Criteria for minimization of spectral abscissa of time-delay systems*

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Abstract Spectral abscissa (SA) is defined as the real part of the rightmost characteristic root(s) of a dynamical system, and it can be regarded as the decaying rate of the system, the smaller the better from the viewpoint of fast stabilization. Based on the Puiseux series expansion of complex-valued functions, this paper shows that the SA can be minimized within a given delay interval at values where the characteristic equation has repeated roots with multiplicity 2 or 3. Four sufficient conditions in terms of the partial derivatives of the characteristic function are established for testing whether the SA is minimized or not, and they can be tested directly and easily.

Key words time delay, stability, parameter tuning, spectral abscissa (SA), decaying speed

Chinese Library Classification O29, O32

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1 Introduction

Time delays are usually inevitable in control applications when digital controllers and digital filters are used, and they are also very common in human-machine interaction systems because humans need time to respond. Time-delay systems are a kind of infinite dimensional systems described by delay differential equations^[1–6]. No matter how small the delay is, the evolution of a time-delay system is determined by the state defined over an interval of the time, not by a single state for the delay-free systems. A very short delay might be neglected in some applications for simplicity in mathematical analysis, but the control design without considering the delay effect might be not reliable and not safe in applications where the stability or control performance depends sharply on the delay changes^[1,7]. In order to dispense the delay effect, Smith predictor control is generally used, and it works well in applications such as cases when stability and rapidity are addressed as a major factor and a minor factor, respectively. In the applications of Smith predictor control, accurate modeling is required, and thus it might not be able to achieve the target control output in the presence of modeling uncertainty and measurement uncertainty^[8]. Hence, the delay effect on the system dynamics and control performance must be carefully studied.

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Stability analysis and tuning of control parameters are two key problems in control design. For recent advances of stability and related topics of time-delay systems, the reader is referred to the review paper^[9]. Parameter tuning is a hard task as stated in Ref. [10] for the tuning problem of the most popularly used simple controls, “there is strong evidence that PI and PID controllers remain poorly understood and, in particular, poorly tuned in many applications”. The tuning of control parameters of time-delay systems becomes even more challenging due to the infinite dimensional properties^[11].

Parameter tuning depends on control objective. When the decaying rate of the control error between the state and the target is addressed, the spectral abscissa (SA for short), the real part of the rightmost characteristic root(s), is used. Let $p(\lambda)$ be the characteristic function of an equilibrium of the time-delay system, then SA is defined by

$$\sigma = \max_{\lambda \in \mathbb{C}} \{\operatorname{Re}(\lambda) : p(\lambda) = 0\}, \quad (1)$$

where $\operatorname{Re}(\lambda)$ stands for the real part of λ . If SA is negative, then the control error decays to zero, or the zero equilibrium of the time-delay system is exponentially stable. When SA is negative, the smaller the value of σ is, the faster the solution of the time-delay system decays to the equilibrium. Thus, SA can be regarded as a representative of the decaying rate for characterizing stability^[5,12–13]. In the case when SA exhibits a change from decreasing to increasing behavior in a given range of a parameter, the proposed time-delay control improves the stability of the time-delay system. At the point where SA is minimized, the time-delay decays to the zero equilibrium at the fastest speed. Results are available in the literature for calculating SA. For example, SA can be found in two steps. The first step is to find an estimation of the SA via integral calculation, and the second step is to refine the estimated value using Newton’s iteration method^[14]. When a parameter varies in a given interval, say $\tau \in [0, \bar{\tau}]$, SA as a function of τ can be obtained by solving several initial value problems of a nonlinear ordinary differential equation resulting from the characteristic equation^[15]. Such a method is named the parameter continuation method, and it provides an overview of SA with respect to the change of delay in a given interval. Other methods or algorithms are referred to Ref. [9]. The parametric continuation method proposed in Ref. [15] finds SA of a time-delay system with a single parameter effectively. However, the method fails at points where the time-delay system has repeated characteristic roots, where multiple branches of the characteristic root coexist. A repeated characteristic root of some time-delay systems might result in minimized SA in a given interval^[12–13,15–16], but there are no general criteria to check this fact. From the viewpoint of parameter tuning, it is highly required to know how to find an approximation of the repeated characteristic roots, how to check that a repeated characteristic root determines SA, and what criteria can be obtained to characterize the minimized SA resulting from repeated characteristic roots.

The problem statement of the paper is presented in Section 2. In Section 3, the Puiseux series expansion is used to approximate the multiple branches of the repeated characteristic root locally and analytically, and a complex contour integral is adopted to test whether the repeated real characteristic root is the SA, and in Section 4, the low-order Puiseux series expansion is applied to derive easy-checked criteria for testing the minimized SA resulting from repeated characteristic roots. Finally in Section 5, some concluding remarks are drawn from the study.

2 Problem statement

A general linear time-delay system of a retarded type can be described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \sum_{i=1}^m \mathbf{B}_i \mathbf{x}(t - \tau_i), \quad \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{A}, \mathbf{B}_i \in \mathbb{R}^{n \times n}, \quad (2)$$

where $\tau_i > 0$ ($i = 1, 2, \dots, m$) are the time-delays. The stability of the unique equilibrium $\mathbf{x} = \mathbf{0}$ can be determined by the root location of the characteristic equation

$$p(\lambda) := \lambda^n + \sum_{i=1}^n a_i(e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \dots, e^{-\lambda\tau_m})\lambda^{n-i} = 0, \quad (3)$$

where $a_i(z_1, z_2, \dots, z_m)$ ($i = 1, 2, \dots, n$) are polynomials with respect to $z_1, z_2, \dots, z_m$. The characteristic function $p(\lambda)$ is analytic in λ , it has a finite number of complex roots in any given bounded closed region, and there is a line parallel to the imaginary axis such that all the characteristic roots stay in the region left to the parallel line^[17]. Thus, for given parameters, the SA σ is well-defined by Eq. (1).

