Surface contact behavior of functionally graded thermoelectric materials indented by a conducting punch*

Xiaojuan TIAN¹, Yueting ZHOU^{1,†}, Lihua WANG¹, Shenghu DING²

1. School of Aerospace Engineering and Applied Mechanics, Tongji University,
Shanghai 200092, China;

2. School of Mathematics and Statistics, Ningxia University,
Yinchuan 750021, China

(Received Dec. 16, 2020 / Revised Mar. 1, 2021)

Abstract The contact problem for thermoelectric materials with functionally graded properties is considered. The material properties, such as the electric conductivity, the thermal conductivity, the shear modulus, and the thermal expansion coefficient, vary in an exponential function. Using the Fourier transform technique, the electro-thermo-elastic problems are transformed into three sets of singular integral equations which are solved numerically in terms of the unknown normal electric current density, the normal energy flux, and the contact pressure. Meanwhile, the complex homogeneous solutions of the displacement fields caused by the gradient parameters are simplified with the help of Euler's formula. After addressing the non-linearity excited by thermoelectric effects, the particular solutions of the displacement fields can be assessed. The effects of various combinations of material gradient parameters and thermoelectric loads on the contact behaviors of thermoelectric materials are presented. The results give a deep insight into the contact damage mechanism of functionally graded thermoelectric materials (FGTEMs).

Key words thermoelectric material, functionally graded property, conducting punch, conjugate complex root, energy flux, contact pressure

Chinese Library Classification O343.3

2010 Mathematics Subject Classification 45B05, 74B05, 74F05, 74M15

1 Introduction

The temperature difference between cold and hot extremities of thermoelectric devices can be directly converted into electrical energy by the Seebeck effect. Conversely, through the Peltier effect, the thermoelectric device can be energized to achieve cooling^[1–2]. Therefore, they have broad applications in the fields of aviation detectors, engine waste heat power generation, thermoelectric refrigeration, and wearable equipment^[3–4].

* Citation: TIAN, X. J., ZHOU, Y. T., WANG, L. H., and DING, S. H. Surface contact behavior of functionally graded thermoelectric materials indented by a conducting punch. *Applied Mathematics and Mechanics (English Edition)*, 42(5), 649–664 (2021) <https://doi.org/10.1007/s10483-021-2732-8>

† Corresponding author, E-mail: zhouyt@tongji.edu.cn

Project supported by the National Natural Science Foundation of China (Nos. 11972257, 11832014, 11762016, and 11472193) and the Fundamental Research Funds for the Central Universities (No. 22120180223)

©The Author(s) 2021

Generally, due to thermal cycling, thermal shock, and mismatch of thermo-electric-elastic properties, internal thermal stress and mechanical stress inevitably occur in thermoelectric devices^[5]. It should be strongly emphasized that functionally graded thermoelectric materials (FGTEMs) can effectively adjust the contact stress distribution on the surface of materials and improve the contact damage resistance, which cannot be achieved by conventional isotropic thermoelectric materials. In 1960, Ioffe first proposed the concept of FGTEMs^[6]. Many researchers made endeavors to improve the thermoelectric efficiency of FGTEMs by increasing the peak efficiency^[7–10]. Many scholars have verified that FGTEMs have excellent properties from the experimental point of view. Dashevsky et al.^[11] proposed and experimentally confirmed the feasibility of preparing FGTEMs for In-doped PbTe single crystals. Anatychuk et al.^[12] showed that the thermoelectric efficiency of Bi₂Te₃-based functionally graded thermoelectric legs was about 15% higher than that of uniform thermoelectric materials. Ju et al.^[13] gave the theory and numerical simulation of FGTEMs, indicating that the selection of reasonable gradient properties can improve the thermoelectric conversion efficiency.

The contact areas of thermoelectric devices are extremely failure-prone^[14]. Specifically, contact damage appearing between the electrodes and the thermoelectric legs during long-term operation can surely degrade the performance of thermoelectric equipment, e.g., impeding the loading transition and reducing the interfacial bonding strength^[15–16]. Researchers found that the main cause of thermoelectric device failure is the cracking at the dissimilar interfaces^[17]. The connection between electrodes (punches) and FGTEMs is a key technique for device integration^[18], which relies on the development of corresponding contact mechanics.

Much attention has been paid to the contact mechanics of various materials, involving electroelastic, thermoelastic, and thermo-electro-elastic coupling. As for the electroelastic coupling, the fretting contact model of a piezoelectric half-plane indented by a rigid insulating cylindrical punch was presented^[19]. Then, the fretting contact between a rigid conducting cylindrical punch and a piezoelectric half-plane with functionally graded piezoelectric materials (FGPMs) coating was studied^[20]. The axisymmetric torsional fretting contact of a piezoelectric half-plane with FGPMs coating acted by a spherical conductive punch was investigated^[21], with the assumption that the elastic constants, piezoelectric constants, and dielectric constants of FGPMs have the same gradient index. Based on a similar contact configuration, FGPMs with arbitrarily varying properties were solved^[22–23]. In the case of thermoelastic contact, the indentation response of coating materials in thermoelasticity was analyzed^[24], and the homogeneous multi-layered model was used to approximate the arbitrarily varying parameters of functionally graded materials^[25]. The thermo-electro-mechanical coupling contact problems with frictional heat generated in homogeneous or graded piezoelectric materials have also been investigated^[26–27]. Recently, Zhou et al.^[28] gave useful formulas predicting the homogeneous thermoelectric parameters through the indentation test. Tian et al.^[29] presented a theoretical contact model to reveal the surface sliding contact damage mechanism of homogeneous thermoelectric materials.

However, to the authors' best knowledge, the accurate contact theory about the FGTEMs and the rigid punch has not been addressed in the literature. In the present paper, we perform a theoretical contact model for the FGTEMs acted by the rigid flat punch which is a perfect electric and thermal conductor. First, relevant formulas of the contact model are given. Then, the Cauchy-type singular integral equations in the electric, temperature, and elastic fields are derived, respectively. The degenerated model is presented to verify the generality of the model. Numerical results are provided to explore the effects of material gradient parameters and thermoelectric loads on the contact behaviors of the FGTEMs.

2 Contact model and theoretical analysis

The contact problem under consideration is described in Fig.1. The rigid flat punch of the length $2a$ interacts with the FGTEMs. The punch is assumed to be a perfectly thermally

conducting and electrically conducting conductor and has the total electric current J_{e0} , the total energy flux J_{u0} , and the external force P without any loss being permitted within the contact area.

Fig. 1 The geometry of the plane contact problem for the FGTEMs (color online)

2.1 Basic equations

The thermo-electric-elastic properties of the FGTEMs are approximated by

$$\gamma(y) = \gamma_0 e^{\delta y}, \quad \lambda(y) = \lambda_0 e^{\chi y}, \quad G(y) = G_0 e^{\alpha y}, \quad \beta(y) = \beta_0 e^{\mu y} \quad (1)$$

for $0 < y < \infty$, where $\gamma(y)$, $\lambda(y)$, $G(y)$, and $\beta(y)$ are the electric conductivity, the thermal conductivity, the shear modulus, and the thermal expansion coefficient, respectively. The corresponding values at the surface $y = 0$ of the FGTEMs are denoted as γ_0 , λ_0 , G_0 , and β_0 . The notations δ , χ , α , and μ are the corresponding gradient parameters. These gradient parameters can be either positive or negative. When the gradient index is positive, the value of the corresponding material property approaches infinity as the depth increases, while for negative gradient indexes, it approaches zero. The arbitrary material properties of FGTEMs may be more realistic. As for classic functionally graded materials, the arbitrary parameters have been investigated^[22–23,25]. The present exponential model of FGTEMs may serve as the solid foundation for the future work of FGTEMs with a further arbitrary variation of properties.

