Appl. Math. Mech. -Engl. Ed., **42**(5), 641–648 (2021)

APPLIED MATHEMATICS AND MECHANICS (ENGLISH EDITION)

https://doi.org/10.1007/s10483-021-2731-5

Complex response analysis of a non-smooth oscillator under harmonic and random excitations*

Shichao MA[1,2], Xin NING[1,2], Liang WANG[3,†],
Wantao JIA[3], Wei XU[3]

1. School of Astronautics, Northwestern Polytechnical University, Xi'an 710072, China;
2. National Key Laboratory of Aerospace Flight Dynamics, Xi'an 710072, China;
3. School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an 710129, China

(Received Nov. 11, 2020 / Revised Mar. 1, 2021)

Abstract It is well-known that practical vibro-impact systems are often influenced by random perturbations and external excitation forces, making it challenging to carry out the research of this category of complex systems with non-smooth characteristics. To address this problem, by adequately utilizing the stochastic response analysis approach and performing the stochastic response for the considered non-smooth system with the external excitation force and white noise excitation, a modified conducting process has proposed. Taking the multiple nonlinear parameters, the non-smooth parameters, and the external excitation frequency into consideration, the steady-state stochastic P-bifurcation phenomena of an elastic impact oscillator are discussed. It can be found that the system parameters can make the system stability topology change. The effectiveness of the proposed method is verified and demonstrated by the Monte Carlo (MC) simulation. Consequently, the conclusions show that the process can be applied to stochastic non-autonomous and non-smooth systems.

Key words non-autonomous system, non-smooth system, random excitation

Chinese Library Classification O322
2010 Mathematics Subject Classification 34A34, 34F05, 60J22

1 Introduction

Non-smooth systems are a major area of interest within the field of non-linear dynamics, which can be applied to the modeling of many nonlinear dynamic phenomena in the fields of aerospace, structural engineering, and mechanical engineering[1–5]. The impact and friction described by these systems are often encountered in the components of the systems or between

∗ Citation: MA, S. C., NING, X., WANG, L., JIA, W. T., and XU, W. Complex response analysis of a non-smooth oscillator under harmonic and random excitations. *Applied Mathematics and Mechanics* (*English Edition*), **42**(5), 641–648 (2021) https://doi.org/10.1007/s10483-021-2731-5

† Corresponding author, E-mail: liangwang1129@nwpu.edu.cn

Project supported by the National Natural Science Foundation of China (Nos. 11872306, 11772256, and 11972289) and the Innovation Foundation for Doctor Dissertation of Northwestern Polytechnical University (No. CX202003)

the component and the constraint surface[6]. At the same time, random perturbations are unavoidable in non-smooth systems. Therefore, it is of significant importance to make it clear whether mechanical systems can work normally under the influence of these random factors[7]. Stochastic P-bifurcations of non-smooth systems refer to the sudden change in the topology of the non-smooth systems with variable parameters[8–10]. Using the limiting averaging principle, Xia et al.[11] studied the abrupt phenomenon of stationary probability density functions (PDFs) of a three-degree-of-freedom nonlinear stochastic system. Wang et al.[12] studied the stochastic P-bifurcation of a smooth and discontinuous smooth and discontinuous (SD) oscillator via the numerical method. Obviously, carrying out the research of the stochastic responses of such systems under random perturbation excitations is urgent and necessary.