Roughly speaking, the stability analysis and parameter tuning are hard problems in general, especially when multiple delays as parameters are involved. Thus, this paper focuses on the system with a single delay only. The parametric continuation method proposed in Ref. [15] calculates the SA of a time-delay system with a single delay τ by solving some initial value problems of a nonlinear ordinary differential equation associated with the characteristic function. The characteristic equation $p(\lambda) = 0$ has infinite roots, and each characteristic root λ is differentiable with respect to τ . To address the dependence on τ , $p(\lambda)$ is rewritten as $p(\lambda, \tau)$, and σ is rewritten as $\sigma(\tau)$. By differentiating on $p(\lambda, \tau) = 0$ with respect to τ , if $\frac{\partial p}{\partial \lambda}(\lambda, \tau) \neq 0$, then one has

$$\frac{d\lambda}{d\tau} = -\frac{\frac{\partial p}{\partial \tau}(\lambda, \tau)}{\frac{\partial p}{\partial \lambda}(\lambda, \tau)}. \quad (4)$$

In the applications of the parametric continuation method, it is required to solve Eq. (4) under three types of initial conditions. The first type of initial conditions is based on the rightmost characteristic root(s) of the system with $\tau = 0$, the second type is based on the pair of pure imaginary characteristic roots satisfying $p(i\omega_0, \tau_0) = 0$, and the third type is based on the application of integral criterion mainly for validation. In some applications, say the ones with short delay, only the first type of initial conditions is required, and in some other applications, especially when the delay is large, two or three types of initial conditions are required.

The parametric continuation method proposed in Ref. [15] might fail at repeated characteristic roots with multiplicity $m \geq 2$ satisfying

$$p(\lambda_0, \tau_0) = 0, \quad \frac{\partial p}{\partial \lambda}(\lambda_0, \tau_0) = 0, \quad \dots, \quad \frac{\partial^{m-1} p}{\partial \lambda^{m-1}}(\lambda_0, \tau_0) = 0, \quad \frac{\partial^m p}{\partial \lambda^m}(\lambda_0, \tau_0) \neq 0. \quad (5)$$

One observation is that in some applications, SA is minimized at repeated characteristic roots. A minimized SA indicates the fastest decaying ratio of the solution of the time-delay system with a fixed initial value around an equilibrium. This is helpful in tuning the control parameters. However, no criteria are available to judge whether the minimized SA results from a repeated characteristic root or not. The next two sections are devoted to establishing easy-checked criteria for the minimization of SA resulting from repeated characteristic roots.

3 Repeated characteristic roots approximated by Puiseux series

At a repeated characteristic root, the differential equation (4) is not well-defined. Thus, in a neighborhood of τ_0 , the characteristic root λ as a function of τ does not have a Taylor series expansion. Instead, the characteristic root λ has a Puiseux series expansion of the following form:

$$\lambda = \sum_{k=0}^{\infty} c_k(\tau - \tau_0)^{k\xi}, \quad (6)$$

where $\xi > 0$, and $c_0 = \lambda_0$. In particular, when $\xi = 1$, Eq. (6) is the well-known Taylor series. The coefficients $c_0, c_1, c_2, \dots$ of the low orders in Eq. (6) can be found by using the method of undetermined coefficients. In a small neighborhood of τ_0 , the larger the order is, the smaller the power of $\tau - \tau_0$ is. Thus, the characteristic root is dominated by the low-order terms such as $c_0, c_1(\tau - \tau_0)^\xi$, and $c_2(\tau - \tau_0)^{2\xi}$. For demonstrating how the low-order terms are determined, two cases with multiplicities 2 and 3 are studied in this section.

It is worth mentioning that the Puiseux series expansion has been used in the stability analysis of time-delay systems when the characteristic equation has repeated purely imaginary characteristic roots^[18], a critical case between stability and instability.

3.1 Characteristic roots with multiplicity 2

In the case when $\lambda = \lambda_0$ is a repeated root with multiplicity $m = 2$, an approximate expression of λ in a neighborhood of τ_0 is assumed to be

$$\lambda = \lambda_0 + c_1(\tau - \tau_0)^\xi + c_2(\tau - \tau_0)^{2\xi} + c_3(\tau - \tau_0)^{3\xi} + c_4(\tau - \tau_0)^{4\xi} + o(|\tau - \tau_0|^{4\xi}), \quad (7)$$

where $\xi > 0$ and $c_1 \neq 0$, and c_2, c_3 , and c_4 are constants to be determined as done below. The two terms $c_1(\tau - \tau_0)^\xi + c_2(\tau - \tau_0)^{2\xi}$ in Eq. (7) are the most dominating ones, defining the skeleton of the two branches of the characteristic root λ in a neighborhood of τ_0 , and the two terms $c_3(\tau - \tau_0)^{3\xi} + c_4(\tau - \tau_0)^{4\xi}$ are used for correcting the approximated characteristic roots. The assumption $c_1 \neq 0$ is not necessary when $\xi = \frac{1}{2}$ is used.

By using the second-order Taylor approximation of $p(\lambda, \tau)$ at (λ_0, τ_0) , the notations $p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0) = \frac{\partial^2 p}{\partial \lambda^2}(\lambda_0, \tau_0)$, $p_{\lambda, \tau}^{(2)}(\lambda_0, \tau_0) = \frac{\partial^2 p}{\partial \lambda \partial \tau}(\lambda_0, \tau_0)$, and so on, one has

$$p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0)L^2 + 2p_{\lambda, \tau}^{(2)}(\lambda_0, \tau_0)LT + p_{\tau, \tau}^{(2)}(\lambda_0, \tau_0)T^2 + 2p'_\tau(\lambda_0, \tau_0)T + \text{h.o.t.} = 0 \quad (8)$$

due to $p_{\lambda, \tau}^{(2)}(\lambda_0, \tau_0) = p_{\tau, \lambda}^{(2)}(\lambda_0, \tau_0)$, where $L = \lambda - \lambda_0$, $T = \tau - \tau_0$, and h.o.t. stands for higher-order terms with respect to the powers of L and T . Equation (8) can be simply denoted as $aL^2 + bLT + cT^2 + dT + \text{h.o.t.} = 0$ with $a = p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0) \neq 0$. Two cases are studied only.