With the parameters introduced above, by considering that both electric charges and energy are conserved, the governing equations and the constitutive equations can be written as

$$\nabla \cdot \mathbf{j}_e = 0, \quad \nabla \cdot \mathbf{j}_u = 0, \quad (2)$$

$$\mathbf{j}_e = -\gamma(y) \nabla V - \gamma(y) \varepsilon \nabla T, \quad \mathbf{q} = \varepsilon T \mathbf{j}_e - \lambda(y) \nabla T, \quad (3)$$

where $\nabla \cdot = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ is a divergence operator. $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})^T$ is a gradient operator. $\mathbf{j}_e = (j_{ex}, j_{ey})^T$ denotes the electric current density. $\mathbf{q} = (q_x, q_y)^T$ represents the thermal flux. The superscript “T” means transpose of a vector or a matrix. The energy flux $\mathbf{j}_u$ can be expressed as $\mathbf{j}_u = \mathbf{q} + \mathbf{j}_e V$. ε , V , and T are the Seebeck coefficient, the electric potential, and the temperature, respectively. To facilitate the solution procedure, the electrochemical potential function F is defined as $F = V + \varepsilon T$. After calculating, the governing equations and the constitutive equations in Eqs. (2) and (3) revealing the thermoelectric gradient properties can be rewritten as follows:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \delta \frac{\partial F}{\partial y} = 0, \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \chi \frac{\partial T}{\partial y} = -\frac{\gamma(y)}{\lambda(y)} \left(\left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 \right), \quad (4)$$

$$\mathbf{j}_e = -\gamma(y) \nabla F, \quad \mathbf{j}_u = -\gamma(y) F \nabla F - \lambda(y) \nabla T. \quad (5)$$

The electrochemical potential function $F(x, y)$ can be solved from the first equation of Eq. (4). The temperature $T(x, y)$ can be obtained from the second equation of Eq. (4) which can further help to solve the elastic field.

The thermoelasticity constitutive relations of the FGTEMs in terms of the temperature $T(x, y)$ as well as the displacement components $u(x, y)$ and $w(x, y)$ are written as

$$\begin{cases} \sigma_{xx} = \frac{G_0 e^{\alpha y}}{\kappa - 1} \left((1 + \kappa) \frac{\partial u}{\partial x} + (3 - \kappa) \frac{\partial w}{\partial y} - 4\beta^* e^{\mu y} T \right), \\ \sigma_{yy} = \frac{G_0 e^{\alpha y}}{\kappa - 1} \left((1 + \kappa) \frac{\partial w}{\partial y} + (3 - \kappa) \frac{\partial u}{\partial x} - 4\beta^* e^{\mu y} T \right), \quad \sigma_{xy} = G_0 e^{\alpha y} \left(\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \right), \end{cases} \quad (6)$$

where $\kappa = 3 - 4\nu$, and $\beta^* = (1 + \nu)\beta_0$ for the considered plane strain case.

The equilibrium equations of the FGTEMs are given as follows:

$$\begin{cases} \frac{\kappa + 1}{\kappa - 1} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2}{\kappa - 1} \frac{\partial^2 w}{\partial x \partial y} + \alpha \left(\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \right) = \frac{4\beta^* e^{\mu y}}{\kappa - 1} \frac{\partial T}{\partial x}, \\ \frac{\kappa + 1}{\kappa - 1} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x^2} + \frac{2}{\kappa - 1} \frac{\partial^2 u}{\partial x \partial y} + \frac{\alpha}{\kappa - 1} \left((3 - \kappa) \frac{\partial u}{\partial y} + (\kappa + 1) \frac{\partial w}{\partial x} \right) = \frac{4\beta^* e^{\mu y}}{\kappa - 1} \left((\alpha + \mu) T + \frac{\partial T}{\partial y} \right). \end{cases} \quad (7)$$

To solve the governing equations in Eqs. (4) and (7), the classical Fourier integral transform technique and the superposition principle are adopted.

2.2 Electric field

The solution to the first equation of Eq. (4) under the regularity condition at infinity ($F(x, \infty) = 0, |x| < \infty$) can be obtained as follows:

$$F(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(s) e^{n(s)y} e^{-isx} ds, \quad (8)$$

where $n(s)$ is negative and given as $n(s) = -\frac{\delta}{2} - \sqrt{\frac{\delta^2}{4} + s^2}$, and $A(s)$ is to be determined.

Due to the problem considered here, the surface normal electric current density $j_{ey}(x, 0)$ can be denoted as

$$j_{ey}(x, 0) = -J_e^*(x), \quad |x| < a, \quad j_{ey}(x, 0) = 0, \quad |x| > a, \quad (9)$$

where $J_e^*(x)$ is the unknown normal electric current density under the punch, and satisfies

$$\int_{-a}^a J_e^*(x) dx = J_{e0}, \quad (10)$$

in which J_{e0} represents the total electric current acted on the punch.

Considering the first equation of Eq. (5) and Eqs. (8) and (9), $A(s)$ can be given as $A(s) = -\frac{1}{\gamma_0 n(s)} \int_{-a}^a J_e^*(x) e^{isx} dx$.

Substituting $A(s)$ into Eq. (8) and performing an asymptotic analysis, one can get

$$\frac{\partial F(x, 0)}{\partial x} = \frac{1}{\pi \gamma_0} \int_{-a}^a J_e^*(r) \left(K_{11}(x, r) - \frac{1}{r - x} \right) dr, \quad (11)$$

where the generalized Fredholm integral kernel $K_{11}(x, r)$ without singularity is

$$K_{11}(x, r) = \int_0^\infty \left(\frac{s}{n(s)} + 1 \right) \sin(s(r - x)) ds. \quad (12)$$

Furthermore, by considering the boundary condition $F(x, 0) = F_0$, $|x| < a$, Eq. (11) becomes

$$\int_{-a}^a J_e^*(r) \left(K_{11}(x, r) - \frac{1}{r-x} \right) dr = 0, \quad (13)$$

where $J_e^*(r)$ satisfies the equilibrium equation (10).

It is noted that $K_{11}(x, r)$ will vanish when the electric conductivity gradient parameter δ equals zero. Then, Eq. (13) becomes one that describes the isotropic thermoelectric materials with the following exact solution^[29]: $J_e^*(x) = \frac{J_{e0}}{\pi\sqrt{a^2-x^2}}$, $|x| < a$, which can verify our derivation.

