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Hedetniemi's Conjecture and Adjoint Functors in Thin Categories

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Abstract We survey results on Hedetniemi's conjecture which are connected to adjoint functors in the "thin" category of graphs, and expose the obstacles to extending these results.

1 Introduction and Terminology

Hedetniemi's conjecture states that the chromatic number of the product of two graphs is equal to the minimum of the chromatic numbers of the factors. This conjecture has caught the attention of some category theorists, who recognize their turf in its setting of morphisms and products in Cartesian closed categories. Yet, the input from category theorists has not gone much beyond reformulations of the problem and rediscovery of basic results.

In our opinion there is potential for deeper input. However, we will argue that the consideration of the graph-theoretic product in terms of its categorial definition is perhaps the wrong approach. The most relevant property of the product seems to be that it is preserved by right adjoint functors. Indeed, significant results have been proved using this fact.

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The purpose of this paper is to expose this point of view in greater detail. However for this purpose, we will need to escape the usual category of graphs and homomorphisms and work instead in thin categories.

1.1 Graphs, Digraphs, Homomorphisms and Chromatic Numbers

It will be useful to consider both finite graphs and finite digraphs. A *digraph G* consists of the following data: 1. a finite set $V(G)$ of *vertices*; 2. a finite set $A(G)$ of *arcs*; 3. two maps $t_G, h_G : A(G) \to V(G)$ that determine the *tail* and the *head* of each arc. For $a \in A(G)$, we say that it is an arc from $t_G(a)$ to $h_G(a)$ and write $a : t_G(a) \to h_G(a)$.

A digraph is *simple* if for any $u, v \in V(G)$ there is at most one arc from u to v. For simple digraphs it is common for an arc $u \to v$ to be called (u, v), and so $A(G)$ is usually viewed as a binary relation on $V(G)$. When $A(G)$ is symmetric, G is called a *graph*; an *edge* of G is a pair $\{u, v\}$ with $(u, v), (v, u) \in A(G)$.

A *homomorphism* f of a digraph G to a digraph H is a pair of maps (f_0, f_1), $f_0 : V(G) \to V(H)$, $f_1 : A(G) \to A(H)$, such that both of the following diagrams commute:

$$
\begin{array}{ccc}
A(G) & \xrightarrow{f_1} & A(H) \\
\downarrow{t_G} & & \downarrow{t_H} \\
V(G) & \xrightarrow{f_0} & V(H)
\end{array}
\qquad
\begin{array}{ccc}
A(G) & \xrightarrow{f_1} & A(H) \\
\downarrow{h_G} & & \downarrow{h_H} \\
V(G) & \xrightarrow{f_0} & V(H)
\end{array}
$$

Homomorphisms are composed component-wise. Where H is simple, f is usually identified with f_0 because f_1 is uniquely determined by it.

The *product* $G \times H$ of two digraphs is given by

$$
V(G \times H) = V(G) \times V(H),
$$
$$
A(G \times H) = A(G) \times A(H),
$$
$$
t_{G \times H} = (t_G, t_H) \text{ and } h_{G \times H} = (h_G, h_H).
$$

This construction, along with the obvious projections, is the product in the category of digraphs and their homomorphisms. Note that symmetry is preserved by the product, that is, the product of two graphs is a graph.

The *complete graph* K_n on n vertices is the (simple) graph defined by

$$
V(K_n) = \{0, 1, \dots, n-1\},
$$
$$
A(K_n) = \{(i, j) : i \neq j\}.
$$

An *n-colouring* of a digraph G is a homomorphism of G to K_n. The *chromatic number* $\chi(G)$ of G is the least n such that G admits an n-colouring. Note that a digraph G can have *loops*, that is, arcs of the form $u \to u$. In this case, G has no n-colourings for any n, and we define $\chi(G) = \infty$.

1.2 Hedetniemi's Conjecture and Related Questions

Hedetniemi's conjecture states the following:

Conjecture 1.1 ([15]) *If G and H are graphs, then*

$$
\chi(G \times H) = \min\{\chi(G), \chi(H)\}.
$$

The inequality $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$ follows immediately from definitions, so Hedetniemi's conjecture is equivalent to the statement that $\chi(G \times H) \geq \min\{\chi(G), \chi(H)\}$. Though no counterexamples have been found in fifty years, there is no categorial reason for Hedetniemi's conjecture to hold, since "arrows go the wrong way". In fact, Hedetniemi's conjecture fails for digraphs, with relatively small examples (see Section 3). The following much weaker version of Hedetniemi's conjecture will be relevant to our discussion:

Conjecture 1.2 (The weak Hedetniemi conjecture) *For every integer n, there exists an integer $f(n)$ such that if G and H are graphs with chromatic number at least $f(n)$, then*

$$\chi(G \times H) \geq n.$$

We will also be interested in the characterization of so-called "multiplicative graphs". A graph K is called *multiplicative* if the existence of a homomorphism of a product $G \times H$ to K implies the existence of a homomorphism of a factor G or H to K. Hence Hedetniemi's conjecture is equivalent to the statement that the complete graphs are multiplicative. However, few graphs have yet been shown to be multiplicative.

1.3 Structure of the Paper

The paper is structured as follows. In Section 2 we present the thin categories of graphs and digraphs, and functors relevant to our discussion. In Section 3 we give an exposition of some of the strange results on Conjecture 1.2 obtained using such functors. In Section 4 we see how attempts to improve on the previous results have led to proving the multiplicativity of some "circular complete graphs"; we then discuss apparent limitations of the method of adjoint functors towards Hedetniemi's conjecture. We provide proof of the results which we can prove using properties of adjoint functors, and we propose open problems throughout the paper.

2 Adjoint Functors in the Thin Categories of Graphs and Digraphs

2.1 Thin Categories

In the context of colourings and Hedetniemi's conjecture, the existence or non-existence of a homomorphism between two graphs or digraphs seems to be more relevant than the structure of homomorphisms between them. Hence we focus our study on thin categories. A *thin category* is a category in which there is at most one morphism between any two objects. The *thin category of digraphs (resp. graphs)* is the category whose objects are digraphs (resp. graphs) and where there is a unique morphism from G to H if there exists a homomorphism of G to H.

If a homomorphism of G to H exists, we write $G \to H$; we write $G \not\to H$ if it does not. Thus $\to$ is a binary relation on the class $\mathcal{D}$ of finite digraphs or on its subclass $\mathcal{G}$ of finite graphs. More precisely, $\to$ is a preorder on $\mathcal{G}$ and on $\mathcal{D}$, and Hedetniemi's conjecture can be naturally stated in terms of the quotient order $\mathcal{G}/\leftrightarrow$. The relation $\leftrightarrow$ is naturally defined by $G \leftrightarrow H$ if $G \to H$ and $H \to G$; G and H are then called *homomorphically equivalent*. Note that G and H are homomorphically equivalent if and only if they are isomorphic in the thin category.