Case 1 $d = 2p'_\tau(\lambda_0, \tau_0) \neq 0$

By using the fourth-order expansion of Puiseux series $L = c_1T^\xi + c_2T^{2\xi} + c_3T^{3\xi} + c_4T^{4\xi} + o(|T|^{4\xi})$, Eq. (8) becomes

$$dT + cT^2 + ac_1^2T^{2\xi} + 2ac_1c_2T^{3\xi} + a(2c_1c_3 + c_2^2)T^{4\xi} + 2a(c_1c_4 + c_2c_3)T^{5\xi} + a(c_3^2 + 2c_2c_4)T^{6\xi} + 2ac_3c_4T^{7\xi} + ac_4^2T^{8\xi} + bc_1T^{1+\xi} + bc_2T^{1+2\xi} + bc_3T^{1+3\xi} + bc_4T^{1+4\xi} + \text{h.o.t.} = 0.$$

In a neighborhood of τ_0 , $|T|$ is assumed small, even very small. In this case, the larger the order is, the smaller the value of the corresponding term is. Thus, the value of the left-hand side is dominated by the low orders. The undetermined constants ξ, c_1, c_2, c_3 , and c_4 are determined when the dominating terms vanish. The lowest-order terms must be found from dT and $ac_1^2T^{2\xi}$. When

$$\xi = \frac{1}{2}, \quad c_1 = \pm \sqrt{-\frac{d}{a}},$$

the two terms dT and $ac_1^2T^{2\xi}$ have the same order, and the lowest-order term $(d + ac_1^2)T$ disappears. Furthermore, Eq. (8) becomes

$$(bc_1 + 2ac_1c_2)T^{3/2} + (ac_2^2 + bc_2 + c + 2ac_1c_3)T^2 + (2ac_2c_3 + bc_3 + 2ac_1c_4)T^{5/2} + o(T^{5/2}) = 0.$$

To determine the coefficients c_2, c_3 , and c_4 by eliminating the low-order terms, one has

$$\begin{cases} bc_1 + 2ac_1c_2 = 0, \\ ac_2^2 + bc_2 + c + 2ac_1c_3 = 0, \\ 2ac_2c_3 + bc_3 + 2ac_1c_4 = 0. \end{cases}$$

It follows that in terms of the partial derivatives of $p(\lambda, \tau)$ at (λ_0, τ_0) , the coefficients c_1 , c_2 , c_3 , and c_4 are found to be

$$\begin{cases} c_1 = \pm \sqrt{-\frac{d}{a}} = \pm \sqrt{-\frac{2p'_\tau(\lambda_0, \tau_0)}{p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0)}}, \\ c_2 = -\frac{b}{2a} = -\frac{p_{\tau, \lambda}^{(2)}(\lambda_0, \tau_0)}{p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0)}, \\ c_3 = -\frac{ac_2^2 + bc_2 + c}{2ac_1} = \frac{(b/a)^2 - 4(c/a)}{8c_1} = \pm \frac{\left(\frac{p_{\lambda, \tau}^{(2)}(\lambda_0, \tau_0)}{p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0)}\right)^2 - \frac{p_{\tau, \tau}^{(2)}(\lambda_0, \tau_0)}{p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0)}}{2\sqrt{-\frac{2p'_\tau(\lambda_0, \tau_0)}{p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0)}}}, \\ c_4 = 0, \end{cases}$$

where the coefficients c_1 and c_2 of the two dominating terms are represented by $a = p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0)$, $b = 2p_{\lambda, \tau}^{(2)}(\lambda_0, \tau_0)$, $d = p'_\tau(\lambda_0, \tau_0)$ only, not by $c = p_{\tau, \tau}^{(2)}(\lambda_0, \tau_0)$. This means that the skeleton of the two branches of the repeated characteristic root is affected by a , b , d only, not by c .

Case 2 $d = 2p'_\tau(\lambda_0, \tau_0) = 0$

In this case, it holds

$$L = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}T + o(T).$$

In order to have more accurate approximation of the repeated characteristic root in a neighborhood of τ_0 , a higher-order Taylor expansion is required. The third-order expansion is

$$AL^3 + BL^2\tau + CLT^2 + DT^3 + aL^2 + bLT + cT^2 + \text{h.o.t.} = 0.$$

Let $L = c_1T + c_2T^2 + o(T^2)$. Then, the coefficients of powers T and T^2 vanish, namely,

$$\begin{cases} ac_1^2 + bc_1 + c = 0, \\ Ac_1^3 + Bc_1^2 + Cc_1 + D + (2ac_1 + b)c_2 = 0. \end{cases}$$

When $2ac_1 + b \neq 0$, it holds

$$c_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad c_2 = -\frac{Ac_1^3 + Bc_1^2 + Cc_1 + D}{2ac_1 + b}.$$

Example 1 Let us consider a time-delay system with the following characteristic function:

$$p(\lambda, \tau) = \lambda^4 + 2\lambda^2 + 2 + ue^{-\lambda\tau} + ve^{-2\lambda\tau}, \quad u, v \in \mathbb{R}, \quad \tau > 0.$$

Solving u and v from $p(\lambda, \tau) = 0$ and $p'_\lambda(\lambda, \tau) = 0$ gives

$$u = -\frac{2}{\tau}(\tau\lambda^4 + 2\lambda^3 + 2\tau\lambda^2 + 2\lambda + 2\tau)e^{\lambda\tau}, \quad v = \frac{1}{\tau}(\tau\lambda^4 + 4\lambda^3 + 2\tau\lambda^2 + 4\lambda + 2\tau)e^{2\lambda\tau}.$$

Let $\lambda = i\omega$. Then, the imaginary parts of u and v must be zeros, which, respectively, lead to

$$\sin(\omega\tau) = \frac{2\omega(\omega^2 - 1)}{\tau(\omega^4 - 2\omega^2 + 2)} \cos(\omega\tau), \quad \sin(2\omega\tau) = \frac{4\omega(\omega^2 - 1)}{\tau(\omega^4 - 2\omega^2 + 2)} \cos(2\omega\tau).$$

From $\tan(2\omega\tau) = \frac{2\tan(\omega\tau)}{1 - \tan^2(\omega\tau)}$ with $\omega \neq 0$, one finds $\omega = \pm 1$. Thus, $\sin(\omega\tau) = 0$ and $\sin(2\omega\tau) = 0$, which give $\tau = \pi, 2\pi, 3\pi, \dots$. In particular, when $\tau = \pi$, one has $u = 2$ and $v = 1$. Hence, when $u = 2$ and $v = 1$, $p(\lambda, \pi)$ has repeated roots $\lambda = \pm i$ with multiplicity 2.