2.3 Temperature field

The distributions of the temperature function $T(x, y)$ and the normal energy flux $j_{uy}(x, y)$ on the surface or at infinity can be expressed as

$$T(x, 0) = T_0, \quad |x| < a, \quad T(x, \infty) = 0, \quad |x| < \infty, \quad (14)$$

$$j_{uy}(x, 0) = -J_u^*(x), \quad |x| < a, \quad j_{uy}(x, 0) = 0, \quad |x| > a, \quad (15)$$

where T_0 and $J_u^*(x)$ are the constant value temperature within the contact area and the unknown normal energy flux inside the contact region, respectively. The prescribed total energy flux J_{u0} above the punch is

$$\int_{-a}^a J_u^*(x) dx = J_{u0}. \quad (16)$$

Indeed, the gradient index of electric conductivity δ and that of thermal conductivity χ for real FGTEMs may be different, which will make it hard to obtain the particular solution of the temperature function in an analytical form when solving the second equation of Eq. (4). Therefore, the assumption that δ and χ have the same values is used to obtain the particular solution of temperature fields in an analytical form. Except for other statements, it is modeled that χ is equal to δ in the following calculation. Now, by employing the method used in the electric field and considering the second equation of Eq. (4) and boundary conditions in Eqs. (14) and (15), the temperature field involving the particular solutions and homogeneous solutions can be obtained as follows:

$$T(x, y) = -\frac{\gamma_0}{2\lambda_0} F^2(x, y) + \frac{1}{2\pi} \int_{-\infty}^{\infty} B(s) e^{n(s)y} e^{-isx} ds, \quad (17)$$

where $B(s)$ is given as $B(s) = -\frac{1}{\lambda_0 n(s)} \int_{-a}^a J_u^*(x) e^{isx} dx$.

The nonlinearity in Eq. (17) is caused by the coupled thermoelectric materials constitutive relations in Eq. (3). The singular integral equation in terms of $J_u^*(r)$ can be expressed as follows:

$$\int_{-a}^a J_u^*(r) \left(K_{11}(x, r) - \frac{1}{r-x} \right) dr = 0 \quad (18)$$

with the equilibrium equation (16) to complete the solution.

The above assumption $\delta = \chi$ is used to obtain the particular solution of temperature fields in an analytical form, which finally leads to the similar equations, i.e., Eqs. (13) and (18). While the total electric current J_{e0} and total energy flux J_{u0} appearing in the equilibrium equations (10) and (16) may make the two similar singular integral equations have different solutions, the results obtained here are primary and can act as the beginning of a more complex situation.

2.4 Mechanical field

Similarly, the Fourier integral transform technique can be used to solve Eq. (7). By considering the displacement boundary conditions at infinity, $u(x, \infty) = 0$, $w(x, \infty) = 0$, $|x| < \infty$, the complex homogeneous solutions to Eq. (7) in the Fourier transform field can be written as

$$\tilde{u}_H(s, y) = \sum_{j=1}^2 C_j e^{m_j y}, \quad \tilde{w}_H(s, y) = -i \sum_{j=1}^2 C_j M_j e^{m_j y}, \quad (19)$$

where $\tilde{u}(s, y) = \int_{-\infty}^{\infty} u(x, y) e^{isx} dx$, and $\tilde{w}(s, y) = \int_{-\infty}^{\infty} w(x, y) e^{isx} dx$. $C_1(s)$ and $C_2(s)$ are to be determined. $M_j = \frac{(\kappa-1)(m_j^2 + \alpha m_j) - s^2(\kappa+1)}{s(2m_j + \alpha(\kappa-1))}$, in which m_j ($j = 1, 2$) with negative real parts are given by

$$m_1 = -\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} + s^2 + i|\alpha||s| \left(\frac{3-\kappa}{\kappa+1}\right)^{1/2}}, \quad m_2 = -\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} + s^2 - i|\alpha||s| \left(\frac{3-\kappa}{\kappa+1}\right)^{1/2}}. \quad (20)$$

It is noticed that the two characteristic roots in Eq. (20) are conjugate complex roots caused by the FGTEMs. Then, m_j ($j = 1, 2$) can be denoted as^[30]

$$m_1 = \overline{m_2} = o_1 + i\tau_1, \quad (21)$$

where the bar represents conjugation. $o_1 = \text{Re}(m_1)$, and $\tau_1 = \text{Im}(m_1)$. $\text{Re}(\cdot)$ and $\text{Im}(\cdot)$ are the real part and the imaginary part, respectively.

By substituting Eq. (21) into Eq. (19) and using Euler's formula, the transformed complex homogeneous solutions can be rewritten in the real homogeneous solutions' form as follows:

$$\begin{cases} \tilde{u}_H(s, y) = e^{o_1 y} (C_1 \cos(\tau_1 y) + C_2 \sin(\tau_1 y)), \\ \tilde{w}_H(s, y) = -ie^{o_1 y} (C_1 (\Gamma_1 \cos(\tau_1 y) - \Theta_1 \sin(\tau_1 y)) + C_2 (\Theta_1 \cos(\tau_1 y) + \Gamma_1 \sin(\tau_1 y))), \end{cases} \quad (22)$$

where $\Gamma_1 = \text{Re}(M_1)$, and $\Theta_1 = \text{Im}(M_1)$.

With the convolution formula, the nonlinear term in Eq. (17) can be addressed. Then, the particular solutions to Eq. (7) under the thermoelectric effects are

$$\begin{cases} \tilde{u}_P(s, y) = -\frac{i\beta^* \gamma_0 e^{\mu y}}{\pi \lambda_0 (\kappa - 1)} \int_{-\infty}^{\infty} \frac{E_1}{\Delta_1} A(s - \xi) A(\xi) e^{\eta_1(s, \xi)y} d\xi + \frac{i4\beta^* e^{\mu y}}{\kappa - 1} \frac{E_2}{\Delta_2} B(s) e^{\eta_2(s)y}, \\ \tilde{w}_P(s, y) = -\frac{\beta^* \gamma_0 e^{\mu y}}{\pi \lambda_0 (\kappa - 1)} \int_{-\infty}^{\infty} \frac{Z_1}{\Delta_1} A(s - \xi) A(\xi) e^{\eta_1(s, \xi)y} d\xi + \frac{4\beta^* e^{\mu y}}{\kappa - 1} \frac{Z_2}{\Delta_2} B(s) e^{\eta_2(s)y}, \end{cases} \quad (23)$$

where $\eta_1(s, \xi) = n(\xi) + n(s - \xi)$, $\eta_2(s) = n(s)$, and E_j , Z_j , and Δ_j ($j = 1, 2$) are given by

$$\begin{cases} E_j = s(\mu + \eta_j + \alpha) \left(H_j + 2\alpha \frac{\kappa - 2}{\kappa - 1} \right) - sV_j, \\ Z_j = (\mu + \eta_j + \alpha) \left(\frac{\kappa - 1}{\kappa + 1} V_j - \frac{4\kappa s^2}{\kappa^2 - 1} \right) + s^2 H_j, \\ \Delta_j = V_j \left(\frac{\kappa - 1}{\kappa + 1} V_j - \frac{4\kappa s^2}{\kappa^2 - 1} \right) + s^2 H_j \left(H_j + 2\alpha \frac{\kappa - 2}{\kappa - 1} \right), \end{cases} \quad (24)$$

in which $V_j = \frac{\kappa+1}{\kappa-1}(\mu + \eta_j)(\mu + \eta_j + \alpha) - s^2$, and $H_j = \frac{(\mu + \eta_j) + \alpha(3 - \kappa)}{\kappa - 1}$.

From Eq. (23), it can be seen that the parts relating to functions A and B represent the effects of the electric field and the temperature field on the displacement field, respectively.

As mentioned before, the gradient parameters of the material properties can be either positive or negative. These two different cases make the kernels of the integral equations sufficiently

different. Whether the gradient parameters are positive or negative, the root $n(s)$ is almost always negative, and the characteristic roots $m_1 = \bar{m}_2$ given in Eq. (20) have negative real parts verified by the numerical tests. Thus, for the half-plane problem with $y > 0$, the displacements and the stresses vanish at infinity by checking Eqs. (19), (22), (23), and (6).