Both $\mathcal{G}/\leftrightarrow$ and $\mathcal{D}/\leftrightarrow$ are distributive lattices with $(G/\leftrightarrow) \wedge (H/\leftrightarrow) = (G \times H)/\leftrightarrow$. (In fact, both $\mathcal{G}/\leftrightarrow$ and $\mathcal{D}/\leftrightarrow$ are Heyting algebras because the categories of graphs and digraphs are Cartesian closed.) An element K of $\mathcal{G}$ (resp. $\mathcal{D}$) is multiplicative if and only if $K/\leftrightarrow$ is meet-irreducible in $\mathcal{G}/\leftrightarrow$ (resp. $\mathcal{D}/\leftrightarrow$). Thus Hedetniemi's conjecture is equivalent to the statement that for every complete graph K_n, $K_n/\leftrightarrow$ is meet-irreducible in $\mathcal{G}/\leftrightarrow$. This reformulation does not simplify matters since the structure of $\mathcal{G}/\leftrightarrow$ is complex, and in particular, dense above K_2 (see [17]).

In thin categories, adjunction comes down to an equivalence between two existential statements. Specifically, two functors L, R between thin categories ($\mathcal{G}$ or $\mathcal{D}$) are respectively left and right adjoints of each other if the following property holds:

$$L(G) \rightarrow H \Leftrightarrow G \rightarrow R(H).$$

This condition is weaker than adjunction in the usual categories with all homomorphisms. Thus all adjoint pairs in these categories (which we present next) are also adjoint pairs in the thin categories, but there are other thin adjoints as we shall see in Section 2.5.

2.2 Pultr Template and Functors

The functors relevant to our discussion are connected to those introduced by Pultr in [22]. Pultr worked, among others, in the usual categories of graphs and digraphs (as defined in Section 1.1). He characterized the adjoint functors in these categories by means of the following construction.

(i)   A *Pultr template* is a quadruple $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$ where P, Q are digraphs and ϵ_1, ϵ_2 homomorphisms of P to Q.

(ii) Given a Pultr template $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$, the *left Pultr functor* $\Lambda_{\mathcal{T}}$ is the following construction: For a digraph G, $\Lambda_{\mathcal{T}}(G)$ contains one copy P_u of P for every vertex u of G, and for every arc $a : u \rightarrow v$ of G, $\Lambda_{\mathcal{T}}(G)$ contains a copy Q_a of Q with $\epsilon_1(P)$ identified with P_u and $\epsilon_2(P)$ identified with P_v.

(iii) Given a Pultr template $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$ the *central Pultr functor* $\Gamma_{\mathcal{T}}$ is the following construction: For a digraph H, the vertices of $\Gamma_{\mathcal{T}}(H)$ are the homomorphisms $f : P \rightarrow H$, and the edges of $\Gamma_{\mathcal{T}}(H)$ are the homomorphisms $g : Q \rightarrow H$; such g is an arc from $f_1 = g \circ \epsilon_1$ to $f_2 = g \circ \epsilon_2$.

For any Pultr template $\mathcal{T}$, $\Lambda_{\mathcal{T}} : \mathcal{D} \rightarrow \mathcal{D}$ and $\Gamma_{\mathcal{T}} : \mathcal{D} \rightarrow \mathcal{D}$ are respectively left and right adjoints of each other (see [7, 22]). We use the adjective "central" rather than "right" for $\Gamma_{\mathcal{T}}$ because in some important cases, $\Gamma_{\mathcal{T}}$ itself admits a right adjoint in the thin category.

The right adjoints into $\mathcal{G}$ need an additional property of the Pultr template: both P and Q must be graphs, and moreover, Q must admit an automorphism q with $q \circ \epsilon_1 = \epsilon_2$ and $q \circ \epsilon_2 = \epsilon_1$. The existence of such an automorphism makes the conditions in (ii) and (iii) symmetric, so that $\Lambda_{\mathcal{T}}(G)$ is well defined, and $\Gamma_{\mathcal{T}}(H)$ is a graph rather than a digraph. (For the adjunction to work in the "non-thin" category of graphs and homomorphisms, we need to allow graphs with multiple edges.)

We now give a simple example of a Pultr template, which will be useful later on. We let $\mathcal{T}(3) = (K_1, P_3, \epsilon_1, \epsilon_2)$, where P_3 is the path with three edges and ϵ_1, ϵ_2 map K_1 to the endpoints of P_3. Then $\Lambda_{\mathcal{T}(3)}(G)$ is obtained from G by replacing each edge by a path with three edges, and $\Gamma_{\mathcal{T}(3)}(H)$ is obtained from H by putting edges between vertices joined by a walk of length 3 in H. In particular, if H is the cycle with five vertices, then $\Gamma_{\mathcal{T}(3)}(H)$ is the complete graph K_5. The adjunction between $\Lambda_{\mathcal{T}(3)}$ and $\Gamma_{\mathcal{T}(3)}$ then reads as follows.

For any graph G, G is 5-colourable if and only if $\Lambda_{\mathcal{T}(3)}(G)$ admits a homomorphism to the five-cycle.

This is a "complexity reduction" of the problem of determining whether a graph is 5-colourable to the problem of determining whether a graph admits a homomorphism to the five-cycle. Since the former problem is NP-complete, so is the latter. Using an arsenal of similar reductions, Hell and Nešetřil [16] eventually proved that for any non-bipartite graph H without loops, the problem of determining whether a graph admits a homomorphism to H is NP-complete.

2.3 "Graph Products" and Variants on Hedetniemi's Conjecture

Other examples of Pultr templates and functors are connected to the various "products" used in graph theory. Indeed there are many useful ways to define an edge set on the Cartesian product $V(G) \times V(H)$ of the vertex sets of simple graphs G and H. We consider the following examples from [18].

- The *direct product* $G \times H$ is the product we introduced earlier: it has edges $\{(u_1, u_2), (v_1, v_2)\}$ such that $\{u_1, v_1\}$ is an edge of G and $\{u_2, v_2\}$ is an edge of H,
- The *Cartesian product* $G \square H$ has edges $\{(u_1, u_2), (v_1, v_2)\}$ whenever $\{u_1, v_1\}$ is an edge of G and $u_2 = v_2$, or $u_1 = v_1$ and $\{u_2, v_2\}$ is an edge of H,
- The *lexicographic product* $G \circ H$ has edges $\{(u_1, u_2), (v_1, v_2)\}$ such that $\{u_1, v_1\}$ is an edge of G and u_2, v_2 are arbitrary, or $u_1 = v_1$ and $\{u_2, v_2\}$ is an edge of H.