In order to find an approximation of the repeated root $\lambda = i$ in a neighborhood of $\tau = \pi$, because $d = 2p'_\tau(i, \pi) = 0$, the third-order Taylor expansion of $p(\lambda, \tau)$ at $(\lambda, \tau) = (i, \pi)$ is given below

$$(\pi^2 - 4)L^2 + 2i\pi LT - T^2 + (4i - \pi^3)L^3 + (2\pi - 3i\pi^2)L^2T + (2i + 3\pi)LT^2 + iT^3 + \text{h.o.t.} = 0.$$

Let $L = c_1T + c_2T^2 + o(T^2)$. Then, one finds

$$c_1 = -\frac{1}{\pi \pm 2}, \quad c_2 = \frac{-i + (\pi^3 - 4i)c_1^3 + (3i\pi^2 - 2\pi)c_1^2 + (-2i - 3\pi)c_1}{2((\pi^2 - 4)c_1 + i\pi)}.$$

Thus, the two branches of the repeated characteristic root $\lambda = i$ at $\tau = \pi$ are

$$\lambda_{1,2} = i - \frac{i}{\pi \pm 2}(\tau - \pi) + \frac{i(\pm 1 + 2i + \pi)}{(\pi \pm 2)^3}(\tau - \pi)^2 + o((\tau - \pi)^2).$$

3.2 Characteristic roots with multiplicity 3

If $\lambda = \lambda_0$ is a repeated root with multiplicity $m = 3$, then in a neighborhood of τ_0 , the third-order Taylor expansion of $p(\lambda, \tau)$ gives

$$aL^3 + bL^2T + cLT^2 + dT^3 + eLT + fT^2 + gT + \text{h.o.t.} = 0, \quad a \neq 0, \quad (9)$$

where $L = \lambda - \lambda_0$, $T = \tau - \tau_0$, $a = p_{\lambda, \lambda, \lambda}^{(3)}(\lambda_0, \tau_0)$, and so on. Because the multiplicity is 3, a low-order Puiseux expansion is assumed to be

$$L = c_1T^{1/3} + c_2T^{2/3} + c_3T + c_4T^{4/3} + c_5T^{5/3} + c_6T^2 + o(T^2). \quad (10)$$

Then, the coefficients $c_1, c_2, \dots, c_6$ are determined from the low-order terms of $T, T^{4/3}, T^{5/3}, T^2, T^{7/3}$, and $T^{8/3}$ that should be zero, respectively, namely,

$$\begin{cases} ac_1^3 + g = 0, \\ 3ac_1^2c_2 + c_1e = 0, \\ 3ac_1^2c_3 + 3ac_1c_2^2 + bc_1^2 + ec_2 = 0, \\ 3ac_1^2c_4 + 6ac_1c_2c_3 + ac_2^3 + 2bc_1c_2 + ec_3 + f = 0, \\ 3ac_1^2c_5 + 6ac_1c_2c_4 + 3ac_1c_3^2 + 3ac_2^2c_3 + 2bc_1c_3 + bc_2^2 + cc_1 + ec_4 = 0, \\ 3ac_1^2c_6 + 6ac_1c_2c_5 + 6ac_1c_3c_4 + 3ac_2^2c_4 + 3ac_2c_3^2 + 2bc_1c_4 + 2bc_2c_3 + cc_2 + ec_5 = 0. \end{cases} \quad (11)$$

If $g = 6p'_\tau(\lambda_0, \tau_0) \neq 0$, then one finds

$$\begin{cases} c_1 = \sqrt[3]{-\frac{g}{a}}, \quad \sqrt[3]{-\frac{g}{a}} \left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \right), \\ c_2 = -\frac{e}{3ac_1}, \\ c_3 = -\frac{3ac_1c_2^2 + bc_1^2 + ec_2}{3ac_1^2} = -\frac{b}{3a}, \\ c_4 = -\frac{9a(3af - be)c_1^3 - e^3}{81a^3c_1^5}, \\ c_5 = -\frac{27a^2(3ac - b^2)c_1^6 + 9ae(3af - be)c_1^3 - e^4}{243a^4c_1^7}, \\ c_6 = 0. \end{cases} \quad (12)$$

Here, three solutions of c_1 correspond to three branches of S , respectively. The coefficients c_1, c_2 , and c_3 of the three dominating terms $c_1T^{1/3} + c_2T^{2/3} + c_3T$ are represented by $a = p_{\lambda, \lambda, \lambda}^{(3)}(\lambda_0, \tau_0)$, $b = 3p_{\lambda, \lambda, \tau}^{(3)}(\lambda_0, \tau_0)$, $e = 6p_{\lambda, \tau}^{(2)}(\lambda_0, \tau_0)$, $g = 6p'_\tau(\lambda_0, \tau_0)$ only, not by $c =$

$3p_{\lambda,\tau}^{(3)}(\lambda_0, \tau_0)$, $d = p_{\tau,\tau}^{(3)}(\lambda_0, \tau_0)$, and $f = 3p_{\tau,\tau}^{(2)}(\lambda_0, \tau_0)$. When $c_1 = \sqrt[3]{-g/a}$, all the coefficients in the Puiseux series expansion are real.

