On the consistency of two-phase local/nonlocal piezoelectric integral model*

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Abstract In this paper, we propose general strain- and stress-driven two-phase local/nonlocal piezoelectric integral models, which can distinguish the difference of nonlocal effects on the elastic and piezoelectric behaviors of nanostructures. The nonlocal piezoelectric model is transformed from integral to an equivalent differential form with four constitutive boundary conditions due to the difficulty in solving integro-differential equations directly. The nonlocal piezoelectric integral models are used to model the static bending of the Euler-Bernoulli piezoelectric beam on the assumption that the nonlocal elastic and piezoelectric parameters are coincident with each other. The governing differential equations as well as constitutive and standard boundary conditions are deduced. It is found that purely strain- and stress-driven nonlocal piezoelectric integral models are ill-posed, because the total number of differential orders for governing equations is less than that of boundary conditions. Meanwhile, the traditional nonlocal piezoelectric differential model would lead to inconsistent bending response for Euler-Bernoulli piezoelectric beam under different boundary and loading conditions. Several nominal variables are introduced to normalize the governing equations and boundary conditions, and the general differential quadrature method (GDQM) is used to obtain the numerical solutions. The results from current models are validated against results in the literature. It is clearly established that a consistent softening and toughening effects can be obtained for static bending of the Euler-Bernoulli beam based on the general strain- and stress-driven local/nonlocal piezoelectric integral models, respectively.

Key words nonlocal piezoelectric integral model, softening effect, toughening effect, general differential quadrature method (GDQM)

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1 Introduction

Since Pan and his co-authors^[1] first reported ZnO piezoelectric nanowires in 2001, piezoelectric nanomaterials and nanostructures have received close attention from the modern sci-

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ence and technology. Piezoelectric nanostructures were widely used in chemical sensors^[2], micro/nano electromechanical^[3], nanogenerators^[4], piezoelectric field effect transistor^[5] and so on due to the outstanding mechanical and electric properties. The experimental tests and atomic simulations^[6–7] showed that the piezoelectric nanostructures have a strong size effect at the nanoscale and exhibit different behaviors from the macro-scale.

Several high-order continuum models such as flexoelectric^[8–9], nonlocal piezoelectric differential model^[10–11], strain gradient theory^[12–13], and couple stress theory^[14–16] are developed to capture accurately the size-dependent behavior of nanoscale piezoelectric structures, since the classic piezoelectric model failed to do so. It is worth noting that the nonlocal piezoelectric differential model, due to the simplicity, has been widely used to address the size-dependent bending, buckling, and vibration of nano-beams, tubes, plates, sheets, and shells^[17–26].

However, several studies showed that there were some inconsistent results while using the nonlocal elastic differential model to study the bar under a tensile load^[27], the cantilever beam under a concentrated load at the end^[28–30], and the clamped beams under a uniformly distributed load^[31]. Therefore, the original nonlocal model in integral form restarts to attract the attention from the scientific community. Romano et al.^[32] found that the transformation for nonlocal elastic model from integral to differential forms should be equipped with two extra constraints from integral constitutive relation, and further pointed out the purely nonlocal integral model would lead to an ill-posed problem, since there was a conflict between the two constitutive constraints and the equilibrium equations^[33].

Wang et al.^[34–35] transformed the two-phase local/nonlocal integral model proposed initially by Eringen^[36] into the equivalent differential form with two constitutive constraints and used it to study the static bending of straight Euler-Bernoulli and Timoshenko beams under different boundary and loading conditions, respectively. With the help of Laplace transform, Zhang et al.^[37] and Zhang and Qing^[38] utilized the two-phase local/nonlocal integral model directly to study the static bending of curved Euler-Bernoulli and Timoshenko beams. Meng et al.^[39] deduced the semi-analytic solution for static bending Euler-Bernoulli beam with the axial force. A consistent softening effect was obtained while using the two-phase strain-driven local/nonlocal integral model.

In order to overcome the ill-posedness of the purely strain-driven nonlocal integral model, Romano and Barretta^[33] proposed a stress-driven nonlocal integral model by swapping the interpretation of source and output fields in Eringen's nonlocal integral model. With the application of stress-derived nonlocal integral model, Oskouie et al.^[40–41] utilized the Ritz-method to bypass constitutive constraints and studied the bending response of Euler-Bernoulli and Timoshenko beams. Based on the Laplace transform, Zhang et al.^[42] deduced the exact and asymptotic bending analysis of straight Euler-Bernoulli and Timoshenko beams, and Zhang et al.^[43] derived the exact solutions for bending of curved functionally graded Timoshenko beam. Barretta et al.^[44–45] used the stress-driven based purely nonlocal and two-phase local/nonlocal integral models to study the static bending of Timoshenko beam, respectively. A consistent toughening effect was obtained while using two-phase stress-driven local/nonlocal integral model.

To the best knowledge of the authors, there is no literature about the application of the nonlocal integral model to the nanoscale piezoelectric structures. The plan for this paper is as follows. In Section 2, we present the Helmholtz-kernel based two-phase local/nonlocal piezoelectric integral models, and transform it from integral equivalently differential forms with constitutive boundary conditions. In Section 3, the differential governing equation and standard boundary conditions for a piezoelectric Euler-Bernoulli beam are derived, and the equivalently nonlocal differential constitutive relations with constitutive boundary conditions are presented. In Section 4, nominal variables are introduced to normalize the governing equations as well as constitutive and standard boundary conditions, and the general differential quadrature method is used to solve the differential equations numerically. In Section 5, several benchmark exam-

ples for the nonlocal piezoelectric Euler-Bernoulli beam under different boundary and loading conditions are illustrated numerically and discussed.

2 Two-phase local/nonlocal piezoelectric integral model

2.1 Nonlocal piezoelectric integral model

According to the classic piezoelectric model^[46], the constitutive relations in stress-charge and strain-voltage forms can be expressed respectively as

$$S_{ij} = c_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \quad \Delta_i = e_{ikl}\varepsilon_{kl} + \mu_{ik}E_k, \quad (1)$$

$$\varepsilon_{kl} = s_{ijkl}S_{ij} + g_{kij}\Delta_i, \quad E_i = -g_{ikl}S_{kl} + \beta_{ik}\Delta_k, \quad (2)$$

where S_{ij} and Δ_i are components for the local stress tensor and the electric displacement vector, respectively. c_{ijkl} , e_{kij} , and μ_{ik} are the elastic, piezoelectric, and dielectric constants, respectively. s_{ijkl} , g_{kij} , and β_{ik} are the elastic compliance constants, alternative forms of the piezoelectric constant, and the permittivity constants, respectively. ε_{kl} and E_i are components of strain tensor and electric field vector, which are defined as

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\varphi_{,i}, \quad (3)$$

where u_i and φ are the displacement component and electric potential, respectively.

From the idea of Romano and Barretta^[33], one-dimensional general strain- and stress-driven two-phase local/nonlocal piezoelectric integral models are expressed respectively as

$$\left\{ \begin{array}{l} \sigma_{ij} = c_{ijkl} \left((1 - \xi_1)\varepsilon_{kl} + \frac{\xi_1}{2\kappa_1} \int_a^b \varepsilon_{kl}(\eta) e^{-|x-\eta|/\kappa_1} d\eta \right) \\ \quad - e_{kij} \left((1 - \xi_2)E_k + \frac{\xi_2}{2\kappa_2} \int_a^b E_k(\eta) e^{-|x-\eta|/\kappa_2} d\eta \right), \\ D_i = e_{ikl} \left((1 - \xi_1)\varepsilon_{kl} + \frac{\xi_1}{2\kappa_1} \int_a^b \varepsilon_{kl}(\eta) e^{-|x-\eta|/\kappa_1} d\eta \right) \\ \quad + \mu_{ik} \left((1 - \xi_2)E_k + \frac{\xi_2}{2\kappa_2} \int_a^b E_k(\eta) e^{-|x-\eta|/\kappa_2} d\eta \right), \end{array} \right. \quad (4)$$

and

$$\left\{ \begin{array}{l} \varepsilon_{kl} = s_{ijkl} \left((1 - \xi_1)S_{ij} + \frac{\xi_1}{2\kappa_1} \int_a^b S_{ij}(\eta) e^{-|x-\eta|/\kappa_1} d\eta \right) \\ \quad + g_{kij} \left((1 - \xi_2)\Delta_i + \frac{\xi_2}{2\kappa_2} \int_a^b \Delta_i(\eta) e^{-|x-\eta|/\kappa_2} d\eta \right), \\ E_i = -g_{ikl} \left((1 - \xi_1)S_{kl} + \frac{\xi_1}{2\kappa_1} \int_a^b S_{kl}(\eta) e^{-|x-\eta|/\kappa_1} d\eta \right) \\ \quad + \beta_{ik} \left((1 - \xi_2)\Delta_k + \frac{\xi_2}{2\kappa_2} \int_a^b \Delta_k(\eta) e^{-|x-\eta|/\kappa_2} d\eta \right), \end{array} \right. \quad (5)$$

where σ_{ij} and D_i are components for the nonlocal stress tensor and the electric displacement vector, and κ_i is the nonlocal parameter. ξ_i is the volume fraction of nonlocal effect. It degenerates into the classic piezoelectric model when $\xi_i = 0$, and the model becomes a purely nonlocal integral model when $\xi_i = 1$.

