Numerical solution to the Falkner-Skan equation: a novel numerical approach through the new rational α -polynomials*

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Abstract The new rational α -polynomials are used to solve the Falkner-Skan equation. These polynomials are equipped with an auxiliary parameter. The approximated solution to the Falkner-Skan equation is obtained by the new rational α -polynomials with unknown coefficients. To find the unknown coefficients and the auxiliary parameter contained in the polynomials, the collocation method with Chebyshev-Gauss points is used. The numerical examples show the efficiency of this method.

Key words Falkner-Skan equation, rational Chebyshev polynomial, mapping parameter, collocation method, singular Sturm-Liouville problem

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1 Introduction

Nonlinear ordinary differential equations play an essential role in scientific, technical, economic, and mathematical modeling. Applications of nonlinear ordinary differential equations include the calculation of trajectories of space gliders and airplanes, forecast calculations, spread of AIDS, and automated parking maneuvering based on mechanical systems^[1]. Boundary layer equations are an important class of nonlinear ordinary differential equations that have many applications in physics and fluid mechanics^[2–3]. Applications of boundary layer equations include metal plate cooling in a cooling bath, aerodynamic extrusion of plastic sheets, plastic film design, metal spinning, metallic plates, insulation materials, and glass and polymer studies^[4]. One class of these boundary layer equations is the stationary Falkner-Skan boundary layer equation. The problem is very old and has already been addressed with many different numerical methods. Furthermore, computational fluid dynamics have already evolved in more advanced topics. Efficient numerical methods for boundary layer equations were developed 50 years ago, and finding a new computational method will be useful for solving new problems. In this paper, the authors aim to present a new method for the Falkner-Skan equation, which may be applicable to other problems.

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The Falkner-Skan equation plays an important role in fluid mechanics, aerospace, and heat transfer. In fluid dynamics, the Falkner-Skan boundary layer equation describes the two-dimensional laminar boundary layer formed on a wedge, where the plate is not parallel to the flow. The Falkner-Skan equation is the generalized state of the Blasius boundary layer equations.

Different methods have been used to solve the Falkner-Skan equation, involving exponential second kind Chebyshev^[5-6], rational Chebyshev functions^[7-8], iterative method^[9], Chebyshev cardinal functions^[10], shifted Chebyshev collocation method^[11], piecewise linear functions^[12], Legendre rational polynomials^[13], Chebyshev finite difference method^[14], Adomian's decomposition method^[15], homotopy analysis method^[16], Sinc-collocation method^[17], finite-difference method^[18], rational Legendre functions^[19], spline functions^[20], group invariance theory^[21], Gegenbauer neural network^[22], and very recently the wavelet method^[23].

In this paper, new a -polynomials equipped with parameter a are used to solve this equation. These polynomials were used in Ref. [24] to solve the Benjamin-Bona-Mahony-Burgers (BBMB) equations and in Ref. [25] to solve an inverse problem.

2 Problem formulation

In this section, we explain how to obtain the Falkner-Skan ordinary differential equation. Assume the following boundary layer equations for steady incompressible flow with constant viscosity and density^[26]:

$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = U \frac{dU}{dx} + \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho} (u - U) \end{cases} \quad (1)$$

with the conditions

$$\begin{cases} u(x, 0) = 0, & v(x, 0) = 0, \\ u = U(x), & y \rightarrow \infty, \\ U(x) = ax. \end{cases} \quad (2)$$

Here, the coordinate system is chosen with x pointing parallel to the plate in the direction of the flow and y -coordinate pointing towards the free stream. u and v are the x - and y -velocity components, respectively. U is the inherent characteristic velocity. σ is the electrical conductivity. ρ is the fluid density. ν is the kinematic viscosity.

We can generalize the Blasius boundary layer by considering a wedge at an angle of attack $\pi\beta/2$ from some uniform velocity field U_0 . We then estimate the outer flow to be of the form

$$u_e(x) = U_0 \left(\frac{x}{l} \right)^m,$$

where l is a characteristic length, and m is a dimensionless constant. In the Blasius solution, $m = 0$, corresponding to an angle of attack of zero radians. Thus, we can write

$$\beta = \frac{2m}{2m+1}.$$

In the Blasius solution, we use a similarity variable η to solve the boundary layer equations. We will obtain^[27]

$$\eta = y \sqrt{\frac{U_0(m+1)}{2\nu l}} \left(\frac{x}{l} \right)^{(m-1)/2}, \quad \psi = \sqrt{\frac{2l\nu U_0}{m+1}} \left(\frac{x}{l} \right)^{(m+1)/2} f(\eta), \quad u = U_0 f', \quad v = -\sqrt{\frac{\nu U_0}{x}} f.$$