3 Solutions of the contact stress

With the transform matrix method, the relationship between the surface displacement and the surface stress can be obtained. Interested readers can find detailed information in Ref. [29]. The normal displacement can then be given as follows:

$$w(x, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (Q_{21}\tilde{\sigma}_{yy}(s, 0) + Q_{22}\tilde{\sigma}_{xy}(s, 0) + L_2(s))e^{-isx}ds, \quad (25)$$

where

$$\begin{cases} \tilde{\sigma}_{yy}(s, 0) = \int_{-\infty}^{\infty} \sigma_{yy}(x, 0)e^{isx}dx, \\ \tilde{\sigma}_{xy}(s, 0) = \int_{-\infty}^{\infty} \sigma_{xy}(x, 0)e^{isx}dx, \\ L_2(s) = R_2(s) - Q_{21}R_3(s) - Q_{22}R_4(s), \end{cases}$$

and

$$\mathbf{Q}(s) = \begin{pmatrix} 1 & 0 \\ -i\Gamma_1 & -i\Theta_1 \end{pmatrix} \cdot \left(\begin{array}{cc} -\frac{iG_0((-3+\kappa)s - (1+\kappa)(\Gamma_1o_1 - \Theta_1\tau_1))}{-1+\kappa} & -\frac{iG_0(1+\kappa)(\Theta_1o_1 + \Gamma_1\tau_1)}{-1+\kappa} \\ G_0(o_1 - \Gamma_1s) & G_0(\tau_1 - \Theta_1s) \end{array} \right)^{-1}. \quad (26)$$

In the above equations,

$$\begin{cases} R_2(s) = \tilde{w}_P(s, 0), \quad R_4(s) = G_0 \frac{\partial}{\partial y} \tilde{u}_P(s, y)|_{y=0} - isG_0 \tilde{w}_P(s, 0), \\ R_3(s) = \frac{G_0(1+\kappa)}{\kappa-1} \frac{\partial}{\partial y} \tilde{w}_P(s, y)|_{y=0} - is \frac{G_0(3-\kappa)}{\kappa-1} \tilde{u}_P(s, 0) - \frac{4G_0\beta^*}{\kappa-1} \tilde{T}(s, 0). \end{cases} \quad (27)$$

Differentiating Eq. (25) with respect to x yields

$$\frac{\partial w(x, 0)}{\partial x} = -\frac{i}{2\pi} \int_{-\infty}^{\infty} (sQ_{21}\tilde{\sigma}_{yy}(s, 0) + sQ_{22}\tilde{\sigma}_{xy}(s, 0) + sL_2(s))e^{-isx}ds. \quad (28)$$

The asymptotic behavior of the integrands for the variable s can be given as

$$\lim_{|s| \rightarrow \infty} sQ_{21} = Q_1^\infty \frac{|s|}{s}, \quad \lim_{|s| \rightarrow \infty} sQ_{22} = iQ_2^\infty, \quad (29)$$

where $Q_1^\infty = -\frac{\kappa+1}{4G_0}$, and $Q_2^\infty = -\frac{\kappa-1}{4G_0}$.

The surface normal stress inside the contact region is unknown, denoted as $p(x)$, while it remains free outside the contact region, i.e.,

$$\sigma_{yy}(x, 0) = -p(x), \quad |x| < a; \quad \sigma_{yy}(x, 0) = 0, \quad |x| > a. \quad (30)$$

The mechanical boundary conditions applied to Eq. (7) can be written as

$$w(x, 0) = w_0, \quad |x| < a; \quad \sigma_{xy}(x, 0) = 0, \quad x \in \mathbf{R}. \quad (31)$$

By separating the leading term in Eq. (28) and considering Eqs. (30) and (31), one can get the singular integral equation corresponding to the contact pressure as follows:

$$\frac{1}{\pi} \int_{-a}^a \left(\frac{1}{r-x} - \frac{4G_0}{1+\kappa} K_{22}(x, r) \right) p(r) dr = \frac{1}{\pi} \frac{4G_0}{1+\kappa} f(x), \quad (32)$$

where the generalized Fredholm integral kernel function $K_{22}(x, r)$ without singularity is

$$K_{22}(x, r) = \int_0^\infty (sQ_{21} - Q_1^\infty) \sin(s(r-x)) ds, \quad (33)$$

and the right-hand side function $f(x)$ revealing the thermoelectric effects and the gradient characteristic of thermoelectric materials is given as $f(x) = \int_0^\infty sL_2(s) \sin(sx) ds$ with $L_2(s) = \int_{-\infty}^\infty Z_1(s, \xi) A(s-\xi) A(\xi) d\xi + Z_2(s) B(s)$, in which

$$Z_1(s, \xi) = -\frac{\beta^* \gamma_0 d_1}{\pi \lambda_0 b_1}, \quad Z_2(s) = \frac{4\beta^* d_2}{b_2} \quad (34)$$

with

$$b_j = (1+\kappa)s^4 + (1+\kappa)(\eta_j + \mu)^2(\alpha + \eta_j + \mu)^2 - s^2((-3+\kappa)\alpha^2 + 2(1+\kappa)(\eta_j + \mu)(\alpha + \eta_j + \mu)), \quad j = 1, 2, \quad (35)$$

$$d_j = (\eta_j + \mu)(\alpha + \eta_j + \mu)^2 - s^2(2\alpha + \eta_j + \mu) + 2G_0s(s^2 - (\eta_j + \mu)^2)(sQ_{21} - iQ_{22}(\alpha + \eta_j + \mu)), \quad j = 1, 2. \quad (36)$$

The obtained right-hand side expression of Eq. (32), which is first given, reveals the noteworthy difference between the contact analysis of classical materials (e.g., piezoelectric materials^[31] and piezomagnetic materials^[32]) and that of FGTEMs considered here. Specifically, from Eq. (32) and Eqs. (34)–(36), one can see that the equivalent force $f(x)$ consists of thermoelectric loads relating to the terms of $A(\cdot)$ in the electric field, $B(\cdot)$ in the temperature field, and the gradient parameters α and μ .

To obtain a physically possible solution, the following equilibrium condition should be satisfied:

$$\int_{-a}^a p(x) dx = P. \quad (37)$$

As for homogeneous thermoelectric materials, there exists $sQ_{21} = Q_1^\infty$ which can lead to $K_{22}(x, r) = 0$. Then, the singular integral equation (32) for FGTEMs can degenerate into the case of isotropic thermoelectric materials. Furthermore, it should be noted that if without thermoelectric loads and gradient parameters, $L_2(s)$ associated with the particular solutions of the displacements will also vanish. Then, the singular integral equation for isotropic materials without thermoelectric effects can be reduced to

$$\frac{1}{\pi} \int_{-a}^a \frac{1}{r-x} p(r) dr = 0 \quad (38)$$

with the aid of Eq. (37), and the exact solution can be obtained as $p(x) = \frac{P}{\pi\sqrt{a^2-x^2}}$, $|x| < a$, which can illustrate the correctness of our derivation again.