With the development of nonlinear disciplines and the upgrading of computers, there are many effective methods to study the responses of such systems, including analytical methods and numerical algorithms[13–16]. Since the non-smooth properties of stochastic non-smooth systems are difficult to be handled, more and more scholars are currently working in the stochastic responses of such non-smooth systems[17–20]. However, due to the existence of non-smooth factors, an approximate transformation is used for many methods to convert the original non-smooth system to a smooth system, and then the system responses are obtained. The drawback is obvious, which may cause the non-smooth characteristics that are unique to this type of system to be destroyed. Unless performing the approximate transformation, there are seldom rigorous analytical methods used to solve non-smooth systems directly. For strong nonlinear systems, the analytical method is difficult to solve and the derivation process is complicated. Relatively speaking, numerical methods possess better calculating performances. The Monte Carlo (MC) simulation method is the most common numerical method that can guarantee the correctness of the results. Unfortunately, this method puts forward higher requirements to the amount of samples and computation time to get the exact responses. Many thanks to Hsu, a generalized cell mapping (GCM) method has been proposed, and can be utilized to solve these problems[21–22]. A large number of references have verified the rationality of the GCM algorithm[23–25]. As a further development, Wang et al.[26] proposed an improved cell mapping algorithm for solving original non-smooth impact systems. However, we find that the algorithm proposed in Ref. [26] is limited to autonomous systems, and cannot solve the external excitation problem of the considered non-autonomous systems. Motivated by the aforementioned observations, the main aim of this paper is to investigate a more optimizing process which is suitable for general non-autonomous impact systems. The main innovations can be summarized as follows. Based on Ref. [22], in order to retain the non-smooth characteristics of the system, no non-smooth transformation is imposed on the vibro-impact system. A special and fast cell mapping method is used to calculate the PDFs of non-autonomous vibro-impact system for the first time. With the MC simulation, the accuracy of the method is illustrated by random steady state bifurcation analysis.

This paper is arranged as follows. Section 2 gives a stochastic elastic impact with the external excitation force as a research object and the corresponding stochastic GCM (SGCM) method. Section 3 analyzes the stochastic P-bifurcation phenomena by variable parameters, including the nonlinear parameters, the non-smooth parameters, and the external excitation frequency, using the optimization cell mapping method. Conclusions are given in Section 4.

2 Model and corresponding method

We study a stochastic non-smooth system with external excitations as shown in Fig. 1. A mass block can be reciprocated by the external excitation and random force. Once the displacement reaches a certain distance, the mass block will collide with the constraint surface. For practical purposes, it is assumed that the mass block and constraint surface are both regarded as elastomers[6]. Thus, the elastic impact needs to satisfy the Hertzian contact law[27–28]. It is a non-autonomous periodic system and can be described by

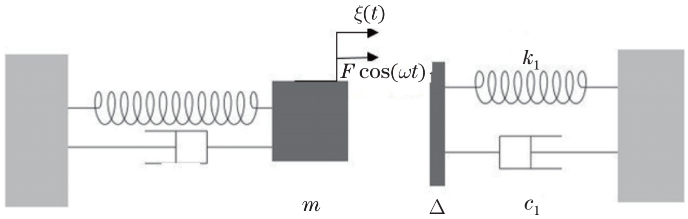

Fig. 1 Schematic of the model

$$\ddot{x} + a_0\dot{x} + b_0x + c_0x^3 + gx^2\dot{x} + f_e(x,\dot{x}) = F\cos(\omega t) + \xi(t), \tag{1}$$

$$f_e(x,\dot{x}) = \begin{cases} k_1(x-q) + c_1\dot{x}, & x \geqslant q, \\ 0, & x < q. \end{cases} \tag{2}$$

Equation (2) means the elastic impact process with the damping constant c_1 and the spring stiffness k_1. When $x \geqslant q$, the elastic impact occurs. The parameters a_0, b_0, and c_0 are fixed, and then this system is perturbed by the Gaussian white noise $\xi(t)$ which satisfies

$$E(\xi(t)) = 0, \quad E(\xi(t)\xi(t+\tau)) = 2\sigma\delta(\tau). \tag{3}$$

For the stochastic elastic impact system with the external excitation, we examine its dynamic behavior by studying its stochastic responses. Now, the corresponding SGCM method is described to gain the stochastic responses. Under the external excitation of period force, the non-autonomous system can be expressed as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(x,t,\xi(t)) = \boldsymbol{f}(x,t+T,\xi(t)). \tag{4}$$

