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A Smashing Subcategory of the Homotopy Category of Gorenstein Projective Modules

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Abstract Let A be an artin algebra of finite CM-type. In this paper, we show that if A is virtually Gorenstein, then the homotopy category of Gorenstein projective A-modules, denote $K(A\text{-}\mathcal{GP})$, is always compactly generated. Based on this result, it will be proved that the homotopy category of projective A-modules, denote $K(A\text{-}\mathcal{P})$, is a smashing subcategory of $K(A\text{-}\mathcal{GP})$ and the corresponding Verdier quotient is also compactly generated. Furthermore, it turns out that the inclusion functor $i : K(A\text{-}\mathcal{P}) \to K(A\text{-}\mathcal{GP})$ induces a recollement of $K(A\text{-}\mathcal{GP})$.

Keywords Gorenstein projective modules · Compactly generated homotopy categories · Smashing subcategory · Recollements

1 Introduction

Let $\mathcal{X}$ be a class of left modules over an associative ring R which is closed under set-indexed coproducts and direct summands. Holm and Jørgensen [13] study the general question of when the homotopy category $K(\mathcal{X})$ of $\mathcal{X}$ is compactly generated. They give a number of sufficient conditions on R and $\mathcal{X}$ which ensure that $K(\mathcal{X})$ is compactly generated.

Let A be an artin algebra and A-Mod the category of A-modules. Denote by $A\text{-}\mathcal{P}$ the full subcategory of projective A-modules, $A\text{-}\mathcal{GP}$ the full subcategory of

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Gorenstein projective A-modules, and $A\text{-}\mathcal{G}proj$ the full subcategory of all finitely-generated Gorenstein projective modules. As is well known, the homotopy category $K(A\text{-}\mathcal{P})$ is compactly generated [15, Theorem 2.4].

Gorenstein projective modules and algebras of finite Cohen–Macaulay type receive a lot of attention (See e.g. [1, 4–6, 8–10, 12, 14, 16, 17, 19]). Recall from [4, 6] that an artin algebra A is of finite Cohen–Macaulay type (simply, CM-type) if there are only finitely many isomorphism classes of finitely-generated indecomposable Gorenstein projective A-modules. We are interested in the compact generatedness of the homotopy category $K(A\text{-}\mathcal{GP})$ of an artin algebra A of finite CM-type.

In Section 2, we first show that if A is virtually Gorenstein of finite CM-type, then $K(A\text{-}\mathcal{GP})$ is compactly generated. Next, based on this result, we show that $K(A\text{-}\mathcal{P})$ is a smashing subcategory of $K(A\text{-}\mathcal{GP})$ and the Verdier quotient $K(A\text{-}\mathcal{GP})/K(A\text{-}\mathcal{P})$ is also compactly generated.

The concept of recollement goes back to the work of Beilinson et al. [2]. In Section 3, we show the existence of recollements of the homotopy category $K(A\text{-}\mathcal{GP})$.

2 Conditions for Compact Generatedness

Our aim in this section is to show that $K(A\text{-}\mathcal{GP})$ is compactly generated provided A is virtually Gorenstein of finite CM-type. So based on the result of Bruns and Herzog [6, Proposition 2.11], and the result of Jørgensen [16], $K(A\text{-}\mathcal{P})$ is a smashing subcategory of $K(A\text{-}\mathcal{GP})$ and the Verdier quotient $K(A\text{-}\mathcal{GP})/K(A\text{-}\mathcal{P})$ is also compactly generated.

Our strategy for the compact generatedness of $K(A\text{-}\mathcal{GP})$ is to give sufficient conditions on A. We will use the following lemma.

Lemma 2.1 [4, Theorem 4.10] *Let A be an artin algebra. Then A is virtually Gorenstein of finite CM-type if and only if any Gorenstein projective A-module is a direct sum of finitely-generated modules.*

Now we are ready to state and prove our first main theorem in this section.