4 Intensity factors

The unknown normal electric current density $J_e^*(r)$, the unknown normal energy flux $J_u^*(r)$, and the unknown contact pressure $p(r)$ all satisfy the first kind Cauchy-type singular integral equation demonstrated in Eqs. (13), (18), and (32), which can be solved numerically with the collocation method. In the normalized interval as $x = a\tilde{x}$ and $r = a\tilde{r}$, the functions $J_e^*(r)$, $J_u^*(r)$, and $p(r)$ are

$$J_e^*(r) = \frac{\tilde{J}_e^*(\tilde{r})}{\sqrt{1-\tilde{r}^2}} = \frac{\sum_{j=0}^{N-1} Y_j \tilde{T}_j(\tilde{r})}{\sqrt{1-\tilde{r}^2}}, \quad (39)$$

$$J_u^*(r) = \frac{\tilde{J}_u^*(\tilde{r})}{\sqrt{1-\tilde{r}^2}} = \frac{\sum_{j=0}^{N-1} U_j \tilde{T}_j(\tilde{r})}{\sqrt{1-\tilde{r}^2}}, \quad (40)$$

$$p(r) = \frac{\tilde{p}(\tilde{r})}{\sqrt{1-\tilde{r}^2}} = \frac{\sum_{j=0}^{N-1} B_j \tilde{T}_j(\tilde{r})}{\sqrt{1-\tilde{r}^2}}, \quad (41)$$

where $\tilde{T}_j(\tilde{r}) = \cos(j \arccos(\tilde{r}))$ is the Chebyshev polynomial of the first kind.

With the help of the Gauss-Chebyshev quadrature method, one can obtain systems of linear algebraic equations as follows:

$$\frac{1}{N} \sum_{n=1}^N \tilde{J}_e^*(\tilde{r}_n) \left(aK_{11}(a\tilde{x}_l, a\tilde{r}_n) - \frac{1}{\tilde{r}_n - \tilde{x}_l} \right) = 0, \quad \frac{1}{N} \sum_{n=1}^N \tilde{J}_e^*(\tilde{r}_n) = \frac{J_{e0}}{a\pi}, \quad (42)$$

$$\frac{1}{N} \sum_{n=1}^N \tilde{J}_u^*(\tilde{r}_n) \left(aK_{11}(a\tilde{x}_l, a\tilde{r}_n) - \frac{1}{\tilde{r}_n - \tilde{x}_l} \right) = 0, \quad \frac{1}{N} \sum_{n=1}^N \tilde{J}_u^*(\tilde{r}_n) = \frac{J_{u0}}{a\pi}, \quad (43)$$

$$\frac{1}{N} \sum_{n=1}^N \tilde{p}(\tilde{r}_n) \left(\frac{1}{\tilde{r}_n - \tilde{x}_l} - \frac{4G_0}{1+\kappa} aK_{22}(a\tilde{x}_l, a\tilde{r}_n) \right) = \frac{1}{\pi} \frac{4G_0}{1+\kappa} f(a\tilde{x}_l), \quad \frac{1}{N} \sum_{n=1}^N \tilde{p}(\tilde{r}_n) = \frac{P}{a\pi}, \quad (44)$$

where $\tilde{r}_n = \cos\left(\frac{2n-1}{2N}\pi\right)$, $n = 1, 2, \dots, N$, and $\tilde{x}_l = \cos\left(\frac{l}{N}\pi\right)$, $l = 1, 2, \dots, N-1$.

Once the coefficients Y_j , U_j , and B_j are determined, the various intensity factors, such as the electric current density intensity factors $K_e(\pm a)$, the energy flux intensity factors $K_u(\pm a)$, and the stress intensity factors $K_I(\pm a)$, can be evaluated as follows:

$$K_e(\pm a) = \lim_{x \rightarrow -a} \sqrt{2\pi(a \mp x)} J_e^*(x) = \sqrt{\pi a} \sum_{j=0}^{N-1} Y_j (\pm 1)^j, \quad (45)$$

$$K_u(\pm a) = \lim_{x \rightarrow -a} \sqrt{2\pi(a \mp x)} J_u^*(x) = \sqrt{\pi a} \sum_{j=0}^{N-1} U_j (\pm 1)^j, \quad (46)$$

$$K_I(\pm a) = \lim_{x \rightarrow -a} \sqrt{2\pi(a \mp x)} p(x) = \sqrt{\pi a} \sum_{j=0}^{N-1} B_j (\pm 1)^j. \quad (47)$$

When the values of the above-calculated intensity factors reach a certain threshold, contact damage will occur due to crack initiation and propagation.

5 Numerical results and discussion

From the solution process, one can see that if without the gradient parameters δ , χ , α , and μ , the FGTEMs considered here will become homogeneous ones.

The numerical analysis is carried out to investigate the singularities near the contact edges including $K_e(\pm a)$, $K_u(\pm a)$, and $K_I(\pm a)$. The distributions of the contact pressure, the normal energy flux, and the normal electric current density below the rigid flat punch are also studied. The material parameters of the Bi_2Te_3 -based thermoelectric material in Refs. [33] and [34] are selected as the surface properties required for this paper. Specifically, the electrical conductivity, the thermal conductivity, the thermal expansion coefficient, and the shear modulus at the surface $y = 0$ are taken as $\gamma_0 = 1 \times 10^5$ S/m, $\lambda_0 = 2.2$ W/(m·K), $\beta_0 = 1.68 \times 10^{-5}$ K $^{-1}$, and $G_0 = 16.786$ GPa, respectively.

5.1 Convergence studies

The convergence analyses of K_{11} in Eq. (12) and K_{22} in Eq. (33) are given in Figs. 2(a) and 2(b), respectively. The variation of the integral value of $K_{11}(a\tilde{x}_l, a\tilde{r}_n)$ with the truncating number N_{t1} under different gradient indexes $a\delta = a\chi = 1/4, 1/2, 2, 4$ is presented in Fig. 2(a). One can see that the magnitudes of K_{11} become steady as N_{t1} increases under $a\delta = a\chi = 1/4, 1/2$, while the magnitudes of K_{11} have a slight fluctuation with $a\delta$ and $a\chi$ equaling 2 or 4. Therefore, the specific values of K_{11} and their relative errors are listed in Table 1, which shows that the relative errors maintain within 4% when N_{t1} reaches 1×10^8 and are less than 2% once N_{t1} is beyond 2×10^8 . Thus, the value of N_{t1} is taken as 2×10^8 in the following calculations.

Fig. 2 Convergence of kernel functions with $\tilde{x} = \cos(\pi/35)$ and $\tilde{r} = \cos(\pi/70)$, (a) $K_{11}(a\tilde{x}, a\tilde{r})$ and (b) $K_{22}(a\tilde{x}, a\tilde{r})$ (color online)

Table 1 Convergence of the kernel function $K_{11}(a\tilde{x}_l, a\tilde{r}_n)$ with $\tilde{x} = \cos(\pi/35)$ and $\tilde{r} = \cos(\pi/70)$, where $K_{11}^{(k+1)}$ represents the value of K_{11} in the $(k+1)$ th row

N_{t1}	$K_{11}(a\tilde{x}_l, a\tilde{r}_n)$		$\frac{ K_{11}^{(k+1)} - K_{11}^k }{K_{11}^{(k+1)}} \times 100\%$	
	$a\delta = a\chi = 4$	$a\delta = a\chi = 2$	$a\delta = a\chi = 4$	$a\delta = a\chi = 2$
0.0	0	0	—	—
5.0×10^7	32 034.20	16 108.20	100	100
1.0×10^8	30 808.34	15 495.30	3.98	3.96
1.5×10^8	30 882.16	15 532.13	0.24	0.24
2.0×10^8	31 270.01	15 725.89	1.24	1.23
2.5×10^8	30 745.37	15 463.74	1.71	1.70
3.0×10^8	31 197.62	15 689.87	1.45	1.44
3.5×10^8	30 935.42	15 558.77	0.85	0.84
4.0×10^8	30 975.55	15 578.91	0.13	0.13

The kernel function K_{22} in Eq. (33) is relevant to the gradient parameter of the shear modulus $a\alpha$. Figure 2(b) demonstrates the effects of the magnitudes of the truncating number N_{t2} on the

convergent behavior of K_{22} when $a\alpha = 0.001, 1/3, 1, 3$. When $a\alpha$ equals 3, the curve fluctuates slightly, and the corresponding relative errors are lower than 2% as N_{t2} is over 1.5×10^8 . Taking the remaining results of Fig. 2(b) into account, one may take N_{t2} as 2×10^8 in the following calculations, which can ensure the accuracy for the selected gradient parameters.

