Effective elastic properties of one-dimensional hexagonal quasicrystal composites*

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Abstract The explicit expression of Eshelby tensors for one-dimensional (1D) hexagonal quasicrystal composites is presented by using Green's function method. The closed forms of Eshelby tensors in the special cases of spheroid, elliptic cylinder, ribbon-like, penny-shaped, and rod-shaped inclusions embedded in 1D hexagonal quasicrystal matrices are given. As an application of Eshelby tensors, the analytical expressions for the effective properties of the 1D hexagonal quasicrystal composites are derived based on the Mori-Tanaka method. The effects of the volume fraction of the inclusion on the elastic properties of the composite materials are discussed.

Key words one-dimensional (1D) hexagonal quasicrystal, Eshelby tensor, Mori-Tanaka method

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1 Introduction

Quasicrystals are a kind of solid ordered phases with amorphous rotational symmetry and long-range quasi-periodic ordering. They have become a new kind of functional materials and structural materials^[1–3].

In recent years, many achievements have been made in the elastic theory of quasicrystals^[4–8]. The properties of quasicrystals are affected by defects such as dislocations, cracks, holes, and inclusions. Li and Fan^[9] considered a straight dislocation in one-dimensional (1D) hexagonal quasicrystals. Fan et al.^[10] studied the problem of a moving screw dislocation in 1D hexagonal quasicrystals. Li^[11] derived the elastic field in a 1D hexagonal quasicrystal infinite medium with planar cracks. Yang et al.^[12] studied the anti-plane problem of three unequal cracks in a circular hole in 1D hexagonal piezoelectric quasicrystals by introducing two displacement functions.

The inclusion problem of quasicrystalline materials has attracted a lot of scholars' attention. Shi^[13] studied the collinear periodic cracks and/or rigid line inclusions of antiplane sliding

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modes in a 1D hexagonal quasicrystal. Gao and Ricoeur^[14] discussed the three-dimensional (3D) elastic problem of two-dimensional (2D) quasicrystals with spheroidal inclusions under uniformly distributed loads at infinity. Wang and Schiavone^[15] derived the thermoelastic field in a decagonal quasicrystalline composite reinforced by an elliptical inclusion. According to the linear elastic theory of piezoelectric quasicrystals, Guo et al.^[16] analyzed the antiplane elastic problem of 1D hexagonal piezoelectric quasicrystals with elliptical inclusions. Guo and Pan^[17] studied the three-phase cylinder model of 1D piezoelectric quasicrystal composites, and predicted the effective moduli of the piezoelectric quasicrystalline composites. Wang and Guo^[18] obtained the exact closed-form solution of phonon, phase, and electric field stress in 1D piezoelectric quasicrystal composites with the confocal elliptic cylinder model.

The effective elastic properties of quasicrystal composites are strongly affected by the microstructure of the composites. Because of the excellent characteristics of quasicrystal composites, it is very important to predict the effects of some influential factors on the effective properties of quasicrystal composites. So far, we have not found any experimental study on the effective elastic constants of quasicrystal composites.

In the present study, Green's function method is used to solve Eshelby tensors for 1D hexagonal quasicrystals^[19]. An analytical formula for the effective elastic properties of quasicrystal materials was obtained by the Mori-Tanaka method^[20–21]. According to 1D hexagonal quasicrystal elastic properties^[22–23], the relationship between the effective properties and the volume fraction of the inclusion is discussed. It provides some theoretical guides for the evaluation of mechanical properties of 1D hexagonal quasicrystal composites.

2 Basic equations of 1D hexagonal quasicrystals

In the rectangular coordinate system (x_1, x_2, x_3) , the atomic arrangement of 1D hexagonal quasicrystals is periodic in the x_1x_2 -plane and quasi-periodic along the x_3 -direction. The equilibrium equations are given by

$$\sigma_{ij,j} = 0, \quad H_{3j,j} = 0, \quad (1)$$

where $i, j = 1, 2, 3$, the comma in the subscript denotes partial differentiation, and the repeated indices represent summation. σ_{ij} is the stress of the phonon field, and H_{3j} is the stress of the phason field.