2.2 Transformation of nonlocal piezoelectric model from integral to differential forms

It is well-known that the integro-differential equations are generally more difficult to solve than the differential equations. The nonlocal integral equations (4) and (5) are transformed into equivalent differential forms with corresponding constitutive boundary conditions in the following.

Proposition 1 *The nonlocal piezoelectric integral model can be generally expressed as*

$$F(x) = \left((1 - \xi_1)G_1(x) + \frac{\xi_1}{2\kappa_1} \int_a^b G_1(\eta) e^{-|x-\eta|/\kappa_1} d\eta \right) + \left((1 - \xi_2)G_2(x) + \frac{\xi_2}{2\kappa_2} \int_a^b G_2(\eta) e^{-|x-\eta|/\kappa_2} d\eta \right), \quad (6)$$

which can be transformed into an equivalent differential equation as

$$\begin{aligned} & \kappa_1^2 \kappa_2^2 F'''' - (\kappa_1^2 + \kappa_2^2) F'' + F \\ &= G_1 - \kappa_2^2 G_1'' - \kappa_1^2 (1 - \xi_1) (G_1'' - \kappa_2^2 G_1''') + G_2 - \kappa_1^2 G_2'' - \kappa_2^2 (1 - \xi_2) (G_2'' - \kappa_1^2 G_2''') \end{aligned} \quad (7)$$

with constitutive boundary conditions at $x = a$ and $x = b$

$$\left\{ \begin{aligned} & F - (\kappa_1 + \kappa_2) F' + \kappa_1 \kappa_2 F'' \\ &= (1 - \xi_1) (G_1 - (\kappa_1 + \kappa_2) G_1' + \kappa_1 \kappa_2 G_1'') \\ &\quad + (1 - \xi_2) (G_2 - (\kappa_1 + \kappa_2) G_2' + \kappa_1 \kappa_2 G_2'') - \xi_1 \kappa_2 G_1 / \kappa_1 - \xi_2 \kappa_1 G_2 / \kappa_2, \\ & (\kappa_1 + \kappa_2) F - (\kappa_1^2 + \kappa_1 \kappa_2 + \kappa_2^2) F' + \kappa_1^2 \kappa_2^2 F'''' \\ &= (1 - \xi_1) ((\kappa_1 + \kappa_2) G_1 - \kappa_1 (\kappa_1 + \kappa_2) G_1' + \kappa_1^2 \kappa_2^2 G_1''') + (1 - \xi_2) ((\kappa_1 + \kappa_2) G_2 \\ &\quad - \kappa_2 (\kappa_1 + \kappa_2) G_2' + \kappa_1^2 \kappa_2^2 G_2''') - \kappa_2^2 G_1' - \kappa_1^2 G_2', \end{aligned} \right. \quad (8)$$

$$\left\{ \begin{aligned} & F + (\kappa_1 + \kappa_2) F' + \kappa_1 \kappa_2 F'' \\ &= (1 - \xi_1) (G_1 + (\kappa_1 + \kappa_2) G_1' + \kappa_1 \kappa_2 G_1'') \\ &\quad + (1 - \xi_2) (G_2 + (\kappa_1 + \kappa_2) G_2' + \kappa_1 \kappa_2 G_2'') - \xi_1 \kappa_2 G_1 / \kappa_1 - \xi_2 \kappa_1 G_2 / \kappa_2, \\ & (\kappa_1 + \kappa_2) F + (\kappa_1^2 + \kappa_1 \kappa_2 + \kappa_2^2) F' - \kappa_1^2 \kappa_2^2 F'''' \\ &= (1 - \xi_1) ((\kappa_1 + \kappa_2) G_1 + \kappa_1 (\kappa_1 + \kappa_2) G_1' - \kappa_1^2 \kappa_2^2 G_1''') + (1 - \xi_2) ((\kappa_1 + \kappa_2) G_2 \\ &\quad + \kappa_2 (\kappa_1 + \kappa_2) G_2' - \kappa_1^2 \kappa_2^2 G_2''') + \kappa_2^2 G_1' + \kappa_1^2 G_2', \end{aligned} \right. \quad (9)$$

in which $(\cdot)' = \frac{d(\cdot)}{dx}$.

Proof Removing the absolute sign in Eq. (6), we get

$$\begin{aligned} F(x) &= (1 - \xi_1) G_1(x) + \frac{\xi_1}{2\kappa_1} \left(e^{-x/\kappa_1} \int_a^x G_1(\eta) e^{\eta/\kappa_1} d\eta + e^{x/\kappa_1} \int_x^b G_1(\eta) e^{-\eta/\kappa_1} d\eta \right) \\ &\quad + (1 - \xi_2) G_2(x) + \frac{\xi_2}{2\kappa_2} \left(e^{-x/\kappa_2} \int_a^x G_2(\eta) e^{\eta/\kappa_2} d\eta + e^{x/\kappa_2} \int_x^b G_2(\eta) e^{-\eta/\kappa_2} d\eta \right). \end{aligned} \quad (10)$$

Performing the first, the second, the third, and the fourth derivatives of Eq. (10), we obtain

$$\begin{aligned} 2\kappa_1^2 \kappa_2^2 F' &= 2\kappa_1^2 \kappa_2^2 (1 - \xi_1) G_1' - \kappa_2^2 \xi_1 \left(\int_a^x G_1 e^{(\eta-x)/\kappa_1} d\eta - \int_x^b G_1 e^{(x-\eta)/\kappa_1} d\eta \right) \\ &\quad + 2\kappa_1^2 \kappa_2^2 (1 - \xi_2) G_2' - \kappa_1^2 \xi_2 \left(\int_a^x G_2 e^{(\eta-x)/\kappa_2} d\eta - \int_x^b G_2 e^{(x-\eta)/\kappa_2} d\eta \right), \end{aligned} \quad (11)$$

$$2\kappa_1^3\kappa_2^3F'' = 2\kappa_1^3\kappa_2^3(1 - \xi_1)G_1'' + \kappa_2^3\xi_1\left(\int_a^b G_1e^{-|x-\eta|/\kappa_1}d\eta - 2\kappa_1G_1\right) \\ + 2\kappa_1^3\kappa_2^3(1 - \xi_2)G_2'' + \kappa_1^3\xi_2\left(\int_a^b G_2e^{-|x-\eta|/\kappa_2}d\eta - 2\kappa_2G_2\right), \quad (12)$$

$$2\kappa_1^4\kappa_2^4F''' = 2\kappa_1^4\kappa_2^4(1 - \xi_1)G_1''' - \kappa_2^4\xi_1\left(\int_a^x G_1e^{(\eta-x)/\kappa_1}d\eta - \int_x^b G_1e^{(x-\eta)/\kappa_1}d\eta + 2\kappa_1^2G_1'\right) \\ + 2\kappa_1^4\kappa_2^4(1 - \xi_2)G_2''' - \kappa_1^4\xi_2\left(\int_a^x G_2e^{(\eta-x)/\kappa_2}d\eta - \int_x^b G_2e^{(x-\eta)/\kappa_2}d\eta + 2\kappa_1^2G_2'\right), \quad (13)$$

$$2\kappa_1^5\kappa_2^5F'''' = 2\kappa_1^5\kappa_2^5(1 - \xi_1)G_1'''' + \kappa_2^5\xi_1\left(\int_a^b G_1e^{-|x-\eta|/\kappa_1}d\eta - \kappa_1G_1 - \kappa_1^3G_1''\right) \\ + 2\kappa_1^5\kappa_2^5(1 - \xi_2)G_2'''' + \kappa_1^5\xi_2\left(\int_a^b G_2e^{-|x-\eta|/\kappa_2}d\eta - \kappa_2G_2 - \kappa_2^3G_2''\right). \quad (14)$$

Combination of Eqs. (6), (10), and (12) gives differential equation (6).