For a fixed graph H and $\star \in \{\times, \square, \circ\}$, let $\mathcal{T}(\star, H)$ be the Pultr template $(H \star K_1, H \star K_2, \epsilon_1, \epsilon_2)$ where $\epsilon_1(u) = (u, 0)$ and $\epsilon_2(u) = (u, 1)$. Then for any graph G, $\Lambda_{\mathcal{T}(\star, H)}(G) = G \star H$. Therefore any product by a fixed graph is a left adjoint, and the equivalence

$$G \star H \to K \Leftrightarrow G \to \Gamma_{\mathcal{T}(\star, H)}(K)$$

holds. In particular, $\Gamma_{\mathcal{T}(\times, H)}(K)$ is the exponential graph usually denoted K^H, which is the exponential object in the category $\mathcal{G}$.

Exponential graphs connect to Hedetniemi's conjecture as follows: We have $G \times H \to K_n$ if and only if $G \to \Gamma_{\mathcal{T}(\times, H)}(K_n) = K_n^H$. If $\chi(H) \le n$, any homomorphism $f : H \to K_n$ corresponds to a loop in K_n^H, so the condition $G \to K_n^H$ is trivially satisfied. Let $\chi(H) > n$. Then

$$\chi(K_n^H) = \max\{\chi(G) : G \times H \to K_n\}$$

because $\chi(G) \le \chi(K_n^H)$ whenever $G \to K_n^H$, that is, whenever $G \times H \to K_n$. Thus, Hedetniemi's conjecture is equivalent to the statement that

$$\chi(K_n^H) \le n \text{ whenever } \chi(H) > n.$$

Similar observations hold in the case of the other graph products: We have $G \square H \to K_n$ if and only if $G \to \Gamma_{\mathcal{T}(\square, H)}(K_n)$. When $\chi(H) > n$, $\Gamma_{\mathcal{T}(\square, H)}(K_n)$ is empty, so $G \square H \to K_n$ only if G is empty. It is easy to show that when $\chi(H) \le n$, $\Gamma_{\mathcal{T}(\square, H)}(K_n)$ is homomorphically equivalent to K_n, so $G \square H \to K_n$ only if $\chi(G) \le n$. Therefore, we have

$$\chi(G \square H) = \max\{\chi(G), \chi(H)\}.$$

This is the *Sabidussi identity*, which is sometimes viewed as a companion formula to Hedetniemi's conjecture.

There is no corresponding formula for the lexicographic product. In general, we have $\chi(G \circ H) = \chi(G \circ K_m)$, where $m = \chi(H)$ (see [9]). Thus $\chi(G \circ H)$ does not depend on the structure of H, just on its chromatic number. However, the structure of G is relevant.

We have $G \circ K_m = \Lambda_{\mathcal{T}(\circ, K_m)}(G) \to K_n$ if and only if $G \to \Gamma_{\mathcal{T}(\circ, K_m)}(K_n)$, and the latter is homomorphically equivalent to the *Kneser graph* $K(n, m)$ defined by

$$V(K(n, m)) = \mathcal{P}_m(V(K_n)),$$
$$A(K(n, m)) = \{(A, B) : A \cap B = \emptyset\}.$$

The name of these graphs is derived from Kneser's conjecture which states that $\chi(K(n, m)) = n - 2m + 2$. Kneser's conjecture was proved by Lovász [19] in a famous paper which introduced the field of "topological bounds" on the chromatic number.

2.4 Products, Functors and Hedetniemi's Conjecture

Our examples so far have situated Hedetniemi's conjecture within the more general problem of determining the chromatic number of graphs of the type $R(K_n)$, where R is some right adjoint in the thin category of graphs. These questions can be easy or hard, but there seems to be nothing special about the fact that $\times$ is the "correct" product in the category of graphs. The special property of $\times$ that will be relevant to our discussion is the fact that thin right adjoints commute with it, up to homomorphic equivalence:

$$R(G \times H) \leftrightarrow R(G) \times R(H).$$

Here, $\times$ cannot be replaced by the various "products" of graph theory. It is a property that can be used to say something meaningful about Hedetniemi's conjecture, when suitable functors are found, as shown in Section 3.

2.5 A Right Adjoint of a Central Pultr Functor

Our last example of this section introduces a functor that we will use in Section 4. It is our first example of a thin right adjoint that is not a central Pultr functor. It can be motivated by the following question.

Does there exist, for every integer n, a graph G_n such that $\chi(G_n) = n$ and G_n admits an n-colouring $f : G_n \to K_n$ where the neighbourhood of each colour class is an independent set?

In [10], Gyárfás, Jensen and Stiebitz present the question as a strengthening of a question of Harvey and Murty. Note that if two adjacent vertices u_1 and u_2 are respective neighbours of two identically coloured vertices u_0 and u_3, then u_0, u_1, u_2, u_3 is a walk of length three between two identically coloured vertices. Therefore the above question can be reformulated in terms of the functor $\Gamma_{\mathcal{T}(3)}$ introduced in the first example of this section.

Does there exist, for every integer n, a graph G_n such that $\chi(G_n) = \chi(\Gamma_{\mathcal{T}(3)}(G_n)) = n$?

It turns out that $\Gamma_{\mathcal{T}(3)}$ has a thin right adjoint $\Omega_{\mathcal{T}(3)}$, introduced in [23]. For a graph G, $\Omega_{\mathcal{T}_3}(G)$ is the graph constructed as follows. The vertices of $\Omega_{\mathcal{T}_3}(G)$ are the couples (u, U) such that $u \in V(H)$, $U \subseteq V(G)$, and every vertex in U is adjacent to u. Two couples $(u, U), (v, V)$ are joined by an edge of $\Omega_{\mathcal{T}_3}(H)$ if $u \in V$, $v \in U$, and every vertex in U is adjacent to every vertex in V. It is easy to see that $\chi(\Omega_{\mathcal{T}(3)}(K_n)) \leq n$. And since the condition $\Gamma_{\mathcal{T}(3)}(G_n) \to K_n$ is equivalent to $G_n \to \Omega_{\mathcal{T}(3)}(K_n)$, a graph G_n exists with the required properties if and only if $G_n = \Omega_{\mathcal{T}(3)}(K_n)$ fits the bill, that is $\chi(\Omega_{\mathcal{T}(3)}(K_n)) = n$. Once again, the question reduces to determining the chromatic number of some thin right adjoint of K_n. The question is answered in the affirmative in [10].

2.6 Excursion: Topological Bounds on Chromatic Numbers

As an aside note, we mention that the method for finding a lower bound on $\chi(\Omega_{\mathcal{T}(3)}(K_n))$ in [10] is the topological method devised by Lovász [19] for Kneser graphs. Thus the "topological bounds" are tight for chromatic numbers of $\Omega_{\mathcal{T}(3)}(K_n)$ and $\Gamma_{\mathcal{T}(\circ, K_m)}(K_n)$. They are also trivially tight for $\Gamma_{\mathcal{T}(\square, H)}(K_n)$, and Hedetniemi's conjecture would imply that they are also tight for $\Gamma_{\mathcal{T}(\times, H)}(K_n)$. Since it is easy to devise graphs for which such topological bounds are not tight, it is intriguing to see it work well in the case of many right adjoints.