If $g = 6p_{\tau}'(\lambda_0, \tau_0) = 0$, but $e = 6p_{\lambda,\tau}^{(2)}(\lambda_0, \tau_0) \neq 0$ and $f = 3p_{\tau,\tau}^{(2)}(\lambda_0, \tau_0) \neq 0$, then from Eq. (11), one finds the coefficients $c_1, c_2, \dots, c_6$ in the Puiseux series expansion to be

$$\begin{cases} c_1 = c_2 = c_4 = c_5 = 0, \\ c_3 = -\frac{f}{e}, \\ c_6 = -\frac{af^3 - bef^2 + ce^2f - de^3}{e^4}. \end{cases}$$

In this case, one has $L = c_3T + c_6T^2 + o(T^2)$. This is only one branch of the repeated root with multiplicity 3, and the other two branches can be found from Eq. (9) by assuming $L = \hat{c}_1T^{1/2} + \hat{c}_2T + \hat{c}_3T^{3/2} + \hat{c}_4T^2 + o(T^2)$. Actually, in this case, the coefficients of the low-order terms $T^{3/2}$, T^2 , $T^{5/2}$, and T^3 in Eq. (9) should be zero, namely,

$$\begin{cases} a\hat{c}_1^3 + e\hat{c}_1 = 0, \\ 3a\hat{c}_1^2\hat{c}_2 + b\hat{c}_1^2 + e\hat{c}_2 + f = 0, \\ 3a\hat{c}_1^2\hat{c}_3 + 3a\hat{c}_1\hat{c}_2^2 + 2b\hat{c}_1\hat{c}_2 + c\hat{c}_1 + e\hat{c}_3 = 0, \\ 3a\hat{c}_1^2\hat{c}_4 + 6a\hat{c}_1\hat{c}_2\hat{c}_3 + a\hat{c}_2^3 + 2b\hat{c}_1\hat{c}_3 + b\hat{c}_2^2 + c\hat{c}_2 + e\hat{c}_4 + d = 0. \end{cases}$$

The first condition gives

$$\hat{c}_1 = 0, \pm\sqrt{-\frac{e}{a}}.$$

When $\hat{c}_1 = 0$, one has $\hat{c}_2 = c_3$, $\hat{c}_3 = 0$, and $\hat{c}_4 = c_6$, the same result obtained above. With $\hat{c}_1 = \pm\sqrt{-\frac{e}{a}}$, one finds the coefficients $\hat{c}_2$, $\hat{c}_3$, and $\hat{c}_4$ of the other two branches.

If $g = 6p_{\tau}'(\lambda_0, \tau_0) = 0$, $e = 6p_{\lambda,\tau}^{(2)}(\lambda_0, \tau_0) = 0$, and $f = 3p_{\tau,\tau}^{(2)}(\lambda_0, \tau_0) = 0$, then a higher-order Taylor expansion is required to find the Puiseux series expansion in a neighborhood of τ_0 .

3.3 Real rightmost characteristic roots checked via complex contour integral

A repeated characteristic root $\lambda_0 \in \mathbb{R}$ is meaningful in parameter tuning if it is a rightmost one, namely, no characteristic roots in $\text{Re}(\lambda) > \lambda_0$. Thus, criteria are required to check whether the repeated characteristic root is the rightmost one. Assume that $p(\lambda, \tau_0) = 0$ has a root λ_0 with multiplicity m . Because $p(\lambda, \tau_0)$ is an analytic function in $\lambda \in \mathbb{C}$, it can be expressed as a Taylor series expansion at $\lambda = \lambda_0$. Thus, there is an analytic function $q(\lambda, \tau_0)$ in $\lambda \in \mathbb{C}$ such that

$$p(\lambda, \tau_0) = (\lambda - \lambda_0)^m q(\lambda, \tau_0), \quad (13)$$

where $q(\lambda, \tau_0)$ is analytic in $\lambda \in \mathbb{C}$, and $q(\lambda_0, \tau_0) \neq 0$. Moreover, $q(\lambda, \tau)$ has no characteristic roots in $\text{Re}(\lambda) \geq \lambda_0$, a fact that can be checked by using the argument principle.

When $\lambda_0 \in \mathbb{R}$, with $R > \varepsilon > 0$, let

$$\mathcal{C}(\varepsilon, R) = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4$$

be the contour of complex integral, shown in Fig. 1, with $\text{Re}(\lambda) = \lambda_0 > 0$, where $\mathcal{C}_1$: $\lambda = \lambda_0 + Re^{i\theta}$ in which θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, $\mathcal{C}_2$: $\lambda = \lambda_0 + i\omega$ in which ω varies from R to ε , $\mathcal{C}_3$: $\lambda = \lambda_0 + \varepsilon e^{i\theta}$ in which θ varies from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$, and $\mathcal{C}_4$: $\lambda = \lambda_0 + i\omega$ in which ω varies from $-\varepsilon$ to $-R$. Then, according to the argument principle, when λ_0 is the rightmost root of $p(\lambda, \tau_0)$, it is necessary and sufficient to have

$$\lim_{\varepsilon \rightarrow +0, R \rightarrow +\infty} \oint_{\mathcal{C}(\varepsilon, R)} \frac{q'(\lambda, \tau_0)}{q(\lambda, \tau_0)} d\lambda = 0. \quad (14)$$

Fig. 1 Integral contour $\mathcal{C}(\varepsilon, R) = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4$ when $\operatorname{Re}(\lambda) = \lambda_0 > 0$ (color online)

4 Minimization of SA at repeated real characteristic roots

From the viewpoint of stability of a time-delay system, it is important to know as the delay τ increases in a given interval, under what conditions, $\sigma(\tau)$ (the SA with respect to τ) is first decreased and then increased such that $\sigma(\tau)$ is minimized at $\tau = \tau_0$. In general, this problem might be complicated. This section is devoted to establishing simple criteria to check whether $\sigma(\tau)$ is minimized at repeated real characteristic roots.