The constitutive equations are

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + R_{ij3l}\omega_{3l}, \quad H_{3j} = R_{kl3j}\varepsilon_{kl} + K_{3j3l}\omega_{3l}, \quad (2)$$

where $k, l = 1, 2, 3$, ε_{kl} is the strain of the phonon field, ω_{3l} is the strain of the phason field, and C_{ijkl} , K_{3j3l} , and R_{ij3l} stand for the elastic moduli in the phonon field, the phason field, and the phonon-phason coupling field, respectively.

Besides, the geometry equations are given by

$$\varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}), \quad \omega_{3l} = w_{3,l}, \quad (3)$$

in which u_k and w_3 represent the displacements in the phonon field and the phason field, respectively.

Equations (1)–(3) can be compactly expressed with the notation of Lothe and Barnett as^[24]

$$Z_{Kl} = \begin{cases} \varepsilon_{kl}, & K = 1, 2, 3, \\ \omega_{3l}, & K = 4, \end{cases} \quad U_K = \begin{cases} u_k, & K = 1, 2, 3, \\ w_3, & K = 4, \end{cases} \quad (4)$$

$$\Xi_{Ij} = \begin{cases} \sigma_{ij}, & I = 1, 2, 3, \\ H_{3j}, & I = 4, \end{cases} \quad F_{IjKl} = \begin{cases} C_{ijkl}, & I = K = 1, 2, 3, \\ R_{ij3l}, & I = 1, 2, 3, \quad K = 4, \\ R_{kl3j}, & I = 4, \quad K = 1, 2, 3, \\ K_{3j3l}, & I = K = 4, \end{cases} \quad (5)$$

where Z_{Kl} , U_K , Ξ_{Ij} , and F_{IjKl} are the matrices of the strain, the displacement, the stress, and the quasicrystal elastic modulus, respectively.

Then, Eqs. (1) and (2) can be, respectively, rewritten as

$$\Xi_{Ij,j} = 0, \quad (6)$$

$$\Xi_{Ij} = F_{IjKl} Z_{Kl}. \quad (7)$$

Inverting these relations, we have

$$Z_{Gh} = E_{GhIj} \Xi_{Ij}, \quad (8)$$

where E_{GhIj} is the inverse matrix of the matrix F_{IjKl} .

With Eqs. (3) and (4), Eq. (7) can be rewritten as

$$\Xi_{Ij} = F_{IjKl} U_{K,l}. \quad (9)$$

Substituting Eq. (9) into Eq. (6) yields

$$F_{IjKl} U_{K,lj} = 0. \quad (10)$$

3 Problem statement

Consider an infinite 1D hexagonal quasicrystal R^3 , containing an inclusion of ellipsoidal shape $\Omega \subset R^3$ (see Fig. 1). The inclusion is defined by $(x_1/a_1)^2 + (x_2/a_2)^2 + (x_3/a_3)^2 = 1$, where a_1 , a_2 , and a_3 are the lengths of the semi-axes of the ellipsoid. The inclusion is under a uniform stress-free strain Z_{Kl}^* . The 1D hexagonal quasicrystal inclusion problem is given by

$$\Xi_{Ij} = F_{IjKl} (Z_{Kl} - T_{Kl}^*(\mathbf{x})), \quad (11)$$

$$F_{IjKl} Z_{Kl} = F_{IjKl} U_{K,l}, \quad (12)$$

where the eigen strain field $T_{Kl}^*(\mathbf{x})$ is as follows:

$$T_{Kl}^*(\mathbf{x}) = \begin{cases} Z_{Kl}^*, & \mathbf{x} \in \Omega, \\ 0, & \mathbf{x} \in R^3 \setminus \Omega. \end{cases} \quad (13)$$

Substituting Eqs. (11) and (12) into Eq. (6), we have

$$F_{IjKl} U_{K,lj} = F_{IjKl} T_{Kl,j}^*(\mathbf{x}). \quad (14)$$

By using the Fourier transformation, the elastic displacement due to the eigen strain field $T_{Kl}^*(\mathbf{x})$ can be determined as^[25–26]

$$U_K(\mathbf{x}) = -F_{IjAb} \int_{\Omega} T_{Ab}^*(\mathbf{x}') G_{KI,j}(\mathbf{x} - \mathbf{x}') d\mathbf{x}', \quad (15)$$