Therefore, Eqs. (1) and (2) can be transformed into the following third-order nonlinear Falkner-Skan differential equation:

$$\frac{d^3 f}{d\eta^3} + f \frac{d^2 f}{d\eta^2} + \beta \left(1 - \left(\frac{df}{d\eta} \right)^2 \right) - M^2 \left(\frac{df}{d\eta} - 1 \right) = 0 \quad (3)$$

with the boundary conditions

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1,$$

in which $M^2 = \sigma B_0^2 / (\rho a)$. No exact analytical solution is known for Eq. (3).

3 Rational a -polynomials

In this section, the concept of a new class of rational polynomials is introduced. Also, by changing the coordinate, the unbounded domain is considered.

Definition^[28] Assume that a is a constant parameter and $A_0(z) = 1$. The a -polynomials as a combination of the Chebyshev polynomials of the second kind, $U_n(z)$, are defined by

$$A_n(z) = azU_{n-1}(z) + U_n(z), \quad z \in [-1, 1], \quad n \geq 1. \quad (4)$$

The following equations are also established:

$$A_{n+1}(z) = 2zA_n(z) - A_{n-1}(z), \quad n \geq 1, \quad (5)$$

$$A_n(z) = \left(1 + \frac{a}{2} \right) U_n(z) + \frac{a}{2} U_{n-2}(z), \quad n \geq 2. \quad (6)$$

See Refs. [24] and [28] for more properties. By changing the variable $z = \frac{\eta-L}{\eta+L}$ for $\eta \in [0, +\infty)$, in which L is an arbitrary large positive number and will be chosen later, the following new rational a -polynomials are obtained from Eq. (4):

$$AL_n(\eta) = a \frac{\eta-L}{\eta+L} U_{n-1} \left(\frac{\eta-L}{\eta+L} \right) + U_n \left(\frac{\eta-L}{\eta+L} \right), \quad \eta \in [0, +\infty), \quad n \geq 1 \quad (7)$$

with $AL_0(\eta) = 1$. The constant L is called the mapping parameter. By using

$$R_n(\eta) = U_n \left(\frac{\eta-L}{\eta+L} \right), \quad (8)$$

according to Eq. (6), we have

$$AL_n(\eta) = \left(1 + \frac{a}{2} \right) R_n(\eta) + \frac{a}{2} R_{n-2}(\eta), \quad n \geq 2. \quad (9)$$

Proposition 1 The following recursive relation holds for the rational a -polynomials:

$$AL_{n+1}(\eta) = 2 \frac{\eta-L}{\eta+L} AL_n(\eta) - AL_{n-1}(\eta), \quad n \geq 1.$$

Proposition 2 Considering the inner product

$$(f, g)_\omega = \int_0^\infty \omega(\eta) f(\eta) g(\eta) d\eta, \quad (10)$$

where $\omega(\eta) = \frac{4L\sqrt{L\eta}}{(\eta+L)^3}$ is the positive weight function, we have

$$(AL_n, AL_m)_\omega = \begin{cases} \frac{\pi}{2}, & n = m = 0, \\ \frac{1}{8}(2+a)^2\pi, & n = m = 1, \\ \frac{1}{4}(2+2a+a^2)\pi, & n = m \geq 2, \\ \frac{a\pi}{4}, & |n-m| = 2, \quad nm = 0, \\ \frac{1}{8}a(2+a)\pi, & |n-m| = 2, \quad nm \neq 0, \\ 0, & \text{otherwise} \end{cases}$$

and

$$(R_n, R_m)_\omega = \frac{\pi}{2}\delta_{n,m}, \quad (11)$$

where $\delta_{n,m}$ is the delta Kronecker number.

Proposition 3 The system $\{AL_{4n}\}_{n=0}^\infty$ is an orthogonal system with respect to the inner product defined in Proposition 2.