For this system, suppose that time is the explicit state variable, and rewrite the system as

$$\dot{x} = y, \quad \dot{y} = F\cos(\omega\cdot\theta) - a_0y - b_0x - c_0x^3 - gx^2y - f_e(x,y) + \xi(t), \quad \dot{\theta} = 1, \tag{5}$$

$$f_e(x,\dot{x}) = \begin{cases} k_1(x-q) + c_1\dot{x}, & x \geqslant q, \\ 0, & x < q. \end{cases} \tag{6}$$

For the systems (5)–(6), we first need to subdivide the state space into smaller intervals. Then, each interval produces a number of random sample trajectories. The one-step transition probability matrix can be derived by calculating the sample probability numbers falling into each interval. Thus, on the basis of the Chapman-Kolmogorov (C-K) equation

$$p(\boldsymbol{x}, nT) = \int p(\boldsymbol{x}, T|\boldsymbol{x}_0, 0)p(\boldsymbol{x}_0, (n-1)T)\mathrm{d}\boldsymbol{x}_0, \tag{7}$$

we choose T as the length of one-step iteration to satisfy one complete impact process. Therefore, the probability density distribution after iterations can be calculated easily by

$$p(n) = P\cdot p(n-1) = P\cdot p(n-2) = \cdots = P^n\cdot p(0), \quad n\in\mathbb{N}. \tag{8}$$

3 Stochastic P-bifurcation analysis

The stochastic P-bifurcation studies the shape of the invariant measure (steady-state response probability density) of the stochastic dynamic system with the change of the parameters, including the significant changes in the number and position of peaks. The PDF of the system can be obtained by using the above process which can study the steady-state stochastic P-bifurcation phenomenon of the non-smooth system under the variable parameters. At the same time, the MC simulation is used to illustrate the validity of the process.

For the systems (5)–(6), the parameters are fixed at $a_0 = -0.1$, $b_0 = -0.5$, $c_0 = 0.5$, $k_1 = 0.8$, $q = 0$, and $F = 0.23$, where the noise intensity is 0.02. The domain $D = (-3 \leqslant x \leqslant 3, -3 \leqslant y \leqslant 3)$ is divided into 2 500 intervals, and each interval produces 200 random sample trajectories. First, the effects of damping constants are considered. Choosing $g = 0.1$ and $\omega = 0.5$, Figs. 2–3 provide the steady-state marginal PDFs with $c_1 = 0.02, 0.06, 0.10, 0.16$, respectively. As shown in Fig. 2, the PDFs of x present a relatively uniform distribution with $c_1 = 0.02$. But as c_1 increases, some parts of the PDFs start to bulge. It indicates that the impact probability of the elastic surface would be increased with the increasing stiffness coefficient of the impact surface. Different from the PDFs of x, the topological structure of marginal PDFs of velocity y is bimodal-shaped when $c_1 = 0.02$ in Fig. 3. But the shape tends to have one peak as the stiffness coefficient increases to $c_1 = 0.16$. Figure 4 demonstrates the joint PDFs with changing damping constants. Among them, the bright areas represent the locations where the

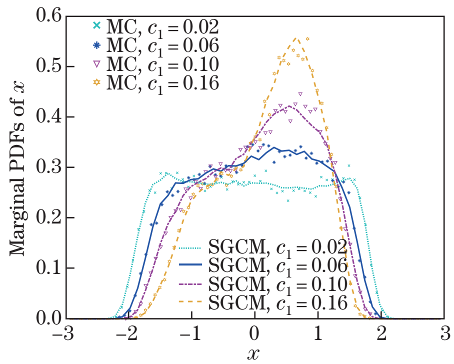

Fig. 2 Marginal PDFs of x for different values of non-smooth damping constant c_1 (color online)

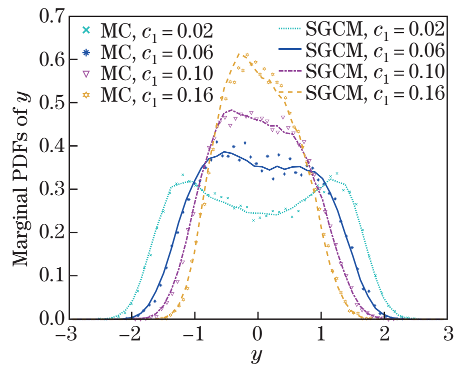

Fig. 3 Marginal PDFs of y for different values of non-smooth damping constant c_1 (color online)