Fig. 1 An ellipsoidal inclusion Ω in infinite 1D hexagonal quasicrystals

where

$$G_{KI}(\mathbf{x} - \mathbf{x}') = \frac{1}{8\pi^2} \int_{S^2} N_{KI}(\bar{\xi}) D^{-1}(\bar{\xi}) \delta(\bar{\xi} \cdot (\mathbf{x} - \mathbf{x}')) dS(\bar{\xi}) \quad (16)$$

is the elastic displacement at $\mathbf{x}$ due to a unit point force at a specific interior point $\mathbf{x}'$, and S^2 is the unit sphere, δ is the Dirac delta function, $N_{KI}(\bar{\xi})$ is the cofactor of H_{KI} , and $D^{-1}(\bar{\xi})$ is the determinant of H_{KI} . The matrix H_{KI} is given by

$$H_{KI} = \bar{\xi}_j \bar{\xi}_l F_{IjKl}. \quad (17)$$

In addition, the following transformations of variables are used:

$$\begin{cases} a_1 \bar{\xi}_1 = \zeta_1, & a_2 \bar{\xi}_2 = \zeta_2, & a_3 \bar{\xi}_3 = \zeta_3, & \zeta_1/\zeta = \bar{\zeta}_1, & \zeta_2/\zeta = \bar{\zeta}_2, \\ \zeta_3/\zeta = \bar{\zeta}_3, & \zeta = (\zeta_1^2 + \zeta_2^2 + \zeta_3^2)^{1/2}, & dS(\bar{\zeta}) = a_1 a_2 a_3 \zeta^{-3} dS(\bar{\xi}). \end{cases} \quad (18)$$

With Eq. (18), we can write the induced displacement within the inclusion as^[26]

$$U_K(\mathbf{x} - \mathbf{x}') = \frac{1}{4\pi} F_{IjAb} Z_{Ab}^* x_l G_{KIjl}, \quad (19)$$

where

$$G_{KIjl} = \int_{-1}^1 \int_0^{2\pi} N_{KI}(\bar{\zeta}) D^{-1}(\bar{\zeta}) \bar{\zeta}_j \bar{\zeta}_l d\theta d\bar{\zeta}_3, \quad (20)$$

in which

$$\bar{\zeta}_1 = (1 - \bar{\zeta}_3^2)^{1/2} \cos \theta, \quad \bar{\zeta}_2 = (1 - \bar{\zeta}_3^2)^{1/2} \sin \theta, \quad \bar{\zeta}_3 = \bar{\zeta}_3, \quad (21)$$

and x_l is the component of an arbitrary point $\mathbf{x}$. Using Eqs.(4), (19), and (20), the strain field Z_{KI} and the eigen strain field of inclusion Z_{Ab}^* can be expressed as the following linear relationship:

$$Z_{KI} = S_{KIAb} Z_{Ab}^* \quad \text{in } \Omega, \quad (22)$$

where S_{KIAb} can be expressed as

$$S_{KIAb} = \begin{cases} \frac{1}{8\pi} F_{IjAb} (G_{kIjl} + G_{lIjk}), & K = k = 1, 2, 3, \\ \frac{1}{4\pi} F_{IjAb} G_{4Ijl}, & K = 4, \end{cases} \quad (23)$$

and S_{KLAb} is the Eshelby tensor for the 1D hexagonal quasicrystal.

Substituting Eq. (5) into Eq. (23) yields

$$\begin{cases} S_{klab} = \frac{1}{8\pi}(C_{ijab}(G_{kijl} + G_{lijl}) + R_{ab3j}(G_{k4jl} + G_{l4jk})), & k = 1, 2, 3, \quad a = 1, 2, 3, \\ S_{kl4b} = \frac{1}{8\pi}(R_{ij3b}(G_{kijl} + G_{lijl}) + K_{3j3b}(G_{k4jl} + G_{l4jk})), & k = 1, 2, 3, \\ S_{4lab} = \frac{1}{4\pi}(C_{ijab}G_{4ijl} + R_{ab3j}G_{44jl}), & a = 1, 2, 3, \\ S_{4l4b} = \frac{1}{4\pi}(R_{ij3b}G_{4ijl} + K_{3j3b}G_{44jl}), \end{cases} \quad (24)$$

in which $l, b = 1, 2, 3$. For convenience, Eq. (2) is given in detail as follows:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \\ H_{33} \\ H_{32} \\ H_{31} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 & R_1 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 & R_1 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 & R_3 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 & 0 & R_5 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 & 0 & 0 & R_5 \\ 0 & 0 & 0 & 0 & 0 & C_{66} & 0 & 0 & 0 \\ R_1 & R_1 & R_3 & 0 & 0 & 0 & K_3 & 0 & 0 \\ 0 & 0 & 0 & R_5 & 0 & 0 & 0 & K_1 & 0 \\ 0 & 0 & 0 & 0 & R_5 & 0 & 0 & 0 & K_1 \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \\ w_{3,3} \\ w_{3,2} \\ w_{3,1} \end{pmatrix}, \quad (25)$$

where $C_{66} = (C_{11} - C_{12})/2$.