Is it a coincidence? In modern terms, the bounds can be defined via functors which associate to a graph a "hom-complex" and eventually an object in a category of topological spaces. This point of view is developed in Section 7 of [4]. Perhaps the effectiveness of the topological bounds can be traced functorially in some way. As far as we know, there is no unifying theory as to when topological bounds on the chromatic number are tight.

3 The arc Graph Construction

3.1 Multiplicative Complete Graphs in $\mathcal{G}$ and $\mathcal{D}$, and the Poljak-Rödl Function

The complete graphs K_1 and K_2 can be shown to be multiplicative both in $\mathcal{G}$ and $\mathcal{D}$ with relatively straightforward arguments. The first nontrivial case of Hedetniemi's conjecture was established by El-Zahar and Sauer in the aptly named paper [5].

Theorem 3.1 ([5]) *K_3 is multiplicative in $\mathcal{G}$.*

However, a few years before, Poljak and Rödl [21] had proved that for every $n \geq 3$, there are digraphs G_n and H_n both with $n + 1$ vertices and chromatic number $n + 1$, such that $\chi(G_n \times H_n) = n$. Thus K_n fails to be multiplicative in $\mathcal{D}$ for any $n \geq 3$. This justifies some skepticism towards Hedetniemi's conjecture.

Let $\phi : \mathbb{N} \to \mathbb{N}$ be defined by

$$\phi(n) = \min\{\chi(G \times H) : G, H \in \mathcal{G} \text{ and } \chi(G), \chi(H) = n\}.$$

Hedetniemi's Conjecture 1.1 is that $\phi(n) = n$ for all $n \in \mathbb{N}$, but so far this has been verified only for $n \leq 4$. The weak Hedetniemi Conjecture 1.2 is that ϕ is unbounded. Poljak and Rödl proved the following result.

Theorem 3.2 ([21]) *If ϕ is bounded, then its least upper bound is at most 16.*

This result and its developments, along with their relationship with adjoint functors, are the topic of this section.

3.2 The Arc Graph Construction and its Chromatic Properties

Let $\vec{P}_i$ be the directed path with i arcs, that is, the digraph defined by $V(\vec{P}_i) = \{0, \ldots, i\}$ and $A(\vec{P}_i) = \{(0, 1), \ldots, (i - 1, i)\}$. Consider the Pultr template $\mathcal{T} = (\vec{P}_1, \vec{P}_2, \epsilon_1, \epsilon_2)$, where ϵ_1 and ϵ_2 map the arc of $\vec{P}_1$ respectively to the first and the second arc of $\vec{P}_2$. Then $\Gamma_{\mathcal{T}} : \mathcal{D} \to \mathcal{D}$ is the *arc graph construction* usually denoted δ: The vertices of $\delta(G) = \Gamma_{\mathcal{T}}(G)$ are the arcs of G, and its arcs correspond to pairs of consecutive arcs of G. We will use the following property of the arc graph construction.

Proposition 3.3 ([14]) *For any digraph G, the following holds.*

(i)   *If $\chi(\delta(G)) \leq n$, then $\chi(G) \leq 2^n$.*

(ii) *If $\chi(G) \leq \binom{n}{\lfloor n/2 \rfloor}$, then $\chi(\delta(G)) \leq n$.*

Proof We present a proof using adjoint functors. It turns out that δ has a thin right adjoint, which we will call δ_R. The vertices of $\delta_R(G)$ are the ordered pairs (U, V) of sets of vertices of G such that $(u, v) \in A(G)$ for all $u \in U$ and $v \in V$. The arcs of $\delta_R(G)$ are the ordered pairs $((U, V), (W, X))$ such that $V \cap W \neq \emptyset$. Note that the vertices of $\delta_R(K_n)$ are the ordered pairs (U, V) of disjoint sets of vertices of K_n. The map $f : \delta_R(K_n) \to \delta_R(K_n)$ defined by $f(U, V) = (U, \overline{U})$ (where $\overline{U}$ is the complement of U) is an endomorphism, with image of size 2^n. Therefore $\chi(\delta_R(K_n)) \leq 2^n$. Using adjunction we then get

$$\chi(\delta(G)) \leq n \Rightarrow \delta(G) \to K_n \Rightarrow G \to \delta_R(K_n) \Rightarrow \chi(G) \leq 2^n.$$

Also, $\delta_R(K_n)$ contains a copy of $K_{\binom{n}{\lfloor n/2 \rfloor}}$ induced by the sets $(U, \overline{U})$ such that $|U| = \lfloor n/2 \rfloor$. Hence

$$\chi(G) \leq \binom{n}{\lfloor n/2 \rfloor} \Rightarrow G \to K_{\binom{n}{\lfloor n/2 \rfloor}} \to \delta_R(K_n) \Rightarrow \delta(G) \to K_n \Rightarrow \chi(\delta(G)) \leq n.$$

$\square$

Therefore we have $\chi(\delta(G)) \simeq \log(\chi(G))$ and similarly $\chi(\delta^n(G)) \simeq \log^n(\chi(G))$ (where the exponents represent composition). These are essentially the only known examples of right adjoints with non-trivial predictable effects on the chromatic number of general graphs. Chromatic numbers of graphs obtained by applying arc-graph-like functors to transitive tournaments have recently been studied in [1, 2]. However, the following is not known.

Problem 3.4 Suppose that $R : \mathcal{D} \to \mathcal{D}$ is a right adjoint such that for some unbounded functions $a, b : \mathbb{N} \to \mathbb{N}$ we have $a(\chi(G)) \leq \chi(R(G)) \leq b(\chi(G))$ for all $G \in \mathcal{D}$. Does it follow that for some n we then have $\delta^n(G) \to R(G)$ for all $G \in \mathcal{D}$?

3.3 The Possible Bounds on the Poljak-Rödl Function

The directed version of Theorem 3.2 is the following.

Proposition 3.5 *Let $\psi : \mathbb{N} \to \mathbb{N}$ be defined by*

$$\psi(n) = \min\{\chi(G \times H) : G, H \in \mathcal{D} \text{ and } \chi(G), \chi(H) = n\}.$$

If ψ is bounded, then its least upper bound is at most 4.