Considering the boundary conditions for Eqs. (10)–(13) at $x = a$, we have

$$2\kappa_1\kappa_2F(a) = 2\kappa_1\kappa_2(1 - \xi_1)G_1(x) + \kappa_2\xi_1\int_a^b G_1(\eta)e^{(a-\eta)/\kappa_1}d\eta \\ + 2\kappa_1\kappa_2(1 - \xi_2)G_2(x) + \kappa_1\xi_2\int_a^b G_2(\eta)e^{(a-\eta)/\kappa_2}d\eta, \quad (15)$$

$$2\kappa_1^2\kappa_2^2F'(a) = 2\kappa_1^2\kappa_2^2(1 - \xi_1)G_1'(a) + \kappa_2^2\xi_1\int_a^b G_1e^{(a-\eta)/\kappa_1}d\eta \\ + 2\kappa_1^2\kappa_2^2(1 - \xi_2)G_2'(a) + \kappa_1^2\xi_2\int_a^b G_2e^{(a-\eta)/\kappa_2}d\eta, \quad (16)$$

$$2\kappa_1^3\kappa_2^3F''(a) = 2\kappa_1^3\kappa_2^3(1 - \xi_1)G_1''(a) + \kappa_2^3\xi_1\left(\int_a^b G_1e^{(a-\eta)/\kappa_1}d\eta - 2\kappa_1G_1(a)\right) \\ + 2\kappa_1^3\kappa_2^3(1 - \xi_2)G_2''(a) + \kappa_1^3\xi_2\left(\int_a^b G_2e^{(a-\eta)/\kappa_2}d\eta - 2\kappa_2G_2(a)\right), \quad (17)$$

$$2\kappa_1^4\kappa_2^4F'''(a) = 2\kappa_1^4\kappa_2^4(1 - \xi_1)G_1'''(a) + \kappa_2^4\xi_1\left(\int_a^b G_1e^{(a-\eta)/\kappa_1}d\eta - 2\kappa_1^2G_1'(a)\right) \\ + 2\kappa_1^4\kappa_2^4(1 - \xi_2)G_2'''(a) + \kappa_1^4\xi_2\left(\int_a^b G_2e^{(a-\eta)/\kappa_2}d\eta - 2\kappa_1^2G_2'(a)\right). \quad (18)$$

Calculating $\int_a^b G_1e^{(a-\eta)/\kappa_1}d\eta$ and $\int_a^b G_2e^{(a-\eta)/\kappa_2}d\eta$ from Eqs. (15) and (16) and substituting them into Eqs. (17) and (18), we obtain the constitutive boundary conditions at $x = a$ listed as Eq. (8). Similarly, we obtain the constitutive boundary conditions (9) at $x = b$.

Setting $\xi_1 = \xi_2 = \xi$ and $\kappa_1 = \kappa_2 = \kappa$, we can obtain the simplified nonlocal piezoelectric integral models. Combination of Eqs. (10)–(12) as well as boundary conditions at $x = a$, e.g., Eqs. (15)–(16), gives

$$F - \kappa^2F'' = G - \kappa^2(1 - \xi)G'', \quad (19)$$

$$F(a) - \kappa F'(a) = (1 - \xi)(G(a) - \kappa G'(a)), \quad (20)$$

where $G = G_1 + G_2$. Similarly, we can express the constitutive boundary conditions at $x = b$ as

$$F(b) + \kappa F'(b) = (1 - \xi)(G(b) + \kappa G'(b)). \quad (21)$$

The equivalent differential equation (19) and constitutive boundary conditions (20)–(21) have the same forms compared with the results of Ref. [34] and Ref. [32], respectively.

3 Static bending analysis of nonlocal piezoelectric Euler-Bernoulli nanobeam

3.1 Governing equation of Euler-Bernoulli beam

Figure 1 shows a piezoelectric nanobeam with length L and cross-sectional area $A = bh$ (thickness h and width b) under the action of a distributed load $q(x)$ as well as the concentrated force $\widehat{Q}$ and the bending moment $\widehat{M}$. The polarization direction of the piezoelectric structure is parallel to the positive z -axis. The displacement field for an Euler-Bernoulli beam theory is

$$u_x = -zw', \quad u_z = w(x). \quad (22)$$

Fig. 1 Schematic illustration of a piezoelectric nanobeam

According to Wang^[47], the electrical potential of piezoelectric beams, which should satisfy the Maxwell equation, can be assumed to be

$$\phi(x, z) = -\cos(\beta z)\phi(x) + 2zV_0/h, \quad (23)$$

where $\beta = \pi/h$, and $\phi(x)$ is the electric potential in the x -direction.

Combining Eqs. (3) with (22)–(23), we express the nonzero components for the strain and electric field as

$$\varepsilon_{11} = -zw'', \quad E_1 = \cos(\beta z)\phi', \quad E_3 = -\beta \sin(\beta z)\phi + 2V_0/h. \quad (24)$$

Combination of Eqs. (1) and (24) gives

$$\begin{cases} S_{11} = -c_{1111}zw'' + e_{311}(\beta \sin(\beta z)\phi - 2V_0/h), \\ \Delta_1 = \mu_{11}\cos(\beta z)\phi', \\ \Delta_3 = -e_{311}zw'' - \mu_{33}(\beta \sin(\beta z)\phi - 2V_0/h). \end{cases} \quad (25)$$

Therefore, the variation of the strain energy of the piezoelectric beams is given as

$$\delta \Pi_s = \int_0^L \int_A \sigma_{11} \delta \varepsilon_{11} - D_1 \delta E_1 - D_3 \delta E_3 dA dx. \quad (26)$$

Substituting Eq. (24) into Eq. (26), after a lengthy simplification, we get

$$\delta \Pi_s = M' \delta w|_0^L - M \delta w'|_0^L - P \delta \phi|_0^L + \int_0^L ((P' + \beta R)\delta \phi - M''\delta w) dx, \quad (27)$$

where

$$M = \int_A z \sigma_{11} dA, \quad P = \int_A \cos(\beta z) D_1 dA, \quad R = \int_A \sin(\beta z) D_3 dA. \quad (28)$$

The variation of external work δW_e due to external load is

$$\delta W_e = \int_0^L q \delta w dx + \widehat{M} \delta w'|_0^L + \widehat{Q} \delta w|_0^L, \quad (29)$$

where q is the distributed load, and $\widehat{M}$ and $\widehat{Q}$ are the bending moment and force applied at beam ends, respectively.

According to the minimum total potential energy principle, the government equations and the standard boundary conditions can be expressed as

$$P' + \beta R = 0, \quad (30a)$$

$$M'' + q = 0, \quad (30b)$$

$$(M' - \widehat{Q}) \delta w|_0^L = 0, \quad (M + \widehat{M}) \delta w'|_0^L = 0, \quad P \delta \phi|_0^L = 0. \quad (31)$$

3.2 Two-phase strain-driven local/nonlocal piezoelectric integral model

Based on simplified strain-driven local/nonlocal piezoelectric integral model, we can express relation between nonlocal stress/electric displacement and strain and electric filed as

$$\begin{cases} \sigma_{11} = (1 - \xi) S_{11}(x) + \frac{\xi}{2\kappa} \int_0^L S_{11}(\eta) e^{-|x-\eta|/\kappa} d\eta, \\ D_1 = (1 - \xi) \Delta_1(x) + \frac{\xi}{2\kappa} \int_0^L \Delta_1(\eta) e^{-|x-\eta|/\kappa} d\eta, \\ D_3(x) = (1 - \xi) \Delta_3(x) + \frac{\xi}{2\kappa} \int_0^L \Delta_3(\eta) e^{-|x-\eta|/\kappa} d\eta. \end{cases} \quad (32)$$