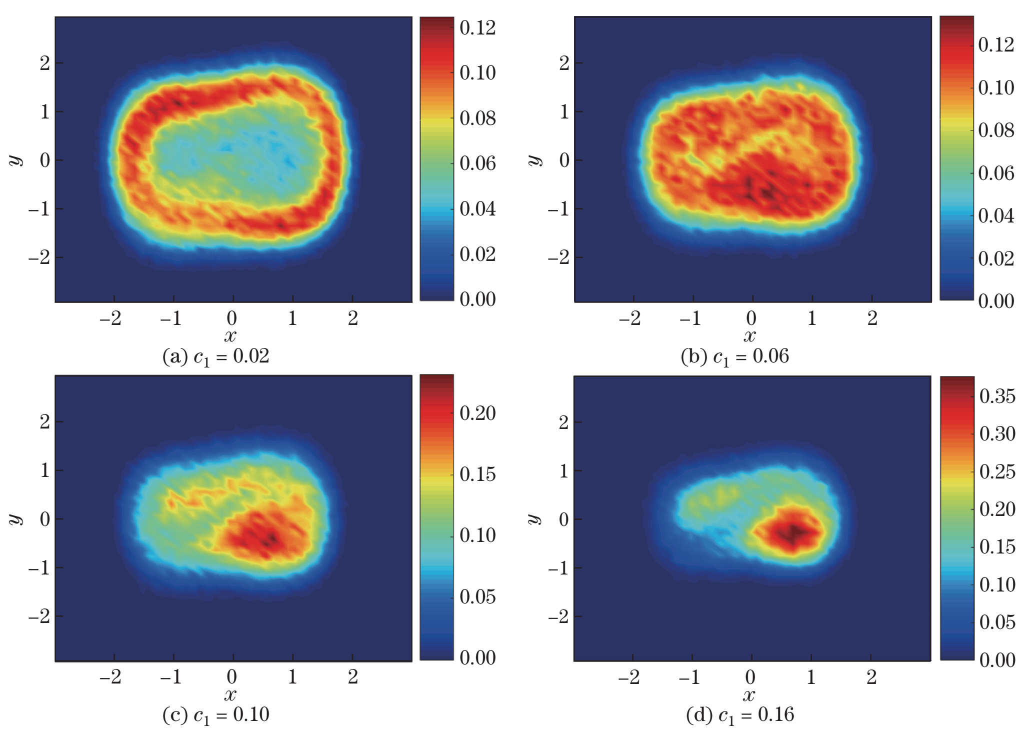

Fig. 4 Contour plots of joint PDFs for different values of damping constant c_1 (color online)

PDF is larger. When $c_1 = 0.02$, the bright areas of joint PDFs exhibit a circle. It is apparent that the bright areas of joint PDFs become a pie distribution with $c_1 = 0.06$. As c_1 continues to increase, the bright areas begin to shrink, and the color becomes darker. Certainly, numerical results are identical with the MC simulation, which verifies the feasibility of the process.

Then, how the adjustable parameter g on a small scale induces the generation of stochastic P-bifurcation comes into consideration under fixed $c_1 = 0.04$ and $\omega = 0.5$. Figure 5 shows the steady-state marginal PDFs of x with $g = 0.07, 0.10, 0.15, 0.20$. When $g = 0.07$, the topological figure of the marginal PDF is bimodal. The distribution of PDFs is relatively uniform with $g = 0.10$. As g continues to increase, the shape of PDFs is a bimodal structure again, but both peaks are getting taller and leaner, which indicates a more concentrated PDFs distribution. In Fig. 6, the marginal PDFs of y have obviously transformed into a single peak with the increase in g. Similarly, the MC simulation results are in good agreement with the method. Figure 7