Proof The proof uses the fact that δ is a right adjoint, hence it commutes with the product. Let b be the least upper bound on ψ. Let n be an integer such that $\psi(n) = b$. Note that ψ is non-decreasing; thus there exist digraphs G, H such that $\chi(G) = \chi(H) = 2^n$ and $\chi(G \times H) = b$. By Lemma 3.3, we have $\chi(\delta(G)) \geq n$ and $\chi(\delta(H)) \geq n$, whence $\chi(\delta(G) \times \delta(H)) \geq b$. However $\delta(G) \times \delta(H) \leftrightarrow \delta(G \times H)$, whence $\chi(\delta(G \times H)) \geq b$. By Lemma 3.3, for any integer m such that $b = \chi(G \times H) \leq \binom{m}{\lfloor m/2 \rfloor}$, we have $b \leq \chi(\delta(G \times H)) \leq m$. The only integers $b \geq 3$ which satisfy this property are 3 and 4. $\square$

The proof of Theorem 3.2 is essentially an adaptation of the proof of Corollary 3.5. Indeed for $G, H \in \mathcal{G}$, let $\vec{G}, \vec{H}$ be orientations of G and H respectively, constructed by selecting one of the arcs (u, v), (v, u) for each edge $\{u, v\}$ of G and H. Then $\chi(G \times H) = \chi(\vec{G} \times H)$, since $\vec{G} \times H$ is an orientation of $G \times H$. Furthermore the arcs of $\vec{G} \times H$ can be partitioned into arcs of $\vec{G} \times \vec{H}$ and arcs of $\vec{G} \times \vec{H}'$, where $\vec{H}'$ is obtained by reversing the arcs of $\vec{H}$. Theorem 3.2 is then proved by adapting the argument of the proof of Corollary 3.5 to the function ψ' defined by

$$\psi'(n) = \min\{\max\{\chi(\vec{G} \times \vec{H}), \chi(\vec{G} \times \vec{H}')\} : \vec{G}, \vec{H} \in \mathcal{D} \text{ and} \chi(\vec{G}), \chi(\vec{H}) = n\}.$$

This accounts for the value $16 = 4^2$ in the statement of Theorem 3.2. These results were improved shortly afterwards, independently by Poljak, Schmerl and Zhu (the latter two unpublished):

Theorem 3.6 ([20]) *(i)   If ψ is bounded, then its least upper bound is 3.*
(ii)   If ϕ is bounded, then its least upper bound is at most 9.

The proof method is simply a finer analysis of the chromatic properties of δ; in particular this accounts for the value $9 = 3^2$ in the second statement. Thus the case of undirected graphs is never dealt with directly. Proof methods have always used directed graphs.

Problem 3.7 Is there a functor $R : \mathcal{G} \to \mathcal{G}$ that allows to provide tighter upper bounds on ϕ, in the case that it is bounded?

A later result gives a further link between ϕ and ψ:

Theorem 3.8 ([24]) *ϕ is bounded if and only if ψ is bounded.*

Thus the weak Hedetniemi Conjecture 1.2 is equivalent for graphs and for digraphs. It would be interesting to connect the case of digraphs more directly to Hedetniemi's conjecture.

Problem 3.9 Is there a way to prove that if ψ grows too slowly, then Hedetniemi's conjecture is false? (Note that if ψ' is sub-quadratic, then Hedetniemi's conjecture is false.)

3.4 Multiplicative Complete Graphs

The last tentative application of δ to Hedetniemi's conjecture is an appealing argument that seems to have been first noticed by Roman Bašic.

Proposition 3.10 *If K_n is multiplicative, then $K_{\binom{n}{\lfloor n/2 \rfloor}}$ is multiplicative.*

Proof The proof uses a strengthening of Lemma 3.3 (ii) in the case of undirected graphs: Sperner's theorem implies that if G is an undirected graph, then $\chi(\delta(G))$ is the largest integer n such that $\chi(G) \leq \binom{n}{\lfloor n/2 \rfloor}$. In other words, for an undirected graph G we have

$$\delta(G) \to K_n \Leftrightarrow G \to K_{\binom{n}{\lfloor n/2 \rfloor}}.$$

Now suppose that $G \times H \to K_{\binom{n}{\lfloor n/2 \rfloor}}$. Then $\delta(G \times H) \to K_n$, that is, $\delta(G) \times \delta(H) \to K_n$. If K_n is multiplicative, this implies $\delta(G) \to K_n$ or $\delta(H) \to K_n$, whence $G \to K_{\binom{n}{\lfloor n/2 \rfloor}}$ or $H \to K_{\binom{n}{\lfloor n/2 \rfloor}}$. Thus if K_n is multiplicative, then $K_{\binom{n}{\lfloor n/2 \rfloor}}$ is multiplicative. $\square$

The attentive reader may have noticed that the above argument switches between $\mathcal{G}$ and $\mathcal{D}$. It correctly uses the multiplicativity of K_n in $\mathcal{D}$ to prove the multiplicativity of $K_{\binom{n}{\lfloor n/2 \rfloor}}$ in $\mathcal{G}$. However, multiplicativity is category sensitive. The precise formulation of Proposition 3.10 is the following.

Proposition 3.11 (The precise formulation of Proposition 3.10) *If K_n is multiplicative in $\mathcal{D}$, then $K_{\binom{n}{\lfloor n/2 \rfloor}}$ is multiplicative in $\mathcal{G}$.*

Thus the result turns out to be trivial, since the only complete graphs that are multiplicative in $\mathcal{D}$ are K_0, K_1 and K_2. Attempts have been made to fix the argument, or to find similar arguments, using other functors. (The symmetrisation of δ does not seem to be the correct functor, either.) In Section 4, we examine the results obtained using the functors related to the Pultr template $\mathcal{T}(3)$ of Section 2, and its generalizations.

3.5 Excursion: Odd Girth of Shift Graphs

Let $\vec{K}_n$ denote the transitive tournament on n vertices, that is,

$$V(\vec{K}_n) = \{0, 1, \ldots, n - 1\},$$
$$A(\vec{K}_n) = \{(i, j) : i < j\}.$$

The *shift graph* $S(n, k)$ is the graph $\delta^{k-1}(\vec{K}_n)$. The shift graphs (or their symmetrisations) are folklore "topology free" examples of graphs with large chromatic number and no short odd cycles. We show how these properties connect to adjoint functors.

Proposition 3.12 *$S(n, k)$ has no odd cycle with fewer than $2k + 1$ vertices, and for $\log_2^{k-1}(n) > m$ we have $\chi(S(n, k)) \geq m$ (where the exponent represents composition).*

Proof We have $\chi(\vec{K}_n) = n$, hence the iterated logarithmic lower bounds on $\chi(S(n, k))$ follow from Proposition 3.3, that is, from the right adjoint of δ. We now show that the absence of short odd cycles follows from the left adjoint of δ. Suppose that some orientation C of an odd cycle admits a homomorphism to $S(n, k) = \delta(S(n, k - 1))$, for some $k > 1$. Then $\delta_L(C)$ admits a homomorphism to $S(n, k - 1)$, where δ_L is the left adjoint of δ. By construction, the number of arcs in $\delta_L(C)$ is equal to the number of vertices in C. Moreover, since C admits a homomorphism to $S(n, k)$ which does not contain an oriented cycle, C must contain a source and a sink. The copies of $\vec{P}_1$ corresponding to sources and sinks of C are pendant; therefore any odd cycle of $\delta_L(C)$ has fewer arcs than C. Since we have $C \not\rightarrow \delta(K_2) \simeq K_2$, we get $\delta_L(C) \not\rightarrow K_2$ whence $\delta_L(C)$ indeed has an odd cycle. The smallest odd cycle in $\vec{K}_n$ has three vertices; iteratively, this implies that the smallest odd cycle in $S(n, k)$ has at least $2k + 1$ vertices. $\qquad\square$

4 Path Functors and Circular Graphs

The natural generalization of Proposition 3.11 is the following.