Combining Eqs. (28) and (32) and taking into account Eqs. (24)–(25), we obtain

$$\begin{cases} M = (1 - \xi)(e_{311}\beta I_2 \phi - c_{1111} I_1 w'') + \frac{\xi}{2\kappa} \int_a^b (e_{311}\beta I_2 \phi - c_{1111} I_1 w'') e^{-|x-\eta|/\kappa} d\eta, \\ 2P = (1 - \xi) \mu_{11} A \phi' + \frac{\xi}{2\kappa} \int_a^b \mu_{11} A \phi' e^{-|x-\eta|/\kappa} d\eta, \\ -2R = (1 - \xi)(2e_{311} I_2 w'' + \mu_{33} \beta A \phi) + \frac{\xi}{2\kappa} \int_a^b (2e_{311} I_2 w'' + \mu_{33} \beta A \phi) e^{-|x-\eta|/\kappa} d\eta, \end{cases} \quad (33)$$

where

$$I_1 = \int_A z^2 dA, \quad I_2 = \int_A z \sin(\beta z) dA. \quad (34)$$

According to Eqs. (19)–(21), the integral constitutive equations can be transformed equivalently into differential forms with the corresponding constitutive constraints

$$\begin{aligned} \kappa^2 M'' - M &= \kappa^2 (1 - \xi)(e_{311}\beta I_2 \phi'' - c_{1111} I_1 w''''') - (e_{311}\beta I_2 \phi - c_{1111} I_1 w''), \\ \kappa M'(0) - M(0) &= (1 - \xi)(e_{311}\beta I_2 (\kappa \phi'(0) - \phi(0)) - c_{1111} I_1 (\kappa w'''(0) - w''(0))), \end{aligned} \quad (35a)$$

$$\kappa M'(L) + M(L) = (1 - \xi)(e_{311}\beta I_2 (\kappa \phi'(L) + \phi(L)) - c_{1111} I_1 (\kappa w'''(L) + w''(L))), \quad (35b)$$

$$\begin{aligned} 2(\kappa^2 P'' - P) &= \mu_{11} A (\kappa^2 (1 - \xi) \phi''' - \phi'), \\ 2(\kappa P'(0) - P(0)) &= (1 - \xi) \mu_{11} A (\kappa \phi''(0) - \phi'(0)), \end{aligned} \quad (36a)$$

$$2(\kappa P'(L) + P(L)) = (1 - \xi) \mu_{11} A (\kappa \phi''(L) + \phi'(L)), \quad (36b)$$

$$\begin{aligned} 2(R - \kappa^2 R'') &= \kappa^2 (1 - \xi)(2e_{311} I_2 w'''' + \mu_{33} \beta A \phi'') - (2e_{311} I_2 w'' + \mu_{33} \beta A \phi), \\ -2(\kappa R'(0) - R(0)) &= (1 - \xi)(2e_{311} I_2 (\kappa w'''(0) - w''(0)) + \mu_{33} \beta A (\kappa \phi'(0) - \phi(0))), \end{aligned} \quad (37a)$$

$$-2(\kappa R'(L) + R(L)) = (1 - \xi)(2e_{311} I_2 (\kappa w'''(L) + w''(L)) + \mu_{33} \beta A (\kappa \phi'(L) + \phi(L))). \quad (37b)$$

Notice that Eqs. (35)–(37) turn traditional nonlocal piezoelectric models if the constitutive boundary conditions are neglected and $\xi = 1$.

Combining Eqs. (30a) with (36a) as well as Eqs. (30b) with (35a), we get

$$P = \mu_{11}A(\phi' - \kappa^2(1 - \xi)\phi''')/2 - \kappa^2\beta R', \quad (38)$$

$$M = (e_{311}\beta I_2\phi - c_{1111}I_1w'') - \kappa^2(1 - \xi)(e_{311}\beta I_2\phi'' - c_{1111}I_1w'''' - \kappa^2q). \quad (39)$$

Substituting P and M from Eqs. (38)–(39) into Eq. (30), we get

$$\begin{cases} (e_{311}\beta I_2\phi'' - c_{1111}I_1w'''' - \kappa^2(1 - \xi)(e_{311}\beta I_2\phi'''' - c_{1111}I_1w'''''' - \kappa^2q'' + q = 0, \\ \mu_{11}A(\phi'' - \kappa^2(1 - \xi)\phi'''' - \beta\kappa^2(1 - \xi)(2e_{311}I_2w'''' + \mu_{33}\beta A\phi'')) \\ + \beta(2e_{311}I_2w'' + \mu_{33}\beta A\phi) = 0. \end{cases} \quad (40)$$

3.3 Two-phase stress-driven local/nonlocal piezoelectric integral model

Based on the simplified stress-driven local/nonlocal piezoelectric integral model, one obtains

$$\begin{cases} S_{11}(x) = (1 - \xi)\sigma_{11}(x) + \frac{\xi}{2\kappa} \int_0^L \sigma_{11}(\eta) e^{-|x-\eta|/\kappa} d\eta, \\ \Delta_1(x) = (1 - \xi)D_1(x) + \frac{\xi}{2\kappa} \int_0^L D_1(\eta) e^{-|x-\eta|/\kappa} d\eta, \\ \Delta_3(x) = (1 - \xi)D_3(x) + \frac{\xi}{2\kappa} \int_0^L D_3(\eta) e^{-|x-\eta|/\kappa} d\eta. \end{cases} \quad (41)$$

Combination of Eq. (28) with Eqs. (24)–(25) gives

$$\begin{cases} (e_{311}\beta I_2\phi - c_{1111}I_1w'') = (1 - \xi)M + \frac{\xi}{2\kappa} \int_a^b M e^{-|x-\eta|/\kappa} d\eta, \\ \mu_{11}A\phi'/2 = (1 - \xi)P + \frac{\xi}{2\kappa} \int_a^b P e^{-|x-\eta|/\kappa} d\eta, \\ -(2e_{311}I_2w'' + \mu_{33}\beta A\phi)/2 = (1 - \xi)R + \frac{\xi}{2\kappa} \int_a^b R e^{-|x-\eta|/\kappa} d\eta. \end{cases} \quad (42)$$

Similarly, we can transform the integral constitutive relations (42) equivalently into differential forms with the corresponding constitutive constraints as

$$\kappa^2(e_{311}\beta I_2\phi'' - c_{1111}I_1w'''' - (e_{311}\beta I_2\phi - c_{1111}I_1w'')) = \kappa^2(1 - \xi)M'' - M, \quad (43a)$$

$$e_{311}\beta I_2(\kappa\phi'(0) - \phi(0)) - c_{1111}I_1(\kappa w'''(0) - w''(0)) = (1 - \xi)(\kappa M'(0) - M(0)), \quad (43b)$$

$$e_{311}\beta I_2(\kappa\phi'(L) + \phi(L)) - c_{1111}I_1(\kappa w'''(L) + w''(L)) = (1 - \xi)(\kappa M'(L) + M(L)), \quad (43b)$$

$$\mu_{11}A(\kappa^2\phi''' - \phi') = 2\kappa^2(1 - \xi)P'' - 2P, \quad (44a)$$

$$\mu_{11}A(\kappa\phi''(0) - \phi'(0)) = 2(1 - \xi)(\kappa P'(0) - P(0)), \quad (44a)$$

$$\mu_{11}A(\kappa\phi''(L) + \phi'(L)) = 2(1 - \xi)(\kappa P'(L) + P(L)), \quad (44b)$$

$$(2e_{311}I_2w'' + \mu_{33}\beta A\phi) - \kappa^2(2e_{311}I_2w'''' + \mu_{33}\beta A\phi'') = 2\kappa^2(1 - \xi)R'' - 2R, \quad (45a)$$

$$-2e_{311}I_2(\kappa w'''(0) - w''(0)) - \mu_{33}\beta A(\kappa\phi'(0) - \phi(0)) = 2(1 - \xi)(\kappa R'(0) - R(0)), \quad (45a)$$

$$-2e_{311}I_2(\kappa w'''(L) + w''(L)) - \mu_{33}\beta A(\kappa\phi'(L) + \phi(L)) = 2(1 - \xi)(\kappa R'(L) + R(L)). \quad (45b)$$

Combination of Eqs. (30a) and (44a) as well as (30b) and (43a) gives

$$P = \mu_{11}A(\phi' - \kappa^2\phi''')/2 - \kappa^2(1 - \xi)\beta R', \quad (46)$$

$$M = (e_{311}\beta I_2\phi - c_{1111}I_1w'') - \kappa^2(e_{311}\beta I_2\phi'' - c_{1111}I_1w'''' - \kappa^2(1 - \xi)q). \quad (47)$$

Substituting P and M from Eqs. (38)–(39) into Eq. (30), we get

$$\begin{cases} \mu_{11}A(\phi'' - \kappa^2\phi''') - \beta(2e_{311}I_2w'' + \mu_{33}\beta A\phi) + \beta\kappa^2(2e_{311}I_2w'''' + \mu_{33}\beta A\phi'') = 0, \\ (e_{311}\beta I_2\phi'' - c_{1111}I_1w''') - \kappa^2(e_{311}\beta I_2\phi'''' - c_{1111}I_1w''''') - \kappa^2(1 - \xi)q'' + q = 0. \end{cases} \quad (48)$$