It should be pointed out that the efficiency of the collocation method sufficiently depends on the value of the gradient index. Table 2 gives the effects of the collocation number N with $a\delta = a\chi = 1/4, 1/2, 2, 4$ on the maximum normal electric current density $j_{ey}(0, 0)/j_{ey0}$ and the maximum normal energy flux $j_{uy}(0, 0)/j_{uy0}$ with $j_{ey0} = J_{e0}/(2a)$ and $j_{uy0} = J_{u0}/(2a)$. It can be seen that the results are closely equal to each other with larger N and the data with $N = 35$ or $N = 50$ are nearly identical. Thereby, $N = 35$ is employed in the thermoelectric calculations.

Table 3 shows the effects of the electric conductivity gradient parameter $a\delta = 1/4, 1, 4$, the thermal conductivity gradient parameter $a\chi = 1/4, 1, 4$, the shear modulus gradient parameter $a\alpha = 0.001, 1, 2$, and the thermal expansion coefficient gradient parameter $a\mu = 1, 2, 4$ on the convergence behavior of the collocation method in the calculation of the maximum contact pressure $\sigma_{yy}(0, 0)/\sigma_0$ with $\sigma_0 = P/(2a)$. The effects of the collocation number on the maximum contact pressure $\sigma_{yy}(0, 0)/\sigma_0$ under different values of the thermoelectric loads J_{u0} and J_{e0} are presented in Table 4. Consider all data given in Tables 3 and 4, and the results become stable with the increase in the collocation number under various gradient parameters. $N = 25$ or $N = 30$ can ensure sufficient accuracy. Thus, $N = 30$ is used in the following calculations of the contact pressure. Consider the four specific gradient parameters taken in Table 3, and the ranges of the gradient parameters for which the accuracy is acceptable are selected as $a\delta, a\chi, a\alpha, a\mu \leq 4$.

Table 2 Effects of N and $a\delta = a\chi$ on $j_{ey}(0, 0)/j_{ey0}$ and $j_{uy}(0, 0)/j_{uy0}$

N	$a\delta = a\chi$			
	1/4	1/2	2	4
4	-0.717 9	-0.741 1	-0.830 6	-0.898 8
8	-0.682 8	-0.709 5	-0.808 4	-0.872 2
12	-0.677 2	-0.705 2	-0.809 8	-0.880 1
16	-0.674 9	-0.702 9	-0.806 4	-0.873 3
20	-0.673 9	-0.702 0	-0.805 8	-0.873 1
25	-0.672 3	-0.700 7	-0.805 9	-0.874 5
30	-0.672 3	-0.701 1	-0.805 1	-0.871 9
35	-0.672 2	-0.700 5	-0.804 6	-0.871 1
50	-0.672 5	-0.700 7	-0.804 8	-0.871 1

Table 3 Effects of the collocation number N and the gradient parameters $a\delta, a\chi, a\alpha$, and $a\mu$ on the maximum contact pressure $\sigma_{yy}(0, 0)/\sigma_0$

N	$J_{e0} = 0.002 \text{ A/m}, J_{u0} = 0.002 \text{ W/m}$						$J_{e0} = 0.008 \text{ A/m}, J_{u0} = 0.008 \text{ W/m}$		
	$a\alpha = 1, a\mu = 0$			$a\delta = a\chi = a\mu = 1$			$a\delta = a\chi = 0, a\alpha = 1$		
	$a\delta = a\chi$			$a\alpha$			$a\mu$		
	1/4	1	4	0.001	1	2	1	2	4
6	-0.855 4	-0.846 3	-0.837 2	-0.672 2	-0.856 0	-0.928 0	-0.907 4	-0.922 8	-0.866 4
12	-0.860 1	-0.850 3	-0.840 5	-0.656 4	-0.860 7	-0.948 8	-0.915 5	-0.932 8	-0.872 5
20	-0.856 0	-0.846 1	-0.836 0	-0.653 1	-0.856 7	-0.940 0	-0.912 7	-0.930 5	-0.868 8
25	-0.856 8	-0.846 7	-0.836 6	-0.651 1	-0.857 5	-0.942 4	-0.914 0	-0.932 0	-0.869 8
30	-0.855 2	-0.845 2	-0.835 2	-0.652 0	-0.855 9	-0.937 4	-0.912 1	-0.930 0	-0.868 1

Table 4 Effects of the collocation number N and the thermoelectric loads J_{e0} and J_{u0} on the maximum contact pressure $\sigma_{yy}(0, 0)/\sigma_0$ ($a\delta = a\chi = a\mu = 1$)

N	$J_{e0} = 0 \text{ A/m}$			$J_{u0} = 0 \text{ W/m}$		
	$J_{u0}/(\text{W}\cdot\text{m}^{-1})$			$J_{e0}/(\text{A}\cdot\text{m}^{-1})$		
	0.001	0.005	0.01	30	70	100
6	-0.845 3	-0.887 8	-0.940 9	-0.825 0	-0.781 7	-0.726 5
12	-0.849 4	-0.894 7	-0.951 4	-0.827 6	-0.781 1	-0.721 9
20	-0.845 1	-0.891 4	-0.949 3	-0.822 8	-0.775 3	-0.714 8
25	-0.845 8	-0.892 5	-0.950 9	-0.823 3	-0.775 4	-0.714 2
30	-0.844 2	-0.890 7	-0.948 9	-0.821 9	-0.774 2	-0.713 4

5.2 Effects of the gradient parameters on thermoelectric fields

Figure 3 reveals the effects of various gradient parameters $a\delta$ and $a\chi$ fixed at $1/4$, $1/2$, 0 , 0.001 , 2 , and 4 on the normalized electric current density $j_{ey}(x, 0)/j_{ey0}$ and the normalized energy flux $j_{uy}(x, 0)/j_{uy0}$. The closed-form results, which can be obtained as

$$j_{ey}(x, 0) = -J_{e0}/(\pi\sqrt{a^2 - x^2}), \quad j_{uy}(x, 0) = -J_{u0}/(\pi\sqrt{a^2 - x^2}),$$

are validated with small values of the gradient parameters $a\delta = a\chi = 0.001$ as the degenerated results shown in Fig. 3. Excellent agreement can be found between the reduced results of $a\delta = a\chi = 0.001$ computed by the program and those of $a\delta = a\chi = 0$ obtained by the closed-form solutions, which further validates the present derivations. Furthermore, there are singularities in $j_{ey}(x, 0)/j_{ey0}$ and $j_{uy}(x, 0)/j_{uy0}$ at both ends of the contact region for the flat punch as shown in Fig. 3. It can be seen that when $a\delta$ and $a\chi$ take different magnitudes, the ranges of $j_{ey}(x, 0)/j_{ey0}$ and $j_{uy}(x, 0)/j_{uy0}$ generated at the contact surface also change. Therefore, by considering the actual operating environment of thermoelectric materials, the appropriate gradient parameters can be selected according to the theoretical analysis of this paper for the rational preparation of thermoelectric materials.