Using Eqs. (19)–(21), (23), and (24), the closed forms of Eshelby tensors in the special cases of spheroid, elliptic, ribbon-like, penny-shaped, and rod-shaped inclusions embedded in the 1D hexagonal quasicrystal matrices are presented in Appendix A. In particular, when the contribution of the phason field is ignored, the degradation results are consistent with those in the existing literature^[26]. In addition, when phason coupling is absent, the expressions in Appendix A are consistent with the previous results of the piezoelectric medium without piezoelectric coupling^[27], verifying the accuracy of the model.

4 Effective properties of 1D hexagonal quasicrystal composites

A two-phase 1D hexagonal quasicrystal composite with phases perfectly bonded is considered. According to Refs [27] and [28], we use the tensorial notations to write

$$\overline{\mathbf{Z}} = v_1 \overline{\mathbf{Z}}_1 + v_2 \overline{\mathbf{Z}}_2, \quad \overline{\mathbf{\Xi}} = v_1 \overline{\mathbf{\Xi}}_1 + v_2 \overline{\mathbf{\Xi}}_2, \quad (26)$$

where $\mathbf{Z}$ and $\mathbf{\Xi}$ are defined in Eqs. (4) and (5), respectively. The subscripts 1 and 2 denote the matrix and the inclusion of the 1D hexagonal quasicrystal composites, respectively. In addition, v_j is the volume fraction of the phase j , $j = 1, 2$, and the overbar denotes the volume average.

$\overline{\mathbf{\Xi}}$ and $\overline{\mathbf{Z}}$ are related, given by

$$\overline{\mathbf{\Xi}} = \mathbf{F} \overline{\mathbf{Z}}, \quad (27)$$

where $\mathbf{F}$ is the effective elastic modulus tensor of 1D hexagonal quasicrystal composites.

Assume that there is a linear relationship between $\overline{\mathbf{Z}}$ and $\overline{\mathbf{Z}}_2^{[27]}$, denoted by

$$\overline{\mathbf{Z}}_2 = \mathbf{A} \overline{\mathbf{Z}}, \quad (28)$$

where $\mathbf{A}$ is the strain-potential gradient concentration matrix, which is obtained by the Mori-Tanaka approach^[20].

Using Eqs. (26)–(28), we readily find that

$$\mathbf{F} = \mathbf{F}_1 + v_2(\mathbf{F}_2 - \mathbf{F}_1)\mathbf{A}. \quad (29)$$

In the Mori-Tanaka approach, we further assume that

$$\overline{\mathbf{Z}}_2 = \mathbf{A}^{\text{dil}}\overline{\mathbf{Z}}_1, \quad (30)$$

where $\mathbf{A}^{\text{dil}}$ is a concentration tensor and given by^[29]

$$\mathbf{A}^{\text{dil}} = (\mathbf{I} + \mathbf{S}\mathbf{F}_1^{-1}(\mathbf{F}_2 - \mathbf{F}_1))^{-1}, \quad (31)$$

in which $\mathbf{S}$ is the 1D hexagonal quasicrystal Eshelby tensor defined in Eq. (23), and $\mathbf{I}$ is the identity matrix.

Using Eqs. (26), (28), and (30), we have

$$\mathbf{A} = \mathbf{A}^{\text{dil}}(v_1\mathbf{I} + v_2\mathbf{A}^{\text{dil}})^{-1}. \quad (32)$$

Substituting Eq. (32) into Eq. (29) yields

$$\mathbf{F} = \mathbf{F}_1 + v_2(\mathbf{F}_2 - \mathbf{F}_1)\mathbf{A}^{\text{dil}}(v_1\mathbf{I} + v_2\mathbf{A}^{\text{dil}})^{-1}. \quad (33)$$

5 Numerical results

In this section, the cylindrical hole is treated as the special cases of the rod-shaped inclusion (see Appendix A) to discuss the effective elastic moduli of 1D hexagonal quasicrystal composites. This indicates that the inclusion has no stiffness that can be considered as zero. We then have that $\mathbf{F}_2 = \mathbf{0}$.