Theorem 2.2 *Let A be a virtually Gorenstein artin algebra of finite CM-type. Then $K(A\text{-}\mathcal{GP})$ is a compactly generated triangulated category.*

Proof Since A is virtually Gorenstein of finite CM-type, we get from Lemma 2.1 that $A\text{-}\mathcal{GP} = \mathrm{Add}(A\text{-}\mathcal{G}proj)$ which means that $A\text{-}\mathcal{GP}$ is contravariantly finite in A-Mod, and also each Gorenstein projective module is pure projective which means that every pure exact sequence of modules from $A\text{-}\mathcal{GP}$ is split exact. This implies that $K(A\text{-}\mathcal{GP})$ is a compactly generated triangulated category by [13, Theorem 3.1]. □

Recall from [11] that a complex $X^\bullet$ is $A\text{-}\mathcal{GP}$-acyclic if the induced complex $\mathrm{Hom}_A(G, X^\bullet)$ is acyclic for each module in $A\text{-}\mathcal{GP}$, and the Gorenstein derived category $D_{gp}(A\text{-Mod})$ of an artin algebra A is defined to be the Verdier quotient of the homotopy category $K(A\text{-mod})$ with respect to the thick subcategory $K_{gpac}(A\text{-Mod})$ which consists of all $A\text{-}\mathcal{GP}$-acyclic complexes.

Corollary 2.3 *Let A be a Gorenstein artin algebra of finite CM-type. Then $D_{gp}(A\text{-Mod})$ is compactly generated.*

Proof By the assumption on A, we see from [3, Corollary 8.3 and Corollary 8.5] that A satisfies the conditions on Theorem 2.2. Hence we get that $K(A\text{-}\mathcal{GP})$ is a compactly generated triangulated category. By [7, Proposition 3.5] there is a triangle-equivalence $D_{gp}(A\text{-Mod}) \cong K(A\text{-}\mathcal{GP})$. This implies that $D_{gp}(A\text{-Mod})$ is compactly generated. □

For our second main theorem we need a definition and some lemmas.

Recall from [18] that a full subcategory $\mathcal{B}$ of a compactly generated triangulated category $\mathcal{T}$ is smashing if the inclusion $\mathcal{B} \to \mathcal{T}$ has a right adjoint which preserves coproducts.

Lemma 2.4 [18, Lemma 4.1] *Let $\mathcal{B}$ be a smashing subcategory of a compactly generated triangulated category $\mathcal{T}$. Then $\mathcal{T}/\mathcal{B}$ is a compactly generated triangulated category.*

Lemma 2.5 [5, Proposition 2.11] *Let $\mathcal{T}$ and $\mathcal{T}'$ be compactly generated triangulated categories, and let $F : \mathcal{T} \to \mathcal{T}'$ be a fully faithful triangle functor which preserves coproducts and compact objects. Then F admits a right adjoint $G : \mathcal{T}' \to \mathcal{T}$ which preserves coproducts.*

So in view of the above lemmas, we have the following theorem.

Theorem 2.6 *Let A be a virtually Gorenstein artin algebra of finite CM-type. Then $K(A\text{-}\mathcal{P})$ is a smashing subcategory of $K(A\text{-}\mathcal{GP})$. Moreover, $K(A\text{-}\mathcal{GP})/K(A\text{-}\mathcal{P})$ is a compactly generated triangulated category.*

Proof By the assumpotion on A, we get from Theorem 2.2 that $K(A\text{-}\mathcal{GP})$ is compactly generated, and from [15, Theorem 2.4] that $K(A\text{-}\mathcal{P})$ is compactly generated and each compact object $P^\bullet$ is exactly the upper bounded complex of finitely-generated projective modules. Let $i : K(A\text{-}\mathcal{P}) \to K(A\text{-}\mathcal{GP})$ be the inclusion functor. Note that i naturally preserves coproducts. Let $\{G_i^\bullet\}_{i\in I}$ be any family objects in $K(A\text{-}\mathcal{GP})$. Then we have $\mathrm{Hom}_{K(A\text{-}\mathcal{GP})}(iP^\bullet, \coprod_{i\in I} G_i^\bullet) = \mathrm{Hom}_{K(A\text{-}\mathcal{GP})}(P^\bullet, \coprod_{i\in I} G_i^\bullet) \cong \coprod_{i\in I} \mathrm{Hom}_{K(A\text{-}\mathcal{GP})}(P^\bullet, G_i^\bullet) = \coprod_{i\in I} \mathrm{Hom}_{K(A\text{-}\mathcal{GP})}(iP^\bullet, G_i^\bullet)$. This implies that i preserves compact objects. Hence by Lemma 2.5 we get that i admits a right adjoint $R : K(A\text{-}\mathcal{GP}) \to K(A\text{-}\mathcal{P})$ which preserves coproducts. This means $K(A\text{-}\mathcal{P})$ is a smashing subcategory of $K(A\text{-}\mathcal{GP})$. This implies by Lemma 2.4 that $K(A\text{-}\mathcal{GP})/K(A\text{-}\mathcal{P})$ is a compactly generated triangulated category. □

3 Recollements for the Homotopy Category $K(A\text{-}\mathcal{GP})$

In this section, let A be an artin algebra. Based on the compact generatedness of the full subcategory $K(A\text{-}\mathcal{P})$ of $K(A\text{-}\mathcal{GP})$, we will apply the arguments of Neeman to prove the existence of a recollement of $K(A\text{-}\mathcal{GP})$.