4 Numerical solution by general differential quadrature method (GDQM)

4.1 Mathematical formulation in normalized form

The following nominal quantities are introduced to normalize the differential equations and boundary conditions as well as constitutive constraints:

$$\begin{cases} \eta = x/L, & W(\eta) = w(x)/L, & \Phi(\eta) = \phi(x)/L, & \lambda = \kappa/L, \\ Q(\zeta) = \frac{L^3}{EI_1}q(x), & \tilde{R}(\zeta) = \frac{R(x)L}{2e_{311}I_2}. \end{cases} \quad (49)$$

For the strain-driven nonlocal piezoelectric integral model, the differential governing equations (37a) and (40) turn into

$$\begin{cases} \lambda^2(1 - \xi)W_\eta^{(6)} - W_\eta^{(4)} + C_1(\Phi_\eta'' - \lambda^2(1 - \xi)\Phi_\eta^{(4)}) = \lambda^2Q_\eta'' - Q, \\ \lambda^2(1 - \xi)\Phi_\eta^{(4)} - \Phi_\eta'' - C_2(\lambda^2(1 - \xi)W_\eta^{(4)} - W_\eta'') - C_3(\lambda^2(1 - \xi)\Phi_\eta'' - \Phi) = 0, \\ \lambda^2(1 - \xi)\tilde{R}_\eta'' - \tilde{R} + (\lambda^2W_\eta^{(4)} - W_\eta'') + C_4(\lambda^2\Phi_\eta'' - \Phi) = 0, \end{cases} \quad (50)$$

where $(\cdot)'_\eta = \frac{d(\cdot)}{d\eta}$, and

$$C_1 = \frac{e_{311}\beta LI_2}{c_{1111}I_1}, \quad C_2 = \frac{2\beta Le_{311}I_2}{A\mu_{11}}, \quad C_3 = \frac{\mu_{33}\beta^2 L^2}{\mu_{11}}, \quad C_4 = \frac{\mu_{33}L\beta A}{2e_{311}I_2}. \quad (51)$$

The constitutive constraints are normalized as

$$\begin{cases} \lambda^2(1 - \xi)(\lambda W_\eta^{(5)} - W_\eta^{(4)}) - \xi\lambda W_\eta''' + \xi W'' + \lambda^2Q_\eta \\ - C_1(\lambda^2(1 - \xi)(\lambda\Phi_\eta''' - \Phi_\eta'') - \xi\lambda\Phi_\eta' + \xi\Phi) = 0, & \eta = 0, \\ \lambda^2(1 - \xi)(\lambda W_\eta^{(5)} + W_\eta^{(4)}) - \xi\lambda W_\eta''' - \xi W'' - \lambda^2Q \\ - C_1(\lambda^2(1 - \xi)(\lambda\Phi_\eta''' + \Phi_\eta'') - \xi\lambda\Phi_\eta' - \xi\Phi) = 0, & \eta = 1, \\ \lambda^2(1 - \xi)(\lambda\Phi_\eta^{(4)} - \Phi_\eta''') - \xi\lambda\Phi_\eta'' + \xi\Phi_\eta' + 2C_2\lambda^2(\lambda\tilde{R}_\eta'' - \tilde{R}_\eta') = 0, & \eta = 0, \\ \lambda^2(1 - \xi)(\lambda\Phi_\eta^{(4)} + \Phi_\eta''') - \xi\lambda\Phi_\eta'' - \xi\Phi_\eta' + 2C_2\lambda^2(\lambda\tilde{R}_\eta'' + \tilde{R}_\eta') = 0, & \eta = 1, \\ (1 - \xi)(\lambda W_\eta''' - W_\eta'') + C_4(1 - \xi)(\lambda\Phi_\eta' - \Phi) + 2(\lambda\tilde{R}_\eta' - \tilde{R}) = 0, & \eta = 0, \\ (1 - \xi)(\lambda W_\eta''' + W_\eta'') + C_4(1 - \xi)(\lambda\Phi_\eta' + \Phi) + 2(\lambda\tilde{R}_\eta' + \tilde{R}) = 0, & \eta = 1. \end{cases} \quad (52)$$

By combining Eqs. (31) with (38) and (39), the normalized standard boundary conditions can be expressed as

$$\begin{cases} ((C_1\Phi_\eta' - W_\eta''') - \lambda^2(1 - \xi)(C_1\Phi_\eta''' - W_\eta^{(5)}) - \lambda^2Q_\eta' - \tilde{Q})\delta W|_0^1 = 0, \\ ((C_1\Phi - W_\eta') - \lambda^2(1 - \xi)(C_1\Phi_\eta'' - W_\eta^{(4)}) - \lambda^2Q + \tilde{M})\delta W_\eta'|_0^1 = 0, \\ (C_4(\Phi_\eta' - \lambda^2(1 - \xi)\Phi_\eta''') - \lambda^2C_3\tilde{R}_\eta')\delta \Phi|_0^1 = 0, \end{cases} \quad (53)$$

where $\tilde{Q} = \frac{L^2}{EI_1}\hat{Q}$, and $\tilde{M} = \frac{L}{EI_1}\hat{M}$.

For the stress-driven nonlocal piezoelectric integral model, the differential governing equations (37a) and (40) turn into

$$\begin{cases} (\lambda^2 W_\eta^{(6)} - W_\eta^{(4)}) - C_1(\lambda^2 \Phi_\eta^{(4)} - \Phi_\eta'') = \lambda^2(1 - \xi)Q_\eta'' - Q, \\ \lambda^2 \Phi_\eta^{(4)} - \Phi_\eta'' + C_2(W_\eta'' - \lambda^2 W_\eta^{(4)}) + C_3(\Phi - \lambda^2 \Phi_\eta'') = 0, \\ (W_\eta'' + C_4\Phi) - \lambda^2(W_\eta^{(4)} + C_4\Phi_\eta'') = \lambda^2(1 - \xi)\widetilde{R}_\eta'' - \widetilde{R}. \end{cases} \quad (54)$$

The corresponding normalized constitutive and standard boundary conditions are

$$\begin{cases} (1 - \xi)\lambda^2(\lambda W_\eta^{(5)} - W_\eta^{(4)}) + \xi\lambda W_\eta''' - \xi W_\eta'' + \lambda^2(1 - \xi)^2 Q \\ - C_1((1 - \xi)\lambda^2(\lambda \Phi_\eta''' - \Phi_\eta'') + \xi\lambda \Phi_\eta' - \xi \Phi_\eta) = 0, \quad \eta = 0, \\ (1 - \xi)\lambda^2(\lambda W_\eta^{(5)} + W_\eta^{(4)}) + \xi\lambda W_\eta''' + \xi W_\eta'' - \lambda^2(1 - \xi)^2 Q \\ - C_1((1 - \xi)\lambda^2(\lambda \Phi_\eta''' + \Phi_\eta'') + \xi\lambda \Phi_\eta' + \xi \Phi_\eta) = 0, \quad \eta = 1. \end{cases} \quad (55)$$

$$\begin{cases} \lambda^2(1 - \xi)(\lambda \Phi_\eta^{(4)} - \Phi_\eta''') + \xi\lambda \Phi_\eta'' - \xi \Phi_\eta' + 2C_2\lambda^2(1 - \xi)^2(\lambda \widetilde{R}_\eta'' - \widetilde{R}_\eta') = 0, \quad \eta = 0, \\ \lambda^2(1 - \xi)(\lambda \Phi_\eta^{(4)} + \Phi_\eta''') + \xi\lambda \Phi_\eta'' + \xi \Phi_\eta' + 2C_2\lambda^2(1 - \xi)^2(\lambda \widetilde{R}_\eta'' + \widetilde{R}_\eta') = 0, \quad \eta = 1. \\ \begin{cases} (\lambda W_\eta''' - W_\eta'') + C_4(\lambda \Phi_\eta' - \Phi) + 2(1 - \xi)(\lambda \widetilde{R}_\eta' - \widetilde{R}) = 0, & \eta = 0, \\ (\lambda W_\eta''' + W_\eta'') + C_4(\lambda \Phi_\eta' + \Phi) + 2(1 - \xi)(\lambda \widetilde{R}_\eta' + \widetilde{R}) = 0, & \eta = 1. \end{cases} \\ \begin{cases} ((\lambda^2 W_\eta^{(5)} - W_\eta''') - C_1(\lambda^2 \Phi_\eta''' - \Phi_\eta') - \lambda^2(1 - \xi)Q_\eta' - \widetilde{Q})\delta W|_0^1 = 0, \\ ((\lambda^2 W_\eta^{(4)} - W_\eta'') - C_1(\lambda^2 \Phi_\eta'' - \Phi) - \lambda^2(1 - \xi)Q + \widetilde{M})\delta W_\eta'|_0^1 = 0, \\ (C_4(\lambda^2 \Phi_\eta''' - \Phi_\eta') + \lambda^2(1 - \xi)C_3\widetilde{R}_\eta')\delta \Phi|_0^1 = 0. \end{cases} \end{cases} \quad (56)$$

It should be noticed that for both strain- and stress-driven two-phase local/nonlocal piezoelectric integral models, Eqs. (50) and (54) are about sixth-order W , fourth-order Φ , and second-order $\widetilde{R}$ ordinary differential equations. There are totally 12 boundary conditions (six standard boundary conditions and six constitutive boundary conditions), which have the same number of total orders for the differential governing equations. Therefore, the static bending of piezoelectric Euler-Bernoulli beam is well-posed for the two-phase local/nonlocal piezoelectric integral model.