Proposition 4.1 *Let $L : \mathcal{A} \rightarrow \mathcal{B}$ and $R : \mathcal{B} \rightarrow \mathcal{A}$ be thin adjoint functors such that $L(G \times H) \leftrightarrow L(G) \times L(H)$. If K is multiplicative in $\mathcal{B}$, then $R(K)$ is multiplicative in $\mathcal{A}$. Moreover, if $L(R(G)) \leftrightarrow G$ for all $G \in \mathcal{B}$, then the converse holds.*

The condition $L(G \times H) \leftrightarrow L(G) \times L(H)$ is satisfied whenever L is a central Pultr functor which admits a right adjoint. In this section, we examine the applications of Proposition 4.1 to functors associated to the template $\mathcal{T}(3)$ of Section 2 and its natural generalizations.

4.1 The Multiplicativity of Odd Cycles

The functor $\Gamma_{\mathcal{T}(3)} : \mathcal{G} \to \mathcal{G}$ of Section 2 admits the right adjoint $\Omega_{\mathcal{T}(3)} : \mathcal{G} \to \mathcal{G}$. Therefore, using the fact that K_3 is multiplicative in $\mathcal{G}$, we first get that $\Omega_{\mathcal{T}(3)}(K_3)$ is also multiplicative in $\mathcal{G}$. Then, recursively, the graphs $\Omega_{\mathcal{T}(3)}^n(K_3)$ are all multiplicative in $\mathcal{G}$. It turns out that $\Omega_{\mathcal{T}(3)}^n(K_3)$ is homomorphically equivalent to the odd cycle $C_{3^{n+1}}$ with 3^{n+1} vertices. So this sequence of derivations yields particular cases of the following result of Häggkvist, Hell, Miller and Neumann Lara.

Theorem 4.2 ([11]) *The odd cycles are all multiplicative in $\mathcal{G}$.*

The proof given in [11] is an adaptation of the proof of Theorem 3.1 given in [5]. However it is possible to derive the multiplicativity of all odd cycles from that of K_3 using the following adjoint functors that generalize $\Gamma_{\mathcal{T}(3)}$ and $\Omega_{\mathcal{T}(3)}$: Let $\mathcal{T}(2k + 1) = (K_1, P_{2k+1}, \epsilon_1, \epsilon_2)$, where P_{2k+1} is the path with $2k + 1$ edges and ϵ_1, ϵ_2 map K_1 to the endpoints of P_{2k+1}. The right adjoint of $\Gamma_{\mathcal{T}(2k+1)}$ is $\Omega_{\mathcal{T}(2k+1)}$ is characterized in [12]. It is defined as follows.

- The vertices of $\Omega_{\mathcal{T}(2k+1)}(G)$ are the $(k + 1)$-tuples $(A_0, \ldots, A_k)$ such that each $A_i \subseteq V(G)$; A_0 is a singleton; every vertex of A_{i-1} is adjacent to every vertex of A_i for $i = 1, \ldots, k$; and A_{i-1} is contained in A_{i+1} for $i = 1, \ldots, k - 1$.
- The edges of $\Omega_{\mathcal{T}(2k+1)}(G)$ join pairs $(A_0, \ldots, A_k)$, $(B_0, \ldots, B_k)$ such that $A_{i-1} \subseteq B_i$ and $B_{i-1} \subseteq A_i$ for $i = 1, \ldots, k$, and every vertex of A_k is adjacent to every vertex of B_k.

To prove Theorem 4.2 with these functors, we use the following.

Lemma 4.3 ([13]) $\Gamma_{\mathcal{T}(n)}(\Omega_{\mathcal{T}(n)}(G)) \leftrightarrow G \leftrightarrow \Gamma_{\mathcal{T}(n)}(\Lambda_{\mathcal{T}(n)}(G))$ *for any odd n and $G \in \mathcal{G}$.*

Theorem 4.2 can then be proved as follows. We note that $\Omega_{\mathcal{T}(2k+1)}(K_3)$ and $\Omega_{\mathcal{T}(3)}(C_{2k+1})$ are both isomorphic to the odd cycle $C_{3(2k+1)}$. Therefore by Proposition 4.1, the multiplicativity of $C_{3(2k+1)}$ follows from the multiplicativity of K_3, and the multiplicativity of C_{2k+1} follows from the multiplicativity of $C_{3(2k+1)}$.

4.2 Circular Complete Graphs and the Circular Chromatic Number

We now extend Theorem 4.2 to some "circular complete graphs" defined as follows. Let $\mathbb{Z}_s = \{0, 1, \ldots, s - 1\}$ denote the cyclic group with s elements. For integers r, s such that $1 \le r \le s/2$, the *circular complete graph* $K_{s/r}$ is the graph with vertex set $\mathbb{Z}_s$ and with edges $\{x, y\}$ such that $y - x \in \{r, r + 1, \ldots, s - r\}$. In particular, $K_{s/1}$ is the complete graph K_s, and $K_{(2r+1)/r}$ is the odd cycle C_{2r+1}.

The conventional notation $K_{s/r}$ is slightly ambiguous, since it confuses two integer parameters s and r with a single rational parameter s/r. We will write s/r for the parameter pair and $\frac{s}{r}$ for the corresponding fraction. The notation is motivated by the following result, which alleviates some of its ambiguity.

Lemma 4.4 ([3]) *(i)* $K_{s/r} \to K_{s'/r'}$ *if and only if* $\frac{s}{r} \le \frac{s'}{r'}$.

(ii)   For any $\frac{s}{r} > 2$, there exists $\frac{s'}{r'} < \frac{s}{r}$ such that $K_{s/r} \setminus \{x\} \to K_{s'/r'}$ for all $x \in V(K_{s/r})$.

The *circular chromatic number* $\chi_c(G)$ of a graph G is defined as the infimum of the values $\frac{s}{r}$ such that $G \to K_{s/r}$. By Lemma 4.4, the infimum is attained. We have $\chi_c(G) \leq \chi(G) < \chi_c(G) + 1$, hence the circular chromatic number is a refinement of the chromatic number.

As in the case of the chromatic number, the inequality $\chi_c(G \times H) \leq \min\{\chi_c(G), \chi_c(H)\}$ follows immediately from definitions. Zhu [27] conjectured that equality always holds. This is a strengthening of Hedetniemi's conjecture. It is equivalent to the statement that the circular complete graphs are multiplicative. In this direction, the following is known.