Proposition 4 U_n is the eigenfunction of the singular Sturm-Liouville problem

$$\left((1-\eta^2)^{-1/2} \frac{d}{d\eta} \left((1-\eta^2)^{3/2} \frac{d}{d\eta}\right) + n(n+2)\right)U_n(\eta) = 0, \quad (12)$$

and R_n is the eigenfunction of the singular Sturm-Liouville problem

$$\left(\frac{(\eta+L)^3}{4L\sqrt{L\eta}} \frac{d}{d\eta} \left(\frac{4\eta\sqrt{L\eta}}{(\eta+L)} \frac{d}{d\eta}\right) + n(n+2)\right)R_n(\eta) = 0, \quad n = 0, 1, 2, \dots \quad (13)$$

Proposition 5 Suppose that $\omega_1(\eta) = \frac{4\eta\sqrt{L\eta}}{\eta+L}$. Then, we have

$$\int_0^\infty \frac{dR_n(\eta)}{d\eta} \frac{dR_m(\eta)}{d\eta} \omega_1(\eta) d\eta = \frac{1}{2}n(n+2)\delta_{n,m}. \quad (14)$$

Remark 1

$$\lim_{\eta \rightarrow \infty} R_n(\eta) = n+1, \quad \lim_{\eta \rightarrow \infty} \frac{dR_n(\eta)}{d\eta} = 0, \quad \lim_{\eta \rightarrow \infty} \left(\frac{4\eta\sqrt{L\eta}}{\eta+L}\right) \frac{dR_n(\eta)}{d\eta} = 0. \quad (15)$$

4 Convergence and stability theorems

4.1 The convergence theorem

Assume $\Lambda = [0, +\infty)$, ω is the weight function in Proposition 2, and $L_\omega^2(\Lambda)$ is the function Hilbert space with the inner product of Eq. (10). Let N be the positive integer, and we will consider

$$\mathcal{R}_N = \text{span}\{R_0, R_1, \dots, R_N\},$$

which is the subspace of $L_\omega^2(\Lambda)$. We define the $L_\omega^2(\Lambda)$ -orthogonal projection as follows:

$$\begin{cases} P_N : L_\omega^2(\Lambda) \rightarrow \mathcal{R}_N, \\ (P_N u)(\eta) = \sum_{i=0}^N c_i R_i(\eta) \end{cases} \quad (16)$$

satisfying

$$(P_N u - u, \varphi)_\omega = 0, \quad \forall \varphi \in \mathcal{R}_N.$$

To estimate $\|P_N u - u\|_\omega$, we have the following space interpolation:

$$H_{\omega, Q}^r(\Lambda) = \{u | u \text{ is measurable and } \|u\|_{r, \omega, Q} < \infty\},$$

where any real $r > 0$, Q is the Sturm-Liouville operator of Chebyshev rational polynomials, and $\|u\|_{r, \omega, Q}$ is in the following norm:

$$Qv(\eta) = -\omega^{-1}(\eta) \frac{d}{d\eta} \left(\frac{4\eta\sqrt{L\eta}}{\eta+L} \frac{d}{d\eta} v(\eta) \right) = -\frac{\eta(\eta+L)^2}{L} \frac{d^2 v}{d\eta^2} - \frac{(\eta+L)(\eta+3L)}{2L} \frac{dv}{d\eta}, \quad (17)$$

$$\|u\|_{r, \omega, Q} = \left(\sum_{i=0}^r \left\| (\eta+L)^{\frac{r}{2}+i} \frac{d^i u}{d\eta^i} \right\|_\omega^2 \right)^{1/2}. \quad (18)$$

Lemma 1 Q^m is a continuous mapping from $H_{\omega, Q}^{2m}(\Lambda)$ to $L_\omega^2(\Lambda)$.

Proof It is easily proved by inducting

$$Q^m v(\eta) = \sum_{k=1}^{2m} (\eta+L)^{m+k} q_k(\eta) \frac{d^k v(\eta)}{d\eta^k}, \quad (19)$$

where q_k is a rational function bounded uniformly on the whole interval Λ . The desired result is obtained from Eqs. (18) and (19).

Theorem 1 For any real $r \geq 0$, $v \in H_{\omega, Q}^r(\Lambda)$, $v = \sum_{n=0}^{\infty} \hat{v}_n R_n(\eta)$, and $c \in \mathbb{R}$, there is

$$\|P_N v - v\|_\omega \leq cN^{-r} \|v\|_{r, \omega, Q}. \quad (20)$$

Proof First, we assume that $r = 2m$. Due to Eqs. (11), (13), and (17) and the integration by parts, we obtain