However, if $\xi = 1$, Eq. (50) turns into fourth-order W , second-order Φ , and second-order $\widetilde{R}$ ordinary differential equations for strain-driven purely nonlocal piezoelectric integral model, and Eq. (54) turns into sixth-order W and fourth-order Φ ordinary differential equations for the stress-driven purely nonlocal piezoelectric integral model. Therefore, the total number of differential orders for governing equations is less than that of boundary conditions. Therefore, the static bending of piezoelectric Euler-Bernoulli beam is ill-posed for the purely stress-driven nonlocal piezoelectric integral models.

Special attention is paid to the traditional nonlocal piezoelectric differential model when we set $\xi = 1$ and neglect the constitutive boundary conditions, in which the total number of differential orders for governing equations is the same as the number of standard boundary conditions. However, it can be seen that the differential governing equations are independent of the nonlocal parameter for a concentrated force and a constantly or linearly distributed load. Taking into account the boundary conditions, we obtain that no size-dependent bending appears while using traditional nonlocal piezoelectric differential model under following cases.

- (i) Linear distributed load: clamped-clamped boundary conditions.
- (ii) Constantly distributed load: clamped-guided and clamped-clamped boundary conditions.

(iii) Concentrated force at beam end: clamped-free, clamped-guided, and simply supported-guided boundary conditions.

In other words, the traditional nonlocal piezoelectric differential model would lead to inconsistent bending response for Euler-Bernoulli piezoelectric beam under different boundary and loading conditions.

4.2 Numerical solution using GDQM

The GDQM shows great advantages in solving differential equations, which is used to solve Eqs. (50) and (54). On the basis of the GDQM and following the Chebyshev-Gauss-Lobatto rule, the beam domain is discretized with n grid nodes as

$$\eta_i = (1 - \cos((i-1)\pi/(N-1)))/2, \quad i = 1, 2, \dots, n, \quad (57)$$

where N is the grid number. The functions R , Φ , and W can be approximated by^[48–50]

$$R(\eta) = \sum_{i=1}^N \varphi_i(\eta) R(\eta_i), \quad (58)$$

$$\Phi(\eta) = \sum_{i=1}^N \varphi_i^\Phi(\eta) \Phi(\eta_i) + \psi_1^\Phi(\eta) \Phi'_\eta(\eta_1) + \psi_N^\Phi(\eta) \Phi'_\eta(\eta_N), \quad (59)$$

$$W(\eta) = \sum_{i=1}^N \varphi_i^W(\eta) W(\eta_i) + \psi_1^W(\eta) W'_\eta(\eta_1) + \psi_N^W(\eta) W'_\eta(\eta_N) \\ + \Gamma_1^W(\eta) W''_\eta(\eta_1) + \Gamma_N^W(\eta) W''_\eta(\eta_N), \quad (60)$$

where $\varphi_i(\eta) = \prod_{\substack{k=1 \\ k \neq i}}^N \frac{\eta - \eta_k}{\eta_i - \eta_k}$ is the Lagrange interpolation function, and

$$\psi_j^\Phi(x) = \varphi_j(\eta)(\eta - \eta_j)(\eta - \eta_{N-j+1})/(\eta_j - \eta_{N-j+1})^2, \\ \varphi_i^\Phi(\eta) = \begin{cases} \frac{\eta - \eta_{N-i+1}}{\eta_i - \eta_{N-i+1}} \varphi_i(\eta) - \left(\varphi'_i(\eta_i) + \frac{1}{\eta_i - \eta_{N-i+1}} \right) \psi_i^\Phi(\eta), & i = j, \\ (\eta - \eta_1)(\eta - \eta_N) \varphi_i(\eta) / ((\eta_i - \eta_1)(\eta_i - \eta_N)), & \text{others,} \end{cases} \quad (61)$$

$$\Gamma_j^W(\eta) = (\eta - \eta_j)^2 (\eta - \eta_{N-j+1})^2 \varphi_j(\eta) / (2(\eta_j - \eta_{N-j+1})^2), \\ \psi_j^W(\eta) = \frac{(\eta - \eta_j)(\eta - \eta_{N-j+1})^2}{(\eta_j - \eta_{N-j+1})^2} \varphi_j(\eta) - \left(2\varphi'_j(\eta_j) + \frac{4}{\eta_j - \eta_{N-j+1}} \right) \Gamma_j^W(\eta), \\ \varphi_i^W(\eta) = \begin{cases} \frac{(\eta - \eta_{N-i+1})^2}{(\eta_i - \eta_{N-i+1})^2} \varphi_i(\eta) - \left(\varphi'_i(\eta_i) + \frac{2}{\eta_i - \eta_{N-i+1}} \right) \psi_i^W(\eta) \\ - \left(\varphi''_i(\eta_i) + \frac{4\varphi'_i(\eta_i)}{\eta_i - \eta_{N-i+1}} + \frac{2}{(\eta_i - \eta_{N-i+1})^2} \right) \Gamma_i^W(\eta), & i = j, \\ (\eta - \eta_1)^2 (\eta - \eta_N)^2 \varphi_i(\eta) / ((\eta_i - \eta_1)^2 (\eta_i - \eta_N)^2), & \text{others,} \end{cases} \quad (62)$$

where $j = 1, N$.

Performing derivative with respect to η on Eqs. (58)–(60), we obtain

$$\Delta_R^{(i)} = \mathbf{X}^{(i)} \Delta^R, \quad (63)$$

$$\Delta_\Phi^{(i)} = \mathbf{Y}^{(i)} \Delta^\Phi, \quad (64)$$

$$\Delta_W^{(i)} = \mathbf{Z}^{(i)} \Delta^W, \quad (65)$$

where $\mathbf{X}^{(i)}$, $\mathbf{Y}^{(i)}$, and $\mathbf{Z}^{(i)}$ are the weighting coefficients of the i th-order derivative, which are defined explicitly^[47–49] as

$$\begin{aligned}\Delta^R &= (R(\eta_1)R(\eta_2)\cdots R(\eta_N))^T, \\ \Delta^\Phi &= (\Phi(\eta_1)\Phi(\eta_2)\cdots\Phi(\eta_N)\Phi'(\eta_1)\Phi'(\eta_N))^T, \\ \Delta^W &= (W(\eta_1)W(\eta_2)\cdots W(\eta_N)W'(\eta_1)W'(\eta_N)W''(\eta_1)W''(\eta_N))^T.\end{aligned}$$

Following the standard GDQM procedure, we obtain the discrete form of the governing differential equations as well as approximate constitutive and standard boundary conditions. The static bending problem for a piezoelectric Euler-Bernoulli beam based on nonlocal integral model turns a set of $3N+6$ linear equations about Δ_j^R , Δ_j^Φ , and Δ_j^W .

5 Numerical results

In this section, the effect of nonlocal parameters ξ and κ on deflections of piezoelectric Euler-Bernoulli beam with $L/h = 20$ is investigated on the basis of strain- and stress-driven nonlocal integral models. It is assumed that the piezoelectric beam is made of the PZT-4 with $c_{1111} = 132$ GPa, $e_{311} = -4.1$ C · m⁻², $\mu_{11} = 5.841 \times 10^{-9}$ C · (V · m)⁻¹, and $\mu_{33} = 7.124 \times 10^{-9}$ C · (V · m)⁻¹^[51].