4.1 Real characteristic root with multiplicity 2

Assume that at $\tau = \tau_0$, λ_0 is a real characteristic root with multiplicity $m = 2$, and $d = 2p'_\tau(\lambda_0, \tau_0) \neq 0$. In a neighborhood of τ_0 , λ has the Puiseux series expansion (7) given by

$$L = c_1 T^{1/2} + c_2 T + c_3 T^{3/2} + o(T^2), \quad (15)$$

where $L = \lambda - \lambda_0$, $T = \tau - \tau_0$, and c_1 , c_2 , and c_3 can be represented by using partial derivatives as shown in Subsection 3.1. In addition, $\operatorname{Im}(c_1) = 0$ if and only if $\operatorname{Im}(c_3) = 0$, because all the partial derivatives in Eq. (15) are real numbers. If $p'_\tau(\lambda_0, \tau_0) \cdot p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0) > 0$, then $c_1, c_3 \in \mathbb{C}$, $c_2 \in \mathbb{R}$, and in a left-neighborhood of τ_0 , where $T < 0$, one has $\operatorname{Re}(c_1 T^{1/2}) = i^2 \operatorname{Im}(c_1)(-T)^{1/2}$, and

$$\begin{aligned} \operatorname{Re}(L) &= -\operatorname{Im}(c_1)(-T)^{1/2} + c_2 T - \operatorname{Im}(c_3)(-T)^{3/2} + o(T^2) \\ &= \pm \sqrt{\frac{2p'_\tau(\lambda_0, \tau_0)}{p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0)}} (-T)^{1/2} + o(|T|^{1/2}). \end{aligned} \quad (16)$$

Besides, the branch with sign “+” decreases, and the branch with sign “−” increases, as T increases to 0. The branch with sign “+” determines $\sigma(\tau)$, because it is over the one with sign “−”. In a right-neighborhood of τ_0 , where $T > 0$, one has

$$\operatorname{Re}(L) = c_2 T + o(T^2) = -\frac{p_{\tau, \lambda}^{(2)}(\lambda_0, \tau_0)}{p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0)} T + o(T^2). \quad (17)$$

In order that as τ increases, $\operatorname{Re}(L) = \operatorname{Re}(\lambda) - \lambda_0$ is first decreased and then increased, it requires that $p_{\tau, \lambda}^{(2)}(\lambda_0, \tau_0) \cdot p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0) < 0$.

If $p'_\tau(\lambda_0, \tau_0) \cdot p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0) < 0$, then $c_1, c_2, c_3 \in \mathbb{R}$. In a left-neighborhood of τ_0 with $T < 0$, one has

$$\operatorname{Re}(L) = c_2 T + o(|T|^{1/2}) = -\frac{p_{\tau, \lambda}^{(2)}(\lambda_0, \tau_0)}{p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0)} T + o(T^2), \quad (18)$$

while in a right-neighborhood of τ_0 with $T > 0$, one has

$$\operatorname{Re}(L) = c_1 T^{1/2} + c_2 T + c_3 T^{3/2} + o(T^2) = \pm \sqrt{-\frac{2p'_\tau(\lambda_0, \tau_0)}{p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0)}} T^{1/2} + o(|T|^{1/2}). \quad (19)$$

In order that as τ increases, $\operatorname{Re}(L) = \operatorname{Re}(\lambda) - \lambda_0$ is first decreased and then increased, it requires that $p_{\tau, \lambda}^{(2)}(\lambda_0, \tau_0) \cdot p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0) > 0$. In summary, one has the following theorems.

Theorem 1 Assume that at $\tau = \tau_0$, λ_0 is a real root with multiplicity $m = 2$, Eq. (14) holds and $d = 2p'_\tau(\lambda_0, \tau_0) \neq 0$. Then, $\sigma(\tau)$ is minimized at $\tau = \tau_0$ in a neighborhood of τ_0 if one of the two conditions holds: (i) $p'_\tau(\lambda_0, \tau_0) \cdot p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0) > 0$ and $p_{\tau, \lambda}^{(2)}(\lambda_0, \tau_0) \cdot p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0) < 0$; or (ii) $p'_\tau(\lambda_0, \tau_0) \cdot p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0) < 0$ and $p_{\tau, \lambda}^{(2)}(\lambda_0, \tau_0) \cdot p_{\lambda, \lambda}^{(2)}(\lambda_0, \tau_0) > 0$.

When $d = p'_\tau(\lambda_0, \tau_0) = 0$, in a neighborhood of τ_0 , $\sigma(\tau)$ is dominated by $\operatorname{Re}(c_1)(\tau - \tau_0)$ with $c_1 = (-b \pm \sqrt{b^2 - 4ac})/(2a)$ and $\operatorname{Re}(c_1) \neq 0$. Because the sign of $\operatorname{Re}(c_1)$ is fixed, the monotony of $\sigma(\tau)$ with respect to the change of τ is kept unchanged in the neighborhood of τ_0 . Thus, $\sigma(\tau)$ is not minimized in the neighborhood of τ_0 , if $d = p'_\tau(\lambda_0, \tau_0) = 0$. Similarly, if $\operatorname{Re}(c_1) = 0$ and $\operatorname{Re}(c_2) \neq 0$, then $\sigma(\tau)$ is dominated by $\operatorname{Re}(c_2)(\tau - \tau_0)^2$, and it has a minimal value λ_0 at $\tau = \tau_0$ if $\operatorname{Re}(c_2) > 0$. If $\operatorname{Re}(c_1) = \operatorname{Re}(c_2) = 0$, then a higher-order expansion is required.

Theorem 2 Assume that at $\tau = \tau_0$, λ_0 is a real root with multiplicity $m = 2$, Eq. (14) holds and $d = 2p'_\tau(\lambda_0, \tau_0) = 0$, and let $L = c_1 T + c_2 T^2 + \dots + c_m T^m + o(|T|^m)$ be the Puiseux series expansion such that $\operatorname{Re}(c_1) = \operatorname{Re}(c_2) = \dots = \operatorname{Re}(c_{m-1}) = 0$ and $\operatorname{Re}(c_m) \neq 0$. Then, $\sigma(\tau)$ is minimized at $\tau = \tau_0$ in a neighborhood of τ_0 only if m is even and $\operatorname{Re}(c_m) > 0$.