Lemma 3.1 [21, Theorem 4.1], [22, Theorem 8.6.1] *Let* $F : \mathcal{T} \to \mathcal{T}^{'}$ *be a triangle functor between triangulated categories* $\mathcal{T}$ *and* $\mathcal{T}^{'}$, *where* $\mathcal{T}$ *is compactly generated.*

(1) *F admits a right adjoint if and only if it preserves all coproducts.*
(2) *F admits a left adjoint if and only if it preserves all products.*

Theorem 3.2 *Let A be an artin algebra. Then the inclusion functor* $i : K(A\text{-}\mathcal{P}) \to K(A\text{-}\mathcal{GP})$ *induces a recollement of the form*

$$K(A\text{-}\mathcal{P}) \underset{R}{\overset{\longleftarrow}{\underset{\longleftarrow}{\xrightarrow{i}}}} K(A\text{-}\mathcal{GP}) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} \operatorname{Ker} R$$

such that $\operatorname{Ker} R \cong K(A\text{-}\mathcal{GP})/K(A\text{-}\mathcal{P})$ *as triangulated categories.*

Proof Since A is an artin algebra, it follows from [15, Theorem 2.4] that $K(A\text{-}\mathcal{P})$ is compactly generated. Note that the inclusion functor i naturally preserves all coproducts and products. Then i admits a right adjoint R, also a left adjoint. Hence by [20, Theorem 2.2] we have a recollement of the form

$$K(A\text{-}\mathcal{P}) \underset{R}{\overset{\longleftarrow}{\underset{\longleftarrow}{\xrightarrow{i}}}} K(A\text{-}\mathcal{GP}) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} \operatorname{Ker} R$$

such that $\operatorname{Ker} R \cong K(A\text{-}\mathcal{GP})/K(A\text{-}\mathcal{P})$ as triangulated categories. □

So in view of the above theorem, we have the following result. Let us begin by recalling some definitions.

Let $\mathcal{T}$ be a triangulated category with the suspension functor Σ. Recall from [5, Section 2] that a torsion pair in $\mathcal{T}$ is a pair of strict full subcategories $(\mathcal{X}.\mathcal{Y})$ of $\mathcal{T}$ satisfying the following conditions: (1) $\mathcal{T}(\mathcal{X}, \mathcal{Y}) = 0$; (2) $\Sigma(\mathcal{X}) \subseteq \mathcal{X}$ and $\Sigma^{-1}(\mathcal{Y}) \subseteq \mathcal{Y}$; (3) For any $T \in \mathcal{T}$ there exists a triangle $X_T \xrightarrow{f_T} T \xrightarrow{g^T} Y^T \xrightarrow{h^T} \Sigma(X_T)$. Then $\mathcal{X}$ is called a torsion class and $\mathcal{Y}$ is called a torsion-free class. A torsion, torsion-free triple, TTF-triple for short, in $\mathcal{T}$ is a triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of full subcategories of $\mathcal{T}$ such that the pairs $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are torsion pairs.

Now we give a TTF-triple in $K(A\text{-}\mathcal{GP})$.

Corollary 3.3 *Let A be an artin algebra. Then there exists a TTF-triple* $(K(A\text{-}\mathcal{P}), \operatorname{Ker} R, (\operatorname{Ker} R)^{\perp})$ *in* $K(A\text{-}\mathcal{GP})$.

Proof By Theorem 3.2 we have the recollement of the form

$$K(A\text{-}\mathcal{P}) \underset{R}{\overset{\longleftarrow}{\underset{\longleftarrow}{\xrightarrow{i}}}} K(A\text{-}\mathcal{GP}) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} \operatorname{Ker} R.$$

Hence by [20, Theorem 2.2] we get that $(K(A\text{-}\mathcal{P}), \operatorname{Ker} R)$ and $(\operatorname{Ker} R, (\operatorname{Ker} R)^{\perp})$ are two torsion pairs in $K(A\text{-}\mathcal{GP})$. This means $K(A\text{-}\mathcal{GP})$ has a TTF-triple $(K(A\text{-}\mathcal{P}), \operatorname{Ker} R, (\operatorname{Ker} R)^{\perp})$. □

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