The elastic constants of 1D hexagonal quasicrystals are shown in Table 1^[22–23].

Table 1 Elastic constants of quasicrystals^[22–23]

Stress field	Elastic constant/GPa				
Phonon	$C_{11} = 234.33,$	$C_{12} = 57.41,$	$C_{13} = 66.63,$	$C_{33} = 232.22,$	$C_{44} = 70.19$
Phason	$K_1 = 24, \quad K_3 = 122$				
Phonon-phason coupling	$R_1 = 0.8846,$	$R_3 = 0.8846,$	$R_5 = 0.8846$		

According to Eqs. (25), (31), (33), and the rod-shaped inclusion in Appendix A, the effective elastic moduli of 1D hexagonal quasicrystal composites can be obtained.

The numerical results of the elastic constants for the effective 1D hexagonal quasicrystal composites in terms of the pore volume fraction v_2 are shown in Fig. 2, where the short notations for the elastic constants are used, in which C_{ijkl}^q , K_{3i3l}^q , and R_{ijkl}^q stand for the elastic moduli in the phonon field, phason field, and phonon-phason coupling field of 1D hexagonal quasicrystal composites, respectively. It can be seen that all the elastic constants decrease as the volume fraction of the cylindrical hole increases. In the research range, the decreases in the phonon constant C_{33}^q and the phason constant K_3^q with the volume fraction are almost linear. The calculation formulae of C_{33}^q and K_3^q are as follows:

$$C_{33}^q = \frac{(2C_{13}^2 - C_{11}C_{33} - C_{12}C_{33})v_2^2 + (2C_{12}C_{33} - 2C_{13}^2)v_2 + (C_{11}C_{33} - C_{12}C_{33})}{(C_{11} + C_{12})v_2 + (C_{11} - C_{12})}, \quad (34)$$

$$K_3^q = \frac{(2R_1^2 - C_{11}K_3 - C_{12}K_3)v_2^2 + (2C_{12}K_3 - 2R_1^2)v_2 + (C_{11}K_3 - C_{12}K_3)}{(C_{11} + C_{12})v_2 + (C_{11} - C_{12})}. \quad (35)$$

When the contribution of the phason field is ignored in the numerical example, the degradation results are consistent with the previous results of the piezoelectric medium without piezoelectric coupling^[29].

Fig. 2 Graphs of (a) and (b) phonon constants, (c) phason constants, and (d) phonon-phason coupling constants versus the volume fraction v_2 in the interval $(0, 0.5)$ (color online)

6 Conclusions

By introducing Eshelby tensors into the quasicrystals, the solution to the ellipsoid inclusion problem of 1D hexagonal quasicrystals is obtained. It is revealed that the Eshelby value of 1D hexagonal quasicrystals depends on the shape of the inclusion and the material constants. The Eshelby tensor of the 1D hexagonal quasicrystal is explicitly determined. The explicit formulation for the effective elastic moduli is obtained with the Mori-Tanaka method. In particular, the numerical examples of 1D hexagonal quasicrystals with cylindrical holes are given. The results show that the effective elastic properties of 1D hexagonal quasicrystal composites decrease with the increase in the porous volume fraction.

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Appendix A

(i) Spheroid ($a_1 = a_2, a_1/a_3 = \rho$)