Example 2 Consider a first-order linear time-delay system described by

$$\dot{x}(t) + ux(t) = -vx(t - \tau) - w\dot{x}(t - \tau), \quad \tau > 0,$$

whose characteristic equation is

$$p(\lambda, \tau) := \lambda + u + (w\lambda + v)e^{-\lambda\tau} = 0.$$

This time-delay system is of retard-type when $v \neq 0$ and $w = 0$, and it is of neutral-type when $w \neq 0$. Assume that λ is a characteristic root with multiplicity 2. Then, eliminating $e^{-\lambda\tau}$ from Eq. (5) with $m = 1$ gives

$$w\tau\lambda^2 + (uw + v)\tau\lambda + u(v\tau - w) + v = 0. \quad (20)$$

When $w = 0$, one has $\lambda = -u + \frac{1}{\tau}$. Thus, it is necessary to have $-\frac{1}{\tau} + ve^{u\tau+1} = 0$. When $w \neq 0$, by solving λ from Eq. (20) and substituting it into $p'_\lambda(\lambda, \tau) = 0$, one obtains a condition governing the existence of repeated characteristic roots. Once the delay values are on hand, the corresponding repeated roots can be obtained from Eq. (20).

For example, when $u = -0.1$, $v = 1$, and $w = 0$, one finds that at $\tau_0 = 0.3822$, $p(\lambda, \tau)$ has a repeated real root $\lambda_0 = -2.5163$ with multiplicity 2. Calculation of the contour integral (14) shows that $\lambda_0 = -2.5163$ is the rightmost characteristic root. In addition, $a = 0.3822$, $b = -5.1327$, $c = 16.5667$, and $d = 6.5836$. Thus, $\sigma(\tau)$ is minimized at $\tau = \tau_0$, according to Theorem 1. When $u = -0.1$, $v = 0.5$, and $w = -0.2$, it can be shown that at $\tau_0 = 0.4702$, $p(\lambda, \tau)$ has a repeated real root $\lambda_0 = -1.2581$ with multiplicity 2. In this case, the contour integral given in Eq. (14) equals 0, which implies that $\lambda_0 = -1.2581$ is the rightmost characteristic root at $\tau_0 = 0.4702$. In addition, $a = 0.6402$, $b = -2.6162$, $c = 2.1495$, and $d = 1.7086$. Thus, $\sigma(\tau)$ is minimized at $\tau = \tau_0$, according to Theorem 1. By using the parametric continuation method^[15], the curves of $\sigma(\tau)$ with respect to $\tau \in [0, 2]$ or to $\tau \in [0, 3]$ can be obtained, as shown in Fig. 2, where it is at $\tau = \tau_0$ that the real repeated characteristic root λ_0 with multiplicity 2 is the minimal value of $\sigma(\tau)$ in $\tau \in [0, 2]$ or in $\tau \in [0, 3]$.

Fig. 2 Plots of $\sigma(\tau)$: (a) $u = -0.1$, $v = 1$, and $w = 0$; (b) $u = -0.1$, $v = 0.5$, and $w = -0.2$ (color online)

4.2 Real characteristic root with multiplicity 3

Assume that at $\tau = \tau_0$, λ_0 is a real characteristic root with multiplicity 3, Eq. (14) holds, and $g = 6p'_\tau(\lambda_0, \tau_0) \neq 0$. In a small neighborhood of τ_0 , $L = \lambda - \lambda_0$ is dominated by

$$L = c_1 T^{1/3} + o(|T|^{1/3}), \quad T = \tau - \tau_0 \quad (21)$$

due to Eqs. (10) and (12), where $a = p_{\lambda, \lambda, \lambda}^{(3)}(\lambda_0, \tau_0) \neq 0$, and

$$c_1 = \sqrt[3]{-\frac{6p'_\tau(\lambda_0, \tau_0)}{p_{\lambda, \lambda, \lambda}^{(3)}(\lambda_0, \tau_0)}}, \quad \sqrt[3]{-\frac{6p'_\tau(\lambda_0, \tau_0)}{p_{\lambda, \lambda, \lambda}^{(3)}(\lambda_0, \tau_0)}} \left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \right).$$

If $p'_\tau(\lambda_0, \tau_0) \cdot p_{\lambda, \lambda, \lambda}^{(3)}(\lambda_0, \tau_0) > 0$, then in a left-neighborhood with $\tau < \tau_0$, or in right-neighborhood of τ_0 with $\tau > \tau_0$, the SA $\sigma(\tau)$ is dominated, respectively, by

$$-\sqrt[3]{\frac{6p'_\tau(\lambda_0, \tau_0)}{p_{\lambda, \lambda, \lambda}^{(3)}(\lambda_0, \tau_0)}}(\tau - \tau_0)^{\frac{1}{3}} \quad \text{or} \quad \frac{1}{2} \sqrt[3]{\frac{6p'_\tau(\lambda_0, \tau_0)}{p_{\lambda, \lambda, \lambda}^{(3)}(\lambda_0, \tau_0)}}(\tau - \tau_0)^{\frac{1}{3}}.$$

If $p'_\tau(\lambda_0, \tau_0) \cdot p_{\lambda, \lambda, \lambda}^{(3)}(\lambda_0, \tau_0) < 0$, then in a left-neighborhood with $\tau < \tau_0$, or in right-neighborhood of τ_0 with $\tau > \tau_0$, the SA $\sigma(\tau)$ is dominated, respectively, by

$$\frac{1}{2} \sqrt[3]{\frac{6p'_\tau(\lambda_0, \tau_0)}{p_{\lambda, \lambda, \lambda}^{(3)}(\lambda_0, \tau_0)}}(\tau - \tau_0)^{\frac{1}{3}} \quad \text{or} \quad -\sqrt[3]{\frac{6p'_\tau(\lambda_0, \tau_0)}{p_{\lambda, \lambda, \lambda}^{(3)}(\lambda_0, \tau_0)}}(\tau - \tau_0)^{\frac{1}{3}}.$$

For both cases, as τ increases in the neighborhood, $\sigma(\tau)$ is first decreased and then increased. Thus, $\sigma(\tau)$ is minimized at $\tau = \tau_0$ in the neighborhood of τ_0 .