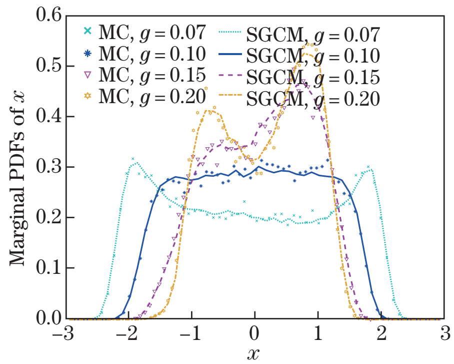

Fig. 5 Marginal PDFs of x for different values of non-linear parameter g (color online)

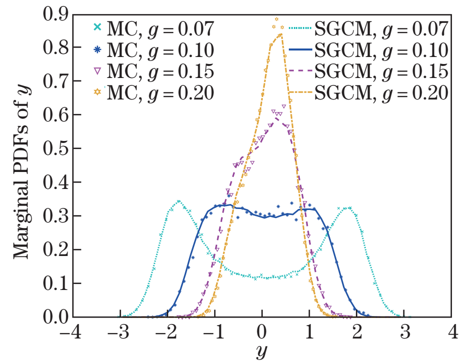

Fig. 6 Marginal PDFs of y for different values of non-linear parameter g (color online)

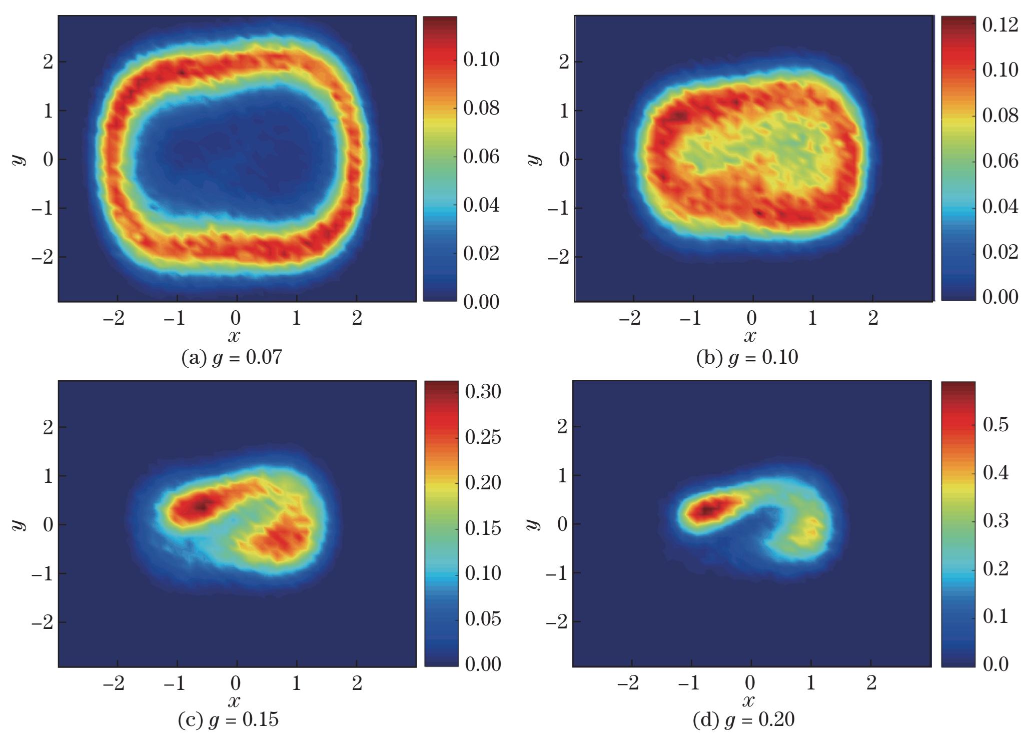

Fig. 7 Contour plots of joint PDFs for different values of non-linear parameter g (color online)

shows the results of the correlational analysis of joint PDFs with $g = 0.07, 0.10, 0.15, 0.20$. Closer inspection of this figure shows that the bright areas of joint PDFs also change from ring to pie and finally to sheet as g increases. The results in this section indicate that the nonlinear parameter g could induce the instability of this non-smooth system on a small scale, and it can also be said that the stochastic P-bifurcation occurs at this time.