$$\begin{aligned}
 S_{1111} &= \frac{1}{4\pi}(C_{11}G_{1111} + C_{12}G_{1221} + C_{13}G_{1331} + R_1G_{1431}), \\
 S_{1122} &= \frac{1}{4\pi}(C_{12}G_{1111} + C_{11}G_{1221} + C_{13}G_{1331} + R_1G_{1431}), \\
 S_{1133} &= \frac{1}{4\pi}(C_{13}G_{1111} + C_{13}G_{1221} + C_{33}G_{1331} + R_3G_{1431}), \\
 S_{1143} &= \frac{1}{4\pi}(R_1G_{1111} + R_1G_{1221} + R_3G_{1331} + K_3G_{1431}), \\
 S_{2211} &= \frac{1}{4\pi}(C_{11}G_{2112} + C_{12}G_{2222} + C_{13}G_{2332} + R_1G_{2432}), \\
 S_{2222} &= \frac{1}{4\pi}(C_{12}G_{2112} + C_{11}G_{2222} + C_{13}G_{2332} + R_1G_{2432}), \\
 S_{2233} &= \frac{1}{4\pi}(C_{13}G_{2112} + C_{13}G_{2222} + C_{33}G_{2332} + R_3G_{2432}), \\
 S_{2243} &= \frac{1}{4\pi}(R_1G_{2112} + R_1G_{2222} + R_3G_{2332} + K_3G_{2432}), \\
 S_{1212} = S_{1221} = S_{2112} = S_{2121} &= \frac{1}{8\pi}(C_{66}(G_{1212} + G_{2211}) + C_{66}(G_{1122} + G_{2121})), \\
 S_{1313} = S_{1331} = S_{3113} = S_{3131} \\
 &= \frac{1}{8\pi}(C_{44}(G_{1313} + G_{3311}) + C_{44}(G_{1133} + G_{3131}) + R_5(G_{1413} + G_{3411})), \\
 S_{1341} = S_{3141} &= \frac{1}{8\pi}(R_5(G_{1313} + G_{3311}) + R_5(G_{1133} + G_{3131}) + K_1(G_{1413} + G_{3411})), \\
 S_{2323} = S_{2332} = S_{3223} = S_{3232} \\
 &= \frac{1}{8\pi}(C_{44}(G_{2323} + G_{3322}) + C_{44}(G_{2233} + G_{3232}) + R_5(G_{3422} + G_{2423})), \\
 S_{2342} = S_{3242} &= \frac{1}{8\pi}(R_5(G_{2323} + G_{3322}) + R_5(G_{2233} + G_{3232}) + K_1(G_{3422} + G_{2423})), \\
 S_{3311} &= \frac{1}{4\pi}(C_{11}G_{3113} + C_{12}G_{3223} + C_{13}G_{3333} + R_1G_{3433}), \\
 S_{3322} &= \frac{1}{4\pi}(C_{12}G_{3113} + C_{11}G_{3223} + C_{13}G_{3333} + R_1G_{3433}), \\
 S_{3333} &= \frac{1}{4\pi}(C_{13}G_{3113} + C_{13}G_{3223} + C_{33}G_{3333} + R_3G_{3433}), \\
 S_{3343} &= \frac{1}{4\pi}(R_1G_{3113} + R_1G_{3223} + R_3G_{3333} + K_3G_{3433}), \\
 S_{4113} = S_{4131} &= \frac{1}{4\pi}(C_{44}H_{3141} + C_{44}H_{1143} + R_5H_{1144}), \\
 S_{4141} &= \frac{1}{4\pi}(R_5G_{4311} + R_5G_{4131} + K_1G_{4411}), \\
 S_{4223} = S_{4232} &= \frac{1}{4\pi}(C_{44}G_{4232} + C_{44}G_{4322} + R_5G_{4422}), \\
 S_{4242} &= \frac{1}{4\pi}(R_5G_{4232} + R_5G_{4322} + K_1G_{4422}),
 \end{aligned}$$

$$\begin{aligned}
S_{4311} &= \frac{1}{4\pi}(C_{11}G_{4113} + C_{12}G_{4223} + C_{13}G_{4333} + R_1G_{4433}), \\
S_{4322} &= \frac{1}{4\pi}(C_{12}G_{4113} + C_{11}G_{4223} + C_{13}G_{4333} + R_1G_{4433}), \\
S_{4333} &= \frac{1}{4\pi}(C_{13}G_{4113} + C_{13}G_{4223} + C_{33}G_{4333} + R_3G_{4433}), \\
S_{4343} &= \frac{1}{4\pi}(R_1G_{4113} + R_1G_{4223} + R_3G_{4333} + K_3G_{4433}).
\end{aligned}$$