Theorem 3 Assume that at $\tau = \tau_0$, λ_0 is a real root with multiplicity $m = 3$, Eq. (14) holds, and $g = 6p'_\tau(\lambda_0, \tau_0) \neq 0$. Then, $\sigma(\tau)$ is minimized at $\tau = \tau_0$ in a neighborhood of τ_0 .

When $g = 6p'_\tau(\lambda_0, \tau_0) = 0$, $e = 6p_{\lambda, \tau}^{(2)}(\lambda_0, \tau_0) \neq 0$, and $f = 3p_{\tau, \tau}^{(2)}(\lambda_0, \tau_0) \neq 0$, the three branches of the repeated characteristic root in a neighborhood of τ_0 are dominated, respectively, by

$$L = -\frac{p_{\tau, \tau}^{(2)}(\lambda_0, \tau_0)}{2p_{\lambda, \tau}^{(2)}(\lambda_0, \tau_0)}T + o(T), \quad L = \pm \sqrt{-\frac{6p_{\lambda, \tau}^{(2)}(\lambda_0, \tau_0)}{p_{\lambda, \lambda, \lambda}^{(3)}(\lambda_0, \tau_0)}}T^{1/2} + o(|T|^{1/2}). \quad (22)$$

If $p_{\lambda,\tau}^{(2)}(\lambda_0, \tau_0) \cdot p_{\lambda,\lambda,\lambda}^{(3)}(\lambda_0, \tau_0) > 0$, then in a right-neighborhood of τ_0 with $T > 0$, $\text{Re}(L)$ is determined by the first relationship in Eq. (22). In order that $\text{Re}(L)$ decreases when $T > 0$, it requires that $p_{\tau,\tau}^{(2)}(\lambda_0, \tau_0) \cdot p_{\lambda,\tau}^{(2)}(\lambda_0, \tau_0) < 0$. Other cases can be discussed similarly. Consequently, the following theorem is true.

Theorem 4 Assume that at $\tau = \tau_0$, λ_0 is a real root with multiplicity $m = 3$, Eq. (14) holds and $g = 6p_{\tau}^{\prime}(\lambda_0, \tau_0) = 0$, but $e = 6p_{\lambda,\tau}^{(2)}(\lambda_0, \tau_0) \neq 0$, and $f = 3p_{\tau,\tau}^{(2)}(\lambda_0, \tau_0) \neq 0$. Then, $\sigma(\tau)$ is minimized at $\tau = \tau_0$ in a neighborhood of τ_0 if (i) $p_{\tau,\tau}^{(2)}(\lambda_0, \tau_0) \cdot p_{\lambda,\tau}^{(2)}(\lambda_0, \tau_0) > 0$ and $p_{\lambda,\tau}^{(2)}(\lambda_0, \tau_0) \cdot p_{\lambda,\lambda,\lambda}^{(3)}(\lambda_0, \tau_0) < 0$; or (ii) $p_{\tau,\tau}^{(2)}(\lambda_0, \tau_0) \cdot p_{\lambda,\tau}^{(2)}(\lambda_0, \tau_0) < 0$ and $p_{\lambda,\tau}^{(2)}(\lambda_0, \tau_0) \cdot p_{\lambda,\lambda,\lambda}^{(3)}(\lambda_0, \tau_0) > 0$.

Example 3 Consider a vibratory system under a delayed position feedback described by

$$\ddot{x}(t) + 2\xi\omega_0\dot{x}(t) + \omega_0^2x(t) = -\omega_0kx(t - \tau),$$

where $\omega_0 > 0$ and $|\xi| < 1$. The characteristic function of the equilibrium $(x, \dot{x}) = (0, 0)$ is

$$p(\lambda; \tau) = \lambda^2 + 2\xi\omega_0\lambda + \omega_0^2 + \omega_0ke^{-\lambda\tau}.$$

By using the method proposed in Ref. [15], it is easy to know that at

$$(\tau, k) = (\tau_0, k_0) = \left(\frac{1}{\sqrt{1-\xi^2}\omega_0}, -2(1-\xi^2)\omega_0 e^{-\frac{\xi+\sqrt{1-\xi^2}}{\sqrt{1-\xi^2}}} \right),$$

the time-delay system has a characteristic root λ_0 with multiplicity $m = 3$, given by

$$\lambda = \lambda_0 = -(\xi + \sqrt{1-\xi^2})\omega_0.$$

Then, $g = p_{\tau}^{\prime}(\lambda_0, \tau_0) = -12\omega_0^3(1-\xi^2)(\xi + \sqrt{1-\xi^2}) < 0$ when $k = k_0$. According to Theorem 3, $\sigma(\tau)$ is minimized at $\tau = \tau_0$ in a neighborhood of τ_0 , if the condition (14) holds. This can be easily checked by using the parameter continuation method as done in Ref. [15].

Here, due to $k_0 < 0$, $u = -\omega_0k_0x(t - \tau)$ is a delayed positive feedback when $|\xi| < 1$. Thus, for the delayed vibration system discussed above, a delayed positive feedback is better than the corresponding delayed negative feedback from the viewpoint of stability, which is in full agreement with the main results proved in Ref. [19].

When $\tau = \tau_0$ is fixed, the same procedure can be done for the case when σ is regarded as a function of k . Then, $\sigma(k)$ is minimized at $k = k_0$.

5 Conclusions

The parameter continuation method works effectively in calculating the SA of a time-delay system. Special attention should be paid to the case when the characteristic equation has repeated roots, for which a Puiseux series expansion is required to identify the multiple branches of the repeated characteristic root. One key observation is that a repeated real characteristic root might result in minimized SA, which can be regarded as the optimized decaying speed of the control error. The main contribution of the paper is the four criteria for testing whether the repeated real root is the minimized SA or not, represented in terms of the partial derivatives of the characteristic function and a contour integral associated with the characteristic function. Further investigation will be made for more general cases.

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