Finally, consider the impact of the external excitation frequency ω. Select the parameters $g = 0.1, c_1 = 0.04$, and the range of stochastic P-bifurcation parameter $\omega = 0.25, 0.75, 1.0$. As shown in Figs. 8–10, the marginal and joint PDFs exhibit the change of the topology. In Figs. 8–9, the marginal PDFs of x and y are both evolved from double peaks to a single peak. It is more like that one of the peaks is falling, and the other peak is rising. For joint PDFs, the bright areas of PDFs are ring-shaped with $\omega = 0.25$, while when $\omega = 0.75$, the bright areas of PDF change into a small area. With the continuous increase in ω, there is a clear trend of the

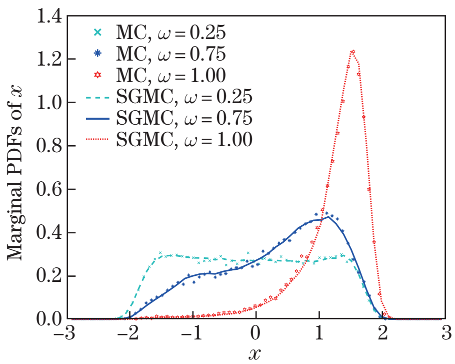

Fig. 8 Marginal PDFs of x for different values of non-linear parameter ω (color online)

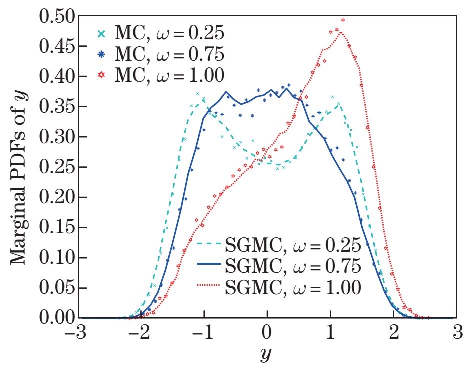

Fig. 9 Marginal PDFs of y for different values of non-linear parameter ω (color online)

