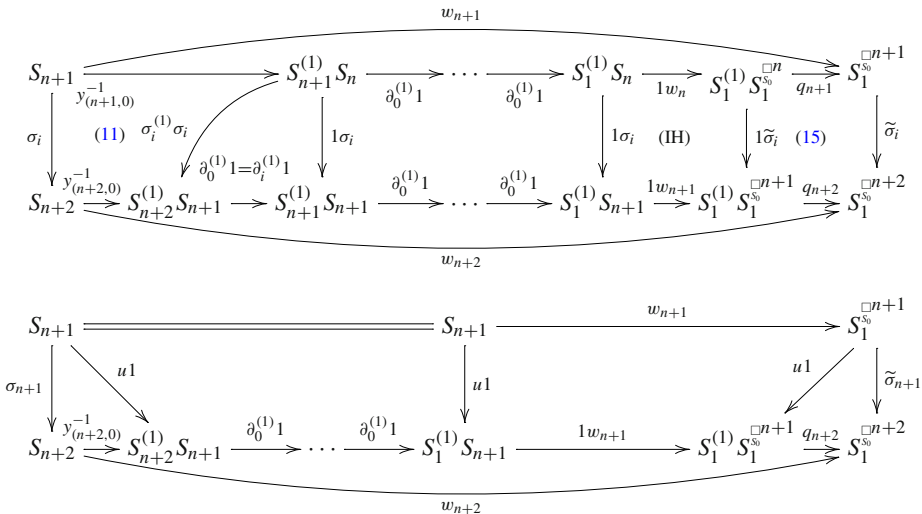


Fig. 1 Multiplicativity of w_n

construction. For $n = 1$ commutativity of the diagrams of (21) is checked as follows.



Now assume that the left-hand diagram of (21) commutes for some $n > 0$ and all $0 \leq i \leq n$. By the commutativity of the diagrams



(where the region marked by (IH) commutes by the induction hypothesis) we conclude that the left-hand diagram of (21) commutes for all $n > 0$ and all $0 \leq i \leq n$.

(a) For all $n > 0$ and for $\widehat{\partial}_i := S_n \xrightarrow{\delta} S_n^2 \xrightarrow{\delta_1} S_n^3 \xrightarrow{1\partial_i 1} S_n S_{n-1} S_n$ for $0 < i \leq n$,

$$\begin{array}{ccc}
 I \xrightarrow{u} S_n & \begin{array}{c} \xrightarrow{\widehat{\partial}_n} \\ \xrightarrow{\widehat{\partial}_{n-1}} \\ \vdots \\ \xrightarrow{\widehat{\partial}_1} \end{array} & S_n S_{n-1} S_n \\
 & \searrow^{1u1\cdot\delta} & \\
 & &
 \end{array} \tag{25}$$

is a joint equalizer in M (that is; the Moore complex of S has length 1, see Examples 2.2 and 2.5).

- (b) $\partial_0 1 \cdot \delta = \partial_0 1 \cdot c \cdot \delta$ and $\partial_1 1 \cdot \delta = \partial_1 1 \cdot c \cdot \delta$.
- (c) The morphisms q_n and $y_{(n,k)}$ of Theorem 4.1 are invertible for $0 \leq k < n$ (equivalently, $q_1, y_{(n,0)}$ and $y_{(n,1)}$ are invertible for all $0 < n$).

morphisms are the simplicial bimonoid morphisms.

As a simple consequence we obtain the following result in [8].

Proposition 4.4 *Let M be a symmetric monoidal category in which equalizers exist and are preserved by taking the monoidal product with any object. Take C to be the category of comonoids in M with the monoidal admissible class S in ([3], Example 2.3) of spans in C . The equivalent categories in ([4], Example 3.12) and in Example 4.3 have equivalent full subcategories as follows.*

- The full subcategory of $CatMon_S(C)$ for whose objects $B \begin{array}{c} \xleftarrow{s} \\ \xleftarrow{i} \\ \xleftarrow{t} \end{array} A \xleftarrow{d} A \square_B A$ both A and B are cocommutative Hopf monoids in M .
- The full subcategory of $Xmod_S(C)$ for whose objects $(B, Y, BY \xrightarrow{l} Y, Y \xrightarrow{k} B)$ both Y and B are cocommutative Hopf monoids in M .
- The full subcategory of $RefGraphMon_S(C)$ for whose objects $B \begin{array}{c} \xleftarrow{s} \\ \xleftarrow{i} \\ \xleftarrow{t} \end{array} A$ the following conditions hold.
 - A and B are cocommutative Hopf monoids (with antipodes z)
 - for the morphisms

$$\vec{s} := A \xrightarrow{\delta} A^2 \xrightarrow{1s} AB \xrightarrow{1z} AB \xrightarrow{1i} A^2 \xrightarrow{m} A, \quad \overleftarrow{t} := A \xrightarrow{\delta} A^2 \xrightarrow{1t} BA \xrightarrow{z1} BA \xrightarrow{1i} A^2 \xrightarrow{m} A$$

the following diagram commutes.

$$\begin{array}{ccccc}
 A^2 & \xrightarrow{\vec{s} \overleftarrow{t}} & A^2 & \xrightarrow{c} & A^2 \\
 \downarrow \overleftarrow{t} & & \downarrow m & & \downarrow m \\
 A^2 & \xrightarrow{m} & A & &
 \end{array} \tag{26}$$

- The full subcategory of $Simp^1 Mon_S(C)$ for whose objects S the following conditions hold.
 - S_n is a cocommutative Hopf monoid in M for all $n \geq 0$.
 - The Moore complex of S has length 1; that is, (25) is a joint equalizer in M for all $n > 1$.

Proof We need to show that the category listed last is a subcategory of the category in Example 4.3. Condition (b) of Example 4.3 becomes trivial thanks to the cocommutativity assumption. Concerning condition (c), the morphisms q_n are invertible by ([4], Proposition 3.13) and the morphisms $y_{(n,k)}$ are invertible by Example 3.4. \square

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