$$\begin{aligned} \hat{v}_n &= \frac{2}{\pi} \int_\Lambda v(\eta) R_n(\eta) \omega(\eta) d\eta = \frac{2}{\pi n(n+2)} \int_\Lambda v(\eta) Q R_n(\eta) \omega(\eta) d\eta \\ &= -\frac{2}{\pi n(n+2)} \int_\Lambda v(\eta) \frac{d}{d\eta} \left(\frac{4\eta\sqrt{L\eta}}{\eta+L} \frac{d}{d\eta} R_n(\eta) \right) d\eta \\ &= \frac{2}{\pi n(n+2)} \int_\Lambda \frac{4\eta\sqrt{L\eta}}{\eta+L} \frac{d}{d\eta} v(\eta) \left(\frac{d}{d\eta} R_n(\eta) \right) d\eta \\ &= -\frac{2}{\pi n(n+2)} \int_\Lambda \frac{d}{d\eta} \left(\frac{4\eta\sqrt{L\eta}}{\eta+L} \frac{d}{d\eta} v(\eta) \right) R_n(\eta) d\eta \\ &= \frac{2}{\pi n(n+2)} \int_\Lambda Q v(\eta) R_n(\eta) \omega(\eta) d\eta \\ &= \cdots = \frac{2}{\pi n^m(n+2)^m} \int_\Lambda Q^m v(\eta) R_n(\eta) \omega(\eta) d\eta. \end{aligned} \quad (21)$$

Now, according to Eqs. (19) and (21) and the definition of $H_{\omega, Q}^r(\Lambda)$, we have

$$\begin{aligned} \|P_N v - v\|_\omega^2 &= \sum_{n=N+1}^{\infty} \hat{v}_n^2 \|R_n\|_\omega^2 \\ &\leq cN^{-4m} \sum_{n=N+1}^{\infty} \left(\frac{\int_\Lambda Q^m v(\eta) R_n(\eta) \omega(\eta) d\eta}{\|R_n\|_\omega} \right)^2 \|R_n\|_\omega^2 \\ &\leq cN^{-4m} \|Q^m v\|_\omega^2 \leq cN^{-4m} \|v\|_{r, \omega, Q}^2. \end{aligned}$$

Next, we put $r = 2m + 1$. With Eq. (13) and the integration by parts, we have

$$\begin{aligned}\hat{v}_n &= \frac{2}{\pi n^m(n+2)^m} \int_{\Lambda} Q^m v(\eta) R_n(\eta) \omega(\eta) d\eta \\ &= -\frac{2}{\pi n^{m+1}(n+2)^{m+1}} \int_{\Lambda} Q^m v(\eta) \frac{d}{d\eta} \left(\frac{4\eta\sqrt{L\eta}}{\eta+L} \frac{d}{d\eta} R_n(\eta) \right) d\eta \\ &= -\frac{2}{\pi n^{m+1}(n+2)^{m+1}} \int_{\Lambda} \frac{d}{d\eta} (Q^m v(\eta)) \frac{d}{d\eta} R_n(\eta) \omega_1(\eta) d\eta.\end{aligned}\quad (22)$$

Now, with Eqs. (14) and (19), we obtain

$$\begin{aligned}\|P_N v - v\|_{\omega}^2 &= \sum_{n=N+1}^{\infty} \hat{v}_n^2 \|R_n\|_{\omega}^2 \\ &= \sum_{n=N+1}^{\infty} \frac{4}{\pi^2(n(n+2))^{2m+2}} \left(\int_{\Lambda} \frac{d}{d\eta} (Q^m v(\eta)) \frac{d}{d\eta} R_n(\eta) \omega_1(\eta) d\eta \right)^2 \\ &= \sum_{n=N+1}^{\infty} \frac{4}{\pi^2(n(n+2))^{2m+2}} \left(\frac{\int_{\Lambda} \frac{d}{d\eta} (Q^m v(\eta)) \frac{d}{d\eta} R_n(\eta) \omega_1(\eta) d\eta}{\left\| \frac{d}{d\eta} R_n \right\|_{\omega_1}^2} \right)^2 \left\| \frac{d}{d\eta} R_n \right\|_{\omega_1}^2 \\ &\leq cN^{-2(2m+1)} \sum_{n=N+1}^{\infty} \left(\frac{\int_{\Lambda} \frac{d}{d\eta} (Q^m v(\eta)) \frac{d}{d\eta} R_n(\eta) \omega_1(\eta) d\eta}{\left\| \frac{d}{d\eta} R_n \right\|_{\omega_1}^2} \right)^2 \left\| \frac{d}{d\eta} R_n \right\|_{\omega_1}^2 \\ &\leq cN^{-2(2m+1)} \left\| \frac{d}{d\eta} (Q^m v) \right\|_{\omega_1}^2 \leq cN^{-2(2m+1)} \left\| \frac{d}{d\eta} (Q^m v)(\eta+L)^{7/2} \right\|_{\omega}^2 \\ &\leq cN^{-2(2m+1)} \|v\|_{r,\omega,Q}^2.\end{aligned}$$

The general result follows from the previous results and space interpolation.