Fig. 3 Effects of various gradient parameters $a\delta$ and $a\chi$ on $j_{ey}(x, 0)/j_{ey0}$ and $j_{uy}(x, 0)/j_{uy0}$, in which $j_{ey}(x, 0)/j_{ey0}$ is equal to $j_{uy}(x, 0)/j_{uy0}$ (color online)

Table 5 shows the effects of different values of the gradient parameters $a\delta$ and $a\chi$ on the intensity factors $K_e(\pm a)/K_0$ and $K_u(\pm a)/K_0$. $K_u(\pm a)/K_0$ is equal to $K_e(\pm a)/K_0$ due to the assumptions of $a\chi = a\delta$ and $J_{e0} = J_{u0}$. Table 5 demonstrates that $K_e(\pm a)/K_0$ and $K_u(\pm a)/K_0$ decrease gradually as the gradient parameters increase. This also proves that the rational design of the gradient parameters of the thermoelectric materials can effectively reduce the singularity of the normal electric current density and the normal energy flux near the contact edges.

Table 5 $K_e(\pm a)/K_0$ and $K_u(\pm a)/K_0$ versus $a\delta$ and $a\chi$ with $K_0 = \sqrt{\pi a}$ and $J_{e0} = J_{u0}$

$a\delta = a\chi$	$\frac{K_e(\pm a)}{K_0} = \frac{K_u(\pm a)}{K_0}$
1/4	3 183.10
1/2	3 182.09
0	2 967.88
0.001	2 795.77
2	2 157.31
4	1 732.35

5.3 Effects of the gradient parameters and thermoelectric loads on the elastic field

The effects of $a\delta$ and $a\chi$ on $\sigma_{yy}(x, 0)/\sigma_0$ are shown in Fig. 4 with $a\alpha = 1$, $a\mu = 0$, $J_{e0} = 0.002$ A/m, and $J_{u0} = 0.002$ W/m. Figure 4 shows that the gradient parameters $a\delta$ and $a\chi$ have no significant effects on the contact pressure. This shows that $a\delta$ and $a\chi$ of thermoelectric materials have limited effects on improving the elastic contact behavior of these materials.

As pointed out before, if without thermoelectric loads, i.e., $J_{e0} = 0$ A/m and $J_{u0} = 0$ W/m, the right-hand side function $f(x)$ of the singular integral equation (32) is expected to vanish. This means that the distribution of contact pressure is only related to the shear modulus gradient parameter $a\alpha$. When $a\alpha = 0$, the closed-form solution of contact pressure can be obtained as

$$p(x) = P/(\pi\sqrt{a^2 - x^2}), \quad |x| < a.$$

Figure 5 shows the effects of $a\alpha$ on $\sigma_{yy}(x, 0)/\sigma_0$ verified with the exact solution. It can be found that the degenerated results of $a\alpha = 0.001$ evaluated by the program agree with those of $a\alpha = 0$ given by the closed-form solutions, which again validates the present derivations. The effects of $a\alpha$ on $\sigma_{yy}(x, 0)/\sigma_0$ are shown in Fig. 6, which depicts that as $a\alpha$ increases, $\sigma_{yy}(x, 0)/\sigma_0$ in the middle part of the contact area significantly increases, while it decreases around the contact edges. Figure 6 also shows that when $a\alpha$ is small, the overall distribution of contact pressure is relatively uniform.

Fig. 4 $\sigma_{yy}(x, 0)/\sigma_0$ beneath the punch with various $a\delta$ and $a\chi$ (color online) **Fig. 5** $\sigma_{yy}(x, 0)/\sigma_0$ beneath the punch with various $a\alpha$ (color online)

Figure 7 exhibits how $a\mu$ affects $\sigma_{yy}(x, 0)/\sigma_0$. It is clear that $\sigma_{yy}(x, 0)/\sigma_0$ in the middle of the contact area slightly decreases as $a\mu$ increases. Figure 8 shows the effects of the total energy flux J_{u0} on $\sigma_{yy}(x, 0)/\sigma_0$. From Fig. 8, one can see that the increment of J_{u0} leads to the increase of $\sigma_{yy}(x, 0)/\sigma_0$ in the middle of the contact area. Figure 9 depicts the effects of the total electric current J_{e0} on $\sigma_{yy}(x, 0)/\sigma_0$, and shows that as J_{e0} increases, the contact pressure in the middle of the contact area may be less compressive.

Fig. 6 $\sigma_{yy}(x,0)/\sigma_0$ beneath the punch with various $a\alpha$ (color online)

Fig. 7 $\sigma_{yy}(x,0)/\sigma_0$ beneath the punch with various $a\mu$ (color online)

Fig. 8 $\sigma_{yy}(x,0)/\sigma_0$ beneath the punch with various J_{u0} (color online)

Fig. 9 $\sigma_{yy}(x,0)/\sigma_0$ beneath the punch with various J_{e0} (color online)

6 Conclusions

This paper proposes an analytical model of the FGTEMs indented by a rigid flat punch. The exponential variation law is used for the properties of the electric conductivity, the thermal conductivity, the shear modulus, and the thermal expansion coefficient. The convergent analysis of kernel functions and collocation methods under different gradient parameters is presented. Numerical tests are obtained to analyze the effects of these gradient parameters and the thermoelectric loads on the near-edge behavior of the FGTEMs. The following conclusions can be drawn.

- (i) The electric conductivity gradient parameter and the thermal conductivity gradient parameter have significant effects on the distributions of the normal electric current density and the normal energy flux, while they have a slight effect on the distribution of the contact pressure.
- (ii) Both a larger shear modulus gradient parameter and an enlarged total energy flux could lead to a higher contact pressure in the middle of the contact region.
- (iii) As the thermal expansion coefficient gradient parameter and the total electric current increase, the contact pressure at the contact edge slightly increases.
- (iv) The quantitative analysis in this paper should be helpful for the design of new FGTEMs and the further understanding of the contact behavior of FGTEMs.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International