Proposition 4.5 ([23]) *The circular complete graphs $K_{s/r}$ with $2 \leq \frac{s}{r} < 4$ are multiplicative.*

The proof uses the functors $\Gamma_{\mathcal{T}(3)}$ and $\Omega_{\mathcal{T}(3)}$ as follows.

(i)   When $\frac{s}{r} < 3$, we have $\Gamma_{\mathcal{T}(3)}(K_{s/r}) \leftrightarrow K_{s/(3r-s)}$.
(ii)   When $\frac{s}{r} < \frac{12}{5}$, we have $\Omega_{\mathcal{T}(3)}(\Gamma_{\mathcal{T}(3)}(K_{s/r})) \leftrightarrow K_{s/r}$.
(iii)   Therefore by Proposition 4.1, when $\frac{s}{r} < \frac{12}{5}$, $K_{s/r}$ is multiplicative if and only if $K_{s/(3r-s)}$ is multiplicative.
(iv)   Starting with the odd cycles $C_{2n+1} = K_{(2n+1)/n}$ which are multiplicative by Proposition 4.2, we can use item (iii) above to infer the multiplicativity of many circular complete graphs $K_{s/r} = \Gamma_{\mathcal{T}(3)}^m C_{2n+1}$. The set of rationals $\frac{s}{r}$ for which $K_{s/r}$ is proved to be multiplicative in this fashion is a dense subset of the interval $(2, 4)$.
(v)   The remaining graphs $K_{s/r}$, $2 < \frac{s}{r} < 4$ are proved to be multiplicative by a density argument: Suppose that

$$\frac{s'}{r'} = \chi_c\{G \times H\} < \min\{\chi_c(G), \chi_c(H)\} = \frac{s}{r}.$$

By item (iv) above there exists $\frac{s'}{r'} < \frac{s''}{r''} < \frac{s}{r}$ such that $K_{s''/r''}$ is multiplicative. But we have $G \times H \to K_{s'/r'} \to K_{s''/r''}$ while $G, H \not\to K_{s''/r''}$, a contradiction.

Corollary 4.6 *The identity $\chi_c(G \times H) = \min\{\chi_c(G), \chi_c(H)\}$ holds whenever $\min\{\chi_c(G), \chi_c(H)\} \leq 4$.*

It turns out that the multiplicativity of K_3 implies that of all the graphs $K_{s/r} : 2 < \frac{s}{r} < 4$. The proof method exposed here does not extend to other circular complete graphs. Property (i) breaks down at $\frac{s}{r} = 3$, since $\Gamma_{\mathcal{T}(3)}(K_{s/r})$ contains loops whenever $\frac{s}{r} \geq 3$. Property (ii) breaks down at $\frac{12}{5}$, we have $\Gamma_{\mathcal{T}(3)}(K_{12/5}) \leftrightarrow K_4$, and $\Omega_{\mathcal{T}(3)}(K_4)$ is 4-chromatic by the result of [10] discussed in Section 2.

4.3 Compositions and Chains of Functors

For odd integers m, n, we consider the functor $L_n^m = \Gamma_{\mathcal{T}(m)} \circ \Lambda_{\mathcal{T}(n)}$ and its right adjoint $R_m^n = \Gamma_{\mathcal{T}(n)} \circ \Omega_{\mathcal{T}(m)}$. In particular, the circular complete graphs $K_{s/r}$ with s an odd multiple of 3 are the graphs $R_n^m(K_3)$ ($\simeq L_n^m(K_3)$) such that $m < 3n$. More generally, for any odd cycle C_k with $k \geq \frac{m}{n}$, we have $L_n^m(C_{2k+1}) \leftrightarrow R_n^m(C_{2k+1}) \leftrightarrow K_{s/r}$, where $s = nk$ and $r = \frac{nk-m}{2}$.

As noted in [13], the circular chromatic number of G can therefore be expressed in terms of the infimum of the values $\frac{m}{n}$ such that $G \to R_m^n(K_3)$. Since R_m^n is the thin right adjoint of L_m^n, we can look instead at the values $\frac{m}{n}$ such that $L_m^n(G)$ admits a homomorphism to K_3. In this characterization, the "base graph" K_3 could be replaced by any of the graphs

$K_{s/r}$ with $2 < \frac{s}{r} < 4$, with an appropriate change of parameters. However, replacing K_3 by K_4 or other graphs yields new graph invariants, defined by homomorphisms into chains of graphs which mimic the circular complete graphs.

At the functorial level, we can order thin functors in an obvious way, putting $F \leq F'$ if $F(G) \rightarrow F'(G)$ for every $G \in \mathcal{G}$.

Proposition 4.7 $L_n^m \leq L_{n'}^{m'}$ *if and only if* $\frac{m}{n} \leq \frac{m'}{n'}$ *and similarly* $R_n^m \leq R_{n'}^{m'}$ *if and only if* $\frac{m}{n} \leq \frac{m'}{n'}$.

Proof The implications $\frac{m}{n} \leq \frac{m'}{n'} \Rightarrow L_n^m \leq L_{n'}^{m'}$ and $R_n^m \leq R_{n'}^{m'}$ are proved in [13]. The converse implications are witnessed by suitably chosen circular complete graphs. □

Up to equivalence, we can assimilate L_n^m and R_n^m to functors $L_{m/n}$ and $R_{m/n}$ corresponding to the rational parameter $\frac{m}{n}$. These are two chains of functors isomorphic to the positive rationals with odd numerator and denominator. This order is in fact isomorphic to the rationals, but our specific labelling leaves holes. In particular, the adjoint functors $L_{1/2}$ and $R_{2/1}$ are not defined. We next see that this specific pair of holes can be filled by Pultr functors.

4.4 A Limit Functor

Consider the Pultr template $\mathcal{T}(2) = (\overline{P}_1, P_1 \square P_2, \epsilon_1, \epsilon_2)$ where P_1 and P_2 are path with vertex-sets $\{0, 1\}$ and $\{0, 1, 2\}$ respectively, $\square$ is the Cartesian product of Section 2, and $(\epsilon_1(0), \epsilon_1(1), \epsilon_2(0), \epsilon_2(1)) = ((0, 0), (1, 0), (1, 2), (0, 2))$. Having $\overline{P}_1$ rather than P_1 as first coordinate of the template makes a difference only for graphs with no edges. In all other cases, we could replace $\overline{P}_1$ by P_1 and get an equivalent functor.