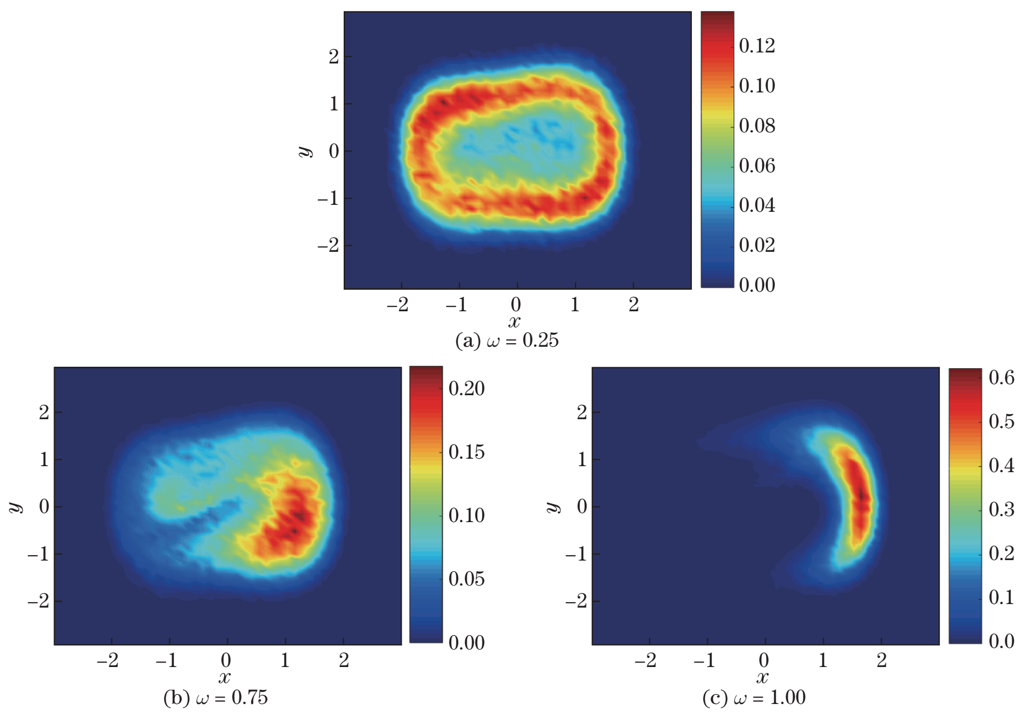

Fig. 10 Contour plots of joint PDFs for different values of external excitation frequency ω (color online)

overall PDF becoming a half-moon structure in Fig. 10. These results indicate that not only the nonlinear and non-smooth parameters of the system but also the external excitation frequency will affect the stability characteristics of the system, the latter of which also make qualitative changes to the system.

4 Conclusions

The improved cell mapping method is used to derive the PDF responses of non-autonomous and non-smooth systems under the external excitation force and noise perturbation. At the same time, it is proven that the method is well coincident with the MC simulation method. Overall, this study strengthens the idea that there is no need to change the one-step transition probability matrix once it is obtained, and thus, we just need to calculate the matrix iteration multiplication to obtain the PDFs. Then, the response calculation time will be greatly reduced compared with the MC simulation method obviously, especially for steady-state responses. In addition, the stochastic P-bifurcation phenomena are also discussed in this paper. By the proposed process, the results reveal that the nonlinear and non-smooth parameters of the system can conduce the stochastic P-bifurcation phenomena, and the change of the external excitation frequency can also cause the topology of PDFs to change. Without any approximate transformation of the original system, this approach will prove useful in expanding the studies for the stochastic responses of non-smooth systems in practical engineering.

References

[1] DI BERNARDO, M., BUDD, C. J., CHAMPNEYS, A. R., KOWALACZYK, P., NORDMARK, A. B., OLIVAR, T. G., and PIIROINEN, P. T. A. Bifurcations in nonsmooth dynamical systems. *SIAM Reviews*, **50**, 629–701 (2008)

[2] WEN, G. L., YIN, S., XU, H. D., ZHANG, S. J., and LV, Z. Y. Analysis of grazing bifurcation from periodic motion to quasi-periodic motion in impact-damper systems. *Chaos, Solitons and Fractals*, **83**, 112–118 (2016)

[3] YIN, S., WEN, G. L., and XU, X. Suppression of grazing-induced instability in single degree-of-freedom impact oscillators. *Applied Mathematics and Mechanics (English Edition)*, **40**(1), 97–110 (2019) https://doi.org/10.1007/s10483-019-2403-6

[4] MONFARED, Z., AFSHARNEZHAD, Z., and ESFAHANI, J. A. Flutter, limit cycle oscillation, bifurcation and stability regions of an airfoil with discontinuous freeplay nonlinearity. *Nonlinear Dynamics*, **90**, 1965–1986 (2017)

[5] ERDODI, I. and HOS, C. Prediction of quarter-wave instability in direct spring operated pressure relief valves with upstream piping by means of CFD and reduced order modelling. *Journal of Fluids and Structures*, **73**, 37–52 (2014)

[6] JIN, D. P. and HU, H. Y. *Vibro-Impact and Control*, Science and Technology Press, Beijing (2005)

[7] ZHU, W. Q. and CAI, G. Q. *Introduction to Stochastic Dynamics*, Science and Technology Press, Beijing (2017)

[8] ARNOLD, L. *Random Dynamical Systems*, Springer, New York (1995)

[9] BASHKIRTSEVA, I., RYAZANOVA, T., and RYASHKO, L. Stochastic bifurcations caused by multiplicative noise in systems with hard excitement of auto-oscillations. *Physical Review E*, **94**(4), 042908 (2015)

[10] YANG, J. H., CAI, X. M., and LIU, X. B. The maximal Lyapunov exponent for a three-dimensional system driven by white noise. *Communications in Nonlinear Science and Numerical Simulations*, **15**(11), 3498–506 (2010)