License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

- [1] BELL, L. E. Cooling, heating, generating power, and recovering waste heat with thermoelectric systems. *Science*, **321**(5895), 1457–1461 (2008)
- [2] POUDEL, B., HAO, Q., MA, Y., LAN, Y., MINNICH, A., YU, B., YAN, X., WANG, D., MUTO, A., VASHAEE, D., CHEN, X., LIU, J., DRESSELHAUS, M. S., CHEN, G., and REN, Z. High-thermoelectric performance of nanostructured bismuth antimony telluride bulk alloys. *Science*, **320**(5876), 634–638 (2008)
- [3] LEONOV, V., TORFS, T., FIORINI, P., and VAN HOOFF, C. Thermoelectric converters of human warmth for self-powered wireless sensor nodes. *IEEE Sensors Journal*, **7**(5), 650–656 (2007)
- [4] LALONDE, A. D., PEI, Y., WANG, H., and SNYDER, J. Lead telluride alloy thermoelectrics. *Materials Today*, **14**(11), 526–532 (2011)
- [5] MEJRI, M., THIMONT, Y., MALARD, B., and ESTOURNÈS, C. Characterization of the thermo-mechanical properties of p-type ($\text{MnSi}_{1.77}$) and n-type ($\text{Mg}_2\text{Si}_{0.6}\text{Sn}_{0.4}$) thermoelectric materials. *Scripta Materialia*, **172**, 28–32 (2019)
- [6] IOFFE, A. F. *Byulleten' Izobretenii (Invention Review)*, USSR 126158, USSR (1960)
- [7] KOIZUMI, M. FGM activities in Japan. *Composites Part B: Engineering*, **28**, 1–4 (1997)
- [8] KUZNETSOV, V. L., KUZNETSOVA, L. A., KALIAZIN, A. E., and ROWE, D. M. High performance functionally graded and segmented Bi_2Te_3 -based materials for thermoelectric power generation. *Journal of Materials Science*, **37**(14), 2893–2897 (2002)
- [9] MAHAN, G. D. Inhomogeneous thermoelectrics. *Journal of Applied Physics*, **70**(8), 4551–4554 (1991)
- [10] MÜLLER, E., DRAŠAR, Č., SCHILZ, J., and KAYSSER, W. A. Functionally graded materials for sensor and energy applications. *Materials Science and Engineering: A*, **362**, 17–39 (2003)
- [11] DASHEVSKY, Z., SHUSTERMAN, S., DARIEL, M. P., and DRABKIN, I. Thermoelectric efficiency in graded indium-doped PbTe crystals. *Journal of Applied Physics*, **92**(3), 1425–1430 (2002)
- [12] ANATYCHUK, L. I., VIKHOR, L. N., STRUTYNSKA, L. T., and TERMENA, I. S. Segmented generator modules using Bi_2Te_3 -based materials. *Journal of Electronic Materials*, **40**(5), 957–961 (2011)
- [13] JU, C., DUI, G., UHL, C. G., CHU, L., WANG, X., and LIU, Y. Performance analysis of a functionally graded thermoelectric element with temperature-dependent material properties. *Journal of Electronic Materials*, **48**(9), 5542–5554 (2019)
- [14] HE, R., SCHIERNING, G., and NIELSCH, K. Thermoelectric devices: a review of devices, architectures, and contact optimization. *Advanced Materials Technologies*, **3**(4), 1700256 (2018)
- [15] HATZIKRANIOU, E., ZORBAS, K. T., SAMARAS, I., KYRATSI, T., and PARASKEVOPOULOS, K. M. Efficiency study of a commercial thermoelectric power generator (TEG) under thermal cycling. *Journal of Electronic Materials*, **39**(9), 2112–2116 (2010)
- [16] BARAKO, M. T., PARK, W., MARCONNET, A. M., ASHEGHI, M., and GOODSON, K. E. Thermal cycling, mechanical degradation, and the effective figure of merit of a thermoelectric module. *Journal of Electronic Materials*, **42**(3), 372–381 (2013)
- [17] HU, X. K., ZHANG, S. M., ZHAO, F., LIU, Y., and LIU, W. S. Thermoelectric device: contact interface and interface materials (in Chinese). *Journal of Inorganic Materials*, **34**(3), 269–278 (2019)
- [18] ZHANG, Q. H., BAI, S. Q., and CHEN, L. D. Technologies and applications of thermoelectric devices: current status, challenges and prospects (in Chinese). *Journal of Inorganic Materials*, **34**(3), 279–293 (2019)

- [19] SU, J., KE, L. L., and WANG, Y. S. Two-dimensional fretting contact analysis of piezoelectric materials. *International Journal of Solids and Structures*, **73-74**, 41–54 (2015)
- [20] SU, J., KE, L. L., and WANG, Y. S. Fretting contact of a functionally graded piezoelectric layered half-plane under a conducting punch. *Smart Materials and Structures*, **25**(2), 25014 (2016)
- [21] SU, J., KE, L. L., WANG, Y. S., and XIANG, Y. Axisymmetric torsional fretting contact between a spherical punch and an FGPM coating. *Applied Mathematical Modelling*, **52**, 576–589 (2017)
- [22] VOLKOV, S. S., VASILIEV, A. S., AIZIKOVICH, S. M., and MITRIN, B. I. Axisymmetric indentation of an electroelastic piezoelectric half-space with functionally graded piezoelectric coating by a circular punch. *Acta Mechanica*, **230**(4), 1289–1302 (2019)
- [23] VASILIEV, A. S. Penetration of a spherical conductive punch into a piezoelectric half-space with a functionally graded coating. *International Journal of Engineering Science*, **142**, 230–241 (2019)
- [24] ZHANG, X., LIU, J., SHEN, H., and WANG, Y. Elastic response of coating materials in thermoelasticity: indented by a hot punch. *Journal of Thermal Stresses*, **42**(4), 475–489 (2019)
- [25] LIU, J., KE, L. L., WANG, Y. S., YANG, J., and ALAM, F. Thermoelastic frictional contact of functionally graded materials with arbitrarily varying properties. *International Journal of Mechanical Sciences*, **63**(1), 86–98 (2012)
- [26] ZHOU, Y. T. and LEE, K. Y. Thermo-electro-mechanical contact behavior of a finite piezoelectric layer under a sliding punch with frictional heat generation. *Journal of the Mechanics and Physics of Solids*, **59**(5), 1037–1061 (2011)
- [27] ZHOU, Y. T., ZHANG, C., ZHONG, Z., and WANG, L. Transient thermo-electro-elastic contact analysis of a sliding punch acting on a functionally graded piezoelectric strip under non-Fourier heat conduction. *European Journal of Mechanics A/Solids*, **73**, 90–100 (2019)
- [28] ZHOU, Y. T., TIAN, X. J., and LI, F. J. On coupling contact analysis of thermoelectric materials. *Applied Mathematical Modelling*, **89**, 1459–1474 (2021)
- [29] TIAN, X. J., ZHOU, Y. T., GUAN, X. F., WANG, L. H., and DING, S. H. The frictional contact problem of a rigid punch sliding over thermoelectric materials. *International Journal of Solids and Structures*, **200-201**(C), 145–157 (2020)
- [30] ZHOU, Y. T. and LEE, K. Y. Thermal response of a partially insulated interface crack in a graded coating-substrate structure under thermo-mechanical disturbance: energy release and density. *Theoretical and Applied Fracture Mechanics*, **56**(1), 22–33 (2011)
- [31] ZHOU, Y. T. and LEE, K. Y. Thermo-electro-mechanical contact behavior of a finite piezoelectric layer under a sliding punch with frictional heat generation. *Journal of the Mechanics and Physics of Solids*, **59**(5), 1037–1061 (2011)
- [32] ZHOU, Y. T., LEE, K. Y., JANG, Y. H., and LI, J. The multiple fields created by two perfectly-conducting punches on piezoelectric/piezomagnetic materials with anisotropy. *Journal of Applied Mathematics and Mechanics*, **97**(8), 946–960 (2017)
- [33] WANG, B. L. and CUI, Y. J. Transient interlaminar thermal stress in multi-layered thermoelectric materials. *Applied Thermal Engineering*, **119**, 207–214 (2017)
- [34] GAO, J. L., DU, Q. G., ZHANG, X. D., and JIANG, X. Q. Thermal stress analysis and structure parameter selection for a Bi₂Te₃-based thermoelectric module. *Journal of Electronic Materials*, **40**(5), 884–888 (2011)