Now, we get the approximation error of $f = \sum_{i=0}^{\infty} c_i AL_i(\eta)$ by $f_N(\eta) = \sum_{i=0}^N c_i AL_i(\eta)$.

Theorem 2 For any real $r > 0$ and $f \in H_{\omega,Q}^r(\Lambda)$, we have

$$\|f_N - f\|_{\omega} \leq \tilde{a}c(N-2)^{-r} \|f\|_{r,\omega,Q}. \quad (23)$$

Proof Using Eq. (9) and Theorem 1, we get the following inequality:

$$\begin{aligned}\|f_N - f\|_{\omega} &= \left\| \sum_{i=N+1}^{\infty} c_i AL_i(\eta) \right\|_{\omega} = \left\| \sum_{i=N+1}^{\infty} c_i \underbrace{\left(\left(1 + \frac{a}{2}\right) R_i(\eta) + \frac{a}{2} R_{i-2}(\eta) \right)}_{\text{Eq. (9)}} \right\|_{\omega} \\ &\leq \left| 1 + \frac{a}{2} \right| \left\| \sum_{i=N+1}^{\infty} c_i R_i(\eta) \right\|_{\omega} + \left| \frac{a}{2} \right| \left\| \sum_{i=N+1}^{\infty} c_i R_{i-2}(\eta) \right\|_{\omega} \\ &\leq \underbrace{\left| 1 + \frac{a}{2} \right| c' N^{-r} \|f\|_{r,\omega,Q}}_{\text{Eq. (20)}} + \underbrace{\left| \frac{a}{2} \right| c'' (N-2)^{-r} \|f\|_{r,\omega,Q}}_{\text{Eq. (20)}} \leq \tilde{a}c(N-2)^{-r} \|f\|_{r,\omega,Q},\end{aligned}$$

where $\tilde{a} = \max\{|1 + \frac{a}{2}|, |\frac{a}{2}|\}$, and $c = \max\{c', c''\}$.

This theorem shows that the rational a -polynomial approximation has exponential convergence.

4.2 The stability theorem

Put $L = 1$. Then, we have $z = \frac{\eta-1}{\eta+1}$. Suppose that $\sigma_{N,j} = \cos(2j\pi/(2N+1))$ are $N+1$ Chebyshev-Gauss points, and

$$\zeta_{N,j} = (1 + \sigma_{N,j})(1 - \sigma_{N,j})^{-1}, \quad \omega(\eta) \frac{d\eta}{dz} = \sqrt{1-z^2}, \quad \rho(z) = \sqrt{1-z^2}.$$

The Chebyshev-Gauss formula implies that

$$\int_{\Lambda} \varphi(\eta) \omega(\eta) d\eta = \int_I \varphi\left(\frac{1+z}{1-z}\right) \rho(z) dz = \sum_{j=0}^N \varphi(\zeta_{N,j}) \omega_j, \quad \forall \varphi \in \mathcal{R}_N, \quad (24)$$

where $\omega_0 = \pi/(2N+1)$ and $\omega_j = \pi/(N+1)$ for $1 \leq j \leq N$. The discrete inner product and norm of the Chebyshev-Gauss rational interpolation points are

$$(u, v)_{\omega, N} = \sum_{j=0}^{N-1} u(\zeta_{N,j}) v(\zeta_{N,j}) \omega_j, \quad \|v\|_{\omega, N} = (v, v)_{\omega, N}^{1/2}.$$

According to Eq. (24), we have

$$(\varphi, \psi)_{\omega, N} = (\varphi, \psi)_{\omega}, \quad \forall \varphi, \psi \in \mathcal{R}_N. \quad (25)$$

The Chebyshev-Gauss rational interpolation operator $I_N f(\eta) : C(\bar{\Lambda}) \rightarrow \mathcal{R}_N$ is

$$I_N f(\zeta_{N,j}) = f(\zeta_{N,j}), \quad 0 \leq j \leq N.$$

The following theorem is related to the stability of the Chebyshev-Gauss rational interpolation.