Proposition 4.8 *For all* $G \in \mathcal{G}$*, we have* $L_n^m(G) \rightarrow \Lambda_{\mathcal{T}(2)}(G)$ *when* $\frac{m}{n} < \frac{1}{2}$*, and* $\Lambda_{\mathcal{T}(2)}(G) \rightarrow L_n^m(G)$ *when* $\frac{m}{n} > \frac{1}{2}$*; as well as* $R_m^n(G) \rightarrow \Gamma_{\mathcal{T}(2)}(G)$ *when* $\frac{n}{m} < 2$*, and* $\Gamma_{\mathcal{T}(2)}(G) \rightarrow R_m^n(G)$ *when* $\frac{n}{m} > 2$*.*

We omit the proof, which is similar to many proofs involving the functors associated with the templates $\mathcal{T}(2k + 1)$. The main point is that $\Lambda_{\mathcal{T}(2)}$ and $\Gamma_{\mathcal{T}(2)}$ fulfill the role of $L_{1/2}$ and $R_{2/1}$. It is not clear which of the other rational holes in the chains $\{L_{m/n}\}_{m,n}$ odd and $\{R_{n/m}\}_{n,m}$ odd can be filled, and how. It can be shown that the irrational holes cannot be filled by finite constructions. Thus, a workable theory of convergence for sequences of thin functors remains to be developed.

One interesting aspect of the template $\mathcal{T}(2)$ is that $\Gamma_{\mathcal{T}(2)}$ has a *partial* right adjoint. Let $\Omega_{\mathcal{T}(2)}(G)$ be the graph defined as follows.

- The vertices of $\Omega_{\mathcal{T}(2)}(G)$ are the ordered pairs $(A, B) \in (\mathcal{P}(V(G)))^2$ such that every vertex of A is adjacent to every vertex of B (i.e., A and B are *completely joined*).
- The edges of $\Omega_{\mathcal{T}(2)}(G)$ are the pairs $\{(A, B), (C, D)\}$ such that A and C are completely joined, B and D are completely joined, $A \cap D \neq \emptyset$, and $B \cap C \neq \emptyset$.

Proposition 4.9 (i) *If* $\Gamma_{\mathcal{T}(2)}(G) \rightarrow H$*, then* $G \rightarrow \Omega_{\mathcal{T}(2)}(H)$*;*
(ii) *if* $G \rightarrow \Omega_{\mathcal{T}(2)}(K_3)$*, then* $\Gamma_{\mathcal{T}(2)}(G) \rightarrow K_3$*;*
(iii) *however,* $K_6 \rightarrow \Gamma_{\mathcal{T}(2)}(\Omega_{\mathcal{T}(2)}(K_4)) \not\rightarrow K_4$*, hence* $\Omega_{\mathcal{T}(2)}$ *is not a right adjoint of* $\Gamma_{\mathcal{T}(2)}$*.*

We again omit the standard proof for the sake of briefness. We note that the result allows the derivation of the multiplicativity of $\Omega_{\mathcal{T}(2)}(K_3) = K_{12/5}$ directly from that of K_3, instead of through density arguments as in the proof of Proposition 4.5. The range of graphs on which $\Omega_{\mathcal{T}(2)}$ acts as a right adjoint of $\Gamma_{\mathcal{T}(2)}$ is not precisely known, but it extends to the odd cycles and allows to reprove the multiplicativity of some circular complete graphs.

In short, adjoint functors allow the extension of the proof of multiplicativity of K_3 to the circular complete graphs $K_{s/r}, 2 < \frac{s}{r} < 4$, sometimes in many ways, using compositions, limits and partial right adjoints. However, the multiplicativity of K_4, the "next case of Hedetniemi's conjecture" remains open. We now present a difficulty in using the adjoint functors method to derive the multiplicativity of other complete graphs from that of K_3.

4.5 Obstacle: A Stronger Form of Multiplicativity

El-Zahar and Sauer actually proved a stronger form of Theorem 3.1:

Theorem 4.10 ([5]) *Let G and H be connected graphs containing odd cycles G', H' respectively. Let L be the subgraph of $G \times H$ induced by $V(G \times H') \cup V(G' \times H)$. If $L \to K_3$, then $G \to K_3$ or $H \to K_3$.*

They conjectured that a similar strong form of multiplicativity would hold for all complete graphs. However this conjecture was refuted in [25]. The counterexamples are of interest to us. Let $G_{m,n}$ be the graph obtained by identifying one end of P_n to a vertex of $K_{7/2}$ and the other end to a vertex of K_m. For $m > 4$, $G_{m,n} \times G_{m,n} \leftrightarrow G_{m,n} \not\to K_4$. Let $L_{m,n}$ be the subgraph of $G_{m,n} \times G_{m,n}$ induced by $V(G_{m,n} \times K_{7/2}) \cup V(K_{7/2} \times G_{m,n})$.

Theorem 4.11 ([25]) *For $n \geq 3$ and m arbitrary, $L_{m,n} \to K_4$.*

Now suppose that for some $\mathcal{T}$, $\Gamma_{\mathcal{T}}$ admits a right adjoint $\Omega_{\mathcal{T}}$ such that $\Omega_{\mathcal{T}}(K_3) = K_4$. Then for $n \geq 3, m \geq 5$, we have $G_{m,n} \not\to K_4$ whence $\Gamma_{\mathcal{T}}(G_{m,n}) \not\to K_3$. However from $L_{m,n} \to K_4 = \Omega_{\mathcal{T}}(K_3)$, we get $\Gamma_{\mathcal{T}}(L_{m,n}) \to K_3$. However $\Gamma_{\mathcal{T}}(L_{m,n})$ contains a copy of $\Gamma_{\mathcal{T}}(G_{m,n}) \times \Gamma_{\mathcal{T}}(K_{7/2}) \cup \Gamma_{\mathcal{T}}(K_{7/2}) \times \Gamma_{\mathcal{T}}(G_{m,n})$. Hence by Theorem 4.10, since $\Gamma_{\mathcal{T}}(G_{m,n}) \not\to K_3$, we must have that $\Gamma_{\mathcal{T}}(K_{7/2})$ is bipartite. It can be shown that this is incompatible with the fact that $\Gamma_{\mathcal{T}}$ admits a right adjoint.

In short, the right adjoints we know seem to transfer a property that is stronger than multiplicativity, and this seems to be an obstacle in deriving the multiplicativity of other complete graphs from that of K_3.

We note that Wrochna [26] has recently proved that all square-free graphs are multiplicative. The functors detailed here can therefore be used to derive from this the multiplicativity of a larger class of graphs. However, Wrochna states that his method also yields the stronger form of multiplicativity for square-free graphs. Hence the multiplicativity of larger complete graphs still seems out of reach.

5 Concluding Comments

We have shown how some important facts concerning the weak Hedetniemi conjecture and the characterization of multiplicative graphs are connected to adjoint functors in the thin category of graphs. In addition, the multi-factor version of the weak Hedetniemi conjecture can also be tackled through adjoint functors, as shown in [6]. The characterization of central

Pultr functors which admit right adjoints is initiated in [8]. However, as we have seen, further developments may depend on functors that escape the mould of Pultr functors. In short, the tools devised so far in this line of study are categorial in nature, and seem to have reached their limit in their present form. Perhaps the contribution of category theory to Hedetniemi's conjecture would be to see the way to sharpen these tools.

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