5.1 Validation

Figures 2–3 illustrate the maximum nominal deflections of nonlocal piezoelectric Euler-Bernoulli beams versus λ for different ξ with millionths of e_{311} , μ_{11} , and μ_{33} , in which, CCU and SSU indicate the clamped-clamped and simply-supported beams under a unit uniformly dis-

Fig. 2 Maximum nominal deflections of (a) CCU-, (b) SSU-, and (c) CFC-Euler-Bernoulli beams based on strain-driven nonlocal model versus λ for different ξ (color online)

tributed load ($Q = 1$), respectively, and CFC, CGC, and SGC indicate clamped-free, clamped-guided, and simply-supported-guided beams under a unit concentrated force ($\tilde{Q} = 1$), respectively. In Figs. 2–3, the solid lines represent the results from current models and the scatters indicate the results for corresponding nonlocal Euler-Bernoulli beams from Wang et al.^[34] and Zhang et al.^[42]. It can be seen from Figs. 2–3 that the results based on current model agree well with those based on nonlocal integral elastic model while piezoelectric and dielectric constants are very small. In other words, the nonlocal piezoelectric integral model can be degenerated to the nonlocal integral elastic model while setting piezoelectric and dielectric constants as zeros.

Fig. 3 Maximum nominal deflections of piezoelectric Euler-Bernoulli beam based on stress-driven nonlocal piezoelectric model versus λ for $\xi = 1$ (color online)

5.2 Effect of nonlocal parameters on Euler-Bernoulli beam bending

Figures 4–5 show the nominal deflection and electric potential of CCU nonlocal piezoelectric Euler-Bernoulli beam versus position for different λ and ξ . It can be seen from Figs. 4–5 that, with the increase in λ and ξ , the nominal deflections as well as the absolute values of the maximum and minimum nominal electric potentials increase and decrease consistently for strain- and stress-driven nonlocal piezoelectric models, respectively.

Fig. 4 Nominal deflections of CCU nonlocal piezoelectric Euler-Bernoulli beam versus nominal position under (a) different λ for $\xi = 0.5$ and (b) different ξ for $\lambda = 0.5$ (color online)

Figures 6–9 show the contour plots of the normalized maximum deflection (NMD), which is defined as the ratio between maximum deflection from nonlocal piezoelectric model and that from classic piezoelectric model, in which λ and ξ range from 0.05 to 1 and 0 to 0.95, respectively. It can be seen that, under different boundary and loading conditions, a consistent softening and toughening response can be observed for strain- and stress-driven nonlocal piezoelectric models, respectively. Meanwhile, the NMDs increase and decrease consistently with the increase of λ

Fig. 5 Nominal electric potentials of CCU nonlocal piezoelectric Euler-Bernoulli beam versus nominal position under (a) different λ for $\xi = 0.5$ and (b) different ξ for $\lambda = 0.5$ (color online)

Fig. 6 Contour plots of NMD of CCU Euler-Bernoulli beam at $\eta = 0.5$ based on (a) strain-driven and (b) stress-driven nonlocal piezoelectric integral models (color online)

Fig. 7 Contour plots of NMD of SSU Euler-Bernoulli beam at $\eta = 0.5$ based on (a) strain-driven and (b) stress-driven nonlocal piezoelectric integral models (color online)

Fig. 8 Contour plots of NMD of CGC Euler-Bernoulli beam at $\eta = 1$ based on (a) strain-driven and (b) stress-driven nonlocal piezoelectric integral models (color online)

Fig. 9 Contour plots of NMD of CFC/SGC Euler-Bernoulli beam at $\eta = 1$ based on (a) strain-driven and (b) stress-driven nonlocal piezoelectric integral models (color online)

and ξ for strain- and stress-driven nonlocal piezoelectric models, respectively. When ξ is small, both the increase and decrease rates of NMDs decrease with the increase in λ for strain- and stress-driven nonlocal piezoelectric integral models, respectively, which is also shown in Fig. 4(a). When λ is small, the increase and decrease rates of NMDs decrease with the increase of ξ for strain- and stress-driven nonlocal piezoelectric integral models, respectively. In addition, along the line $\lambda = \xi$, the increase and decrease rates of NMDs increase with the increase in λ for both strain- and stress-driven nonlocal piezoelectric integral models.

6 Conclusions

General strain- and stress-driven two-phase local/nonlocal piezoelectric integral models are proposed in this paper, which can distinguish the nonlocal effect on elastic and piezoelectric behaviors. The integral nonlocal model is transformed equivalently into differential form with four constitutive boundary conditions. The simplified strain- and stress-driven local/nonlocal piezoelectric integral models can be obtained when the nonlocal parameters for elastic and

piezoelectric behaviors are the same. On the basis of the minimum total potential energy principle, the governing differential equations as well as standard and constitutive boundary conditions are derived for static bending of Euler-Bernoulli beam for general strain- and stress-driven two-phase local/nonlocal piezoelectric integral models.

It is found that the traditional piezoelectric differential model, purely strain- and stress-driven nonlocal piezoelectric integral models are not applicable for static bending analysis of Euler-Bernoulli beam, because that the numbers of differential order of governing equations and boundary conditions are different. Several nominal variables are introduced to normalize the expression for governing equations and boundary conditions, and the GDQM with different order interpolations are used to obtain the numerical solution.

The numerical study shows that, by setting e_{311} , μ_{11} , and μ_{33} as very small value, the strain-driven local/nonlocal piezoelectric integral model can be degenerated to strain-driven local/nonlocal integral elastic model, and stress-driven local/nonlocal piezoelectric integral model can be generated to purely stress-driven nonlocal elastic model for $\xi = 0$. A consistent softening and toughening effect can be obtained for strain- and stress-driven local/nonlocal piezoelectric integral models, respectively. For a fixed value of $\lambda(\xi)$, both the increase and decrease rates of NMDs decrease with the increase in $\xi(\lambda)$ for strain- and stress-driven nonlocal piezoelectric integral models, respectively.

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References

- [1] PAN, Z. W., DAI, Z. R., and WANG, Z. L. Nanobelts of semiconducting oxides. *Science*, **291**, 1947–1949 (2001)
- [2] WAN, Q., LI, Q. H., CHEN, Y. J., WANG, T. H., HE, X. L., LI, J. P., and LIN, C. L. Fabrication and ethanol sensing characteristics of ZnO nanowire gas sensors. *Applied Physics Letters*, **84**, 3654–3656 (2004)
- [3] LAZARUS, A., THOMAS, O., and DEU, J. F. Finite element reduced order models for nonlinear vibrations of piezoelectric layered beams with applications to NEMS. *Finite Elements in Analysis and Design*, **49**, 35–51 (2012)
- [4] SU, W. S., CHEN, Y. F., HSIAO, C. L., and TU, L. W. Generation of electricity in GaN nanorods induced by piezoelectric effect. *Applied Physics Letters*, **90**, 063110 (2007)
- [5] WANG, X. D., ZHOU, J., SONG, J. H., LIU, J., XU, N. S., and WANG, Z. L. Piezoelectric field effect transistor and nanoforce sensor based on a single ZnO nanowire. *Nano Letters*, **6**, 2768–2772 (2006)
- [6] LI, C., GUO, W., KONG, Y., and GAO, H. Size-dependent piezoelectricity in zinc oxide nanofilms from first-principles calculations. *Applied Physics Letters*, **90**, 033108 (2007)
- [7] HADJESFANDIARI, A. R. Size-dependent piezoelectricity. *International Journal of Solids and Structures*, **50**, 2781–2791 (2013)
- [8] SHEN, S. and HU, S. A theory of flexoelectricity with surface effect for elastic dielectrics. *Journal of the Mechanics and Physics of Solids*, **58**, 665–677 (2010)
- [9] HU, S. D., LI, H., and TZOU, H. S. Distributed flexoelectric structural sensing: theory and experiment. *Journal of Sound and Vibration*, **348**, 126–136 (2015)
- [10] BURSIA, E. V. and TRUNOV, N. N. Nonlocal piezoelectric effect. *Soviet Physics Solid State*, **16**, 760–762 (1974)