Theorem 3 For any $f \in H_{\omega, Q}^1$, we have

$$\|I_N f\|_{\omega} \leq c(\|f\|_{\omega} + N^{-1} \|(\eta+1)^{1/2}(\eta)^{1/4} \partial_{\eta} f\|_{\omega}).$$

Proof Let $\eta = (1 + \cos \theta)/(1 - \cos \theta)$, $\hat{f}(\theta) = f((1 + \cos \theta)/(1 - \cos \theta))$, and

$$\frac{d\eta}{d\theta} = -\frac{2 \sin \theta}{(1 - \cos \theta)^2} = \frac{1}{2}(\eta+1)\sqrt{\eta}.$$

Put $K_j = (2j\pi/(2N+1), (2j+1)\pi/(2N+1))$ for $j = 0, 1, \dots, N$. Then, by Eq. (25), there is

$$\begin{aligned} \|I_N f\|_{\omega}^2 &= \|I_N f\|_{\omega, N}^2 = \sum_{j=0}^N f^2(\zeta_{N,j}) \omega_j \\ &= \sum_{j=0}^N \hat{f}^2(\theta_{N,j}) \omega_j \leq \frac{\pi}{N+1} \sum_{j=0}^{N-1} \sup_{\theta \in K_j} \hat{f}^2(\theta). \end{aligned}$$

By the following inequality (13.7 in Ref. [29]):

$$\max_{a \leq \eta \leq b} |g(\eta)| \leq c \left(\frac{1}{\sqrt{b-a}} \|g\|_{L^2(a,b)} + \sqrt{b-a} \|\partial_{\eta} g\|_{L^2(a,b)} \right), \quad \forall g \in H^1(a,b),$$

applying it to each interval K_j , yields

$$\begin{aligned} \|I_N f\|_{\omega}^2 &\leq c \sum_{j=0}^N (\|\hat{f}(\theta)\|_{L^2(K_j)}^2 + N^{-2} \|\partial_{\theta} \hat{f}(\theta)\|_{L^2(K_j)}^2) \\ &\leq c (\|\hat{v}(\theta)\|_{L^2(0,\pi)}^2 + N^{-2} \|\partial_{\theta} \hat{f}(\theta)\|_{L^2(0,\pi)}^2) \\ &\leq c (\|\hat{f}(\eta)\|_{L_{\omega}^2(\Lambda)}^2 + N^{-2} \|(\eta+1)^{1/2}(\eta)^{1/4} \partial_{\eta} \hat{f}(\eta)\|_{L_{\omega}^2(\Lambda)}^2), \end{aligned}$$

which implies the desired result.

5 Method implementation

The solution to the Falkner-Skan equation is obtained by using the present method. To find the solution, the approximate solution to the equation is estimated as

$$f_N(\eta) = \eta + \sum_{i=0}^N c_i A L_i(\eta), \quad (26)$$

due to Remark 1 and the boundary condition $f'(\infty) = 1$. Considering the Chebyshev-Gauss points, the collocation points of the present method are defined as $z_j = \frac{\cos(j\pi/N)}{N}$ for $j = 0, 1, \dots, N$.

Now, we consider the collocation method for the residual of Eq. (3) with the collocation points η_j ($j = 0, 1, \dots, N$), where $z_j = \frac{\eta_j - L}{\eta_j + L}$. Therefore, the unknown coefficients $\{c_j\}_{j=0}^N$ and the parameters a and L are obtained by solving the following nonlinear system of equations:

$$\begin{cases} R(\eta_j) = 0, & j = 0, 1, \dots, N, \\ f(0) = 0, \\ f'(0) = 0, \end{cases} \quad (27)$$

where

$$R(\eta) = \frac{d^3 f_N}{d\eta^3} + f_N \frac{d^2 f_N}{d\eta^2} + \beta \left(1 - \left(\frac{df_N}{d\eta} \right)^2 \right) - M^2 \left(\frac{df_N}{d\eta} - 1 \right).$$

In Table 1, the results of the method are compared with those of the spline function method^[20], the Chebyshev polynomial approximation^[30], the Chebyshev finite difference method^[14], and the piecewise linear functions method^[12]. The results show good approximation for the solution to the equation.