(ii) Elliptic cylinder ($a_1/a_2 = a, a_3 \rightarrow \infty$)

$$\begin{aligned}
S_{1111} &= \frac{a}{2(a+1)^2} \left(\frac{2C_{11} + C_{12}}{C_{11}} + \frac{a+2}{a} \right), \quad S_{1122} = \frac{a}{2(a+1)^2} \left(\frac{(a+2)C_{12}}{aC_{11}} - 1 \right), \\
S_{1133} &= \frac{C_{13}}{(a+1)C_{11}}, \quad S_{1143} = \frac{R_1}{(a+1)C_{11}}, \quad S_{2211} = \frac{a}{2(a+1)^2} \left(\frac{(1+2a)C_{12}}{C_{11}} - 1 \right), \\
S_{2222} &= \frac{a}{2(a+1)^2} \left(\frac{3C_{11} + C_{12}}{C_{11}} + 2a \right), \quad S_{2233} = \frac{aC_{13}}{(a+1)C_{11}}, \quad S_{2243} = \frac{aR_1}{(a+1)C_{11}}, \\
S_{1212} &= S_{1221} = S_{2112} = S_{2121} = \frac{a}{2(a+1)^2} \left(\frac{a^2 + a + 1}{a} - \frac{C_{12}}{C_{11}} \right), \\
S_{1313} &= S_{1331} = S_{3113} = S_{3131} = \frac{1}{2(a+1)}, \quad S_{4141} = \frac{1}{a+1}, \\
S_{2323} &= S_{2332} = S_{3223} = S_{3232} = \frac{a}{2(a+1)}, \quad S_{4242} = \frac{a}{a+1}.
\end{aligned}$$

(iii) Ribbon-like ($a_1 \ll a_2, a_1/a_2 = a, a_3 \rightarrow \infty$)

$$\begin{aligned}
S_{1111} &= 1 - \frac{a(C_{11} - C_{12})}{2C_{11}}, \quad S_{1122} = \frac{C_{12}}{C_{11}} - \frac{a(C_{11} + 3C_{12})}{2C_{11}}, \quad S_{1133} = (1-a)\frac{C_{13}}{C_{11}}, \\
S_{1143} &= (1-a)\frac{R_1}{C_{11}}, \quad S_{2211} = -a\frac{C_{11} - C_{12}}{2C_{11}}, \quad S_{2222} = a\frac{3C_{11} + C_{12}}{2C_{11}}, \\
S_{2233} &= a\frac{C_{13}}{C_{11}}, \quad S_{2243} = a\frac{R_1}{C_{11}}, \\
S_{1212} &= S_{1221} = S_{2112} = S_{2121} = \frac{1}{2} - \frac{a(C_{11} + C_{12})}{2C_{11}}, \\
S_{1313} &= S_{1331} = S_{3113} = S_{3131} = \frac{1-a}{2}, \quad S_{4141} = 1-a, \\
S_{2323} &= S_{2332} = S_{3223} = S_{3232} = \frac{a}{2}, \quad S_{4242} = a.
\end{aligned}$$

(iv) Penny-shaped ($a_1 = a_2 \gg a_3, a_3 \rightarrow 0$)

$$\begin{aligned}
S_{1313} &= S_{1331} = S_{3113} = S_{3131} = S_{2323} = S_{2332} = S_{3223} = S_{3232} = \frac{1}{2}, \\
S_{1341} &= S_{3141} = S_{2342} = S_{3242} = \frac{R_5}{2C_{44}}, \quad S_{3311} = S_{3322} = \frac{C_{13}K_3 - R_1R_3}{C_{33}K_3 - R_3^2}, \\
S_{3333} &= S_{4343} = 1, \quad S_{4311} = S_{4322} = \frac{C_{33}R_1 - C_{13}R_3}{C_{33}K_3 - R_3^2}.
\end{aligned}$$

(v) Rod-shaped ($a_1 = a_2, a_3 \rightarrow \infty$)

$$\begin{aligned}
S_{1111} &= S_{2222} = \frac{5C_{11} + C_{12}}{8C_{11}}, \quad S_{1122} = S_{2211} = \frac{3C_{12} - C_{11}}{8C_{11}}, \\
S_{1133} &= S_{2233} = \frac{C_{13}}{2C_{11}}, \quad S_{1143} = S_{2243} = \frac{R_1}{2C_{11}}, \\
S_{1212} &= S_{1221} = S_{2112} = S_{2121} = \frac{3C_{11} - C_{12}}{8C_{11}}, \\
S_{1313} &= S_{1331} = S_{3113} = S_{3131} = S_{2323} = S_{2332} = S_{3223} = S_{3232} = \frac{1}{4}, \\
S_{4141} &= S_{4242} = \frac{1}{2}.
\end{aligned}$$