- [11] ERINGEN, A. C. Theory of nonlocal piezoelectricity. *Journal of Mathematical Physics*, **25**, 717–727 (1984)
- [12] ARANI, A. G., ABDOLLAHIAN, M., and KOLAHCHI, R. Nonlinear vibration of a nanobeam elastically bonded with a piezoelectric nanobeam via strain gradient theory. *International Journal of Mechanical Sciences*, **100**, 32–40 (2015)
- [13] LI, Y. S. and FENG, W. J. Microstructure-dependent piezoelectric beam based on modified strain gradient theory. *Smart Materials and Structures*, **23**, 095004 (2014)
- [14] DEHKORDI, S. F. and BENI, Y. T. Electro-mechanical free vibration of single-walled piezoelectric/flexoelectric nano cones using consistent couple stress theory. *International Journal of Mechanical Sciences*, **128**, 125–139 (2017)
- [15] LI, Y. S. and PAN, E. Static bending and free vibration of a functionally graded piezoelectric microplate based on the modified couple-stress theory. *International Journal of Engineering Science*, **97**, 40–59 (2015)
- [16] MALIKAN, M. Electro-mechanical shear buckling of piezoelectric nanoplate using modified couple stress theory based on simplified first order shear deformation theory. *Applied Mathematical Modelling*, **48**, 196–207 (2017)
- [17] AREFI, M. Analysis of a doubly curved piezoelectric nano shell: nonlocal electro-elastic bending solution. *European Journal of Mechanics A-Solids*, **70**, 226–237 (2018)
- [18] ZHANG, L., GUO, J., and XING, Y. Bending deformation of multilayered one-dimensional hexagonal piezoelectric quasicrystal nanoplates with nonlocal effect. *International Journal of Solids and Structures*, **132**, 278–302 (2018)
- [19] ZHANG, L., GUO, J., and XING, Y. Bending analysis of functionally graded one-dimensional hexagonal piezoelectric quasicrystal multilayered simply supported nanoplates based on nonlocal strain gradient theory. *Acta Mechanica Sinica*, **34**, 237–251 (2020)
- [20] LI, Y. D., BAO, R., and CHEN, W. Buckling of a piezoelectric nanobeam with interfacial imperfection and van der Waals force: is nonlocal effect really always dominant? *Composite Structures*, **194**, 357–364 (2018)
- [21] SUN, J., WANG, Z., ZHOU, Z., XU, X., and LIM, C. W. Surface effects on the buckling behaviors of piezoelectric cylindrical nanoshells using nonlocal continuum model. *Applied Mathematical Modelling*, **59**, 341–356 (2018)
- [22] CHEN, L., KE, L. L., JIE, Y., KITIPORNCHAI, S., and WANG, Y. S. Nonlinear vibration of piezoelectric nanoplates using nonlocal Mindlin plate theory. *Mechanics of Advanced Materials and Structures*, **25**, 1252–1264 (2018)
- [23] MAO, J. J., LU, H. M., ZHANG, W., and LAI, S. K. Vibrations of graphene nanoplatelet reinforced functionally gradient piezoelectric composite microplate based on nonlocal theory. *Composite Structures*, **236**, 111813 (2020)
- [24] ZENG, S., WANG, K., WANG, B., and WU, J. Vibration analysis of piezoelectric sandwich nanobeam with flexoelectricity based on nonlocal strain gradient theory. *Applied Mathematics and Mechanics (English Edition)*, **41**, 859–880 (2020) <https://doi.org/10.1007/s10483-020-2620-8>
- [25] ATKINSON, C. A remark on non-local theories of elasticity, piezoelectric materials etc. *International Journal of Engineering Science*, **97**, 95–97 (2015)
- [26] GHAYESH, M. H. and FAROKHI, H. Nonlinear broadband performance of energy harvesters. *International Journal of Engineering Science*, **147**, 103202 (2020)
- [27] BENVENUTI, E. and SIMONE, A. One-dimensional nonlocal and gradient elasticity: closed-form solution and size effect. *Mechanics Research Communications*, **48**, 46–51 (2013)
- [28] CHALLAMEL, N. and WANG, C. M. The small length scale effect for a non-local cantilever beam: a paradox solved. *Nanotechnology*, **19**, 345703 (2008)
- [29] FERNANDEZ-SAEZ, J., ZAERA, R., LOYA, J. A., and REDDY, J. N. Bending of Euler-Bernoulli beams using Eringen's integral formulation: a paradox resolved. *International Journal of Engineering Science*, **99**, 107–116 (2016)
- [30] LI, C., YAO, L. Q., CHEN, W. Q., and LI, S. Comments on nonlocal effects in nano-cantilever beams. *International Journal of Engineering Science*, **87**, 47–57 (2015)

-
- [31] REDDY, J. N. and PANG, S. D. Nonlocal continuum theories of beams for the analysis of carbon nanotubes. *Journal of Applied Physics*, **103**, 023511 (2008)
 - [32] ROMANO, G., BARRETTA, R., DIACO, M., and DE SCIARRA, F. M. Constitutive boundary conditions and paradoxes in nonlocal elastic nanobeams. *International Journal of Mechanical Sciences*, **121**, 151–156 (2017)
 - [33] ROMANO, G. and BARRETTA, R. Nonlocal elasticity in nanobeams: the stress-driven integral model. *International Journal of Engineering Science*, **115**, 14–27 (2017)
 - [34] WANG, Y. B., ZHU, X. W., and DAI, H. H. Exact solutions for the static bending of Euler-Bernoulli beams using Eringen's two-phase local/nonlocal model. *AIP Advances*, **6**, 085114 (2016)
 - [35] WANG, Y. B., HUANG, K., ZHU, X., and LOU, Z. Exact solutions for the bending of Timoshenko beams using Eringen's two-phase nonlocal model. *Mathematics and Mechanics of Solids*, **24**, 559–572 (2019)
 - [36] ERINGEN, A. C. Theory of nonlocal elasticity and some applications. *Res Mechanica*, **21**, 313–342 (1987)
 - [37] ZHANG, P., QING, H., and GAO, C. Theoretical analysis for static bending of circular Euler-Bernoulli beam using local and Eringen's nonlocal integral mixed model. *Zeitschrift für Angewandte Mathematik und Mechanik*, **99**, e201800329 (2019)
 - [38] ZHANG, P. and QING, H. Exact solutions for size-dependent bending of Timoshenko curved beams based on a modified nonlocal strain gradient model. *Acta Mechanica*, **231**, 5251–5276 (2020)
 - [39] MENG, L. C., ZOU, D. J., LAI, H., GUO, Z. L., HE, X. Z., XIE, Z. J., and GAO, C. F. Semi-analytic solution of Eringen's two-phase local/nonlocal model for Euler-Bernoulli beam with axial force. *Applied Mathematics and Mechanics (English Edition)*, **39**, 1805–1824 (2018) <https://doi.org/10.1007/s10483-018-2395-9>
 - [40] OSKOUIE, M. F., ANSARI, R., and ROUHI, H. Bending of Euler-Bernoulli nanobeams based on the strain-driven and stress-driven nonlocal integral models: a numerical approach. *Acta Mechanica Sinica*, **34**, 871–882 (2018)
 - [41] OSKOUIE, M. F., ANSARI, R., and ROUHI, H. Stress-driven nonlocal and strain gradient formulations of Timoshenko nanobeams. *European Physical Journal Plus*, **133**, 336 (2018)
 - [42] ZHANG, J., QING, H., and GAO, C. Exact and asymptotic bending analysis of microbeams under different boundary conditions using stress-derived nonlocal integral model. *Zeitschrift für Angewandte Mathematik und Mechanik*, **100**, e201900148 (2020)
 - [43] ZHANG, P., QING, H., and GAO, C. F. Exact solutions for bending of Timoshenko curved nanobeams made of functionally graded materials based on stress-driven nonlocal integral model. *Composite Structures*, **245**, 112362 (2020)
 - [44] BARRETTA, R., LUCIANO, R., DE SCIARRA, F. M., and RUTA, G. Stress-driven nonlocal integral model for Timoshenko elastic nano-beams. *European Journal of Mechanics A-Solids*, **72**, 275–286 (2018)
 - [45] BARRETTA, R., CAPORALE, A., FAGHIDIAN, S. A., LUCIANO, R., DE SCIARRA, F. M., and MEDAGLIA, C. M. A stress-driven local-nonlocal mixture model for Timoshenko nano-beams. *Composites Part B-Engineering*, **164**, 590–598 (2019)
 - [46] YANG, J. S. *The Mechanics of Piezoelectric Structures*, World Scientific Publishing Company, Singapore (2006)
 - [47] WANG, Q. On buckling of column structures with a pair of piezoelectric layers. *Engineering Structures*, **24**, 199–205 (2002)
 - [48] CHEN, C. N. The Timoshenko beam model of the differential quadrature element method. *Computational Mechanics*, **24**, 65–69 (1999)
 - [49] WU, T. Y. and LIU, G. R. The generalized differential quadrature rule for fourth-order differential equations. *International Journal for Numerical Methods in Engineering*, **50**, 1907–1929 (2001)
 - [50] WANG, X. Novel differential quadrature element method for vibration analysis of hybrid nonlocal Euler-Bernoulli beams. *Applied Mathematics Letters*, **77**, 94–100 (2018)
 - [51] KE, L. L. and WANG, Y. S. Thermoelectric-mechanical vibration of piezoelectric nanobeams based on the nonlocal theory. *Smart Materials & Structures*, **21**, 025018 (2012)