Table 1 Comparison of $f''(0)$ for different methods at $f(0) = 0$ and $M = 0$

β	N	L	a	Present method	Ref. [20]	Ref. [30]	Ref. [14]	Ref. [12]
0	20	2.000 0	0.501 6	0.469 675 01	0.469 600 0	0.469 600	0.469 600 12	0.469 600
0.3	16	2.242 6	2.049 1	0.774 758 92	0.774 782 7	0.774 783	0.774 782 74	—
0.5	16	2.007 7	2.493 7	0.927 680 32	0.927 805 4	0.927 805	0.927 805 39	0.927 680
1	16	1.769 1	2.043 3	1.232 588 18	—	1.232 588	—	1.232 588
2	16	1.781 4	2.410 0	1.687 218 67	1.687 225 6	1.687 226	1.687 225 86	1.687 222
10	16	2.149 8	3.849 8	3.675 234 03	3.675 213 0	3.675 234	3.675 234 31	—
15	16	2.064 5	3.623 5	4.491 486 86	4.491 643 0	4.491 487	4.491 486 88	—

In the nonlinear system of Eq. (27), instead of the initial condition $f(0) = 0$, put a more general condition $f(0) = \alpha$. Table 2 provides the results of the method for the initial condition $f(0) = \alpha$ compared with the homotopy analysis method (HAM)^[16]. The results are in good approximation for the solution to the equation.

Tables 3 and 4 show the results of the present method for the numerical solution of the MHD Falkner-Skan flow in $\beta = \frac{4}{3}$ ($m = 2$) and $\beta = -3$ ($m = -\frac{3}{5}$) cases, respectively. Table 3 compares the results of the present method for $\beta = \frac{4}{3}$ ($m = 2$) by using the HAM^[31], the Sinc-collocation method^[17], and the Gegenbaer neural network method^[22]. According to Table 3, we can conclude that the results are acceptable compared with other methods. In Table 4, the results of the present method are compared for $\beta = -3$ ($m = -\frac{3}{5}$) by using the HAM^[31], the Sinc-collocation method^[17], and the Gegenbaer neural network method^[22].

Table 2 Comparison of $f''(0)$ for different methods at $f(0) = \alpha$ and $M = 0$

β	α	N	L	a	Present method	Ref. [16]
0	1	20	2.768 6	0.864 8	1.283 56	1.283 63
0	2	20	1.827 4	0.618 2	2.194 46	2.194 15
0	3	20	5.652 9	2.444 7	3.145 04	3.145 10
1	1	16	5.730 0	1.038 1	1.889 31	1.889 31
1	2	16	2.879 2	0.919 5	2.670 06	2.670 06
1	3	16	2.810 2	1.598 0	3.526 64	3.526 64
2	1	16	4.123 1	0.543 3	3.309 82	2.309 82
2	2	16	2.412 4	4.885 6	3.039 61	3.039 61
2	3	16	2.800 5	0.169 6	3.846 14	3.846 14

Table 3 Comparison of $f''(0)$ for different methods at $f(0) = 0$ and $\beta = \frac{4}{3}$ ($m = 2$)

M	N	L	a	Present method	Ref. [22]	Ref. [17]	Ref. [31]
2	16	2.043 13	1.602 34	2.439 498 79	2.439 498 33	2.439 498 33	2.439 498 70
5	16	1.966 59	0.996 68	5.190 959 77	5.190 959 45	5.196 856 70	5.190 959 80
10	16	1.998 99	0.451 63	10.096 775 79	10.096 775 45	10.096 775 45	10.096 775 75
50	24	0.576 00	1.000 09	50.019 440 80	50.019 440 71	50.019 440 71	50.019 440 84

Table 4 Comparison of $f''(0)$ for different methods at $f(0) = 0$ and $\beta = -3$ ($m = -\frac{3}{5}$)

M	N	L	a	Present method	Ref. [22]	Ref. [17]	Ref. [31]
5	16	1.853 38	2.784 04	4.600 752 31	4.600 754 94	4.600 754 94	4.600 752 28
10	16	0.963 45	0.132 15	9.806 462 98	9.806 464 20	9.806 464 20	9.806 463 00
15	16	0.983 55	0.116 75	14.871 674 01	14.871 674 84	14.871 674 84	14.871 674 01
20	16	0.995 22	0.224 66	19.903 936 21	19.903 937 01	19.903 937 01	19.903 936 26

To verify the accuracy and convergence of the method, $\|R\|_2$ is used. According to Table 5, the results of the present method are more accurate than those of the other methods. Moreover, the increase in the collocation points increases the accuracy of the solution and reduces $\|R\|_2$. The results of Table 5 confirm the convergence Theorem 2.