[11] XIA, L., HUAN, R. H., ZHU, W. Q., and ZHU, C. X. Reshaping of the probability density function of nonlinear stochastic systems against abrupt changes. *Journal of Vibration and Control*, **26**(7), 532–539 (2020)

[12] WANG, L., XUE, L. L., XU, W., and YUE, X. L. Stochastic P-bifurcation analysis of a fractional smooth and discontinuous oscillator via the generalized cell mapping method. *International Journal of Non-Linear Mechanics*, **96**, 56–63 (2017)

[13] CHONG, A. S. E., YUE, Y., PAVLOVSKAIA, E., and WIERCIGROCH, M. Global dynamics of a harmonically excited oscillator with a play: numerical studies. *Internatioanl Journal of Non-Linear Mechanics*, **94**, 98–109 (2017)

[14] LI, G. F. and DING, W. D. Global behavior of a vibro-impact system with asymmetric clearances. *Journal of Sound and Vibration*, **423**, 180–194 (2018)

[15] LI, G. F., SUN, J., and DING, W. C. Dynamics of a vibro-impact system by the global analysis method in parameter-state space. *Nonlinear Dynanics*, **97**, 541–557 (2019)

[16] YIN, S., JI, J. C., and WEN, G. L. Complex near-grazing dynamics in impact oscillators. *International Journal of Mechanics Sciences*, **156**, 106–122 (2019)

[17] ZHURAVLEV, V. F. A method for analyzing vibration-impact systems by means of special functions. *Mechanics of Solids*, **11**, 23–27 (1976)

[18] RONG, H. W., WANG, X. D., XU, W., and FANG, T. Subharmonic response of a single-degree-of-freedom nonlinear vibro-impact system to a narrow-band random excitation. *Physical Review E*, **80**, 026604 (2009)

[19] PAOLA, M. D. and BUCHER, C. Ideal and physical barrier problems for non-linear systems driven by normal and Poissonian white noise via path integral method. *International Journal of Non-Linear Mechanics*, **81**, 274–282 (2016)

[20] CHEN, L. C., ZHU, H. S., and SUN, J. Q. Novel method for random vibration analysis of single-degree-of-freedom vibroimpact systems with bilateral barriers. *Applied Mathematics and Mechanics* (*English Edition*), **40**(12), 1759–1776 (2019) https://doi.org/10.1007/s10483-019-2543-5

[21] HSU, C. S. A theory of cell-to-cell mapping dynamical systems. *Journal of Applied Mechanics — Transactions of the ASME*, **47**(4), 931–939 (1980)

[22] HSU, C. S. *Cell-to-Cell Mapping: a Method of Global Analysis for Nonlinear System*, Springer, New York (1987)

[23] HONG, L. and XU, J. X. Crises and chaotic transients studied by the generalized cell mapping digraph method. *Physics Letters A*, **262**, 361–375 (1999)

[24] SUN, J. Q. and HSU, C. S. A statistical study of generalized cell mapping. *Journal of Applied Mechanics — Transactions of the ASME*, **55**(3), 694-701 (1988)

[25] ZHANG, H. G., ZHANG, Y. X., and LUO, G. W. Bsains of coexisting multi-dimensional tori in a vibro-impact system. *Nonlinear Dynanics*, **79**, 2177–2185 (2015)

[26] WANG, L., MA, S. C., SUN, C. Y., JIA, W. T., and XU, W. Stochastic response of a class of impact systems calculated by a new strategy based on generalized cell mapping method. *Journal of Applied Mechanics — Transactions of the ASME*, **85**, 054502 (2018)

[27] HESS, D. P., SOOM, A., and KIM, C. H. Normal vibrations and friction at a Hertzian contact under random excitation: theory and experiments. *Journal of Sound and Vibration*, **153**(3), 491–508 (1992)

[28] HESS, D. P. and SOOM, A. Normal vibrations and friction at a Hertzian contact under random excitation: perturbation solution. *Journal of Sound and Vibration*, **164**(2), 317–326 (1993)