Table 5 Comparison of $f''(0)$ for present method at $f(0) = 0$ and different values β , M , and N

N	$(\beta, M) = (1, 0)$		$(\beta, M) = (\frac{4}{3}, 5)$		$(\beta, M) = (-3, 15)$	
	$f''(0)$	$\ R\ _2$	$f''(0)$	$\ R\ _2$	$f''(0)$	$\ R\ _2$
4	1.191 448 78	$1.438 24 \times 10^{-1}$	1.890 009 95	$4.990 15 \times 10^{-1}$	0.005 569 60	$1.190 98 \times 10^{-2}$
8	1.232 400 42	$1.352 23 \times 10^{-2}$	5.190 460 06	$6.984 00 \times 10^{-3}$	14.820 297 09	$1.008 32 \times 10^{-1}$
12	1.232 601 93	$1.039 11 \times 10^{-3}$	5.190 958 40	$4.575 03 \times 10^{-5}$	14.871 650 02	$1.118 67 \times 10^{-3}$
16	1.232 588 18	$1.332 24 \times 10^{-4}$	5.190 959 77	$2.338 79 \times 10^{-6}$	14.871 674 01	$1.868 76 \times 10^{-5}$
20	1.232 587 49	$5.089 71 \times 10^{-5}$	5.190 959 80	$9.688 30 \times 10^{-9}$	14.871 674 01	$4.063 01 \times 10^{-7}$

In Table 6, the results of $f''(0)$ for the a -polynomial method at $f(0) = \alpha$ and $M = 0$ are displayed. In this table, the results of the residual error of the equation are compared in rational a -polynomials and a -polynomial methods, and it can be seen that the superiority of the rational a -polynomial method is evident. Besides, CPU time of the mentioned methods is acceptable.

Table 6 Results of $f''(0)$ for a -polynomial method at $f(0) = \alpha$ and $M = 0$ and comparison of $\|R\|_2$ for present method and a -polynomial method

β	α	N	a	$f''(0)$	a -polynomial method	Present method
					$\ R\ _2$ (CPU time/s)	$\ R\ _2$ (CPU time/s)
0	1	20	1.41820×10^6	1.283 635	5.80249×10^{-5} (2)	7.92882×10^{-6} (6)
0	2	20	9.40653×10^7	2.194 530	2.93219×10^{-4} (2)	3.36447×10^{-6} (6)
0	3	20	1.92207×10^9	3.145 199	2.43328×10^{-4} (2)	1.06890×10^{-5} (6)
1	1	16	9.12232×10^2	1.889 286	1.94615×10^{-3} (1)	9.47916×10^{-5} (2)
1	2	16	1.87921×10^4	2.670 021	7.08507×10^{-4} (1)	1.17158×10^{-5} (2)
1	3	16	8.97651×10^4	3.526 436	5.65119×10^{-3} (1)	1.37253×10^{-5} (2)
2	1	16	3.28932×10^4	2.309 973	9.31681×10^{-4} (1)	1.53240×10^{-5} (2)
2	2	16	2.61471×10^5	3.037 975	2.52814×10^{-2} (1)	2.15517×10^{-5} (2)
2	3	16	6.52909×10^5	3.833 574	9.51944×10^{-2} (1)	9.04868×10^{-7} (2)

In Fig. 1, the $f'(\eta)$ curves are plotted for the values $\beta = 0, 1, 2$ under the initial condition $f(\eta) = 0$ and $M = 0$. It shows that the results are easily observable and comparable. Figure 2 depicts the $f'(\eta)$ curves for $\beta = 1$ under the initial conditions $f(0) = \alpha = 0, 1, 2, 3$ and $M = 0$. In Fig. 3, the $f'(\eta)$ curves are plotted for the values $M = 2, 5, 10, 50$ under the initial condition $f(0) = 0$ and $\beta = \frac{4}{3}$ ($m = 2$). In Fig. 4, the $f'(\eta)$ curves are plotted for $M = 5, 10, 15, 20$ under the initial condition $f(0) = 0$ and $\beta = -3$ ($m = -\frac{3}{5}$).

Fig. 1 $f'(\eta)$ for different values of β at $\alpha = 0$ and $M = 0$ (color online) **Fig. 2** $f'(\eta)$ for different values of α at $\beta = 1$ and $M = 0$ (color online) **Fig. 3** $f'(\eta)$ for different values of M at $\alpha = 0$ and $\beta = \frac{4}{3}$ ($m = 2$) (color online) **Fig. 4** $f'(\eta)$ for different values of M at $\alpha = 0$ and $\beta = -3$ ($m = -\frac{3}{5}$) (color online)

6 Conclusions

A novel method is presented for solving the classical Falkner-Skan equation. It uses two parameters to optimize and improve the approximation accuracy of the solution. The results are acceptable in comparison with other methods. In fact, in many cases, the results obtained by the proposed method are better than the findings of other papers. The present method includes all types of the Falkner-Skan equation. According to the theorems proved, the convergence of the method is guaranteed.

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